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Ancient solutions of exterior problem of parabolic Monge–Ampère equations

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Abstract
We use Perron method to prove the existence of ancient solutions of exterior problem for a kind of parabolic Monge–Ampère equation $-u_t \det D^2u = f$ with prescribed asymptotic behavior at infinity outside some certain bowl-shaped domain in the lower half space for $n \geq 3$, where f is a perturbation of 1 at infinity. We raise this problem for the first time and construct a new subsolution to it. We also use similar method to prove the existence of the entire solutions.

Keywords Parabolic Monge–Ampère equation · Exterior problem · Ancient solution · Asymptotics

Mathematics Subject Classification 35K96 · 35A01

1 Introduction

A celebrated result of Jörgens ($n = 2$ [11]), Calabi ($n \leq 5$ [5]) and Pogorelov ($n \geq 2$ [16]) stated that any entire classical convex solutions to the Monge–Ampère equation

$$\det D^2u = 1 \tag{1}$$

must be a quadratic polynomial. A simpler and more analytical proof was given by Cheng and Yau [7]. Jost and Xin [12] showed another geometric proof. Caffarelli [2] extended this result to viscosity solutions.

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Denote

$$\mathcal{A}_0 := \{A : A \text{ is a real } n \times n \text{ symmetric positive definite matrix with } \det A = 1\}.$$

In [3], Caffarelli and Li extended the theorem of Jörgens–Calabi–Pogorelov. They proved that for $n \geq 3$, the asymptotic behavior of viscosity solutions of (1) outside a bounded set must be

$$\limsup_{x\to\infty} |x|^{n-2}\Big|u(x) - \left(\frac{1}{2}x'Ax + b\cdot x + c\right)\Big| < \infty, \tag{2}$$

where $A \in \mathcal{A}_0$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. They also studied the existence of solutions of the corresponding exterior Dirichlet problem.

Theorem 1.1 (Caffarelli-Li [3]) *Let D be a smooth, bounded, strictly convex open subset of $\mathbb{R}^n$, $n \geq 3$, and let $\varphi \in C^2(\partial D)$. Then for any given $b \in \mathbb{R}^n$ and any given $A \in \mathcal{A}_0$, there exists some constant c^*, depending only on n, D, φ, b, and A, such that for every $c > c^*$ there exists a unique function $u \in C^\infty(\mathbb{R}^n \setminus \overline{D}) \cap C^0(\overline{\mathbb{R}^n \setminus D})$ that satisfies (2) and*

$$\begin{cases} \det D^2u = 1, \ D^2u > 0, & in\ \mathbb{R}^n \setminus \overline{D}, \\ u = \varphi, & on\ \partial D. \end{cases} \tag{3}$$

In [1], Bao, Li and Zhang extended [3] to

$$\det D^2u = f, \tag{4}$$

where f is a perturbation of 1 near infinity:

$$f(x) = 1 + O(|x|^{-\beta}),\ |x| \to \infty,\ \beta > 2,$$

with the asymptotic behavior

$$\begin{cases} \limsup\limits_{|x|\to\infty} |x|^{\min\{n,\beta\}-2}\big|u(x) - (\frac{1}{2}x'Ax + b\cdot x + c)\big| < \infty, & \beta \neq n, \\ \limsup\limits_{|x|\to\infty} |x|^{n-2}(\ln|x|)^{-1}\big|u(x) - (\frac{1}{2}x'Ax + b\cdot x + c)\big| < \infty, & \beta = n. \end{cases} \tag{5}$$

See also [6] for the classification of the asymptotic behavior. Recently, Li and Lu [14] continued to study the critical value c^* and proved the existence and nonexistence of the solutions.

For f being a periodic function, Li [13] proved the existence of entire solutions for (4). Caffarelli and Li [4] proved that the solution u has to be a parabola plus a periodic function. Teixeira–Zhang [17] proved the asymptotic behavior for f being asymptotically close to a periodic function with the dimension $n \geq 4$.

The study of elliptic Monge–Ampère equation was soon extended to parabolic cases. The operator $\mathcal{P}$ of a parabolic version

$$\mathcal{P}u = -u_t \det D^2u = f(x,t) \tag{6}$$

arises as the Jacobian determinant of the map

$$(x,t) \in \mathbb{R}^n \times \mathbb{R} \mapsto \Phi(x,t) := (D_xu(x,t), x\cdot D_xu(x,t) - u) \in \mathbb{R}^n \times \mathbb{R},$$

and has appeared in parabolic maximum principles, curvature flows and quantitative stochastic homogenization over the past few decades.

Recall that a solution is called ancient if it is defined on a time interval of the form $(-\infty, T]$ for some T. There are many results on ancient solutions of parabolic Monge–Ampère equations.

Let

$$\mathbb{R}^{n+1}_{-} = \{(x,t) \big| \, x \in \mathbb{R}^n, \ t \le 0\}.$$

The famous Jörgens, Calabi and Pogorelov theorem was extended by Gutiérrez and Huang [9] to ancient solutions of (6). Their main result is

Theorem 1.2 (Gutiérrez-Huang [9]) *Let $u \in C^{4,2}(\mathbb{R}^{n+1}_{-})$ be a parabolically convex solution to the parabolic Monge–Ampère equation*

$$-u_t \det D^2 u = 1, \quad in \ \mathbb{R}^{n+1}_{-} \tag{7}$$

such that there exist positive constants m_1 and m_2 with

$$-m_1 \le u_t(x,t) \le -m_2, \quad for\ all\ (x,t) \in \mathbb{R}^{n+1}_{-}. \tag{8}$$

Then u must have the form $u(x,t) = \tau t + p(x)$ where $\tau < 0$ is a constant and p is a convex quadratic polynomial.

They also gave an example of viscosity solutions to (7) that does not satisfy the given form above. Later, Xiong and Bao [21] extended this theorem to more general parabolic Monge–Ampère equations

$$u_t = \rho(\log\det D^2 u) \ \text{ in } \mathbb{R}^{n+1}_{-},$$

where $\rho = \rho(z) \in C^2(\mathbb{R})$.

Recently, using the method of perturbation, the study of parabolic equations emphasized on $C^{2,1}$ solutions. Zhang and Bao [23] extended the Liouville theorem of Caffarelli and Li [4] to the parabolic Monge–Ampère equation

$$-u_t \det D^2 u = f(x), \quad \text{in } \mathbb{R}^{n+1}_{-}, \tag{9}$$

where f is a positive periodic function, and $u \in C^{2,1}(\mathbb{R}^{n+1}_{-})$ satisfies (8). Zhang, Wang and Bao [24] considered the equation

$$-u_t \det D^2 u = f(x,t) \ \text{ in } \mathbb{R}^{n+1}_{-}, \tag{10}$$

where $f \in C^0(\mathbb{R}^{n+1}_{-})$ satisfies

$$0 < \inf_{\mathbb{R}^{n+1}_{-}} f \le \sup_{\mathbb{R}^{n+1}_{-}} f < +\infty$$

and

$$\text{supp}(f-1) \text{ is bounded.}$$

Denote

$$\mathcal{A} := \{A : \ A \text{ is a real } n \times n \text{ symmetric positive definite matrix}\}.$$

Their result is

Theorem 1.3 (Zhang–Wang–Bao [24]) *Let $n \geq 3$ and $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ be a parabolically convex solution to (10) such that (8) holds. Then there exist $\tau < 0$, $A \in \mathcal{A}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ satisfying $-\tau \det A = 1$ such that*

$$\limsup_{|x|^2-t\to\infty} e^{\tau t}(1+|x|^2)^{\frac{n-2}{2}}|u(x,t) - \left(\tau t + \frac{1}{2}x'Ax + b\cdot x + c\right)| < +\infty.$$

In [18], Wang and Bao proved the asymptotic behavior of solutions to the parabolic Monge–Ampère equation

$$u_t - \log\det D^2u = f(x,t) \;\text{ in } \mathbb{R}^{n+1}_-,\; n \geq 4,$$

with $\mathrm{supp}(f-a)$ being bounded for $a \in \mathbb{R}$.

The results above on asymptotics of ancient solutions naturally bring a question whether these ancient solutions exist or not. To our knowledge, there is no answer on this question so far.

In this paper, we first consider the exterior boundary value problem of the parabolic Monge–Ampère equation

$$\begin{cases} -u_t \det D^2u = f(x,t), \;\text{ in } \mathbb{R}^{n+1}_-\backslash\overline{D} & (11)\\ u = \varphi(x,t), \quad\quad \text{on } \partial_p D, & (12)\end{cases}$$

where $f \in C^0(\mathbb{R}^{n+1}_-)$ is positive and satisfies

$$f(x,t) = 1 + O(|x|^2-t)^{-\frac{\beta}{2}},\; |x|^2 - t \to \infty, \tag{13}$$

$\beta > 2$ is a constant, and

$$D = \{(x,t)\big|\, Q(x) < t \leq 0\},\; \partial_p D = \{(x,t)\big|\, Q(x) = t \leq 0\},$$

where $Q(x)$ is a strictly convex second-order differentiable function such that D is not empty and bounded.

Our first main theorem is

Theorem 1.4 *Let $n \geq 3$. For any $A \in \mathcal{A}$, $b \in \mathbb{R}^n$, and $\varphi \in C^2(\overline{D})$, there exists $c^* \in \mathbb{R}$, depending on n, A, b, φ, such that for any $c > c^*$, there exists a unique viscosity solution $u_c \in C^0(\mathbb{R}^{n+1}_-\backslash D)$ of (11), (12) and*

$$\begin{cases} \limsup\limits_{|x|^2-t\to+\infty} (|x|^2-t)^{\frac{\min\{n,\beta\}-2}{2}}\left|u_c(x,t) - (\tau t + \frac{1}{2}x'Ax + b\cdot x + c)\right| < \infty, & \beta \neq n,\\ \limsup\limits_{|x|^2-t\to+\infty} (|x|^2-t)^{\frac{n-2}{2}}(\ln(|x|^2-t)^{\frac{1}{2}})^{-1}\left|u_c(x,t) - (\tau t + \frac{1}{2}x'Ax + b\cdot x + c)\right| < \infty, & \beta = n.\end{cases} \tag{14}$$

where $\tau = -\frac{1}{\det A}$.

As an additional result, we use similar method to prove the existence of entire solutions. We consider the problem

$$-\, u_t \det D^2u = f(x,t) \;\text{ in } \mathbb{R}^{n+1}_-, \tag{15}$$

where $f \in C^0(\mathbb{R}^{n+1}_-)$ is positive and satisfies (13). Our second result is

Theorem 1.5 *Let $n \geq 3$. For any $A \in \mathcal{A}$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$, there exists a unique parabolically convex viscosity solution u to (15) satisfying (14).*

Remark 1.6 The assumption $\beta > 2$ is sharp in Theorems 1.4 and 1.5 . Let $r = |x|$, and f be a radial, smooth, positive function such that $f(r) \equiv 1$ for $r \in [0, 1]$ and $f(r) = 1 + r^{-2}$ for $r > 2$. Let

$$u(x,t) = -t + \int_0^{|x|} \left(\int_0^s ny^{n-1} f(y)\,\mathrm{d}y \right)^{\frac{1}{n}} \mathrm{d}s.$$

It is easy to check that

$$-u_t \det D^2 u = f \quad \text{in } \mathbb{R}^{n+1}_-.$$

Moreover,

$$u(x,t) = -t + \frac{1}{2}|x|^2 + O(\log |x|), \quad |x| \to \infty, \ \forall t \le 0,$$

which contradicts to (14).

In addition to ancient solutions defined in the lower half space, there are also results focusing on the time interval $(0, T]$ for some $T > 0$. In [19,20], the authors studied solutions of the interior first initial boundary value problem

$$\begin{cases} -u_t \det D^2 u = f(x,t), & \text{in } Q := \Omega \times (0, T], \\ u = \varphi(x,t), & \text{on } \partial_p Q := (\partial\Omega \times (0, T]) \cup (\Omega \times \{t = 0\}). \end{cases}$$

where Ω is a C^2 bounded convex domain in $\mathbb{R}^n$, and f, φ satisfies some certain conditions. In [8], Dai proved the existence of the viscosity solutions for the exterior initial-boundary problem of the equation

$$-u_t \det D^2 u = f(x), \quad \text{in } (\mathbb{R}^n \setminus \Omega) \times (0, T],$$

where f is a perturbation of 1 near infinity.

Note also that the domains in [8,19,20] are cylinders, which is a special case of bowl-shaped domains. General bowl-shaped domains can be time-varying domains leading to more difficulties.

The paper is arranged as follows. In Sect. 2, we give some useful lemmas. In Sects. 3 and 4, we prove Theorems 1.4 and 1.5, respectively, using Perron method.

2 Notations and lemmas

We begin with some notations. Given a bounded set $\Omega \subset \mathbb{R}^{n+1}_-$ and $t \in \mathbb{R}$, we denote

$$\Omega(t) = \{x : (x,t) \in \Omega\}.$$

Let $t_0 = \inf\{t : \Omega(t) \neq \emptyset\}$. The parabolic boundary of the bounded domain Ω is defined by

$$\partial_p \Omega = (\overline{\Omega}(t_0) \times \{t_0\}) \cup \Big(\bigcup_{t \in \mathbb{R}} (\partial\Omega(t) \times \{t\}) \Big),$$

where $\overline{\Omega}$ denotes the closure of Ω and $\partial\Omega(t)$ denotes the boundary of $\Omega(t)$. We say that the set $\Omega \subset \mathbb{R}^{n+1}$ is a bowl-shaped domain if $\Omega(t)$ is convex for each t and $\Omega(t_1) \subset \Omega(t_2)$ for $t_1 \le t_2$.

We say a function $u \in C^{k,j}(\Omega)$ which means that u is k-th continuous differentiable with spatial variables $x \in \mathbb{R}^n$ and j-th continuous differentiable with time variable t for

$(x,t)\in\Omega$. A function u is called locally parabolically convex if u is locally convex in x and nonincreasing in t.

The following is the definition of viscosity solutions [20,22].

Definition 2.1 Let u be a locally parabolically convex upper-semicontinuous (USC for short) (resp. lower-semicontinuous (LSC for short)) function in Ω. u is called a viscosity subsolution (supersolution) of

$$-u_t\det D^2u=f(x,t),\ \ \text{in }\Omega, \tag{16}$$

if for any point $(\bar{x},\bar{t})\in\Omega$ and any function $h\in C^{2,1}(Q_r(\bar{x},\bar{t}))$ and satisfying

$$u(x,t)-h(x,t)\le(\ge)u(\bar{x},\bar{t})-h(\bar{x},\bar{t}),\ \forall(x,t)\in Q_r(\bar{x},\bar{t}),$$

where

$$Q_r(\bar{x},\bar{t}):=\{(x,t)\big||x-\bar{x}|<r,\bar{t}-r^2<t\le\bar{t}\}\subset\Omega,$$

we have

$$-h_t(\bar{x},\bar{t})\det(D^2h(\bar{x},\bar{t}))\ge(\le)f(\bar{x},\bar{t}).$$

For the supersolution, we also require that $D^2h(\bar{x},\bar{t})>0$ in the matrix sense.

A function $u\in C^0(\Omega)$ is called a viscosity solution of (16), if it is both a viscosity subsolution and supersolution of (16).

Definition 2.2 A function u is called a viscosity subsolution (supersolution) of the problem (11), (12), if u is a viscosity subsolution (supersolution) of (11), and $u(x,t)\le(\ge)\varphi(x,t)$ on ∂_pD.

A function $u\in C^0(\mathbb{R}^{n+1}_-\backslash D)$ is called a viscosity solution of (11), (12), if u is a viscosity solution of (11), and $u(x,t)=\varphi(x,t)$ on ∂_pD.

Next we prove some useful lemmas.

Lemma 2.3 *Let $n\ge3$, $\varphi\in C^2(\overline{D})$. Then there exist some positive constants c_0 and C_0, where c_0 depends only on n, φ, D and C_0 depends only on n, φ, D and $\bar{c}$, such that, for any $(\xi,\lambda)\in\partial_pD$, there exists $\bar{x}(\xi,\lambda)\in\mathbb{R}^n$ satisfying*

$$|\bar{x}(\xi,\lambda)|\le C_0$$

and

$$w_{\xi,\lambda}(x,t)<\varphi(x,t)\text{ on }\partial_pD\backslash\{(\xi,\lambda)\},$$

where

$$w_{\xi,\lambda}(x,t)=\varphi(\xi,\lambda)-\bar{c}(t-\lambda)+\frac{1}{2}|x-\bar{x}|^2-\frac{1}{2}|\xi-\bar{x}|^2,\ (x,t)\in\mathbb{R}^{n+1}_-$$

for any $\bar{c}>c_0$.

Proof Denote

$$I:=\{x\in\mathbb{R}^n\Big|\ Q(x)\le0\}.$$

Let $(\xi,\lambda)\in\partial_pD$. By the mean value theorem, for $x\in I$, there exist $\xi_1,\ \xi_2\in I$ such that

$$\begin{aligned}Q(x)&=Q(\xi)+DQ(\xi_1)\cdot(x-\xi),\\ Q(x)&=Q(\xi)+DQ(\xi)\cdot(x-\xi)+\frac{1}{2}(x-\xi)'D^2Q(\xi_2)(x-\xi).\end{aligned}$$

Let

$$M_1 = \max_{x\in I} |DQ(x)|,\ M_2 = \frac{1}{2}\min_{x\in I} |D^2Q(x)|.$$

Then we have

$$\begin{aligned}|Q(x) - Q(\xi)| &\le M_1|x-\xi|,\\ Q(x) &\ge Q(\xi) + DQ(\xi)\cdot(x-\xi) + M_2|x-\xi|^2.\end{aligned}$$

Again by the mean value theorem, for $(x,t)\in\partial_p D$,

$$\begin{aligned}&\varphi(x,t)\\ &= \varphi(\xi,\lambda) + D_{x,t}\varphi(\xi,\lambda)\cdot\Big((x,t)-(\xi,\lambda)\Big) + \frac{1}{2}\Big((x,t)\\ &\quad - (\xi,\lambda)\Big)' D^2_{x,t}\varphi(\bar\xi,\bar\lambda)\Big((x,t)-(\xi,\lambda)\Big)\\ &\ge \varphi(\xi,\lambda) + D_x\varphi(\xi,\lambda)\cdot(x-\xi) + \varphi_t(\xi,\lambda)(t-\lambda) - C\Big(|x-\xi|^2 + (t-\lambda)^2\Big)\\ &= \varphi(\xi,\lambda) + D_x\varphi(\xi,\lambda)\cdot(x-\xi) + \varphi_t(\xi,\lambda)(Q(x)-Q(\xi))\\ &\quad - C\Big(|x-\xi|^2 + (Q(x)-Q(\xi))^2\Big),\end{aligned}$$

where $(\bar\xi,\bar\lambda)\in\overline{D}$, and $C = \frac{1}{2}(\max_{\bar D}|D^2_{x,t}\varphi| + \max_{\bar D}|\varphi_t|)$.

Define

$$w_{\xi,\lambda}(x,t) = \varphi(\xi,\lambda) - \bar c(t-\lambda) + \frac{1}{2}|x-\bar x|^2 - \frac{1}{2}|\xi-\bar x|^2,\ (x,t)\in\mathbb{R}^{n+1}_-,$$

where

$$\bar x(\xi,\lambda) = -D_x\varphi(\xi,\lambda) + \xi - (\bar c + \varphi_t(\xi,\lambda))DQ(\xi).$$

Then on $\partial_p D$,

$$\begin{aligned}w_{\xi,\lambda}(x,t) &= \varphi(\xi,\lambda) - \bar c(t-\lambda) + \frac{1}{2}\Big(|x|^2 - |\xi|^2\Big) - (x-\xi)\cdot\bar x\\ &= \varphi(\xi,\lambda) - \bar c(Q(x)-Q(\xi)) + \frac{1}{2}|x-\xi|^2 + D_x\varphi(\xi,\lambda)\cdot(x-\xi)\\ &\quad + (\bar c + \varphi_t(\xi,\lambda))DQ(\xi)\cdot(x-\xi).\end{aligned}$$

Thus for $\bar c \ge \max_{\bar D}|\varphi_t|$,

$$\begin{aligned}(w_{\xi,\lambda}-\varphi)(x,t) &\le (-\bar c - \varphi_t(\xi,\lambda))(Q(x)-Q(\xi))\\ &\quad + \frac{1}{2}|x-\xi|^2 + C\Big(|x-\xi|^2 + (Q(x)-Q(\xi))^2\Big)\\ &\quad + (\bar c + \varphi_t(\xi,\lambda))DQ(\xi)\cdot(x-\xi)\\ &\le (-\bar c - \varphi_t(\xi,\lambda))M_2|x-\xi|^2 + \frac{1}{2}|x-\xi|^2 + C\Big(|x-\xi|^2 + M_1^2|x-\xi|^2\Big)\\ &= \Big[(-\bar c - \varphi_t(\xi,\lambda))M_2 + \frac{1}{2} + C(1+M_1^2)\Big]|x-\xi|^2.\end{aligned}$$

Set $c_0 = \frac{1}{M_2}\left(\frac{1}{2} + C(1+M_1^2)\right) + 2C$, then for $\bar{c} > c_0$,

$$(-\bar{c} - \varphi_t(\xi,\lambda))M_2 + \frac{1}{2} + C(1+M_1^2) < 0,$$

and

$$(w_{\xi,\lambda} - \varphi)(x,t) < 0 \text{ on } \partial_p D\backslash\{(\xi,\lambda)\}.$$

□

We say Ω is an open set in the parabolic sense if $\Omega = \overline{\Omega}\backslash\partial_p\Omega$.

Lemma 2.4 *Let $\Omega_1 \subset \Omega_2$ be two open subsets in $\mathbb{R}^{n+1}$ in the parabolic sense. Suppose $u \in USC(\Omega_2)$ and $v \in USC(\overline{\Omega}_1)$ are locally parabolically convex and satisfy*

$$-u_t \det D^2 u \geq f(x,t) \text{ in } \Omega_2, \tag{17}$$

and

$$-v_t \det D^2 v \geq f(x,t) \text{ in } \Omega_1 \tag{18}$$

in the viscosity sense, respectively. Furthermore, assume

$$u \leq v \text{ in } \Omega_1, \quad u = v \text{ on } \partial\Omega_1 \setminus (\partial\Omega_1 \cap \partial\Omega_2).$$

Let

$$w(x,t) = \begin{cases} v(x,t), & (x,t) \in \Omega_1, \\ u(x,t), & (x,t) \in \Omega_2\backslash\Omega_1. \end{cases}$$

Then $w \in USC(\Omega_2)$ is locally parabolically convex and satisfy

$$-w_t \det D^2 w \geq f(x,t) \text{ in } \Omega_2$$

in the viscosity sense.

Proof Let $h \in C^{2,1}(\Omega_2)$ and $(\bar{x},\bar{t}) \in \Omega_2$ satisfying

$$w(x,t) - h(x,t) \leq w(\bar{x},\bar{t}) - h(\bar{x},\bar{t}), \ \forall (x,t) \in Q_r(\bar{x},\bar{t}),$$

for some $Q_r(\bar{x},\bar{t}) \subset \Omega_2$.

If $(\bar{x},\bar{t}) \in \Omega_1$, then for some $Q_{r_1}(\bar{x},\bar{t}) \subset Q_r(\bar{x},\bar{t}) \cap \Omega_1$,

$$v(x,t) - h(x,t) = w(x,t) - h(x,t) \leq w(\bar{x},\bar{t}) - h(\bar{x},\bar{t}) = v(\bar{x},\bar{t}) - h(\bar{x},\bar{t}), \ \forall (x,t) \in Q_{r_1}(\bar{x},\bar{t}).$$

By (18), we have

$$-h_t(\bar{x},\bar{t}) \det D^2 h(\bar{x},\bar{t}) \geq f(x,t).$$

If $(\bar{x},\bar{t}) \in \Omega_2\backslash\Omega_1$, then

$$u(x,t) - h(x,t) \leq w(x,t) - h(x,t) \leq w(\bar{x},\bar{t}) - h(\bar{x},\bar{t}) = u(\bar{x},\bar{t}) - h(\bar{x},\bar{t}), \ \forall (x,t) \in Q_r(\bar{x},\bar{t}).$$

By (17), we have

$$-h_t(\bar{x},\bar{t}) \det D^2 h(\bar{x},\bar{t}) \geq f(x,t).$$

□

Now we give the comparison principle below.

Lemma 2.5 (Comparison principle) *Let Ω be a bounded open set in $\mathbb{R}^{n+1}$ in the parabolic sense. Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ satisfy*

$$-u_t \det D^2 u \geq f(x,t) \ \text{ in } \Omega$$

and

$$-v_t \det D^2 v \leq f(x,t) \ \text{ in } \Omega$$

in the viscosity sense, respectively. Then we have

$$\sup_{\Omega}(u-v) \leq \sup_{\partial_p \Omega}(u-v). \tag{19}$$

Under the assumptions $u, v \in C^0(\overline{\Omega})$ and Ω being a cylindrical domain, the lemma was proved by Wang and Wang [20]. Based on their result, we can then directly obtain our comparison principle.

To introduce the Perron method for parabolic equations, we first define weak viscosity solutions which do not satisfy (semi) continuous properties.

Definition 2.6 Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set in parabolic sense. We say a function u is a weak viscosity subsolution of

$$-u_t \det D^2 u = f(x,t) \ \text{ in } \Omega$$

if the USC envelope of u, namely,

$$u^*(x,t) = \lim_{r \to 0} \sup_{(y,s) \in B_r(x,t)} u(y,s)$$

is finite and a viscosity subsolution, where

$$B_r(x,t) := \{(y,s) \big| \ |x-y|^4 + |t-s|^2 < r^2\} \subset \Omega.$$

Similarly, one uses LSC envelope $u_* = -(-u)^*$ for supersolutions. If u is a weak viscosity sub- and supersolution, we call u a weak viscosity solution.

We can also define weak viscosity solutions of the problem (11), (12) by giving the boundary condition like Definition 2.2.

Similar to the process by Zhan [22], we have the two lemmas below. We give the proof here for completeness.

Lemma 2.7 *Let Ω be an open set in $\mathbb{R}^{n+1}$ in the parabolic sense. Let $\mathcal{S}$ denote any nonempty set of weak viscosity subsolutions of*

$$-v_t \det D^2 v = f(x,t) \ \text{ in } \Omega. \tag{20}$$

Set

$$u(x,t) = \sup\{v(x,t) \big| \ v \in \mathcal{S}\} \ \text{ for } (x,t) \in \Omega.$$

Suppose $u^*(x,t) < \infty$ for $(x,t) \in \Omega$, then u is a weak viscosity subsolution of (20).

Proof By the definition of weak viscosity subsolutions, we need to prove that for all function $\varphi \in C^{2,1}(\Omega)$, if there exists $(\bar{x}, \bar{t}) \in \Omega$ such that

$$\max_{\overline{Q}_r}(u^* - \varphi) = (u^* - \varphi)(\bar{x}, \bar{t}),$$

for some $Q_r := Q_r(\bar{x}, \bar{t})$, then

$$-\varphi_t \det D^2\varphi(\bar{x}, \bar{t}) \geq f(\bar{x}, \bar{t}).$$

Without loss of generality, we can assume that $(u^* - \varphi)(\bar{x}, \bar{t}) = 0$.
Set

$$\psi(x, t) = \varphi(x, t) + |x - \bar{x}|^4 + |t - \bar{t}|^2,$$

then $u^* - \psi$ attains its strict maximum in $\overline{Q}_r$ at $(\bar{x}, \bar{t})$. So in Q_r,

$$(u^* - \psi)(x, t) + |x - \bar{x}|^4 + |t - \bar{t}|^2 = (u^* - \varphi)(x, t) \leq 0 = (u^* - \psi)(\bar{x}, \bar{t}),$$

and

$$(u^* - \psi)(x, t) \leq -|x - \bar{x}|^4 - |t - \bar{t}|^2.$$

By the definition of u, for any k, there is a $v_k \in S$ such that

$$u(\bar{x}, \bar{t}) - \frac{1}{k} < v_k(\bar{x}, \bar{t}).$$

Since $v_k^* - \psi \in USC(\Omega)$, it attains its maximum at (y_k, s_k) in some compact neighborhood $B \subset Q_r$ of $(\bar{x}, \bar{t})$. Noting that

$$(v_k^* - \psi)(x, t) \leq (u^* - \psi)(x, t) \leq -|x - \bar{x}|^4 - |t - \bar{t}|^2,$$

we have,

$$-\frac{1}{k} = (u^* - \psi)(\bar{x}, \bar{t}) - \frac{1}{k} \leq (v_k^* - \psi)(\bar{x}, \bar{t}) \leq (v_k^* - \psi)(y_k, s_k) \leq -|y_k - \bar{x}|^4 - |s_k - \bar{t}|^2 \leq 0.$$

Let $k \to \infty$, we have

$$\lim_{k\to\infty}(y_k, s_k) = (\bar{x}, \bar{t}).$$

Since v_k^* is a viscosity subsolution of (20), and $v_k^* - \psi$ attains its local maximum at (y_k, s_k), then

$$-\psi_t \det D^2\psi(y_k, s_k) \geq f(y_k, s_k).$$

Let $k \to \infty$, we have

$$-\psi_t \det D^2\psi(\bar{x}, \bar{t}) \geq f(\bar{x}, \bar{t}).$$

The lemma follows since at $(\bar{x}, \bar{t})$, $\psi_t = \varphi_t$, $D_x\psi = D_x\varphi$, and $D_{xx}^2\psi = D_{xx}^2\varphi$. □

Lemma 2.8 *Let g be a weak viscosity supersolution of* (20)*. Let*

$$S_g := \{v \big|\ v \text{ is a weak viscosity subsolution of (20) and } v \leq g\}$$

and

$$u(x, t) := \sup\{v(x, t) \big|\ v \in S_g\}.$$

If S_g is not empty, then u is a weak viscosity solution of (20)*.*

Proof By Lemma 2.7, u is a weak viscosity subsolution. If u is not a weak viscosity supersolution, there exists a function $\varphi \in C^{2,1}(\Omega)$, and a point $(\bar{x}, \bar{t}) \in \Omega$ such that

$$\min_{\overline{Q}_r}(u_* - \varphi) = (u_* - \varphi)(\bar{x}, \bar{t}) = 0$$

for some $Q_r := Q_r(\bar{x}, \bar{t})$, and

$$-\varphi_t \det D^2\varphi(\bar{x}, \bar{t}) > f(\bar{x}, \bar{t}).$$

We may assume here

$$(u_* - \varphi)(x, t) \geq |x - \bar{x}|^4 + |t - \bar{t}|^2$$

for $(x, t) \in Q_r$ since the function φ can be modified as $\varphi - |x - \bar{x}|^4 - |t - \bar{t}|^2$ if necessary.

Clearly $\varphi \leq u_* \leq g_*$ in Q_r, so $u_*(\bar{x}, \bar{t}) = \varphi(\bar{x}, \bar{t}) < g_*(\bar{x}, \bar{t})$, otherwise it would contradict the fact that g is a weak viscosity supersolution of (20).

Since f and φ are continuous, for $\delta > 0$ small enough, we have

$$-\varphi_t \det D^2\varphi(x, t) \geq f(x, t), \tag{21}$$

$$\varphi(x, t) + \delta^2 \leq g_*(x, t)$$

for $(x, t) \in B_{2\delta} = \overline{B((\bar{x}, \bar{t}), 2\delta)} \subset Q_r$, and $B((x, t), \delta) = \{(y, s) \in Q_r \big| |x - y|^4 + |t - s|^2 < \delta^2\}$.

(21) indicates that the function $\varphi(x, t) + \delta^2$ is a subsolution in $B_{2\delta}$; furthermore we have

$$u(x, t) \geq u_*(x, t) \geq \varphi(x, t) + |x - \bar{x}|^4 + |t - \bar{t}|^2 \geq \varphi(x, t) + \delta^2 \text{ in } B_{2\delta} \setminus B_\delta.$$

Define $w(x, t)$ by

$$w(x, t) = \begin{cases} \max\{\varphi(x, t) + \delta^2, u(x, t)\}, & (x, t) \in B_\delta, \\ u(x, t), & (x, t) \in \Omega \setminus B_\delta, \end{cases}$$

then

$$w(x, t) = \max\{\varphi(x, t) + \delta^2, u(x, t)\}, \quad (x, t) \in B_{2\delta}.$$

By Lemmas 2.4 and 2.7, w is a weak viscosity subsolution of (20) over Ω. Since $w \leq g$, $w \in S_g$. By the definition of u, we have $u \geq w$.

On the other hand, since

$$0 = (u_* - \varphi)(\bar{x}, \bar{t}) = \lim_{l \to 0} \inf_{(y,s) \in B_l} (u - \varphi)(y, s),$$

there is a point $(z, s) \in B_\delta$ such that $u(z, s) - \varphi(z, s) < \delta^2$ and $u(z, s) < w(z, s)$, which leads to a contradiction. □

Let $H > 0$, and $D_H = \{(x, t) \big| \frac{1}{2}|x|^2 - H^2 < t \leq 0\}$. We prove a parabolic version of Lemma 2.1 in [10].

Lemma 2.9 *Assume* $\varphi(h) \in C^2[0, H)$ *with* $\varphi'(0) = 0$. *Then for* $v(x, t) = \varphi(h)$ *where* $h = \sqrt{-t + \frac{1}{2}|x|^2} < H$, *we have that* $v(x, t) \in C^{2,1}(D_H)$.

Proof For $(x,t)\neq 0$,

$$\frac{\partial v}{\partial x_i}(x,t)=\varphi'(h)\frac{x_i}{2h},$$
$$\frac{\partial^2 v}{\partial x_i\partial x_j}(x,t)=\varphi''(h)\frac{x_ix_j}{4h^2}-\varphi'(h)\frac{x_ix_j}{4h^3}+\varphi'(h)\frac{\delta_{ij}}{2h},$$
$$\frac{\partial v}{\partial t}=\varphi'(h)\frac{-1}{2h}.$$

Since $\varphi'(0)=0$, we have

$$\lim_{(x,t)\to 0}\frac{\partial v}{\partial x_i}(x,t)=\lim_{(x,t)\to 0}\left(\frac{\varphi'(h)-\varphi'(0)}{h-0}\right)\frac{x_i}{2}=\varphi''(0)\cdot 0=0,$$
$$\lim_{(x,t)\to 0}\frac{\partial^2 v}{\partial x_i\partial x_j}(x,t)=\lim_{(x,t)\to 0}\left(\left(\varphi''(h)-\frac{\varphi'(h)}{h}\right)\frac{x_ix_j}{4h^2}+\left(\frac{\varphi'(h)}{h}\frac{\delta_{ij}}{2}\right)\right.$$
$$=\varphi''(0)\frac{\delta_{ij}}{2},$$
$$\lim_{(x,t)\to 0}\frac{\partial v}{\partial t}=\lim_{(x,t)\to 0}\left(\frac{\varphi'(h)-\varphi'(0)}{h-0}\right)\frac{-1}{2}=\varphi''(0)\frac{-1}{2}.$$

Define

$$\frac{\partial v}{\partial x_i}(0)=0,\ \frac{\partial^2 v}{\partial x_i\partial x_j}(0)=\varphi''(0)\frac{\delta_{ij}}{2},\ \frac{\partial v}{\partial t}(0)=\varphi''(0)\frac{-1}{2}.$$

Then $v\in C^{2,1}(D_H)$. □

3 Proof of Theorem 1.4

We first prove the uniqueness. Let $D_H=\{(x,t)\big|\frac{1}{2}|x|^2-H^2<t\leq 0\}$. Suppose u and v are both viscosity solutions of (11), (12) and satisfy (14). Then $\forall\varepsilon>0$, there exists $H>0$ such that $D\subset D_H$ and

$$u(x,t)+\varepsilon\geq v(x,t),\ \ \mathbb{R}^{n+1}_-\backslash D_H.$$

By the comparison principle in Lemma 2.5, we have $u+\varepsilon\geq v$ in $D_H\backslash D$. Thus $u+\varepsilon\geq v$ in $\mathbb{R}^{n+1}_-\backslash D$. Sending $\varepsilon\to 0$, we have $u\geq v$ in $\mathbb{R}^{n+1}_-\backslash D$. Similarly, we can also prove $u\leq v$ in $\mathbb{R}^{n+1}_-\backslash D$. Then the uniqueness is followed.

Next we prove the existence. By an affine transformation, we can only prove the case $A=I$, $b=0$. Without loss of generality, we can assume that $D_{H_1}\subset\subset D\subset\subset D_{H_2}$, where $H_2>H_1$. We divide the proof into five steps.

Step 1. Construct a viscosity subsolution $\underline{u}$ of (11) satisfying (12), (14) and a supersolution u_+ of (11), (12) satisfying (14) such that $\underline{u}\leq u_+$.

Let $F:=\sup_{\mathbb{R}^{n+1}_-}f(x,t)$. By Lemma 2.3, for any $(\xi,\lambda)\in\partial_pD$, there exist $c_0>0$ and $\bar{x}(\xi,\lambda)\in\mathbb{R}^n$, $|\bar{x}(\xi,\lambda)|<\infty$ such that

$$w_{\xi,\lambda}(x,t)<\varphi(x,t)\text{ on }\partial_pD\backslash\{(\xi,\lambda)\},$$

where

$$w_{\xi,\lambda}(x,t)=\varphi(\xi,\lambda)-\bar{c}(t-\lambda)+\frac{1}{2}|x-\bar{x}|^2-\frac{1}{2}|\xi-\bar{x}|^2,\ (x,t)\in\mathbb{R}^{n+1}_-,$$

and

$$\bar{c} = \max\{c_0, F\}.$$

Then

$$-(w_{\xi,\lambda})_t \det D^2 w_{\xi,\lambda} = \bar{c} \geq F \geq f(x,t),\ (x,t) \in \mathbb{R}^{n+1}_-.$$

Set

$$w(x,t) = \max_{(\xi,\lambda)\in\partial_p D} w_{\xi,\lambda}(x,t),\ (x,t) \in \mathbb{R}^{n+1}_-.$$

Then w is a locally Lipschitz function in $\mathbb{R}^{n+1}_-$,

$$w(x,t) = \varphi(x,t),\ (x,t) \in \partial_p D, \tag{22}$$

and by Lemma 2.7,

$$-w_t \det D^2 w \geq f(x,t),\ (x,t) \in \mathbb{R}^{n+1}_- \tag{23}$$

in the viscosity sense.

Let $r = |x|$, and $h(r,t) = \sqrt{-t + \frac{1}{2}r^2}$. By the definition of f, we can find continuous functions $\overline{f}(h)$ and $\underline{f}(h)$ satisfying

$$0 < \underline{f}(h) \leq f(x,t) \leq \overline{f}(h),\ \ \mathbb{R}^{n+1}_- \backslash D,$$
$$\underline{f}(h) \leq 1 \leq \overline{f}(h),\ \ h > 0,$$

$\underline{f}$ is monotonically nondecreasing in h for $h > 0$, and there exist $C, H > 0$ such that

$$\underline{f}(h) = 1 - Ch^{-\beta},\ \overline{f}(h) = 1 + Ch^{-\beta},\ \ \forall h \geq H.$$

We want to construct a subsolution $u_-(x,t) = U(h)$ to (11) that satisfies

$$-(u_-)_t \det D^2 u_- \geq \overline{f}(h) \text{ in } \mathbb{R}^{n+1}_- \setminus \overline{D_{H_1}}. \tag{24}$$

Since $h(r,t) = \sqrt{-t + \frac{1}{2}r^2} > H_1$ in $\mathbb{R}^{n+1}_- \backslash \overline{D_{H_1}}$, we have

$$h_t = \frac{-1}{2h},\ h_r = \frac{r}{2h},\ \ h_{rr} = \frac{-t}{2h^3},$$

and

$$(u_-)_t = U'h_t,\ (u_-)_r = U'h_r,\ (u_-)_{rr} = U''(h_r)^2 + U'h_{rr}.$$

Then we compute

$$\begin{aligned}
-(u_-)_t \det D^2 u_- &= -(u_-)_t (\frac{(u_-)_r}{r})^{n-1}(u_-)_{rr} \\
&= -U'h_t\Big(\frac{U'h_r}{r}\Big)^{n-1}(U''(h_r)^2 + U'h_{rr}) \\
&= \Big(\frac{U'}{2h}\Big)^n\Big(U''\frac{r^2}{4h^2} - U'\frac{t}{2h^3}\Big) \\
&= \frac{U'}{2h^3}\Big[\frac{U''}{2}(\frac{U'}{2h})^{n-1}\frac{r^2}{2} - (\frac{U'}{2h})^n t\Big] \\
&= \frac{U'}{2h^3}\Big[\frac{U''}{2}\Big(\frac{U'}{2h}\Big)^{n-1}h^2 + \Big(\frac{U'}{2h}\Big)^{n-1}\Big(\frac{U''}{2} - \frac{U'}{2h}\Big)t\Big].
\end{aligned}$$

For the thinking of transforming the partial differential inequality into an ordinary one, we expect the following conditions hold:

$$U' \geq 2h, \quad U'' - \frac{U'}{h} \leq 0. \tag{25}$$

Then we could obtain

$$-(u_-)_t \det D^2 u_- \geq \frac{U''}{2}\Big(\frac{U'}{2h}\Big)^{n-1} \text{ in } \mathbb{R}^{n+1}_- \setminus \overline{D_{H_1}}.$$

We know that solutions of

$$\frac{U''}{2}\Big(\frac{U'}{2h}\Big)^{n-1} = \overline{f}(h)$$

must be

$$U(h) = 2\int_0^h \Big(\int_0^s ny^{n-1}\overline{f}(y)\,\mathrm{d}y + C_1\Big)^{\frac{1}{n}}\,\mathrm{d}s + C_2,$$

where $C_1 \geq 0$, C_2 are constants. We expect $u_-(x,t) = U(h)$ with proper C_1 and C_2 to be the subsolution we want.

Now we check the conditions (25). We have

$$U'(h) = 2\Big(\int_0^h ny^{n-1}\overline{f}(y)\,\mathrm{d}y + C_1\Big)^{\frac{1}{n}}, \tag{26}$$

$$U''(h) = 2h^{n-1}\overline{f}(h)\Big(\int_0^h ny^{n-1}\overline{f}(y)\,\mathrm{d}y + C_1\Big)^{\frac{1}{n}-1}. \tag{27}$$

The first condition is obvious since we require $\overline{f} \geq 1$. We claim here that if C_1 is large enough, the second condition also holds, i.e.,

$$\overline{f}(h)h^n - \int_0^h ny^{n-1}\overline{f}(y)\,\mathrm{d}y \leq C_1, \quad \forall h > H_1.$$

In fact, when h is small, the inequality is obvious. So we can only consider the case $h \geq H$,

$$\begin{aligned}
&\overline{f}(h)h^n - \int_0^h ny^{n-1}\overline{f}(y)\,\mathrm{d}y \\
&= (\overline{f}(h)-1)h^n - \int_0^h ny^{n-1}(\overline{f}(y)-1)\,\mathrm{d}y \\
&= Ch^{n-\beta} - \int_0^H ny^{n-1}(\overline{f}(y)-1)\,\mathrm{d}y - \int_H^h ny^{n-1}Cy^{-\beta}\,\mathrm{d}y \\
&= \begin{cases} -\frac{C\beta}{n-\beta}h^{n-\beta} + \frac{Cn}{n-\beta}H^{n-\beta} - \int_0^H ny^{n-1}(\overline{f}(y)-1)\,\mathrm{d}y, & \beta \neq n, \\ C - Cn\ln h + Cn\ln H - \int_0^H ny^{n-1}(\overline{f}(y)-1)\,\mathrm{d}y, & \beta = n, \end{cases} \\
&\leq CH^{n-\beta} - \int_0^H ny^{n-1}(\overline{f}(y)-1)\,\mathrm{d}y.
\end{aligned}$$

Then if C_1 is large enough, the inequality follows.

Due to the process above, for $C_1, C_2 > 0$, we define functions

$$u_-(x,t) = 2\int_{H_2}^{\sqrt{-t+\frac{1}{2}|x|^2}} \Big(\int_0^s ny^{n-1}\overline{f}(y)\,\mathrm{d}y + C_1\Big)^{\frac{1}{n}}\,\mathrm{d}s + \inf_{D_{H_2}} w, \ (x,t) \in \mathbb{R}^{n+1}_-, \tag{28}$$

and

$$u_+(x,t) = 2\int_{H_1}^{\sqrt{-t+\frac{1}{2}|x|^2}} \left(\int_0^s ny^{n-1}\underline{f}(y)\,\mathrm{d}y\right)^{\frac{1}{n}} \mathrm{d}s + C_2,\ (x,t)\in\mathbb{R}^{n+1}_-.$$

Then we see that

$$u_-(x,t) \le 2\int_{H_2}^{H_2} \left(\int_0^s ny^{n-1}\overline{f}(y)\,\mathrm{d}y + C_1\right)^{\frac{1}{n}} \mathrm{d}s + \inf_{D_{H_2}} w \le w(x,t),\ \ \text{on } \partial_p D. \tag{29}$$

Choose $H_3 = H_2 + 1$ and sufficiently large C_1, C_2 such that the following three inequalities hold at the same time

$$u_-(x,t) = 2\int_{H_2}^{H_3} \left(\int_0^s ny^{n-1}\overline{f}(y)\,\mathrm{d}y + C_1\right)^{\frac{1}{n}} \mathrm{d}s + \inf_{D_{H_2}} w \ge w(x,t),\ \ \text{on } \partial_p D_{H_3}, \tag{30}$$

$$u_+(x,t) = 2\int_{H_1}^{H_3} \left(\int_0^s ny^{n-1}\underline{f}(y)\,\mathrm{d}y\right)^{\frac{1}{n}} \mathrm{d}s + C_2 \ge w(x,t),\ \ \text{on } \partial_p D_{H_3}, \tag{31}$$

$$u_+(x,t) = C_2 \ge w(x,t) \ge u_-(x,t),\ \ \text{on } \partial_p D_{H_1}. \tag{32}$$

We have already proved that if C_1 is large enough,

$$-(u_-)_t \det D^2 u_- \ge \overline{f}(h),\ \ \text{in } \mathbb{R}^{n+1}_- \backslash \overline{D_{H_1}}. \tag{33}$$

Using the expressions (26) and (27) for $U'(h)$ and $U''(h)$, similar to the computations of u_-, we can easily obtain that

$$-(u_+)_t \det D^2 u_+ \le \underline{f}(h) + \frac{1}{h^3}\left(\int_0^h ny^{n-1}\underline{f}(y)\,\mathrm{d}y\right)^{\frac{1}{n}} \left[\underline{f}(h) - \frac{1}{h^n}\left(\int_0^h ny^{n-1}\underline{f}(y)\,\mathrm{d}y\right)\right] t.$$

Since $\underline{f}$ is monotonically nondecreasing in h, we have

$$\int_0^h ny^{n-1}(\underline{f}(h) - \underline{f}(y))\,\mathrm{d}y \ge 0.$$

Thus we have

$$-(u_+)_t \det D^2 u_+ \le \underline{f}(h),\ \ \text{in } \mathbb{R}^{n+1}_- \backslash \overline{D_{H_1}}. \tag{34}$$

Now we consider the asymptotics. As $s \to +\infty$, the integrand function in (28)

$$\begin{aligned}
&\left(\int_0^s ny^{n-1}\overline{f}(y)\,\mathrm{d}y + C_1\right)^{\frac{1}{n}} \\
&= (s^n + C_1 + \int_0^H ny^{n-1}(\overline{f}(y) - 1)\,\mathrm{d}y + \int_H^s ny^{n-1}Cy^{-\beta}\,\mathrm{d}y)^{\frac{1}{n}} \\
&= \begin{cases} (s^n + d_1 + d_2 s^{n-\beta})^{\frac{1}{n}}, & \beta \ne n, \\ (s^n + d_3 + d_4 \ln s)^{\frac{1}{n}}, & \beta = n, \end{cases} \\
&= \begin{cases} s(1 + d_1 s^{-n} + d_2 s^{-\beta})^{\frac{1}{n}}, & \beta \ne n, \\ s(1 + d_3 s^{-n} + d_4 s^{-n}\ln s)^{\frac{1}{n}}, & \beta = n, \end{cases} \\
&= \begin{cases} s + O(s^{1-\min\{n,\beta\}}), & \beta \ne n, \\ s + O(s^{1-n}\ln s), & \beta = n, \end{cases}
\end{aligned}$$

where

$$d_1 = C_1 + \int_0^H ny^{n-1}(\overline{f}(y) - 1)\,\mathrm{d}y - \frac{Cn}{n-\beta}H^{n-\beta},\ d_2 = \frac{Cn}{n-\beta},$$
$$d_3 = C_1 + \int_0^H ny^{n-1}(\overline{f}(y) - 1)\,\mathrm{d}y - Cn\ln H,\ d_4 = Cn.$$

Thus, as $|x|^2 - t \to \infty$,

$$\begin{aligned} u_-(x,t) &= 2\int_{H_2}^{\sqrt{-t+\frac{1}{2}|x|^2}} \left(\int_0^s ny^{n-1}\overline{f}(y)\,\mathrm{d}y + C_1\right)^{\frac{1}{n}} \mathrm{d}s + \inf_{D_{H_2}} w \\ &= 2\int_{H_2}^{\sqrt{-t+\frac{1}{2}|x|^2}} \left[\left(\int_0^s ny^{n-1}\overline{f}(y)\,\mathrm{d}y + C_1\right)^{\frac{1}{n}} - s\right] \mathrm{d}s - t + \frac{1}{2}|x|^2 - H_2{}^2 + \inf_{D_{H_2}} w \\ &= \begin{cases} -t + \frac{1}{2}|x|^2 + \mu(C_1) + O\Big((|x|^2 - t)^{\frac{2-\min\{n,\beta\}}{2}}\Big), & \beta \neq n, \\ -t + \frac{1}{2}|x|^2 + \mu(C_1) + O\Big((|x|^2 - t)^{\frac{2-n}{2}} \ln(|x|^2 - t)^{\frac{1}{2}}\Big), & \beta = n, \end{cases} \end{aligned}$$

where

$$\mu(C_1) = 2\int_{H_2}^{\infty} \left[\left(\int_0^s ny^{n-1}\overline{f}(y)\,\mathrm{d}y + C_1\right)^{\frac{1}{n}} - s\right] \mathrm{d}s - H_2{}^2 + \inf_{D_{H_2}} w < +\infty.$$

Similarly, as $|x|^2 - t \to +\infty$,

$$\begin{aligned} u_+(x,t) &= 2\int_{H_1}^{\sqrt{-t+\frac{1}{2}|x|^2}} \left(\int_0^s ny^{n-1}\underline{f}(y)\,\mathrm{d}y\right)^{\frac{1}{n}} \mathrm{d}s + C_2 \\ &= \begin{cases} -t + \frac{1}{2}|x|^2 + \nu(C_2) + O\Big((|x|^2 - t)^{\frac{2-\min\{n,\beta\}}{2}}\Big), & \beta \neq n, \\ -t + \frac{1}{2}|x|^2 + \nu(C_2) + O\Big((|x|^2 - t)^{\frac{2-n}{2}} \ln(|x|^2 - t)^{\frac{1}{2}}\Big), & \beta = n, \end{cases} \end{aligned}$$

where

$$\nu(C_2) = 2\int_{H_1}^{\infty} \left[\left(\int_0^s ny^{n-1}\underline{f}(y)\,\mathrm{d}y\right)^{\frac{1}{n}} - s\right] \mathrm{d}s - H_1^2 + C_2 < +\infty.$$

We can see that $\mu(C_1)$ and $\nu(C_2)$ are continuous and strictly increasing in $(0, +\infty)$ and

$$\lim_{C_1 \to +\infty} \mu(C_1) = +\infty, \quad \lim_{C_2 \to +\infty} \nu(C_2) = +\infty.$$

Then there exists c^* large enough such that for any $c > c^*$, there exists $C_1(c)$ and $C_2(c)$ satisfying $\mu(C_1(c)) = c$ and $\nu(C_2(c)) = c$. Therefore, we have, as $|x|^2 - t \to +\infty$,

$$u_-(x,t) = u_+(x,t) = \begin{cases} -t + \frac{1}{2}|x|^2 + c + O\Big((|x|^2 - t)^{\frac{2-\min\{n,\beta\}}{2}}\Big), & \beta \neq n, \\ -t + \frac{1}{2}|x|^2 + c + O\Big((|x|^2 - t)^{\frac{2-n}{2}} \ln(|x|^2 - t)^{\frac{1}{2}}\Big), & \beta = n. \end{cases} \tag{35}$$

Then by (32), (33), (34), (35), and Lemma 2.5,

$$u_-(x,t) \le u_+(x,t), \quad \text{in } \mathbb{R}^{n+1}_- \backslash D_{H_1}. \tag{36}$$

For $c > c^*$, define

$$\underline{u}(x,t) = \begin{cases} \max\{w(x,t), u_-(x,t)\}, & D_{H_3}\backslash D_{H_1}, \\ u_-(x,t), & \mathbb{R}^{n+1}_-\backslash Q_{H_3}. \end{cases}$$

By (30), we know that $\underline{u} \in C^0(\mathbb{R}^{n+1}_-\backslash D)$ is locally parabolically convex. And by (23), (33) and Lemma 2.4, 2.7, $\underline{u}$ satisfies

$$-\underline{u}_t \det D^2\underline{u} \geq f(x,t), \quad \text{in } \mathbb{R}^{n+1}_-\backslash \overline{D} \tag{37}$$

in the viscosity sense. By (22), (29),

$$\underline{u}(x,t) = w(x,t) = \varphi(x,t), \quad (x,t) \in \partial_p D. \tag{38}$$

Then $\underline{u}$ is a viscosity subsolution of (11), (12). And by (35), as $|x|^2 - t \to +\infty$,

$$\underline{u}(x,t) = \begin{cases} -t + \frac{1}{2}|x|^2 + c + O\Big((|x|^2 - t)^{\frac{2-\min\{n,\beta\}}{2}}\Big), & \beta \neq n, \\ -t + \frac{1}{2}|x|^2 + c + O\Big((|x|^2 - t)^{\frac{2-n}{2}} \ln(|x|^2 - t)^{\frac{1}{2}}\Big), & \beta = n. \end{cases} \tag{39}$$

Furthermore, by (23), (31), (32), and Lemma 2.5,

$$w(x,t) \leq u_+(x,t), \quad \text{in } D_{H_3}\backslash D_{H_1}.$$

Then combining with (36), we have

$$\underline{u}(x,t) \leq u_+(x,t) \ \text{ in } \mathbb{R}^{n+1}_-\backslash D. \tag{40}$$

Step 2. Define the Perron solution of (11), (12).

For $X = \mathbb{R}^{n+1}_-\backslash D$, let $S_{c,X}$ denote the set of locally parabolically convex functions v which are weak viscosity subsolutions of (11), (12) satisfying

$$v(x,t) \leq u_+(x,t), \quad \text{in } \mathbb{R}^{n+1}_-\backslash D.$$

According to Step 1, $\underline{u} \in S_{c,X}$. So $S_{c,X} \neq \emptyset$. Define

$$u_c(x,t) = \sup\{v(x,t) : \ v \in S_{c,X}\}, \ (x,t) \in \mathbb{R}^{n+1}_-\backslash D.$$

Step 3. We prove that u_c has the asymptotic behavior at infinity.

First, by the definition of u_c, we have

$$u_c(x,t) \leq u_+(x,t), \quad \text{in } \mathbb{R}^{n+1}_-\backslash D. \tag{41}$$

And since $\underline{u} \in S_{c,X}$, by (35), (39), as $|x|^2 - t \to +\infty$,

$$u_c(x,t) = \begin{cases} -t + \frac{1}{2}|x|^2 + c + O\Big((|x|^2 - t)^{\frac{2-\min\{n,\beta\}}{2}}\Big), & \beta \neq n, \\ -t + \frac{1}{2}|x|^2 + c + O\Big((|x|^2 - t)^{\frac{2-n}{2}} \ln(|x|^2 - t)^{\frac{1}{2}}\Big), & \beta = n. \end{cases} \tag{42}$$

Step 4. We prove that $u_c(x,t) = \varphi(x,t), \ (x,t) \in \partial_p D$.

For any $(\xi,\tau) \in \partial_p D$, on the one hand, by (38), we have

$$\liminf_{(x,t)\to(\xi,\tau)} u_c(x,t) \geq \lim_{(x,t)\to(\xi,\tau)} \underline{u}(x,t) = \varphi(\xi,\tau).$$

On the other hand, we have

$$\limsup_{(x,t)\to(\xi,\tau)} u_c(x,t) \leq \varphi(\xi,\tau).$$

Indeed, for every $v \in S_{c,X}$, by the definition of viscosity solutions, we have, in the viscosity sense,

$$\begin{cases} -v_t^* + \Delta v^* \geq 0, & (x,t) \in D_{H_2} \backslash \overline{D}, \\ v^* \leq \varphi, & (x,t) \in \partial_p D, \\ v^* \leq \sup\limits_{\partial_p D_{H_2}} u_+ =: B, & (x,t) \in \partial_p D_{H_2}. \end{cases}$$

Let $w^+ \in C^{2,1}(D_{H_2} \backslash \overline{D}) \cap C^0(\overline{D_{H_2}} \backslash D)$ be the solution of the problem [15]

$$\begin{cases} -w_t^+ + \Delta w^+ = 0, & (x,t) \in D_{H_2} \backslash \overline{D}, \\ w^+ = \varphi, & (x,t) \in \partial_p D, \\ w^+ = B, & (x,t) \in \partial_p D_{H_2}. \end{cases}$$

By the comparison principle for the heat conduction equation, which can be proved directly by the definition of viscosity solutions, we have $v \leq v^* \leq w^+, \ (x,t) \in \overline{D_{H_2}} \backslash D$. So $u_c \leq w^+, \ (x,t) \in \overline{D_{H_2}} \backslash D$, and

$$\limsup_{(x,t) \to (\xi,\tau)} u_c(x,t) \leq \lim_{(x,t) \to (\xi,\tau)} w^+(x,t) = \varphi(\xi,\tau).$$

Step 5. We prove that u_c is a viscosity solution of (11).

By the definition of u_c and Lemma 2.8, we can prove that u_c is a weak viscosity solution of (11). Then by Lemma 2.5 and the asymptotic behavior, $u_c^* \leq u_{c*}$. By the definition of u_c^* and u_{c*}, $u_c^* \geq u_{c*}$. So $u_c^* = u_{c*} = u_c$, then u_c is continuous and a viscosity solution.

4 Proof of Theorem 1.5

The uniqueness part can be proved easily by comparison principle and the asymptotic behavior (14). We only prove the existence part and assume $A = I$, $b = 0$ and $c = 0$.

For C_3 large enough, define

$$u_-(x,t) = 2 \int_0^h \left(\int_0^s n y^{n-1} \bar{f}(y)\, \mathrm{d}y + C_3 \right)^{\frac{1}{n}} \mathrm{d}s - \gamma(C_3),$$

and

$$u_+(x,t) = 2 \int_0^h \left(\int_0^s n y^{n-1} \underline{f}(y)\, \mathrm{d}y \right)^{\frac{1}{n}} \mathrm{d}s - C_4,$$

where

$$\gamma(C_3) = 2 \int_0^\infty \left[\left(\int_0^s n y^{n-1} \overline{f}(y)\, \mathrm{d}y + C_3 \right)^{\frac{1}{n}} - s \right] \mathrm{d}s < +\infty,$$

and

$$C_4 = 2 \int_0^\infty \left[\left(\int_0^s n y^{n-1} \underline{f}(y)\, \mathrm{d}y \right)^{\frac{1}{n}} - s \right] \mathrm{d}s < +\infty,$$

Then by computation in Sect. 3, definition of viscosity solutions, and Lemma 2.9, we have

$$-(u_-)_t \det D^2 u_- \geq f \ \text{ in } \mathbb{R}_-^{n+1}$$

and

$$-(u_+)_t \det D^2 u_+ \le f \quad \text{in } \mathbb{R}^{n+1}_-$$

in viscosity sense. And

$$u_-(x,t) = u_+(x,t) = -t + \frac{1}{2}|x|^2 + \begin{cases} O(|x|^2 - t)^{\frac{2-\min\{n,\beta\}}{2}}, & \beta \ne n, \\ O(|x|^2 - t)^{\frac{n-2}{2}} \ln(|x|^2 - t)^{\frac{1}{2}}, & \beta = n. \end{cases}$$

By comparison principle and the asymptotic behavior, we can obtain that $u_- \le u_+$.

Let S denote the set of parabolically convex functions v which are weak viscosity subsolutions of (15), satisfying

$$v(x,t) \le u_+(x,t), \quad \text{in } \mathbb{R}^{n+1}_-.$$

Since $u_- \in S$, $S \ne \emptyset$. Define

$$u(x,t) = \sup\{v(x,t) : \ v \in S\}, \ (x,t) \in \mathbb{R}^{n+1}_-.$$

By the definition of u, we have

$$u_-(x,t) \le u(x,t) \le u_+(x,t), \quad \text{in } \mathbb{R}^{n+1}_-.$$

Then as $|x|^2 - t \to +\infty$,

$$u(x,t) = -t + \frac{1}{2}|x|^2 + \begin{cases} O(|x|^2 - t)^{\frac{2-\min\{n,\beta\}}{2}}, & \beta \ne n, \\ O(|x|^2 - t)^{\frac{n-2}{2}} \ln(|x|^2 - t)^{\frac{1}{2}}, & \beta = n. \end{cases}$$

Using Perron method, comparison principle and the asymptotic behavior as Step 5 in Sect. 3, we can prove that u is a viscosity solution of (15).

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