Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 259 (2015) 344–370

Journal of Differential Equations

www.elsevier.com/locate/jde

Asymptotic behavior on a kind of parabolic Monge–Ampère equation

Bo Wang, Jiguang Bao *

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China

Received 30 April 2014; revised 21 January 2015

Available online 3 March 2015

Abstract

In this paper, we apply level set and nonlinear perturbation methods to obtain the asymptotic behavior of the solution to a kind of parabolic Monge–Ampère equation at infinity. The Jörgens–Calabi–Pogorelov theorem for parabolic and elliptic Monge–Ampère equation can be regarded as special cases of our result.

MSC: 35J60; 35J15

Keywords: Parabolic Monge–Ampère equation; Asymptotic behavior; Level set method; Nonlinear perturbation method

1. Introduction and main results

In affine geometry, a well known theorem of Jörgens ($n = 2$ [11]), Calabi ($n \leq 5$ [4]) and Pogorelov ($n \geq 2$ [14]) asserts that a convex improper affine hypersurface is an elliptic paraboloid. This theorem can also be stated as follows: any classical convex solution of the elliptic Monge–Ampère equation

$$\det(D^2u) = 1 \quad \text{in } \mathbb{R}^n$$

must be a quadratic polynomial.

* Corresponding author.
E-mail addresses: wangbo89630@mail.bnu.edu.cn (B. Wang), jgbao@bnu.edu.cn (J. Bao).

http://dx.doi.org/10.1016/j.jde.2015.02.029

Along the lines of affine geometry, a simpler and more analytic proof was given by Cheng and Yau in [6]. Jost and Xin also give another proof of this result in [12]. Caffarelli proved that Jörgens–Calabi–Pogorelov theorem remains valid for viscosity solution in [2].

In [8], Gutiérrez and Huang established Jörgens–Calabi–Pogorelov theorem to the following kind of parabolic Monge–Ampère equation

$$-u_t \det(D^2u) = 1 \quad \text{in } \mathbb{R}^{n+1}_- := \mathbb{R}^n \times (-\infty, 0], \quad -M_1 \le u_t \le -M_2,$$

where M_1 and M_2 are two positive constants. In [17], Xiong and Bao extended Jörgens–Calabi–Pogorelov theorem to more general parabolic Monge–Ampère equations of the form

$$u_t = \rho\left(\log(\det(D^2u))\right) \quad \text{in } \mathbb{R}^{n+1}_-,$$

which covers the results in [8].

In [3], Caffarelli and Li obtained the asymptotic behavior of convex viscosity solutions of

$$\det(D^2u) = f(x) \quad \text{in } \mathbb{R}^n,$$

where $f \in C^0(\mathbb{R}^n)$ satisfies

$$0 \le \inf_{\mathbb{R}^n} f \le \sup_{\mathbb{R}^n} f < \infty,$$
$$\text{support } (f-1) \text{ is bounded.}$$

Recently, Zhang, Wang and Bao [18] have extended the above result to the following parabolic Monge–Ampère equation

$$-u_t \det(D^2u) = f(x,t) \quad \text{in } \mathbb{R}^{n+1}_-.$$

In this paper, we investigate classical solutions to the parabolic Monge–Ampère equation

$$u_t - \log\left(\det(D^2u)\right) = f(x,t) \quad \text{in } \mathbb{R}^{n+1}_-, \tag{1}$$

such that there exist two constants c_0 and C_0 with

$$c_0 \le u_t \le C_0 \quad \text{in } \mathbb{R}^{n+1}_-, \tag{2}$$

where $f \in C^0(\mathbb{R}^{n+1}_-)$ and there exists $a \in \mathbb{R}^1$ such that

$$\text{support } (f-a) \text{ is bounded.} \tag{3}$$

In the following theorem, we obtain the asymptotic behavior of solutions of (1) and (2) under the hypothesis (3).

Theorem 1.1. *Let $n \geq 4$ and $f \in C^0(\mathbb{R}^{n+1}_-)$ satisfy (3). Assume that $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ is a solution of (1) and (2) which is convex in x. Then there exist an $n \times n$ symmetric positive definite matrix A, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^1$ satisfying*

$$u(x,t) = \tau t + \frac{x^T Ax}{2} + b^T x + c, \quad x \in \mathbb{R}^n, t \leq t^* \tag{4}$$

$$u(x,t) = \tau t + \frac{x^T Ax}{2} + b^T x + c + O(|x|^{2-n}), \quad |x| \to \infty, t^* \leq t \leq 0, \tag{5}$$

where $\tau := \log\det A + a$ and $t^* := \sup\{t \leq 0 : f(x,s) = a, \forall x \in \mathbb{R}^n, s \leq t\} > -\infty$.

By taking $a = 0$ and $f \equiv 0$ in Theorem 1.1, we can obtain the following theorem.

Corollary 1.2. *Let $n \geq 4$ and $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ be a solution to*

$$u_t = \log\left(\det(D^2u)\right) \quad (x,t) \in \mathbb{R}^{n+1}_-, \tag{6}$$

such that there exist two constants c_0 and C_0 with

$$c_0 \leq u_t \leq C_0 \quad (x,t) \in \mathbb{R}^{n+1}_-, \tag{7}$$

under the assumption that u is convex in x. Then there exist an $n \times n$ symmetric positive definite matrix A, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^1$ such that

$$u(x,t) = \tau t + \frac{x^T Ax}{2} + b^T x + c, \quad (x,t) \in \mathbb{R}^{n+1}_-,$$

where $\tau := \log\det A$.

Remark 1.1. It should be remarked that Xiong and Bao have obtained Corollary 1.2 for $n \geq 2$ and $u \in C^{4,2}(\mathbb{R}^{n+1}_-)$ in [17].

Now we consider an elliptic Monge–Ampère equation

$$\det(D^2u) = \exp\{\frac{1}{2}Du \cdot x - u\} \quad \text{in } \mathbb{R}^n. \tag{8}$$

Since an entire solution to (8) is a self-shrinking solution to Lagrangian mean curvature flow in pseudo-Euclidean space, we can obtain an analogous result with the Jörgens–Calabi–Pogorelov theorem on (8) below.

Corollary 1.3. *Let $n \geq 4$ and $u \in C^2(\mathbb{R}^n)$ be a convex solution to (8) satisfying*

$$\limsup_{|x| \to +\infty} |\frac{1}{2}Du(x) \cdot x - u(x)| < \infty. \tag{9}$$

Then there exist an $n \times n$ symmetric positive definite matrix A and $c \in \mathbb{R}^1$ such that

$$u(x) = \frac{x^T Ax}{2} + c, \quad x \in \mathbb{R}^n.$$

Remark 1.2. By replacing the condition (9) with

$$\liminf_{|x|\to+\infty} |x|^2 D^2 u(x) > 2(n-1)I,$$

Huang and Wang obtained Corollary 1.3 for $n \geq 2$ in [10]. Independently, Chau, Chen and Yuan obtained the same result by using a different method in [5].

Proof of Corollary 1.3. First of all, we define

$$v(x,t) = (1-t)u(\frac{x}{\sqrt{1-t}}) \quad \text{in } \mathbb{R}^{n+1}_-.$$

Then we have that v is a solution to

$$v_t = \log\left(\det(D^2 v)\right) \quad \text{in } \mathbb{R}^{n+1}_-.$$

Due to (9), we can also deduce that $|v_t|$ is bounded in $\mathbb{R}^{n+1}_-$. By Corollary 1.2, there exist an $n \times n$ symmetric positive definite matrix A, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^1$ and $\tau \in \mathbb{R}^1$ satisfying

$$\tau = \log \det A,$$

such that

$$v(x,t) = \tau t + \frac{x^T Ax}{2} + b^T x + c \quad \text{in } \mathbb{R}^{n+1}_-.$$

Lastly, by taking $t = 0$, we have

$$u(x) = \frac{x^T Ax}{2} + b^T x + c \quad \text{in } \mathbb{R}^n.$$

Since u is the solution to (8), it is easy to check that $b = 0$. □

Our paper is organized as follows. In Section 2, we will state some preliminaries and simplifications. In Section 3, the proof of Theorem 1.1 is given by being divided into five steps.

2. Preliminaries and simplifications

We begin this section by introducing some notations. We denote by $D^2u(x,t)$ the matrix of second derivatives of u with respect to x and $Du(x,t)$ the gradient of u with respect to x.

Let $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ be a solution of (1) and (2) which is convex in x. Without loss of generality, we can assume that $C_0 < 0$. Indeed, by defining

$$\widetilde{u}(x,t) = u(x,t) - (1 + C_0)t \quad \text{in } \mathbb{R}^{n+1}_-,$$

we can see that $\widetilde{u} \in C^{2,1}(\mathbb{R}^{n+1}_-)$ is a solution to the equation

$$\widetilde{u}_t - \log\left(\det(D^2\widetilde{u})\right) = \widetilde{f}(x,t) \quad \text{in } \mathbb{R}^{n+1}_-, \tag{10}$$

such that there exist two positive constants $\widetilde{c_0} = c_0 - C_0 - 1$ and $\widetilde{C_0} = -1$ with

$$\widetilde{c_0} \le \widetilde{u}_t \le \widetilde{C_0} \quad \text{in } \mathbb{R}^{n+1}_-, \tag{11}$$

where $\widetilde{f} = f - (C_0 + 1)$, $\widetilde{a} = a - (C_0 + 1)$, $\widetilde{t^*} = t^*$ and

$$\text{support } (\widetilde{f} - \widetilde{a}) \text{ is bounded.}$$

Once we have proved that for such $\widetilde{u}$, there exist an $n \times n$ symmetric positive definite matrix A, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^1$ and $\widetilde{\tau} \in \mathbb{R}^1$ satisfying

$$\widetilde{\tau} - \log\det A = \widetilde{a}$$

such that

$$\widetilde{u}(x,t) = \widetilde{\tau}t + \frac{x^TAx}{2} + b^Tx + c, \quad x \in \mathbb{R}^n, t \le \widetilde{t^*},$$
$$\widetilde{u}(x,t) = \widetilde{\tau}t + \frac{x^TAx}{2} + b^Tx + c + O(|x|^{2-n}), \quad |x| \to \infty, \widetilde{t^*} \le t \le 0,$$

then by denoting $\tau = 1 + C_0 + \widetilde{\tau}$ and $t^* = 1 + C_0 + \widetilde{t^*}$, we have

$$u(x,t) = \tau t + \frac{x^TAx}{2} + b^Tx + c, \quad x \in \mathbb{R}^n, t \le t^*,$$
$$u(x,t) = \tau t + \frac{x^TAx}{2} + b^Tx + c + O(|x|^{2-n}), \quad |x| \to \infty, t^* \le t \le 0.$$

We say a function $u : \mathbb{R}^{n+1}_- \to \mathbb{R}$, $(x,t) \mapsto u(x,t)$, is called parabolically convex if it is continuous, convex in x and non-increasing in t. By this definition, it is easy to see that the assumption $C_0 < 0$ yields that u is actually parabolically convex.

Throughout the paper, we will always assume that

$$u(0,0)=0, \quad Du(0,0)=0, \tag{12}$$

and

$$u \geq 0 \quad \text{in } \mathbb{R}^{n+1}_-. \tag{13}$$

In fact, let $\hat{u}(x,t)=u(x,t)-u(0,0)-Du(0,0)\cdot x$. Then we have $\hat{u}(0,0)=0$, $D\hat{u}(0,0)=0$, and

$$\hat{u}_t-\log\left(\det(D^2\hat{u})\right)=f(x,t), \quad c_0 \leq \hat{u}_t \leq C_0 \quad \text{in} \quad \mathbb{R}^{n+1}_-,$$

which show that assumption (12) is reasonable. By (12) and the definition of parabolically convex function we can get that

$$\hat{u}(x,t) \geq \hat{u}(x,0) \geq \hat{u}(0,0)=0, \quad \forall (x,t) \in \mathbb{R}^{n+1}_-.$$

This completes the proof of assumption (13).

Let $D \subset \mathbb{R}^{n+1}_-$ be a bounded set and $t \leq 0$; then we denote

$$D(t) := \{x \in \mathbb{R}^n : (x,t) \in D\},$$

and $t_0=\inf\{t : D(t) \neq \emptyset\}$. The parabolic boundary of the bounded domain D is defined by

$$\partial_p D := \left(\overline{D(t_0)} \times \{t_0\}\right) \cup \left(\bigcup_{t\in\mathbb{R}} (\partial D(t) \times \{t\})\right),$$

where $\overline{D(t_0)}$ denotes the closure of $D(t_0)$ and $\partial D(t)$ denotes the boundary of $D(t)$. We say that the set $D \subset \mathbb{R}^{n+1}_-$ is a bowl-shaped domain if $D(t)$ is convex for each t and $D(t_1) \subset D(t_2)$ for $t_1 \leq t_2$.

At the end of this section, we will list three lemmas that are needed throughout this paper. The proof of Lemma 2.1 and Lemma 2.2 can be found in [18]. And we will only prove Lemma 2.3.

Throughout the paper, we will always denote

$$B_r(0) := \{x \in \mathbb{R} : |x| < r\},$$
$$P_r(A,\tau,\bar{x}) := \{(x,t) \in \mathbb{R}^{n+1}_- : \frac{1}{2}(x-\bar{x})^T A(x-\bar{x})+\tau t < r\},$$

where $r>0$, $\tau<0$, $\bar{x} \in \mathbb{R}^n$ and A is an $n \times n$ symmetric positive definite matrix.

Lemma 2.1. *Let $n \geq 1$ and U be an $(n+1)\times(n+1)$ real upper-triangular matrix. Assume that the diagonals of U are nonnegative and for some $0<\epsilon<1$,*

$$(1-\epsilon)P_1(I,-1,0) \subset U(P_1(I,-1,0)) \subset (1+\epsilon)P_1(I,-1,0). \tag{14}$$

Then

$$\|U - I\| \le C(n)\sqrt{\epsilon}. \tag{15}$$

Lemma 2.2. *Let $n \ge 3$ and $A = (a_{ij}(x,t))$ be a real $n \times n$ symmetric positive definite matrix with*

$$|a_{ij}(x,t) - \delta_{ij}| \le C(|x|^2 + |t|)^{-\frac{\varepsilon}{2}}, \quad (x,t) \in \mathbb{R}^{n+1}_-, \tag{16}$$

and $a_{ij} \in C^{1+\alpha, \frac{1+\alpha}{2}}(\mathbb{R}^{n+1}_-)$, where $\varepsilon, \alpha \in (0,1)$ are constants. Then there exists a positive solution $u \in C^\infty(\mathbb{R}^{n+1}_-)$ of

$$u_t - a_{ij} D_{ij} u = e^t \left((1 + |x|^2)^{\frac{2-n}{2}} + n(n-2)(1 + |x|^2)^{-\frac{n+2}{2}} \right) \ge 0, \quad (x,t) \in \mathbb{R}^{n+1}_-, \tag{17}$$

satisfying

$$0 \le u(x,t) \le C(n) e^t (1 + |x|^2)^{\frac{2-n}{2}}, \quad (x,t) \in \mathbb{R}^{n+1}_-. \tag{18}$$

Lemma 2.3. *Let $g \in C^{2,1}(\mathbb{R}^{n+1}_- \setminus P_1(I, \tau, 0))$ satisfy*

$$g_s - \log\Big(\det(I + D^2 g)\Big) = 0,\ I + D^2 g > 0,\ c_0 < g_s + \tau < C_0, \quad (y,s) \in \mathbb{R}^{n+1}_- \setminus P_1(I,\tau,0),$$
$$|g(y,s)| \le \beta \left(\frac{|y|^2}{2} + \tau s \right)^{\frac{2-\epsilon}{2}}, \quad (y,s) \in \mathbb{R}^{n+1}_- \setminus P_1(I,\tau,0), \tag{19}$$

where τ, c_0, $C_0 < 0$ and β, $\epsilon > 0$. Then there exists some constant $r = r(n, \beta, \epsilon) \ge 2$ such that

$$|D^2 g(y,s)| + |g_s(y,s)| \le C \left(\frac{|y|^2}{2} + \tau s \right)^{-\frac{\epsilon}{2}}, \quad (y,s) \in \mathbb{R}^{n+1}_- \setminus P_r(I,\tau,0),$$

where C depends on n, β, ϵ, c_0 and C_0.

Proof. For $(y,s) \in \partial_p P_{R^2}(I,\tau,0)$, $R > 2$, let

$$\eta_R(z,\iota) := (\frac{4}{R})^2 \left(\frac{1}{2}|y + \frac{R}{4}z|^2 - (s + \frac{R^2}{16}\iota) + g(y + \frac{R}{4}z, s + \frac{R^2}{16}\iota) \right), \quad (z,\iota) \in P_{\frac{9}{4}}(I,\tau,0).$$

Since

$$\begin{aligned} \frac{1}{2}|y + \frac{R}{4}z|^2 + \tau(s + \frac{R^2}{16}\iota) &= \frac{1}{2}(|y|^2 + \frac{R}{2}y^T z + \frac{R^2}{16}|z|^2) + \tau(s + \frac{R^2}{16}\iota) \\ &\ge \frac{1}{2}|y|^2 + \tau s + \frac{R}{4}y^T z \\ &\ge R^2 - \frac{R}{4}|y||z| \end{aligned}$$

$$\begin{aligned}&\geq R^2-\frac{R}{4}(\sqrt{2}R)(\frac{3}{\sqrt{2}})\\&=\frac{R^2}{4}\\&>1,\end{aligned}$$

we can see that η_R is well defined in $P_{\frac{9}{4}}(I,\tau,0)$.

By the decay hypothesis (19) on g, we have

$$\begin{aligned}\eta_R(z,\iota)&\leq\frac{16}{R^2}R^2+\frac{9}{4}+\frac{4}{R}(\sqrt{2}R)(\frac{3}{\sqrt{2}})+(\frac{4}{R})^2\beta\left(\frac{1}{2}|y+\frac{R}{4}z|^2+\tau(s+\frac{R^2}{16}\iota)\right)^{-\frac{\epsilon}{2}}\\&\leq\frac{121}{4}+\frac{162^\epsilon\beta}{R^{2+\epsilon}},\end{aligned}$$

we can take r_1 satisfying $\frac{162^\epsilon\beta}{r_1^{2+\epsilon}}=1$, then if $R\geq\max\{r_1,2\}$, we have

$$\eta_R(z,\iota)\leq\frac{125}{4}.$$

Similarly, we also have

$$\begin{aligned}\eta_R(z,\iota)&\geq\frac{16}{R^2}R^2+\frac{4}{R}y^Tz-(\frac{4}{R})^2\beta\left(\frac{1}{2}|y+\frac{R}{4}z|^2+\tau(s+\frac{R^2}{16}\iota)\right)^{-\frac{\epsilon}{2}}\\&\geq16-\frac{4}{R}(\sqrt{2}R)(\frac{3}{\sqrt{2}})-\frac{162^\epsilon\beta}{R^{2+\epsilon}}\\&=4-\frac{162^\epsilon\beta}{R^{2+\epsilon}}.\end{aligned}$$

Then if $R\geq\max\{r_1,2\}$, we have

$$\eta_R(z,\iota)\geq3.$$

In conclusion, there exists some $r=r(n,\beta,\varepsilon)=\max\{r_1,2\}\geq1$ such that for $(y,s)\in\partial_pP_{R^2}(I,\tau,0)$, $R>r$,

$$3\leq\eta_R(z,\iota)\leq\frac{125}{4},\quad(z,\iota)\in P_{\frac{9}{4}}(I,\tau,0).$$

Since η_R satisfies

$$\eta_{R\iota}-\log\left(\det(D^2\eta_R)\right)=\tau,\quad D^2\eta_R>0,\quad c_0<\eta_{R\iota}<C_0,\quad(z,\iota)\in P_{\frac{9}{4}}(I,\tau,0),$$

by the interior estimates of Pogorelov, Evans–Krylov for parabolic equations, we have

$$\|\eta_R\|_{C^{2,1}(\overline{P_1(I,\tau,0)})}\leq C,\ C^{-1}I\leq D^2\eta_R\leq CI\ \text{ in } P_1(I,\tau,0).$$

Here and in the following, $C \geq 1$ denotes some constant depending on n, τ, c_0 and C_0 unless otherwise stated.

Clearly,

$$g_R(z,\iota) := (\frac{4}{R})^2 g(y + \frac{R}{4}z, s + \frac{R^2}{16}\iota)$$

satisfies

$$g_{R\iota} - a_{ij}D_{ij}g_R = 0, \quad \text{in } P_{\frac{9}{4}}(I,\tau,0),$$

and

$$\|g_R\|_{C^{2,1}(\overline{P_1(I,\tau,0)})} \leq C, \; C^{-1}I \leq I + D^2 g_R \leq CI \; \text{ in } P_1(I,\tau,0), \tag{20}$$

where $\big(a_{ij}(z,\iota)\big) = \int_0^1 (I + \theta D^2 g_R)^{-1} d\theta$ satisfies, in view of (20), that

$$\|a_{ij}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{P_1(I,\tau,0)})} \leq C, \; C^{-1}I \leq (a_{ij}) \leq CI \; \text{ in } P_1(I,\tau,0).$$

By interior Schauder theory (see [13]) and (19),

$$|D^2 g_R(0,0)| + |g_{R\iota}(0,0)| \leq C\|g_R\|_{L^\infty(P_1(I,\tau,0))} \leq CR^{-\varepsilon}.$$

The result of Lemma 2.3 follows. □

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into the following five steps. And in this section we will always assume that $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ be a parabolically convex solution of (1) and (2) with the assumption $C_0 < 0$ without loss of generality, which also satisfies normalizations (12) and (13).

3.1. Normalization of level sets and solutions

Given $H > 0$, let the level set of u be defined as

$$Q_H = \{(x,t) \in \mathbb{R}^{n+1}_- : u(x,t) < H\},$$

and for every $t \leq 0$, we define

$$Q_H(t) = \{x \in \mathbb{R}^n : u(x,t) < H\}.$$

Let x_H and E denote the mass center of $Q_H(0)$ and the ellipsoid of minimum volume containing $Q_H(0)$ with center x_H respectively. By a normalization lemma of John-Cordoba and Gallegos (see [7]), there exists some affine transformation

$$T_H(x) = a_H x + b_H, \tag{21}$$

where a_H is an $n \times n$ matrix and $b_H \in \mathbb{R}^n$ satisfying

$$\det(a_H) = 1, \tag{22}$$

$$T_H(E) = B_R(0), \quad \text{for some } R = R(H) > 0, \tag{23}$$

and

$$B_{\alpha_n R}(0) \subset T_H(Q_H(0)) \subset B_R(0), \tag{24}$$

where $\alpha_n = n^{-\frac{3}{2}}$.

The following proposition gives us the relationship between R and H.

Proposition 3.1. *There exists some constant C depending on n, c_0, C_0, $\sup\limits_{\mathbb{R}^{n+1}_-} f$ and $\inf\limits_{\mathbb{R}^{n+1}_-} f$ such that*

$$C^{-1} \leq \frac{H}{R^2} \leq C.$$

Proof. Since

$$u_t(x,0) - \log\left(\det D^2 u(x,0)\right) = f(x,0), \quad x \in \mathbb{R}^n,$$

we have

$$\det D^2 u(x,0) = \exp\{u_t(x,0) - f(x,0)\}.$$

Let $M_1 = \exp\{\inf\limits_{\mathbb{R}^n}(u_t(x,0) - f(x,0))\}$, then

$$M_1 \leq \det D^2 u(x,0).$$

Now we consider the function

$$w_1(y) = M_1^{-\frac{1}{n}} u(T_H^{-1}(y), 0), \quad y \in \mathbb{R}^n.$$

Then,

$$\det D^2 w_1 \geq 1 \quad \text{in } B_{\alpha_n R}(0) \quad \text{and} \quad w_1 \leq M_1^{-\frac{1}{n}} H \quad \text{on } \partial B_{\alpha_n R}(0).$$

And we also consider the comparison function

$$v_1(y) = \frac{1}{2}(|y|^2 - \alpha_n^2 R^2) + M_1^{-\frac{1}{n}} H, \quad y \in \mathbb{R}^n.$$

It is obvious that v_1 is a C^2 convex function satisfying

$$\det D^2 v_1 = 1 \quad \text{in } B_{\alpha_n R}(0) \quad \text{and} \quad v_1 = M_1^{-\frac{1}{n}} H \quad \text{on } \partial B_{\alpha_n R}(0).$$

By comparison principle, we have $w_1 \le v_1$ in $B_{\alpha_n R}(0)$. In particular,

$$0 \le w_1(0) \le -\frac{1}{2}\alpha_n^2 R^2 + M_1^{-\frac{1}{n}} H.$$

It follows that

$$\frac{1}{2}\alpha_n^2 M_1^{\frac{1}{n}} \le \frac{H}{R^2}.$$

Similarly, we can obtain

$$\frac{H}{R^2} \le \frac{1}{2} M_2^{\frac{1}{n}}, \quad \text{for } M_2 = \exp\{\sup_{\mathbb{R}^n}(u_t(x,0) - f(x,0))\}.$$

Therefore, by taking $C = \max\{\frac{1}{2} M_2^{\frac{1}{n}}, 2\alpha_n^{-2} M_1^{-\frac{1}{n}}\}$, we have

$$C^{-1} \le \frac{H}{R^2} \le C. \qquad \square$$

Proposition 3.2. *For some positive constant C depending on n, c_0, C_0, $\sup_{\mathbb{R}_-^{n+1}} f$ and $\inf_{\mathbb{R}_-^{n+1}} f$,*

$$C^{-1} R \le dist\left(T_H(Q_{\frac{H}{2}}(0)), \partial T_H(Q_H(0))\right) \le 2R. \tag{25}$$

Consequently,

$$B_{\frac{R}{C}}(0) \subset a_H(Q_H(0)) \subset B_{2R}(0). \tag{26}$$

Proof. Once estimate (25) has been established, estimate (26) can be deduced from (25) and the fact

$$dist\left(T_H(0), \partial T_H(Q_H(0))\right) = dist\left(0, \partial a_H(Q_H(0))\right).$$

Since $T_H(Q_{\frac{H}{2}}(0)) \subset T_H(Q_H(0)) \subset B_R(0)$, it is obvious that

$$dist\left(T_H\left(Q_{\frac{H}{2}}(0)\right), \partial T_H(Q_H(0))\right) \le 2R.$$

So in order to obtain estimate (25), we only need to prove the first inequality in (25).

Let us consider the function

$$w(y) = \frac{M_2^{-\frac{1}{n}}}{R^2}\left(u\left(T_H^{-1}(Ry), 0\right) - H\right), \quad y \in O_H(0) := \frac{1}{R} T_H(Q_H(0)),$$

where $M_2 = \exp\{\sup_{\mathbb{R}^n}(u_t(x,0) - f(x,0))\}$. Then by (24), we have

$$B_{\alpha_n}(0) \subset O_H(0) \subset B_1(0), \quad w = 0 \quad \text{on } \partial O_H(0),$$

and

$$\det(D^2 w) \le 1 \quad \text{in } O_H(0).$$

It follows from Lemma 1 in [1] that

$$w(y) \ge -C(n) dist(y, \partial O_H(0))^{\frac{2}{n}}, \quad y \in O_H(0).$$

For every $\bar{y} \in T_H(Q_{\frac{H}{2}}(0))$, let $\bar{x} = \frac{1}{R}\bar{y}$; we then have

$$-\frac{M_2^{-\frac{1}{n}} H}{2R^2} \ge \frac{M_2^{-\frac{1}{n}}}{R^2}(\frac{H}{2}) - \frac{M_2^{-\frac{1}{n}} H}{R^2} \ge w(\bar{x}) \ge -C(n) dist(\bar{x}, \partial O_H(0))^{\frac{2}{n}},$$

i.e.

$$dist(\bar{y}, \partial T_H(Q_H(0)) \ge R\left(\frac{M_2^{-\frac{1}{n}} H}{2R^2 C(n)}\right)^{\frac{n}{2}}.$$

By Proposition 3.1, we obtain

$$dist(\bar{y}, \partial T_H(Q_H(0)) \ge C^{-1} R,$$

where $C = C(n, \sup_{\mathbb{R}^n}(u_t(x,0) - f(x,0)), \inf_{\mathbb{R}^n}(u_t(x,0) - f(x,0)))$. □

Proposition 3.3.

$$\varepsilon_0 a_H^{-1}(B_{\frac{R}{C}}(0)) \times [-\varepsilon_1 H, 0] \subset Q_H \subset a_H^{-1}(B_{2R}(0)) \times [-\varepsilon_2 H, 0], \tag{27}$$

where the constant C is the same as in Proposition 3.2 and ε_0, ε_1, ε_2 are positive constants depending on n, c_0, C_0, $\sup_{\mathbb{R}_-^{n+1}} f$ and $\inf_{\mathbb{R}_-^{n+1}} f$.

Proof. Since $u_t(x,t) \le C_0$ for $t \le 0$, we have $u(x,t) \ge u(x,0) + C_0 t$. By (13) and (26), we then obtain $u(x,t) \ge H$ for $t < \frac{H}{C_0}$ or $x \notin a_H^{-1}(B_{2R}(0))$. So if we take $\varepsilon_2 = -\frac{1}{C_0}$, we have

$$Q_H \subset a_H^{-1}(B_{2R}(0)) \times [-\varepsilon_2 H, 0].$$

Due to (12), $Q_H(0)$ is a section of the convex function $u(x,0)$ at $x=0$. Particularly, from (26) and Lemma 2.1 of [9] we have that for any $\varepsilon_0 \in (0,1)$,

$$\varepsilon_0(\frac{1}{2C}a_H^{-1}(B_{2R}(0))) = \varepsilon_0 a_H^{-1}(B_{\frac{R}{C}}(0)) \subset \varepsilon_0 Q_H(0) \subset Q_{(1-\frac{1-\varepsilon_0}{4C})H}(0).$$

If $(x,t) \in \varepsilon_0 a_H^{-1}(B_{\frac{R}{C}}(0)) \times [-\varepsilon_1 H, 0]$, then

$$u(x,t) = u(x,0) - \int_t^0 u_t(x,\tau)d\tau \le (1 - \frac{1-\varepsilon_0}{4C})H + c_0 t \le (1 - \frac{1-\varepsilon_0}{4C} - c_0\varepsilon_1)H < H.$$

Therefore, if we take ε_0 and ε_1 sufficiently small, we can obtain $u(x,t) < H$,

$$\varepsilon_0 a_H^{-1}(B_{\frac{R}{C}}(0)) \times [-\varepsilon_1 H, 0] \subset Q_H,$$

which completes the proof of (27). □

Let

$$\Gamma_H(x,t) := (\frac{a_H x}{R}, \frac{t}{R^2}) \quad \text{and} \quad Q_H^* := \Gamma_H(Q_H).$$

By Proposition 3.3 and Proposition 3.1, we have

$$B_{\frac{\varepsilon_0}{C}}(0) \times [-\varepsilon_1 C, 0] \subset Q_H^* \subset B_2(0) \times [-\varepsilon_2 C^{-1}, 0].$$

Now we define the normalized function of u

$$v(y,s) = \frac{1}{R^2}u(\Gamma_H^{-1}(y,s)) = \frac{1}{R^2}u(Ra_H^{-1}y, R^2 s), \quad (y,s) \in \overline{Q_H^*}.$$

It is easy to verify that

$$v_s - \log\left(\det(D^2 v)\right) = f(Ra_H^{-1}y, R^2 s), \quad c_0 \le v_s \le C_0 \quad \text{in } Q_H^*,$$

and by Proposition 3.1,

$$v = \frac{H}{R^2} \in (C^{-1}, C) \quad \text{on } \partial_p Q_H^*.$$

3.2. Nonlinear perturbation

By [16], there exists a unique parabolically convex solution $\bar{v} \in C^0(\overline{Q_H^*}) \bigcap C^\infty(Q_H^*)$ of

$$\begin{cases} \bar{v}_s - \log\left(\det(D^2\bar{v})\right) = a & \text{in } Q_H^*, \\ \bar{v} = \frac{H}{R^2} & \text{on } \partial_p Q_H^*, \\ c_0 \le \bar{v}_s \le C_0 & \text{in } Q_H^*. \end{cases}$$

From the interior estimates, for every $\delta > 0$, there exists some positive constant $C = C(\delta)$ such that for all $(y, s) \in Q_H^*$ and $dist_p((y, s), \partial_p Q_H^*) \ge \delta$, we have

$$C^{-1}I \le D^2\bar{v}(y,s) \le CI, \quad |D^3\bar{v}(y,s)| \le C, \quad |D\bar{v}_s(y,s)| \le C, \quad |\bar{v}_{ss}(y,s)| \le C. \tag{28}$$

Lemma 3.4. *For some positive constant C dependent on n, $\sup_{\mathbb{R}^{n+1}_-} f$, $\inf_{\mathbb{R}^{n+1}_-} f$ and meas$\{f - a\}$, we have*

$$|v - \bar{v}| \le CR^{-\frac{n+2}{n+1}} \quad \text{in } Q_H^*.$$

Proof. Let

$$S^+ = \{(y,s) \in Q_H^* : (v - \bar{v})_s(y,s) < 0, D^2(v - \bar{v})(y,s) > 0\}.$$

Since on S^+,

$$\det\left(D^2(v - \bar{v})\right) \le \det\left(D^2 v\right),$$

we have

$$\begin{aligned} -(v - \bar{v})_s + \log\left(\det D^2(v - \bar{v})\right) &\le -v_s + \log\left(\det(D^2 v)\right) \\ &= -f(Ra_H^{-1}y, R^2 s), \end{aligned}$$

and

$$f(Ra_H^{-1}y, R^2 s) = v_s - \log\left(\det(D^2 v)\right) < \bar{v}_s - \log\left(\det(D^2\bar{v})\right) = a.$$

By the Alexandrov–Bakelman–Pucci estimate for parabolic equations (see [15]), we have

$$\sup_{Q_H^*}(v-\bar{v}) \le \max_{\partial_p Q_H^*}(v-\bar{v}) + C(n)\left(\int_{S^+} -(v-\bar{v})_s \det\left(D^2(v-\bar{v})\right) dyds\right)^{\frac{1}{n+1}}$$

$$\le C(n)\left(\int_{S^+} \exp\{-(v-\bar{v})_s\}\det\left(D^2(v-\bar{v})\right) dyds\right)^{\frac{1}{n+1}}$$

$$= C(n)\left(\int_{S^+} \exp\{-(v-\bar{v})_s + \log\left(\det\left(D^2(v-\bar{v})\right)\right)\} dyds\right)^{\frac{1}{n+1}},$$

from which it follows that

$$\sup_{Q_H^*}(v-\bar{v}) \le C(n)\left(\int_{S^+} \exp\{-f\left(Ra_H^{-1}y, R^2 s\right)\} dyds\right)^{\frac{1}{n+1}}$$

$$= C(n)\left(\int_{\Gamma_H^{-1}(S^+)} \exp\{-f(x,t)\}\frac{\det a_H}{R^{n+2}} dxdt\right)^{\frac{1}{n+1}}$$

$$\le C(n)R^{-\frac{n+2}{n+1}}\left(\int_{\{f<a\}} \exp\{-f(x,t)\} dxdt\right)^{\frac{1}{n+1}}$$

$$\le CR^{-\frac{n+2}{n+1}}.$$

Similarly, we can obtain that

$$\sup_{Q_H^*}(\bar{v}-v) \le CR^{-\frac{n+2}{n+1}}.$$

Lemma 3.4 is established. □

3.3. Rough asymptotic behavior

Let $(\bar{y}, 0)$ be the unique minimum point of $\bar{v}$ on $\overline{Q_H^*}$, $\widetilde{H} \in (\bar{v}(\bar{y}, 0), H)$ and $m > 0$. For simplification, we will denote that

$$P_{\widetilde{H}}(0,0) := P_{\widetilde{H}}(D^2\bar{v}(\bar{y},0), \bar{v}_s(\bar{y},0), 0)$$
$$S_{\widetilde{H}}(0,0) := \partial_p P_{\widetilde{H}}(0,0)$$
$$P_{\widetilde{H}}(\bar{y},0) := P_{\widetilde{H}}(D^2\bar{v}(\bar{y},0), \bar{v}_s(\bar{y},0), \bar{y})$$
$$S_{\widetilde{H}}(\bar{y},0) := \partial_p P_{\widetilde{H}}(\bar{y},0)$$
$$mP_{\widetilde{H}}(0,0) := \{(y,s) \in \mathbb{R}^{n+1}_- : (\frac{y}{m}, \frac{s}{m^2}) \in P_{\widetilde{H}}(0,0)\}$$

$$mP_{\widetilde{H}}(\bar{y},0) := \{(y,s) \in \mathbb{R}^{n+1}_{-} : (\frac{y}{m}, \frac{s}{m^2}) \in P_{\widetilde{H}}(\bar{y},0)\}$$
$$mQ_H := \{(y,s) \in \mathbb{R}^{n+1}_{-} : (\frac{y}{m}, \frac{s}{m^2}) \in Q_H\}.$$

Lemma 3.5. *There exist $\bar{k}$ and $\overline{C}$, depending on n, c_0, C_0, $\sup_{\mathbb{R}^{n+1}_{-}} f$, $\inf_{\mathbb{R}^{n+1}_{-}} f$ and meas$\{f-a\}$, such that for $\epsilon = \frac{1}{3}$, $H = 2^{(1+\epsilon)k}$ and $2^{k-1} \le H' \le 2^k$, we have*

$$(\frac{H'}{R^2} - \overline{C}2^{-\frac{3\epsilon k}{2}})^{\frac{1}{2}} P_1(\bar{y},0) \subset \Gamma_H(Q_{H'}) \subset (\frac{H'}{R^2} + \overline{C}2^{-\frac{3\epsilon k}{2}})^{\frac{1}{2}} P_1(\bar{y},0), \quad \forall k \ge \bar{k}. \tag{29}$$

Proof.

$$\Gamma_H(Q_{H'}) = \{v < \frac{H'}{R^2}\} := \{(y,s) \in Q^*_H : v(y,s) < \frac{H'}{R^2}\}.$$

By Lemma 3.4, the level surface of v can be well approximated by the level surface of $\bar{v}$:

$$\{\bar{v} < \frac{H'}{R^2} - \frac{C}{R^{\frac{n+2}{n+1}}}\} \subset \{v < \frac{H'}{R^2}\} \subset \{\bar{v} < \frac{H'}{R^2} + \frac{C}{R^{\frac{n+2}{n+1}}}\},$$

and

$$-\frac{C}{R^{\frac{n+2}{n+1}}} \le v(\bar{y},0) - \frac{C}{R^{\frac{n+2}{n+1}}} \le \bar{v}(\bar{y},0) \le \bar{v}(0,0) \le v(0,0) + \frac{C}{R^{\frac{n+2}{n+1}}} = \frac{C}{R^{\frac{n+2}{n+1}}}, \tag{30}$$

since $v \ge 0$ and $v(0,0) = 0$. We also have that

$$C^{-1}I \le D^2\bar{v}(\bar{y},0) \le CI \quad \bar{v}_s(\bar{y},0) \le C^{-1}, \tag{31}$$

and

$$|\bar{v}(y,s) - \bar{v}(\bar{y},0) - \bar{v}_s(\bar{y},0)s - \frac{1}{2}(y-\bar{y})^T D^2\bar{v}(\bar{y},0)(y-\bar{y})| \le C(|y-\bar{y}|^2 + |s|)^{\frac{3}{2}}, \tag{32}$$

for $dist_p((y,s),(\bar{y},0)) < \frac{1}{C}$ by (28) and $D\bar{v}(\bar{y},0) = 0$.

It is clear by Proposition 3.1 that

$$C^{-1}2^{-\epsilon k} \le \frac{H'}{R^2} \le C2^{-\epsilon k}, \quad C^{-1}2^{\frac{(1+\epsilon)k}{2}} \le R \le C2^{\frac{(1+\epsilon)k}{2}}. \tag{33}$$

Next we will prove the two relations in (29) respectively. On one side, we shall take a positive constant C_1 such that

$$(\frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2}})^{\frac{1}{2}} P_1(\bar{y},0) \subset \{\bar{v} < \frac{H'}{R^2} - \frac{C}{R^{\frac{n+2}{n+1}}}\}. \tag{34}$$

For $(y,s) \in (\frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2}})^{\frac{1}{2}} P_1(\bar{y},0)$, we have

$$\bar{v}_s(\bar{y},0)s+\frac{1}{2}(y-\bar{y})^T D^2\bar{v}(\bar{y},0)(y-\bar{y})<\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2}},$$
$$|y-\bar{y}|^2+|s|<C(\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2}}).$$

Thus, it follows from (30) and (32) that

$$\begin{aligned}\bar{v}(y,s)&\le \bar{v}(\bar{y},0)+\bar{v}_s(\bar{y},0)s+\frac{1}{2}(y-\bar{y})^T D^2\bar{v}(\bar{y},0)(y-\bar{y})+C(|y-\bar{y}|^2+|s|)^{\frac{3}{2}}\\ &\le \frac{C}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2}}+C^{\frac{5}{2}}(\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2}})^{\frac{3}{2}}\\ &\le \frac{C}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2}}+C^4 2^{-\frac{3}{2}\epsilon k}.\end{aligned}$$

We can take $C_1=1+C^4$ such that

$$\frac{2C}{R^{\frac{n+2}{n+1}}}\le 2C^{\frac{2n+3}{n+1}}2^{-\frac{(1+\epsilon)(n+2)k}{2(n+1)}}<2^{-\frac{3\epsilon k}{2}},$$

if $k\ge k_1$, k_1 large enough. Therefore, we obtain

$$\bar{v}(y,s)\le \frac{C}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+2^{-\frac{3\epsilon k}{2}}<\frac{H'}{R^2}-\frac{C}{R^{\frac{n+2}{n+1}}},$$

and finish the proof of (34).

On the other side, we shall take a positive constant C_2 such that

$$\{\bar{v}<\frac{H'}{R^2}+\frac{C}{R^{\frac{n+2}{n+1}}}\}\subset(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}})^{\frac{1}{2}}P_1(\bar{y},0).$$

By using the fact

$$(\bar{y},0)\in\{\bar{v}<\frac{H'}{R^2}+\frac{C}{R^{\frac{n+2}{n+1}}}\}\bigcap(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}})^{\frac{1}{2}}P_1(\bar{y},0),$$

we only need to prove

$$(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}})^{\frac{1}{2}}S_1(\bar{y},0)\subset\{\bar{v}<\frac{H'}{R^2}+\frac{C}{R^{\frac{n+2}{n+1}}}\}^C.$$

For $(y,s)\in(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}})^{\frac{1}{2}}S_1(\bar{y},0)$, then

$$\bar{v}_s(\bar{y},0)s+\frac{1}{2}(y-\bar{y})^T D^2\bar{v}(\bar{y},0)(y-\bar{y})=\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}},$$
$$|y-\bar{y}|^2+|s|<C(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}).$$

Take k_2 large enough satisfying for $k \geq k_2$,

$$|y-\bar{y}|^2+|s| < C(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}) \leq \frac{1}{C^2}.$$

Thus,

$$\begin{aligned}
\bar{v}(y,s) &\geq \bar{v}(\bar{y},0)+\bar{v}_s(\bar{y},0)s+\frac{1}{2}(y-\bar{y})^T D^2\bar{v}(\bar{y},0)(y-\bar{y})-C(|y-\bar{y}|^2+|s|)^{\frac{3}{2}} \\
&\geq -\frac{C}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}-C^{\frac{5}{2}}(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}})^{\frac{3}{2}} \\
&\geq -\frac{C}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}-C^{\frac{5}{2}}(2\frac{H'}{R^2})^{\frac{3}{2}} \\
&\geq -\frac{C}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}-C^4 2^{\frac{3}{2}} 2^{-\frac{3}{2}\epsilon k}.
\end{aligned}$$

We can take $C_2 = 1+2^{\frac{3}{2}}C^4$, then

$$\frac{2C}{R^{\frac{n+2}{n+1}}} \leq 2C^{\frac{2n+3}{n+1}} 2^{-\frac{(1+\epsilon)(n+2)k}{2(n+1)}} < 2^{-\frac{3\epsilon k}{2}},$$

if $k \geq k_2$, k_2 large enough. Therefore, we obtain

$$\bar{v}(y,s) \geq -\frac{C}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+2^{-\frac{3\epsilon k}{2}} > \frac{H'}{R^2}+\frac{C}{R^{\frac{n+2}{n+1}}}.$$

In conclusion, if we take $\overline{C} > \max\{C_1, C_2\}$ and $\bar{k} = \max\{k_1, k_2\}$, then (29) holds. □

Proposition 3.6. *There exist $\bar{k}$ and $\overline{C}$, depending on n, c_0, C_0, $\sup_{\mathbb{R}^{n+1}_-} f$, $\inf_{\mathbb{R}^{n+1}_-} f$ and meas$\{f-a\}$, such that for $\epsilon = \frac{1}{3}$, $H = 2^{(1+\epsilon)k}$ and $2^{k-1} \leq H' \leq 2^k$, we have*

$$(\frac{H'}{R^2}-\overline{C}2^{-\frac{3\epsilon k}{2}})^{\frac{1}{2}} P_1(0,0) \subset \Gamma_H(Q_{H'}) \subset (\frac{H'}{R^2}+\overline{C}2^{-\frac{3\epsilon k}{2}})^{\frac{1}{2}} P_1(0,0), \quad \forall k \geq \bar{k}. \tag{35}$$

Proof. In order to obtain (35), we first show that

$$\partial Q^*_{\widetilde{H}+\bar{v}(\bar{y},0)}(\bar{v}) \subset N_{\delta_1}(S_{\widetilde{H}}(\bar{y},0)), \quad 0 < \widetilde{H} \leq \frac{H}{R^2}-\bar{v}(\bar{y},0), \quad \delta_1 \leq C\widetilde{H}^{\frac{1}{2}}, \tag{36}$$

and neighborhood N is measured by parabolic distance

$$dist_p[(y_1,s_1),(y_2,s_2)] := (|y_1-y_2|^2+|s_1-s_2|)^{\frac{1}{2}}.$$

In fact, for $(y,s) \in \partial Q^*_{\widetilde{H}+\bar{v}(\bar{y},0)}(\bar{v})$, by the Mean Theorem, we have

$$\begin{aligned}\widetilde{H} &= \bar{v}(y,s) - \bar{v}(\bar{y},0)\\ &= \bar{v}(y,s) - \bar{v}(y,0) + \bar{v}(y,0) - \bar{v}(\bar{y},0)\\ &= \bar{v}_s(y,s')s + \frac{1}{2}(y-\bar{y})^T D^2\bar{v}(y',0)(y-\bar{y})\\ &\geq \frac{1}{2C}(|s| + |y-\bar{y}|^2),\end{aligned}$$

where $(y', s') \in Q^*_{\widetilde{H}+\bar{v}(\bar{y},0)}$. Writing

$$\begin{aligned}\widetilde{H} &= \bar{v}(y,s) - \bar{v}(\bar{y},0)\\ &= \bar{v}_s(\bar{y},0)s + (\bar{v}_s(y,s') - \bar{v}_s(\bar{y},0))s + \frac{1}{2}(y-\bar{y})^T D^2\bar{v}(\bar{y},0)(y-\bar{y})\\ &\quad + \frac{1}{2}(y-\bar{y})^T (D^2\bar{v}(y',0) - D^2\bar{v}(\bar{y},0))(y-\bar{y}),\end{aligned}$$

for $(y,s) \in \partial Q^*_{\widetilde{H}+\bar{v}(\bar{y},0)}(\bar{v})$, then

$$\begin{aligned}&|\widetilde{H} - \bar{v}_s(\bar{y},0)s - \frac{1}{2}(y-\bar{y})^T D^2\bar{v}(\bar{y},0)(y-\bar{y})|\\ &= |(\bar{v}_s(y,s') - \bar{v}_s(\bar{y},0))s + \frac{1}{2}(y-\bar{y})^T (D^2\bar{v}(y',0) - D^2\bar{v}(\bar{y},0))(y-\bar{y})|\\ &\leq C|s| + C|y-\bar{y}|^2\\ &\leq C\widetilde{H}.\end{aligned}$$

For any $(y,s) \in \partial Q^*_{\widetilde{H}+\bar{v}(\bar{y},0)}(\bar{v})$ and any $(\widetilde{y},\widetilde{s}) \in S_{\widetilde{H}}(\bar{y},0)$, by the above inequality, we have

$$|\bar{v}_s(\bar{y},0)\widetilde{s} + \frac{1}{2}(\widetilde{y}-\bar{y})^T D^2\bar{v}(\bar{y},0)(\widetilde{y}-\bar{y}) - \bar{v}_s(\bar{y},0)s - \frac{1}{2}(y-\bar{y})^T D^2\bar{v}(\bar{y},0)(y-\bar{y})| \leq C\widetilde{H}.$$

For $s = \widetilde{s}$, take $\widetilde{y}$, $\bar{y}$, y on the same line l with $\widetilde{y}$ and y on the same side of the line l with respect to $\bar{y}$; rotating the coordinates again so that l is parallel to some axis, we have

$$||\widetilde{y}-\bar{y}|^2 - |y-\bar{y}|^2| \geq |y-\widetilde{y}|^2.$$

Then

$$\frac{1}{2C}||\widetilde{y}-\bar{y}|^2 - |y-\bar{y}|^2| \leq C\widetilde{H}.$$

In fact, there exists an orthogonal matrix O such that $D^2\bar{v}(\bar{y},0) = O^T diag\{\lambda_1, \cdots, \lambda_n\}O$, and the length of a vector in Euclidean space is invariant in orthogonal transformation.

Therefore, we obtain

$$|y-\widetilde{y}|^2 \leq C\widetilde{H}.$$

Similarly, for $y = \widetilde{y}$,

$$|\bar{v}_s(\bar{y}, 0)\widetilde{s} - \bar{v}_s(\bar{y}, 0)s| \leq C\widetilde{H}.$$

So we get

$$|s - \widetilde{s}| \leq C\widetilde{H}.$$

This completes the proof of (36).

Next we estimate the distance between $(0, 0)$ and $(\bar{y}, 0)$. By Lemma 3.4, we have

$$\begin{aligned} 0 &\leq \bar{v}(0,0) - \bar{v}(\bar{y}, 0) \\ &= (\bar{v}(0,0) - v(0,0)) + (v(0,0) - v(\bar{y}, 0)) + (v(\bar{y}, 0) - \bar{v}(\bar{y}, 0)) \\ &\leq 2CR^{-\frac{n+2}{n+1}}, \end{aligned}$$

so $(0, 0) \in Q^*_{2CR^{-\frac{n+2}{n+1}}+\bar{v}(\bar{y},0)}(\bar{v})$, and by (36) (taking $\widetilde{H} = 2CR^{-\frac{n+2}{n+1}}$), we have

$$\partial(Q^*_{2CR^{-\frac{n+2}{n+1}}+\bar{v}(\bar{y},0)}(\bar{v})) \subset N_{\delta_1}(S_{2CR^{-\frac{n+2}{n+1}}}(\bar{y}, 0)), \quad \delta_1 \leq C(2CR^{-\frac{n+2}{n+1}})^{1/2},$$

thus we obtain

$$|\bar{y}| = dist_p((0,0), (\bar{y}, 0)) \leq C(2CR^{-\frac{n+2}{n+1}})^{1/2}.$$

So by (29), we have for $k \geq \bar{k}$

$$\begin{aligned} (\frac{H'}{R^2} - C_3 2^{-\frac{3\epsilon k}{2}} - 2C^3 R^{-\frac{n+2}{n+1}})^{\frac{1}{2}} P_1(0,0) &\subset \{v < \frac{H'}{R^2}\} \\ &\subset (\frac{H'}{R^2} + C_3 2^{-\frac{3\epsilon k}{2}} + 2C^3 R^{-\frac{n+2}{n+1}})^{\frac{1}{2}} P_1(0,0). \end{aligned}$$

Since $2^{-\frac{3\epsilon k}{2}} \gg R^{-\frac{n+2}{n+1}}$ and letting $\overline{C} = C^2 + C_3$, then we can obtain (35). □

Proposition 3.7. *There exist positive constants $\widehat{k}$, $\widehat{C}$ depending on n, c_0, C_0, $\sup_{\mathbb{R}^{n+1}_-} f$, $\inf_{\mathbb{R}^{n+1}_-} f$ and $meas\{f - a\}$, some real invertible upper-triangular matrices $\{T_k\}_{k \geq \widehat{k}}$ and negative number $\{\tau_k\}_{k \geq \widehat{k}}$ such that*

$$\tau_k - \log\left(\det(T_k^T T_k)\right) = a, \quad \|T_k T_{k-1}^{-1} - I\| \leq \widehat{C} 2^{-\frac{\epsilon k}{4}}, \quad |\tau_k \tau_{k-1}^{-1} - 1| \leq \widehat{C} 2^{-\frac{\epsilon k}{4}}, \tag{37}$$

and

$$\begin{aligned} &(1 - \widehat{C} 2^{-\frac{\epsilon k}{4}})\sqrt{H'} P_1(I, -1, 0) \subset \Sigma_k(Q_{H'}) \subset (1 + \widehat{C} 2^{-\frac{\epsilon k}{4}})\sqrt{H'} P_1(I, -1, 0), \\ &\quad \forall H' \in [2^{k-1}, 2^k], \end{aligned} \tag{38}$$

where $\Sigma_k = (T_k, -\tau_k)$*. Consequently, for some invertible upper-triangular matrix* T *and* $\tau < 0$*,*

$$\tau - \log\left(\det(T^T T)\right) = a, \quad \|T_k - T\| \le \widehat{C} 2^{-\frac{\epsilon k}{4}}, \quad |\tau_k - \tau| \le \widehat{C} 2^{-\frac{\epsilon k}{4}}. \tag{39}$$

Proof. Let $H = 2^{(1+\epsilon)k}$ and $2^{k-1} \le H' \le 2^k$. By Proposition 3.6 and the definition of Γ_H, there exist some positive constants $\overline{C}$ and $\bar{k}$ depending on n, a, c_0, C_0 and f such that

$$(H' - \overline{C} 2^{-\frac{3\epsilon k}{2}} R^2)^{\frac{1}{2}} P_1(0,0) \subset (a_H, id)(Q_{H'}) \subset (H' + \overline{C} 2^{-\frac{3\epsilon k}{2}} R^2)^{\frac{1}{2}} P_1(0,0), \quad \forall k \ge \bar{k}.$$

Since

$$C^{-1} 2^{-\epsilon k} \le \frac{H'}{R^2} \le C 2^{-\epsilon k},$$

we can obtain

$$(1 - \overline{C} C 2^{-\frac{\epsilon k}{2}})^{\frac{1}{2}} \sqrt{H'} P_1(0,0) \subset (a_H, id)(Q_{H'}) \subset (1 + \overline{C} C 2^{-\frac{\epsilon k}{2}})^{\frac{1}{2}} \sqrt{H'} P_1(0,0).$$

In conclusion, if we take $\widehat{C}$ and $\widehat{k}$ large enough, then

$$(1 - \widehat{C} 2^{-\frac{\epsilon k}{2}}) \sqrt{H'} P_1(0,0) \subset (a_H, id)(Q_{H'}) \subset (1 + \widehat{C} 2^{-\frac{\epsilon k}{2}}) \sqrt{H'} P_1(0,0), \quad k \ge \widehat{k}. \tag{40}$$

Let Q be the real symmetric positive definite matrix satisfying $Q^2 = Q^T Q = D^2 \bar{v}(\bar{y}, 0)$ and O be an orthogonal matrix such that

$$T_k := O Q a_H \quad \text{is upper-triangular,}$$

and we also define $\tau_k = \bar{v}_s(\bar{y}, 0)$ and $\Sigma_k = (T_k, -\tau_k)$. Clearly,

$$\tau_k - \log\left(\det(T_k^T T_k)\right) = \bar{v}_s(\bar{y}, 0) - \log\left((\det a_H)^2 \det D^2 \bar{v}(\bar{y}, 0)\right) = a,$$

and $P_1(I, -1, 0) = (OQ, -\tau_k) P_1(0,0)$.

From (40), we have

$$(1 - \widehat{C} 2^{-\frac{\epsilon k}{2}}) \sqrt{H'} P_1(I, -1, 0) \subset \Sigma_k(Q_{H'}) \subset (1 + \widehat{C} 2^{-\frac{\epsilon k}{2}}) \sqrt{H'} P_1(I, -1, 0), \quad k \ge \widehat{k},$$

that is (38).

By taking $H = 2^{(1+\epsilon)k}$, $H' = 2^{k-1}$ and $H = 2^{(1+\epsilon)(k-1)}$, $H' = 2^{k-1}$, we can obtain

$$(1 - \widehat{C} 2^{-\frac{\epsilon k}{2}}) \sqrt{2^{k-1}} P_1(I, -1, 0) \subset \Sigma_k(Q_{2^{k-1}}) \subset (1 + \widehat{C} 2^{-\frac{\epsilon k}{2}}) \sqrt{2^{k-1}} P_1(I, -1, 0), \tag{41}$$

$$(1 - \widehat{C} 2^{-\frac{\epsilon(k-1)}{2}}) \sqrt{2^{k-1}} P_1(I, -1, 0) \subset \Sigma_{k-1}(Q_{2^{k-1}}) \subset (1 + \widehat{C} 2^{-\frac{\epsilon(k-1)}{2}}) \sqrt{2^{k-1}} P_1(I, -1, 0), \tag{42}$$

respectively. Then

$$(1-\widehat{C}2^{-\frac{\epsilon(k-1)}{2}})\sqrt{2^{k-1}}\Sigma_k\Sigma_{k-1}^{-1}P_1(I,-1,0)\subset\Sigma_k(Q_{2^{k-1}})$$
$$\subset(1+\widehat{C}2^{-\frac{\epsilon(k-1)}{2}})\sqrt{2^{k-1}}\Sigma_k\Sigma_{k-1}^{-1}P_1(I,-1,0), \tag{43}$$

by taking k sufficiently large. Similarly,

$$(1-3\widehat{C}2^{-\frac{\epsilon k}{2}})P_1(I,-1,0)\subset\Sigma_k\Sigma_{k-1}^{-1}P_1(I,-1,0).$$

So by taking k sufficiently large, we have

$$(1-3\widehat{C}2^{-\frac{\epsilon k}{2}})P_1(I,-1,0)\subset\Sigma_k\Sigma_{k-1}^{-1}P_1(I,-1,0)\subset(1+3\widehat{C}2^{-\frac{\epsilon k}{2}})P_1(I,-1,0).$$

Since $\Sigma_k\Sigma_{k-1}^{-1}$ is still upper-triangular, we apply Lemma 2.1 (with $U=\Sigma_k\Sigma_{k-1}^{-1}$) to obtain that

$$\|\Sigma_k\Sigma_{k-1}^{-1}-I\|\le C(n)\sqrt{3\widehat{C}2^{-\frac{\epsilon k}{2}}}\le\widehat{C}2^{-\frac{\epsilon k}{4}}.$$

Estimate (37) has been established. The existence of T, τ and (39) follow by elementary consideration. □

We can deduce from (38) and (39) that on one side,

$$\Sigma(Q_{H'})-\Sigma_k(Q_{H'})\subset\widehat{C}2^{-\frac{\epsilon k}{4}}\sqrt{H'}P_1(I,-1,0),$$
$$\Sigma(Q_{H'})\subset(1+2\widehat{C}2^{-\frac{\epsilon k}{4}})\sqrt{H'}P_1(I,-1,0),$$

and on the other side,

$$\Sigma_k(Q_{H'})-\Sigma(Q_{H'})\subset\widehat{C}2^{-\frac{\epsilon k}{4}}\sqrt{H'}P_1(I,-1,0),$$
$$(1-2\widehat{C}2^{-\frac{\epsilon k}{4}})\sqrt{H'}P_1(I,-1,0)\subset\Sigma(Q_{H'}).$$

In particular, if we take $H'=2^k$, $k\ge\widehat{k}$, then

$$(1-2\widehat{C}(H')^{-\frac{\epsilon}{4}})\sqrt{H'}P_1(I,-1,0)\subset\{(y,s)\in\mathbb{R}^{n+1}_-:u(T^{-1}y,-\frac{s}{\tau})<H'\}$$
$$\subset(1+2\widehat{C}(H')^{-\frac{\epsilon}{4}})\sqrt{H'}P_1(I,-1,0).$$

So we have

$$(1-2\widehat{C}(u(T^{-1}y,-\frac{s}{\tau}))^{-\frac{\epsilon}{4}})^2u(T^{-1}y,-\frac{s}{\tau})<-s+\frac{1}{2}|y|^2$$
$$<(1+2\widehat{C}(u(T^{-1}y,-\frac{s}{\tau}))^{-\frac{\epsilon}{4}})^2u(T^{-1}y,-\frac{s}{\tau}),$$

$$
\begin{aligned}
&-4\widehat{C}(u(T^{-1}y,-\frac{s}{\tau}))^{1-\frac{\epsilon}{4}}-4\widehat{C}^2(u(T^{-1}y,-\frac{s}{\tau}))^{1-\frac{\epsilon}{2}}\\
&\quad< u(T^{-1}y,-\frac{s}{\tau})-(-s+\frac{1}{2}|y|^2)\\
&\quad< 4\widehat{C}(u(T^{-1}y,-\frac{s}{\tau}))^{1-\frac{\epsilon}{4}}-4\widehat{C}^2(u(T^{-1}y,-\frac{s}{\tau}))^{1-\frac{\epsilon}{2}}.
\end{aligned}
$$

Thus

$$
\left|u(T^{-1}y,-\frac{s}{\tau})-(-s+\frac{1}{2}|y|^2)\right|<\widehat{\widehat{C}}(u(T^{-1}y,-\frac{s}{\tau}))^{1-\frac{\epsilon}{4}}.
$$

Consequently, by the fact $C^{-1}u(T^{-1}y,-\frac{s}{\tau})\leq\frac{1}{2}|y|^2-s$, we obtain

$$
\left|u(T^{-1}y,-\frac{s}{\tau})-(-s+\frac{1}{2}|y|^2)\right|\leq C(\frac{1}{2}|y|^2-s)^{\frac{4-\epsilon}{4}},\quad (y,s)\in\mathbb{R}^{n+1}_-\setminus P_{2^{2\bar{k}}}(I,\tau,0).
$$

Let us define $w(y,s)=u(T^{-1}y,s)$; then

$$
w_s-\log\left(\det(D^2w)\right)=\log\left(\det(T^TT)\right)+a=\tau,\quad (y,s)\in\mathbb{R}^{n+1}_-\setminus P_{2^{2\bar{k}}}(I,\tau,0),
$$

where $\widetilde{\Sigma}(Q_H):=(T,id)Q_H$ and

$$
|w(y,s)-(\tau s+\frac{1}{2}|y|^2)|\leq C(\frac{1}{2}|y|^2+\tau s)^{\frac{4-\epsilon}{4}},\quad (y,s)\in\mathbb{R}^{n+1}_-\setminus P_{2^{2\bar{k}}}(I,\tau,0). \tag{44}
$$

We call the above inequality the asymptotic behavior of w.

3.4. Explicit asymptotic behavior

In this section, we will obtain the explicit asymptotic behavior of w.

Proposition 3.8. *There exist $\widetilde{b}\in\mathbb{R}^n$, $\widetilde{c}\in\mathbb{R}$ and some positive constant C depending on n, c_0, C_0, $\sup_{\mathbb{R}^{n+1}_-} f$, $\inf_{\mathbb{R}^{n+1}_-} f$ and meas$\{f-a\}$ such that*

$$
|w(y,s)-\tau s-\frac{|y|^2}{2}-\widetilde{b}^Ty-\widetilde{c}|\leq Ce^{-\tau s}(1+|y|^2)^{-\frac{n-2}{2}},\quad (y,s)\in\mathbb{R}^{n+1}_-\setminus P_{2^{2\bar{k}}}(I,\tau,0).
$$

Proof. Let

$$
g(y,s):=w(y,s)-(\tau s+\frac{|y|^2}{2}),\quad (y,s)\in\mathbb{R}^{n+1}_-\setminus P_{2^{2\bar{k}}}(I,\tau,0),
$$

and by (44) and Lemma 2.3,

$$
|D^2g(y,s)|+|g_s(y,s)|\leq C(\frac{1}{2}|y|^2-s)^{-\frac{\epsilon}{4}}.
$$

It follows that

$$g_s - \widehat{a}_{ij} D_{ij} g = 0 \quad \text{in } \mathbb{R}^{n+1}_- \setminus P_{2^{2\bar{k}}}(I, \tau, 0), \tag{45}$$

where $(\widehat{a}_{ij}(y,s)) = \int_0^1 (I + \theta D^2 g)^{-1} d\theta$.

Let $e \in \mathbb{R}^n$ be a unit vector; now we apply D_s, D_e and D_{ee} to the following equation

$$g_s - \log\Big(\det(I + D^2 g)\Big) = 0, \quad \text{in } \mathbb{R}^{n+1}_- \setminus P_{2^{2\bar{k}}}(I, \tau, 0),$$

in view of the concavity of log(det), we have

$$(g_s)_s - B_{ij} D_{ij}(g_s) = 0, \quad \text{in } \mathbb{R}^{n+1}_- \setminus P_{2^{2\bar{k}}}(I, \tau, 0), \tag{46}$$

$$(D_e g)_s - B_{ij} D_{ij}(D_e g) = 0, \quad \text{in } \mathbb{R}^{n+1}_- \setminus P_{2^{2\bar{k}}}(I, \tau, 0), \tag{47}$$

and

$$(D_{ee} g)_s - B_{ij}(D_{ee} g)_{ij} \le 0, \quad \text{in } \mathbb{R}^{n+1}_- \setminus P_{2^{2\bar{k}}}(I, \tau, 0), \tag{48}$$

where $(B_{ij}) = (I + D^2 g)^{-1}$. We claim that

$$|B_{ij} - \delta_{ij}| \le C(\frac{1}{2}|y|^2 - s)^{-\frac{\epsilon}{4}}.$$

In fact, let $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the eigenvalues of $D^2 g$; then

$$\begin{aligned}
|\frac{1}{1+\lambda_i} - 1| &= \frac{|\lambda_i|}{|1+\lambda_i|}, \\
&\le \frac{|\lambda_i|}{1 - |\lambda_i|}, \\
&= \frac{1}{|\lambda_i|^{-1} - 1}, \\
&\le \frac{1}{\frac{(\frac{1}{2}|y|^2 - s)^{\epsilon/4}}{C} - 1} \\
&\le \frac{1}{\frac{(\frac{1}{2}|y|^2 - s)^{\epsilon/4}}{C} - \frac{(\frac{1}{2}|y|^2 - s)^{\epsilon/4}}{2C}} \\
&\le C\left(\frac{1}{2}|y|^2 - s\right)^{-\frac{\epsilon}{4}},
\end{aligned}$$

this completes the proof of the claim.

By Lemma 2.2, for such coefficients, there exists a positive solution $G(y,s)$ of

$$G_s - B_{ij} D_{ij} G = e^{-\tau s}(1 + |y|^2)^{-\frac{n}{2}-1}\Big(-\tau(1+|y|^2)^2 + n(n-2)\Big) \ge 0 \quad \text{in } \mathbb{R}^{n+1}_-. \tag{49}$$

By (46), (48), (49) and the maximum principle, we have

$$-g_s(y,s) \le G(y,s) \le Ce^{-\tau s}(1+|y|^2)^{-\frac{n-2}{2}}, \tag{50}$$

and

$$D_{ee}g(y,s) \le G(y,s) \le Ce^{-\tau s}(1+|y|^2)^{-\frac{n-2}{2}}. \tag{51}$$

Thus

$$|D^2 g(y,s)| \le Ce^{-\tau s}(1+|y|^2)^{-\frac{n-2}{2}}.$$

Then for $1 \le m \le n$, there exists $\widetilde{b}_m \in \mathbb{R}$ satisfying $D_m g(y,s) - \widetilde{b}_m \to 0$, as $|y|^2 - s \to 0$ (see [18]) and $D_m g - \widetilde{b}_m$ satisfies

$$(D_m g - \widetilde{b}_m)_s - B_{ij}(D_m g - \widetilde{b}_m)_{ij} = 0, \quad (y,s) \in \mathbb{R}^{n+1}_- \setminus P_{2^{2\bar{k}}}(I,\tau,0),$$

from (47). Let $\widetilde{b} = (\widetilde{b}_1, \widetilde{b}_2 \cdots, \widetilde{b}_n)^T$; so we can also deduce from the maximum principle that

$$|Dg(y,s) - \widetilde{b}| \le Ce^{-\tau s}(1+|y|^2)^{-\frac{n-2}{2}}$$

by combining (49).

Similarly, there exists $\widetilde{c} \in \mathbb{R}$ such that

$$|g(y,s) - \widetilde{b}^T y - \widetilde{c}| \le Ce^{-\tau s}(1+|y|^2)^{-\frac{n-2}{2}}, \quad \forall (y,s) \in \mathbb{R}^{n+1}_- \setminus P_{2^{2\bar{k}}}(I,\tau,0). \qquad \square$$

3.5. Finishing proof of Theorem 1.1

Proof. Since $w(y,s) = u(T^{-1}y,s)$, then letting $x = T^{-1}y$ and $t = s$, by Proposition 3.8, we have

$$|u(x,t) - \tau t - \frac{x^T T^T T x}{2} - \widetilde{b}^T T x - \widetilde{c}| \le Ce^{-\tau t}(1+|x|^2)^{\frac{2-n}{2}}, \quad \frac{1}{2}|x|^2 - t \text{ is large enough.}$$

By taking $A = T^T T$, $b = T^T \widetilde{b}$, $c = \widetilde{c}$, we have

$$|u(x,t) - \tau t - \frac{x^T A x}{2} - b^T x - c| \le Ce^{-\tau t}(1+|x|^2)^{\frac{2-n}{2}}, \quad \frac{1}{2}|x|^2 - t \text{ is large enough.} \tag{52}$$

If $t^* \le t \le 0$, then by (52), we have

$$u(x,t) = \tau t + \frac{x^T A x}{2} + b^T x + c + O(|x|^{2-n}), \quad |x| \to \infty, t^* \le t \le 0.$$

If $t \le t^*$, by defining

$$u^*(x,t) = u(x, t + t^*) \quad \text{in } \mathbb{R}^{n+1}_-,$$

we have that $c_0 \le u^*_t \le C_0$ in $\mathbb{R}^{n+1}_-$,

$$u^*_t - \log\Big(\det(D^2u^*)\Big) = f(x, t + t^*) = a \quad \text{in } \mathbb{R}^{n+1}_-$$

and

$$|u^*(x,t) - \tau(t + t^*) - \frac{x^T Ax}{2} - b^T x - c| \le Ce^{-\tau t}(1 + |x|^2)^{\frac{2-n}{2}}, \quad \frac{1}{2}|x|^2 - t \to +\infty.$$

Now we define

$$E^*(x,t) := u^*(x,t) - \tau(t + t^*) - \frac{x^T Ax}{2} - b^T x - c,$$

then

$$E^*(x,t) \to 0, \quad \text{as } \frac{1}{2}|x|^2 - t \to +\infty.$$

Since

$$E^*_t - \widehat{a}_{ij} D_{ij} E^* = 0 \quad \text{in } \mathbb{R}^{n+1}_-,$$

where

$$(\widehat{a}_{ij}) = \int_0^1 (\theta D^2u + (1 - \theta)A)^{-1} d\theta,$$

by the maximum principle, $E^* \equiv 0$ in $\mathbb{R}^{n+1}_-$, i.e.,

$$u^*(x,t) = \tau(t + t^*) + \frac{x^T Ax}{2} + b^T x + c \quad \text{in } \mathbb{R}^{n+1}_-.$$

In conclusion, $u(x,t) = \tau t + \frac{x^T Ax}{2} + b^T x + c - \tau t^*$ in $\mathbb{R}^{n+1}_-$. □

Acknowledgments

The first author is supported in part by the scholarship from China Scholarship Council under the Grant CSC No. 201406040131. The research of the second author is partially supported by Beijing Municipal Commission of Education for the Supervisor of Excellent Doctoral Dissertation (20131002701). All authors were partially supported by NNSF (11371060) and the Fundamental Research Funds for the Central Universities.

References

[1] L. Caffarelli, A localization property of viscosity solutions to the Monge–Ampère equation and their strict convexity, Ann. of Math. 131 (1990) 129–134.

[2] L. Caffarelli, Topics in PDEs: the Monge–Ampère equation, Graduate course, Courant Institute, New York University, 1995.

[3] L. Caffarelli, Y.Y. Li, An extension to a theorem of Jorgens, Calabi, and Pogorelov, Comm. Pure Appl. Math. 56 (2003) 549–583.

[4] E. Calabi, Improper affine hypersurfaces of convex type and a generalization of a theorem by K. Jörgens, Michigan Math. J. 5 (1958) 105–126.

[5] A. Chau, J.Y. Chen, Y. Yuan, Rigidity of entire self-shrinking solutions to curvature flows, J. Reine Angew. Math. 664 (2012) 229–239.

[6] S.Y. Cheng, S.T. Yau, Complete affine hypersurfaces, 1. The completeness of the affine metrics, Comm. Pure Appl. Math. 39 (1986) 839–866.

[7] M. de Guzmén, Differentiation of Integrals in $\mathbb{R}^n$, Lecture Notes in Math., vol. 481, Springer, Berlin, New York, 1975.

[8] C.E. Gutiérrez, Q.B. Huang, A generalization of a theorem by Calabi to the parabolic Monge–Ampère equation, Indiana Univ. Math. J. 47 (4) (1998) 1459–1480.

[9] C.E. Gutiérrez, Q.B. Huang, Geometric properties of the sections of solutions to the Monge–Ampère equation, Trans. Amer. Math. Soc. 352 (9) (2000) 4381–4396.

[10] R.L. Huang, Z.Z. Wang, On the entire self-shrinking solutions to Lagrangian mean curvature flow, Calc. Var. Partial Differential Equations 41 (3–4) (2011) 321–339.

[11] K. Jörgens, Über die Lösungen der Differentailgleichung $rt - s^2 = 1$, Math. Ann. 127 (1954) 130–134.

[12] J. Jost, Y.L. Xin, Some aspects of the global geometry of entire space-like submanifolds, Results Math. 40 (2001) 233–245.

[13] G.M. Lieberman, Second Order Parabolic Differential Equations, World Scientific, River Edge, NJ, 1996.

[14] A.V. Pogorelov, The Minkowski Multidimensional Problem, John Wiley and Sons, Washington, DC, 1978.

[15] K. Tso, On an Aleksandrov–Bakelman type maximum principle for second-order parabolic equations, Comm. Partial Differential Equations 10 (1985) 543–553.

[16] R.H. Wang, G.L. Wang, On another kind of parabolic Monge–Ampère equation: the existence, uniqueness and regularity of the viscosity solution, Northeast. Math. J. 10 (4) (1994) 434–454.

[17] J.G. Xiong, J.G. Bao, On Jörgens, Calabi, and Pogorelov type theorem and isolated singularities of parabolic Monge–Ampère equations, J. Differential Equations 250 (2011) 367–385.

[18] W. Zhang, B. Wang, J.G. Bao, An extension of Jörgens, Calabi, and Pogorelov type theorem to parabolic Monge–Ampère equations, submitted for publication.