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Calculus of Variations

An extension of Jörgens–Calabi–Pogorelov theorem to parabolic Monge–Ampère equation

Wei Zhang[1,2] · **Jiguang Bao**[3] · **Bo Wang**[4]

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Abstract We extend a theorem of Jörgens, Calabi and Pogorelov on entire solutions of elliptic Monge–Ampère equation to parabolic Monge–Ampère equation, and obtain delicate asymptotic behavior of solutions at infinity. For the dimension $n \geq 3$, the work of Gutiérrez and Huang in Indiana Univ. Math. J. **47**, 1459–1480 (1998) is an easy consequence of our result. And along the line of approach in this paper, we can treat other parabolic Monge–Ampère equations.

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1 Introduction

A celebrated result of Jörgens ($n = 2$ [13]), Calabi ($n \leq 5$ [5]) and Pogorelov ($n \geq 2$ [20]) states that any classical convex solutions to the Monge–Ampère equation

$$\det D^2 u = 1 \text{ in } \mathbb{R}^n \tag{1.1}$$

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✉ Jiguang Bao
jgbao@bnu.edu.cn

Wei Zhang
zw9_27@126.com; wzhang@math.uni-hannover.de

Bo Wang
wangbo89630@bit.edu.cn

1 Department of Mathematics, Beijing Technology and Business University, Beijing 100048, China

2 Institut für Analysis, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

3 School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China

4 School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China

must be a quadratic polynomial. A simpler and more analytical proof was given by Cheng and Yau [6]. Jost and Xin showed a quite different proof in [14]. Caffarelli [2] extended above result for classical solutions to viscosity solutions. Caffarelli and Li [3] considered

$$\det D^2 u = f \text{ in } \mathbb{R}^n, \tag{1.2}$$

where f is a positive continuous function and is not equal to 1 only on a bounded set. They proved that for $n \geq 3$, the convex viscosity solution u is very close to quadratic polynomial at infinity. More precisely, for $n \geq 3$, there exist $c \in \mathbb{R}$, $b \in \mathbb{R}^n$ and an $n \times n$ symmetric positive definite matrix A with $\det A = 1$, such that

$$\limsup_{|x|\to\infty} |x|^{n-2} \left| u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + c \right) \right| < \infty.$$

In a subsequent work [4], Caffarelli and Li proved that if f is periodic, then u must be the sum of a quadratic polynomial and a periodic function. In recent paper [12], a similar theorem for a Monge–Ampère equation in half space was established by Jian and Wang.

Above famous Jörgens, Calabi and Pogorelov theorem was extended by Gutiérrez and Huang [9] to solutions of the following parabolic Monge–Ampère equation

$$- u_t \det D^2 u = 1, \tag{1.3}$$

where $u = u(x, t)$ is parabolically convex, i.e., u is convex in x and nonincreasing in t, and $D^2 u$ denotes the Hessian of u with respect to the variable x. They got

Theorem 1.1 *Let $u \in C^{4,2}(\mathbb{R}^{n+1}_-)$ be a parabolically convex solution to the parabolic Monge–Ampère equation (1.3) in $\mathbb{R}^{n+1}_- := \mathbb{R}^n \times (-\infty, 0]$, such that there exist positive constants m_1 and m_2 with*

$$- m_1 \leq u_t(x, t) \leq -m_2, \quad \forall (x, t) \in \mathbb{R}^{n+1}_-. \tag{1.4}$$

Then u must have the form $u(x, t) = C_1 t + p(x)$, where $C_1 < 0$ is a constant and p is a convex quadratic polynomial on x.

and they gave an example to show that viscosity solutions to (1.3) may not be of the form given by above theorem. Recently, Bao and Xiong [26] extended this theorem to general parabolic Monge–Ampère equations.

This type of parabolic Monge–Ampère operator was first introduced by Krylov [15]. Owing to its importance in stochastic theory, he further considered it in [16–18]. This operator is relevant in the study of deformation of a surface by Gauss–Kronecker curvature [8]. Indeed, Tso [23] solved this problem by noting that the support function to the surface that is deforming satisfies an initial value problem involving that parabolic operator. And the operator plays an important role in a maximum principle for parabolic equations [22].

For the parabolic Monge–Ampère equation, there are many results about existence and regularity. For example, Wang and Wang [24] proved the existence of viscosity solutions to (1.3) with an initial boundary value by the approximation procedure and the nonlinear perturbation method, and $C^{2+\alpha,1+\alpha/2}$ regularity of the viscosity solutions. Later, they [25] developed a geometric measure theory associated with above parabolic Monge–Ampère operator, and then used this theory to prove the existence of a viscosity solution to an initial boundary value problem. Gutiérrez and Hang [11] obtained that the interior $W^{2,p}$ estimates for (1.3). Recently, Tang [21] obtained the same estimates under weaker conditions.

In this paper, we extend the theorem of Caffarelli and Li [3] to above parabolic Monge–Ampère equation, and obtain asymptotic behavior at infinity.

Theorem 1.2 *Let $n \geq 3$ and $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ be a parabolically convex solution to the parabolic Monge–Ampère equation*

$$-u_t \det D^2 u = f(x,t) \quad in\ \mathbb{R}^{n+1}_-, \tag{1.5}$$

such that (1.4) holds, where $f \in C^0(\mathbb{R}^{n+1}_-)$ satisfies

$$0 < \inf_{\mathbb{R}^{n+1}_-} f \leq \sup_{\mathbb{R}^{n+1}_-} f < +\infty \tag{1.6}$$

and

$$support\ (f-1)\ is\ bounded. \tag{1.7}$$

Then there exist $\tau < 0$, an $n \times n$ symmetric positive definite matrix A, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ satisfying $-\tau \det A = 1$ such that $E(x,t) := u(x,t) - (\tau t + \frac{x^T Ax}{2} + b^T x + c)$ satisfies

$$\limsup_{|x|^2 - t \to +\infty} e^{\tau t}(1+|x|^2)^{\frac{n-2}{2}} |E(x,t)| < +\infty. \tag{1.8}$$

Moreover, u is C^∞ in the complement of the support of $(f-1)$ and

$$\limsup_{|x|^2 - t \to +\infty} \left(\frac{|x|^2}{2} - t\right)^{\frac{n-2+k}{2}} |D_x^i D_t^j E(x,t)| < +\infty,\ i + 2j = k,\ \forall\, k \geq 1. \tag{1.9}$$

For $n \geq 3$, the theorem of Jörgens, Calabi and Pogorelov to (1.3) is an easy consequence of Theorem 1.2.

Corollary 1.3 *Let $n \geq 3$ and $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ be a parabolically convex solution to (1.3) such that (1.4) holds. Then u must have the form $u(x,t) = C_1 t + p(x)$, where $C_1 < 0$ is a constant and p is a convex quadratic polynomial on x.*

Proof By Theorem 1.2, for some $\tau < 0$, symmetric positive definite matrix A with $-\tau \det A = 1$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$, we have

$$E(x,t) := u(x,t) - \tau t - \frac{x^T Ax}{2} - b^T x - c \to 0, \quad \text{as } \sqrt{|x|^2 - t} \to +\infty.$$

Denote $F(a, M) = -a \det M$. Since

$$F(\tau + E_t, A + D^2 E) - F(\tau, A) = 1 - 1 = 0,$$

it follows that

$$\widehat{a}_1 E_t + \widehat{a}_{ij} D_{ij} E = 0 \quad \text{in } \mathbb{R}^{n+1}_-,$$

where

$$\widehat{a}_1(x,t) = \int_0^1 F_1(-1 + \theta E_t, A + \theta D^2 E) d\theta,\ \widehat{a}_{ij}(x,t) = \int_0^1 F_{ij}(-1 + \theta E_t, A + \theta D^2 E) d\theta.$$

By the maximum principle, $E(x,t) \equiv 0$, i.e.,

$$u(x,t) = \tau t + \frac{x^T Ax}{2} + b^T x + c.$$

□

Remark 1.1 For $n \geq 3$, the same result of Theorem 1.1 is obtained under weaker regularity on u. Precisely, by nonlinear perturbation method developed by Caffarelli, we only need $u \in C^{2,1}$.

Throughout the paper we work on the parabolic Monge–Ampère equation (1.5), but our methods can be applied to other parabolic Monge–Ampere equations, such as

$$\begin{aligned} u_t &= (\det D^2 u)^{\frac{1}{n}} + f(x,t), \\ u_t &= \log \det D^2 u + f(x,t). \end{aligned} \tag{1.10}$$

Taking the (1.10) for example, we get

Corollary 1.4 *Let $f \in C^0(\mathbb{R}^{n+1}_-)$, satisfy (1.6) and (1.7), and let $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ be a convex solution to (1.10) satisfying*

$$\hat{m} \leq u_t \leq \hat{M} \tag{1.11}$$

Then there exist $\tau, c \in \mathbb{R}$, $b \in \mathbb{R}^n$ and a symmetric positive definite $n \times n$ matrix A with $\tau - \log \det A = 1$, such that $E(x,t) := u(x,t) - [\tau t + \frac{1}{2}x^T Ax + b \cdot x + c]$ satisfies

$$\limsup_{|x|^2 - t \to +\infty} e^{\tau t}(1 + |x|^2)^{\frac{n-2}{2}} |E(x,t)| < +\infty.$$

Moreover, u is C^∞ in the complement of the support of $(f-1)$ and

$$\limsup_{|x|^2 - t \to +\infty} \left(\frac{|x|^2}{2} - t \right)^{\frac{n-2+k}{2}} |D_x^i D_t^j E(x,t)| < +\infty, \ i + 2j = k, \ \forall\, k \geq 1.$$

Proof Let

$$\overline{u}(x,t) = u(x,t) - (1 + \hat{M})t.$$

Then $\overline{u} \in C^{2,1}(\mathbb{R}^{n+1}_-)$ is a solution to

$$\overline{u}_t = \log \det D^2 \overline{u} + \overline{f},$$

where $\hat{m} - 1 - \hat{M} \leq \overline{u}_t \leq -1$, $\overline{f} = f - (1 + \hat{M})$. Then support $(\overline{f} - \hat{M})$ is bounded. Now following the same line of the proof of above theorem, we get the asymptotic behavior of $\overline{u}$. Finally, we have the estimates for u. □

Recently, the first and second authors [27] classify all solutions to

$$-u_t \det D^2 u = f(x) \ in \ \mathbb{R}^{n+1}_-,$$

where $f \in C^\alpha(\mathbb{R}^n)$ is a positive periodic function in x. More precisely, if u is a solution to above equation, then u is the sum of a convex quadratic polynomial in x, a periodic function in x and a linear function of t. Indeed, from the regularity theorem obtained by the first author [28], we are able to get the above theorem under the weaker condition $f \in VMO^\psi(\mathbb{R}^n)$.

The paper is organized as follows. Section 2 lists some notations and lemmas used in the proof of Theorems 1.1 and 1.2. The proof of Theorem 1.1 is carried out again by our notations in Sect. 3. Finally, we give the proof of Theorem 1.2 in Sect. 4.

2 Preliminary

We begin to introduce some notations. Let $\mathbb{R}^{n+1}_{-} = \mathbb{R}^n \times (-\infty, 0]$. A function $u : \mathbb{R}^{n+1}_{-} \to \mathbb{R}$, $(x, t) \mapsto u(x, t)$, is called parabolically convex if it is continuous, convex in x and non-increasing in t. We denote by $D^2u(x, t)$, $Du(x, t)$ the matrix of second derivatives and the gradient of u with respect to x respectively. We use the notation $C^{2k,k}(\mathbb{R}^{n+1}_{-})$ to denote the class of functions u such that the derivatives $D^i_x D^j_t u$ are continuous in $\mathbb{R}^{n+1}_{-}$ for $i+2j \le 2k$.

Let $D \subset \mathbb{R}^{n+1}_{-}$ be a bounded set and $t \le 0$, then we denote

$$D(t) = \{x \in \mathbb{R}^n : (x, t) \in D\},$$

and $t_0 = \inf\{t : D(t) \neq \emptyset\}$. The parabolic boundary of the bounded domain D is defined by

$$\partial_p D = (\overline{D(t_0)} \times \{t_0\}) \cup \bigcup_{t\in\mathbb{R}} (\partial D(t) \times \{t\}),$$

where $\overline{D}$ denotes the closure of D and $\partial D(t)$ denotes the boundary of $D(t)$. We say that the set $D \subset \mathbb{R}^{n+1}_{-}$ is a bowl-shaped domain if $D(t)$ is convex for each t and $D(t_1) \subset D(t_2)$ for $t_1 \le t_2$.

We recall the definition of cross section of a convex function. Let $\hat{u} : \mathbb{R}^n \to \mathbb{R}$ be a convex function that for simplicity is assumed smooth. A cross section of $\hat{u}$ at the point $x_0 \in \mathbb{R}^n$ and with height $H > 0$ is the convex set defined by

$$S_{\hat{u}}(x_0, H) = \{x : \hat{u}(x) < \hat{u}(x_0) + D\hat{u}(x_0) \cdot (x - x_0) + H\}.$$

Throughout the following proof of Theorems1.1 and 1.2, we will always assume that

$$u(0,0) = 0, \quad Du(0,0) = 0, \quad D^2u(0,0) = Id, \quad u_t(0,0) = -1, \tag{2.1}$$

and

$$u(x,t) \ge 0, \quad \forall (x,t) \in \mathbb{R}^{n+1}_{-}. \tag{2.2}$$

In fact, we first show that we can assume $u_t(0,0) = -1$. Let $\xi(x,t) = u(\beta x, \alpha t)$, where β and α are two positive numbers. Then $-\xi_t(x,t)\det D^2\xi(x,t) = \alpha\beta^{2n}$, and we can pick β and α such that $\alpha\beta^{2n} = 1$ and $\xi_t(0,0) = \alpha u_t(0,0) = -1$.

Secondly, we show that we can also assume $u(0,0) = 0$ and $Du(0,0) = 0$. Let $\varphi(x,t) = \xi(x,t) - \xi(0,0) - D\xi(0,0)\cdot x$. Then we have $\varphi(0,0) = 0$, $D\varphi(0,0) = 0$, $\varphi_t(0,0) = -1$, and $-\varphi_t(x,t)\det D^2\varphi(x,t) = 1$.

Thirdly, we give the reason for the assumption of $D^2u(0,0) = Id$. Since $\xi(x,t)$ is parabolically convex, $\varphi(x,t)$ is parabolically convex. There exists an orthogonal matrix O such that

$$O^T D^2\varphi(0,0) O = diag\{d_1, d_2, \ldots, d_n\},$$

where $d_i > 0$, $i = 1, 2, \ldots, n$. Let $\psi(x,t) = \varphi(O diag\left\{\frac{x_i}{\sqrt{d_i}}\right\}, t)$. Then

$$D\psi(x,t) = diag\left\{\frac{1}{\sqrt{d_i}}\right\} O^T (D\varphi)\left(O diag\left\{\frac{x_i}{\sqrt{d_i}}\right\}, t\right),$$

$$D^2\psi(x,t) = diag\left\{\frac{1}{\sqrt{d_i}}\right\} O^T D^2\varphi\left(O diag\left\{\frac{x_i}{\sqrt{d_i}}\right\}, t\right) O diag\left\{\frac{1}{\sqrt{d_i}}\right\},$$

and hence $\psi(0,0) = 0$, $D\psi(0,0) = 0$ and $D^2\psi(0,0) = Id$. Since

$$\det D^2\varphi(0,0) = -\frac{1}{\varphi_t(0,0)} = 1,$$

we get that $d_1 d_2 \cdots d_n = 1$ and we can obtain

$$\begin{aligned}
&-\psi_t(x,t)\det D^2\psi(x,t)\\
&= -\varphi_t\left(O\left(\frac{x_1}{\sqrt{d_1}}, \frac{x_2}{\sqrt{d_2}}, \ldots, \frac{x_n}{\sqrt{d_n}}\right), t\right)\\
&\frac{1}{d_1 d_2 \cdots d_n}\det D^2\varphi\left(O\left(\frac{x_1}{\sqrt{d_1}}, \frac{x_2}{\sqrt{d_2}}, \ldots, \frac{x_n}{\sqrt{d_n}}\right), t\right)\\
&= 1.
\end{aligned}$$

This completes the proof of assumption (2.1). By (2.1) and the definition of parabolically convex function we can get that

$$u(x,t) \geq u(x,0) \geq u(0,0) = 0, \quad \forall (x,t) \in \mathbb{R}^{n+1}_-.$$

This completes the proof of assumption (2.2).

In the rest of this section, we would like to give some lemmas that will be useful in the proof of Theorems 1.1 and 1.2.

Lemma 2.1 *Let U be an $(n+1)\times(n+1)$ real upper-triangular matrix. Assume that the diagonals of U are nonnegative and for some $0 < \epsilon < 1$,*

$$(1-\epsilon)\widetilde{E} \subset U(\widetilde{E}) \subset (1+\epsilon)\widetilde{E}, \tag{2.3}$$

where $\widetilde{E} = \{(y,s) \in \mathbb{R}^{n+1}_- : \frac{1}{2}|y|^2 - s < 1\}$ and $(1+\epsilon)\widetilde{E} = \{(y,s) \in \mathbb{R}^{n+1}_- : \frac{1}{2}|y|^2 - s < (1+\epsilon)^2\}$. Then for some constant $C = C(n)$,

$$\|U - I\| \leq C\sqrt{\epsilon}. \tag{2.4}$$

Proof Let $U = (U_{ij})$, we know that $U_{ij} = 0$ for $i < j$. Since $U(\widetilde{E})$ contains an open neighborhood of $\mathbb{R}^{n+1}$, U is invertible. Therefore $U_{ii} > 0$, $i = 1, 2, \ldots, n, n+1$. Write $U^{-1} = (U^{ij})$; then U^{-1} is also upper-triangular, $U^{ii} = \frac{1}{U_{ii}}$, $i = 1, 2, \ldots, n, n+1$. For $1 \leq k \leq n+1$, let e_k denote the unit vector with the kth component equal to 1 and the others equal to zero. By (2.3), it is easy to check that

$$\begin{aligned}
&U(\sqrt{2}e_k) \in (1+\epsilon)\overline{\widetilde{E}}, \quad k = 1, 2, \ldots, n,\\
&U(-e_{n+1}) \in (1+\epsilon)\overline{\widetilde{E}},
\end{aligned}$$

then we have

$$\sqrt{\sum_{j=1}^{n} U_{jk}^2} \leq 1+\epsilon, \quad k = 1, 2, \ldots, n, \tag{2.5}$$

and

$$\frac{1}{2}\sum_{j=1}^{n} U_{j,n+1}^2 + U_{n+1,n+1} \leq (1+\epsilon)^2. \tag{2.6}$$

In particular, $U_{kk} \leq 1+\epsilon$, $1 \leq k \leq n+1$. The same argument can be applied to U^{-1}, so

$$\frac{1}{U_{kk}} = U^{kk} \leq \frac{1}{1-\epsilon}, \quad 1 \leq k \leq n+1.$$

We deduce from the two above estimates that

$$1-\epsilon \leq U_{kk} \leq 1+\epsilon, \quad 1 \leq k \leq n+1. \tag{2.7}$$

It follows from (2.5) and (2.7) that

$$\sum_{j\neq k} U_{jk}^2 \leq (1+\epsilon)^2 - (1-\epsilon)^2 = 4\epsilon, \quad 1 \leq k \leq n. \tag{2.8}$$

It also follows from (2.6) and (2.7) that

$$\begin{aligned}\sum_{j\neq n+1} U_{j,n+1}^2 &\leq 2(1+\epsilon)^2 - 2U_{n+1,n+1}\\ &\leq 2(1+\epsilon)^2 - 2(1-\epsilon)\\ &\leq 8\epsilon.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|U - I\| &= \sqrt{\sum_{j=1}^{n+1}(U_{jj}-1)^2 + \sum_{j\neq k} U_{jk}^2}\\ &\leq \sqrt{\sum_{j=1}^{n+1}\epsilon^2 + 8(n^2-n)\epsilon}\\ &\leq \sqrt{(n+1)\epsilon + 8(n^2-n)\epsilon}\\ &= C(n)\sqrt{\epsilon}.\end{aligned}$$

□

We recall that $u : D \to \mathbb{R}$ is continuous, then the parabolic normal mapping of u is the set valued function $\mathcal{P}_u : D \to \{E : E \in \mathbb{R}^{n+1}\}$ defined by

$$\begin{aligned}\mathcal{P}(x_0, t_0) = \{(p, H) :\ & u(x,t) \geq u(x_0,t_0) + p\cdot(x-x_0),\\ & \forall x \in D(t), t \leq t_0,\ H = p\cdot x_0 - u(x_0,t_0)\}.\end{aligned}$$

If $D' \subset D$, then $\mathcal{P}_u(D') = \bigcup_{(x,t)\in D'} \mathcal{P}_u(x,t)$. And the parabolic Monge–Ampère measure associated with u defined by $|\mathcal{P}_u(D')|_{n+1}$ is a Borel measure, where $|\cdot|_{n+1}$ is the Lebesgue measure in $\mathbb{R}^{n+1}$. The following lemma is an extension to the parabolic case of a result first proved by Alexandrov.

Lemma 2.2 ([9], Theorem 2.1) *Let $D \subset \mathbb{R}^{n+1}$ be an open bounded bowl-shaped domain and $u \in C(\overline{D})$ a parabolically convex function with $u = 0$ on $\partial_p D$. If $(x_0, t_0) \in D$ then*

$$|u(x_0,t_0)|^{n+1} \leq C(n) dist(x_0, \partial D(t_0)) diam(D(t_0))^{n-1} |\mathcal{P}_u(D_{t_0})|_{n+1},$$

where $D_{t_0} = D \cap \{(x,t) : t \leq t_0\}$.

Lemma 2.3 ([11], Proposition 4.1) *Let Q be a normalized bowl-shaped domain in $\mathbb{R}^{n+1}$, which definition will be given in Sect. 3, and u a parabolically convex function in Q satisfying $0 < -u_t \det D^2 u \leq \Lambda$ in Q, $\min_{\overline{Q}} u = 0$, $-m_1 \leq u_t < 0$ in Q, and $u = 1$ on $\partial_p Q$. If $u(X_0) < 1-\varepsilon$, $\varepsilon \in (0,1)$ then $dist(X_0, \partial_p Q) \geq C\varepsilon^{n+1}$, where $X_0 = (x_0, t_0)$ and $C = C(n, \Lambda, m_1)$.*

Lemma 2.4 *Let $n \geq 3$ and $A = (a_{ij}(x,t))$ is a real $n\times n$ symmetric positive definite matrix with*

$$|a_{ij} - \delta_{ij}| \leq \frac{C}{(|x|^2+|t|)^{\varepsilon}}, \quad (x,t) \in \mathbb{R}_-^{n+1}, \tag{2.9}$$

and $a_{ij}(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\mathbb{R}^{n+1}_-)$*, where* $\varepsilon, \alpha \in (0,1)$ *are constants. Then there exists a positive solution to*

$$u_t - a_{ij}D_{ij}u = e^t g(x) =: e^t\left[\frac{1}{(1+|x|^2)^{\frac{n-2}{2}}} + \frac{n(n-2)}{(1+|x|^2)^{\frac{n+2}{2}}}\right] \geq 0, \quad in \quad \mathbb{R}^{n+1}_-, \tag{2.10}$$

satisfying

$$0 \leq u(x,t) \leq \frac{C(n,\varepsilon)e^t}{(1+|x|^2)^{\frac{n-2}{2}}}, \quad in \quad \mathbb{R}^{n+1}_-. \tag{2.11}$$

The idea of the proof comes from Elmar Schrohe and the first author. The desired existence is established by a convergence argument, and (2.11) is an easy consequence of maximum principle.

Proof Firstly, we prove the existence of u. Denote $E_m = \{(x,t) \in \mathbb{R}^{n+1}_- : |x|^2 - t < m^2\}$, $m = 1, 2, \ldots$. Considering

$$\begin{cases} (u_m)_t - a_{ij}D_{ij}u_m = e^t g(x), & \text{in } E_m, \\ u_m = \frac{e^t}{(1+|x|^2)^{\frac{n-2}{2}}} & \text{on } \partial_p E_m, \end{cases} \tag{2.12}$$

we see from [19] that there exists $u_m \in C^{2+\alpha,1+\alpha/2}(E_m) \cap C(\overline{E_m})$ satisfying (2.12). Since $e^t g(x) > 0$ and $u_m|_{\partial_p E_m} > 0$, by maximum principle,

$$u_m > 0. \tag{2.13}$$

Let $w(x,t) = \sup_{\partial_p E_m} u_m + e^t \sup_{E_m} g(x)$, we then have

$$w_t - a_{ij}D_{ij}w = e^t \sup_{E_m} g(x) \geq e^t g(x) = (u_m)_t - a_{ij}D_{ij}u_m,$$

and

$$w|_{\partial_p E_m} \geq u_m|_{\partial_p E_m}.$$

By maximum principle,

$$u_m \leq \sup_{\partial_p E_m} u_m + e^t \sup_{E_m} g(x). \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$|u_m|_{L^\infty(E_m)} \leq \sup_{\partial_p E_m} u_m + e^t \sup_{E_m} g(x) \leq (n-1)^2 + 1. \tag{2.15}$$

By the Schauder interior estimates, we get

$$|u_m|_{C^{2+\alpha,1+\alpha/2}(\overline{E_1})} \leq \frac{C'}{dist(E_1, \partial_p E_m)}(|u_m|_{L^\infty(E_m)} + |e^t g(x)|_{C^{\alpha,\alpha/2}(\overline{E_m})}) \leq C, \quad \forall m > 1,$$

where C depends on $dist(E_1, \partial_p E_m)$, not on m. Therefore there is a subsequence $\{u_m^{(1)}\}$, such that

$$u_m^{(1)} \to u^{(1)} \in C^{2+\alpha,1+\alpha/2}(\overline{E_1}), \quad in \quad C^{2,1}(\overline{E_1}), \quad as\ m \to \infty.$$

For $E_k \subset \mathbb{R}^{n+1}_-$,

$$|u_m^{(k-1)}|_{C^{2+\alpha,1+\alpha/2}(\overline{E_k})} \leq C, \quad m > k,$$

where C depends on k and $dist(E_k, \partial_p E_m)$, not on m. So there is a subsequence $\{u_m^{(k)}\}$, such that

$$u_m^{(k)} \to u^{(k)} \in C^{2+\alpha,1+\alpha/2}(\overline{E_k}), \quad in \quad C^{2,1}(\overline{E_k}), \quad as\ m \to \infty,$$

and $u^{(k)} = u^{(j)}$ in E_j, $j = 1, 2, \dots, k-1$.

Define $u(x,t) = u^{(k)}(x,t)$, if $(x,t) \in E_k$, then $u(x,t)$ is defined on $\mathbb{R}_-^{n+1}$. Consider sequence $\{u_m^{(m)}\}$ in diagram, for any E_k,

$$u_m^{(m)} \to u \in C^{2+\alpha,1+\alpha/2}(\overline{E_k}), \quad in \quad C^{2,1}(\overline{E_k}), \quad as\, m \to \infty \quad (\{u_m^{(m)}\} \subset \{u_m^{(k)}\},\ if\, m > k),$$

Since

$$(u_m^{(m)})_t - a_{ij} D_{ij} u_m^{(m)} = e^t g(x), \quad in\ E_k, \quad \forall\, m > k,$$

we then find u is the solution of

$$u_t - a_{ij} D_{ij} u = e^t g(x),$$

as $m \to \infty$.

Next, the proof of (2.11) is given. It is easy to check that $v = \frac{e^t}{(1+|x|^2)^{\frac{n-2}{2}}}$ is the unique positive smooth solution of

$$v_t - \Delta v = e^t g(x) \quad in\ \mathbb{R}_-^{n+1}. \tag{2.16}$$

Hence,

$$v_t - a_{ij} D_{ij} v = e^t g(x) + (\delta_{ij} - a_{ij}) D_{ij} v \quad in\ \mathbb{R}_-^{n+1}. \tag{2.17}$$

From $|D^2 v| \le \frac{Ce^t}{(1+|x|^2)^{\frac{n}{2}}}$ and (2.9), there exists a $L > 0$ such that

$$|(\delta_{ij} - a_{ij}) D_{ij} v| \le \frac{1}{2} e^t g(x) \quad in\ \mathbb{R}_-^{n+1} \backslash E_L. \tag{2.18}$$

For $(x,t) \in \bar{E}_L$, by (2.15), it is easy to see

$$\begin{aligned} u_m(x,t) &\le ((n-1)^2 + 1) \frac{v(x,t)}{\min_{\bar{E}_L} v} \\ &\le \max\left\{2, (1+L^2)^{\frac{n-2}{2}} e^{L^2} [(n-1)^2 + 1]\right\} v(x,t) \\ &:= C_L v(x,t), \quad m > L. \end{aligned} \tag{2.19}$$

Therefore we obtain

$$(C_L v)_t - a_{ij} D_{ij} (C_L v) \ge \frac{C_L}{2} e^t g(x) \ge (u_m)_t - a_{ij} D_{ij} u_m, \ E_m \backslash E_L,$$

and

$$C_L v \ge u_m \quad on\ \partial_p E_m \cup \partial_p E_L.$$

Using maximum principle, we get

$$u_m \le C_L v \quad in\ E_m \backslash E_L.$$

Let $m \to \infty$,

$$u \le C_L v \quad in\ \mathbb{R}_-^{n+1} \backslash E_L. \tag{2.20}$$

Combining above inequality and (2.19), we finish the proof. □

3 Proof of Theorem 1.1

Given $H > 0$, let

$$Q_H = \{(x,t) \in \mathbb{R}^{n+1}_- \ : \ u(x,t) < H\}, \quad Q_H(t) = \{x \in \mathbb{R}^n \ : \ u(x,t) < H\} \ for \ t \le 0. \tag{3.1}$$

Let x_H be the mass center of $Q_H(0)$, E the ellipsoid of minimum volume containing $Q_H(0)$ with center x_H. By a normalization lemma of John-Cordoba and Gallegos (see [7]), there exists some affine transformation

$$T_H(x) = a_H x + b_H, \tag{3.2}$$

where a_H is an $n \times n$ matrix satisfying

$$\det a_H = 1, \tag{3.3}$$

and $b_H \in \mathbb{R}^n$ such that

$$T_H(E) = B_R(0), \quad \text{for some } R = R(H) > 0, \tag{3.4}$$

and

$$B_{\alpha_n R}(0) \subset T_H(Q_H(0)) \subset B_R(0), \tag{3.5}$$

where $\alpha_n = n^{-\frac{3}{2}}$. By Lemma 3.1 in [9], there exist constants ε_0, ε_1, and ε_2 depending on n, m_1 and m_2 such that for all $H > 0$,

$$\varepsilon_0 E \times [-\varepsilon_1 H, 0] \subset Q_H \subset E \times [-\varepsilon_2 H, 0]. \tag{3.6}$$

Thus, we have

$$B_{\varepsilon_0 R}(0) \times [-\varepsilon_1 H, 0] \subset (T_H, id) Q_H \subset B_R(0) \times [-\varepsilon_2 H, 0]. \tag{3.7}$$

Proposition 3.1 *Let $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ be a parabolically convex solution of (1.3) that also satisfies (1.4), normalizations (2.1) and (2.2). Then there exists some constant $C \ge 1$ depending only on n, m_1 and m_2 such that*

$$C^{-1} H \le R^2 \le CH. \tag{3.8}$$

Proof We have, by (3.7)

$$\left\{(y,s) \in \mathbb{R}^{n+1}_- : s > \frac{\varepsilon_1 H}{\varepsilon_0^2 R^2}(|y|^2 - \varepsilon_0^2 R^2)\right\}$$
$$\subset (T_H, id) Q_H \subset \left\{(y,s) \in \mathbb{R}^{n+1}_- : s > \frac{\varepsilon_2 H}{R^2}(|y|^2 - 2R^2)\right\}.$$

Let us consider

$$w(y,s) = u(T_H^{-1}(y), s) = u(a_H^{-1}(y - b_H), s), \quad (y,s) \in \mathbb{R}^{n+1}_-.$$

On one hand,

$$-w_s \det D^2 w = 1 \quad \text{in} \ \left\{(y,s) \in \mathbb{R}^{n+1}_- : s > \frac{\varepsilon_1 H}{\varepsilon_0^2 R^2}(|y|^2 - \varepsilon_0^2 R^2)\right\}$$

and

$$w \le H \quad \text{on } \partial_p \left\{ (y,s) \in \mathbb{R}^{n+1}_- : s > \frac{\varepsilon_1 H}{\varepsilon_0^2 R^2}(|y|^2 - \varepsilon_0^2 R^2) \right\}.$$

If we take

$$\zeta(y,s) = \frac{\varepsilon_0^{\frac{2n}{n+1}} R^{\frac{2n}{n+1}}}{2^{\frac{n}{n+1}} \varepsilon_1^{\frac{n}{n+1}} H^{\frac{n}{n+1}}} \left(-s + \frac{\varepsilon_1 H}{\varepsilon_0^2 R^2}(|y|^2 - \varepsilon_0^2 R^2) \right) + H,$$

then

$$-\zeta_s \det D^2 \zeta = 1 \quad \text{in } \left\{ (y,s) \in \mathbb{R}^{n+1}_- : s > \frac{\varepsilon_1 H}{\varepsilon_0^2 R^2}(|y|^2 - \varepsilon_0^2 R^2) \right\}$$

and

$$\zeta = H \quad \text{on } \partial_p \left\{ (y,s) \in \mathbb{R}^{n+1}_- : s > \frac{\varepsilon_1 H}{\varepsilon_0^2 R^2}(|y|^2 - \varepsilon_0^2 R^2) \right\}.$$

By the comparison principle, Proposition 2.2 in [25], we have

$$w \le \zeta \quad \text{in } \left\{ (y,s) \in \mathbb{R}^{n+1}_- : s > \frac{\varepsilon_1 H}{\varepsilon_0^2 R^2}(|y|^2 - \varepsilon_0^2 R^2) \right\},$$

in particular,

$$0 \le w(0,0) \le \zeta(0,0) = \frac{\varepsilon_0^{\frac{2n}{n+1}} R^{\frac{2n}{n+1}}}{2^{\frac{n}{n+1}} \varepsilon_1^{\frac{n}{n+1}} H^{\frac{n}{n+1}}}(-\varepsilon_1 H) + H,$$

thus, we can obtain that

$$R \le \frac{\sqrt{2}}{\varepsilon_1^{\frac{1}{2n}} \varepsilon_0} H^{\frac{1}{2}}.$$

Similarly, we can show that

$$R \ge \frac{1}{2^{\frac{1}{2n}} \varepsilon_2^{\frac{1}{2n}}} H^{\frac{1}{2}}.$$

So taking $C = \max \left\{ 2^{\frac{1}{2n}} \varepsilon_2^{\frac{1}{2n}}, \frac{\sqrt{2}}{\varepsilon_1^{\frac{1}{2n}} \varepsilon_0} \right\}$, we have

$$C^{-1} H^{\frac{1}{2}} \le R \le C H^{\frac{1}{2}}.$$

□

Proposition 3.2 *Let* $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ *be a parabolically convex solution of (1.3) that also satisfies (1.4), normalizations (2.1) and (2.2). Then for some positive constant* $C = C(n, m_1, m_2)$,

$$C^{-1} R \le dist\left(T_H\left(Q_{\frac{H}{2}}(0)\right), \partial T_H(Q_H(0)) \right) \le 2R. \tag{3.9}$$

Consequently,

$$B_{\frac{R}{C}}(0) \subset a_H(Q_H(0)) \subset B_{2R}(0), \tag{3.10}$$

and

$$\varepsilon_0' a_H^{-1}\left(B_{\frac{R}{C}}(0)\right) \times [-\varepsilon_1' H, 0] \subset Q_H \subset a_H^{-1}(B_{2R}(0)) \times [-\varepsilon_2' H, 0], \tag{3.11}$$

where ε_0', ε_1', ε_2' are positive constants independent of H.

Proof Since $T_H(Q_{\frac{H}{2}}(0)) \subset T_H(Q_H(0)) \subset B_R(0)$, it is clearly that

$$dist\left(T_H\left(Q_{\frac{H}{2}}(0)\right), \partial T_H(Q_H(0)\right) \le 2R.$$

Let w be defined on $O_H(0) := \frac{1}{R} T_H(Q_H(0))$ by

$$w(y) = \frac{m_2^{\frac{1}{n}}}{R^2}(u(T_H^{-1}(Ry), 0) - H), \quad y \in O_H(0).$$

Then

$$B_{\alpha_n}(0) \subset O_H(0) \subset B_1(0),$$

and

$$\det(D^2 w) \le 1 \ \text{ in } O_H(0), \quad w = 0 \ \text{ on } \partial O_H(0).$$

It follows from Lemma 1 in [1] that

$$w(y) \ge -C(n) dist(y, \partial O_H(0))^{\frac{2}{n}}, \quad y \in O_H(0).$$

For $\overline{y} \in T_H(Q_{\frac{H}{2}}(0))$, let $\overline{x} = \frac{1}{R}\overline{y}$, we then have

$$-\frac{m_2^{\frac{1}{n}} H}{2R^2} = \frac{m_2^{\frac{1}{n}}}{R^2}\left(\frac{H}{2} - H\right) \ge w(\overline{x}) \ge -C(n) dist(\overline{x}, \partial O_H(0))^{\frac{2}{n}},$$

$$C(n) dist(\overline{y}, \partial T_H(Q_H(0)) \ge \frac{m_2^{\frac{1}{2}} H^{\frac{n}{2}}}{2^{\frac{n}{2}} R^{n-1}}.$$

By Proposition 3.1, we obtain

$$dist(\overline{y}, \partial T_H(Q_H(0)) \ge C^{-1} R,$$

where $C = C(n, m_1, m_2) \ge 1$. Estimate (3.9) is established.

Estimate (3.10) follows from (3.9),

$$T_H(0) \in T_H\left(Q_{\frac{H}{2}(0)}\right) \subset T_H(Q_H(0)) \subset B_R(0),$$

and

$$dist(T_H(0), \partial T_H(Q_H(0)) = dist(0, \partial a_H(Q_H(0)).$$

Since $u_t(x,t) \le -m_2$ for $t \le 0$, we have $u(x,t) \ge u(x,0) - m_2 t$. By $u(x,0) \ge 0$ for all x, we then obtain $u(x,t) \ge H$ for $t < -\frac{H}{m_2}$ or $x \overline{\in} a_H^{-1}(B_{2R}(0))$. So if $\varepsilon_2' = \frac{1}{m_2}$, we have

$$Q_H \subset a_H^{-1}(B_{2R}(0)) \times [-\varepsilon_2' H, 0].$$

Due to (2.1), we have that $Q_H(0)$ is a cross section of the convex function $u(x,0)$ at $x = 0$ and with height H, i.e., $Q_H(0) = S_{u(x,0)}(0, H)$. Particularly, from (3.10) and Lemma 2.1 of [10] we have that

$$\varepsilon_0'\left(\frac{1}{2C} a_H^{-1}(B_{2R}(0))\right) = \varepsilon_0' a_H^{-1}\left(B_{\frac{R}{C}}(0)\right) \subset \varepsilon_0' Q_H(0) \subset Q_{\left(1 - \frac{1-\varepsilon_0'}{4C}\right)H}(0)$$

for $0 < \varepsilon_0' < 1$. If $(x, t) \in \varepsilon_0' a_H^{-1}(B_{\frac{R}{C}}(0)) \times [-\varepsilon_1' H, 0]$, then

$$u(x,t) = u(x,0) - \int_t^0 u_t(x,\tau)d\tau \le \left(1 - \frac{1-\varepsilon_0'}{4C}\right) H - m_1 t$$
$$\le \left(1 - \frac{1-\varepsilon_0'}{4C} + m_1\varepsilon_1'\right) H < H.$$

Therefore, taking ε_0' and ε_1' sufficiently small, we can get

$$\varepsilon_0' a_H^{-1} \left(B_{\frac{R}{C}}(0)\right) \times [-\,\varepsilon_1' H, 0] \subset Q_H.$$

□

Proposition 3.3 *Let $u \in C^{4,2}(\mathbb{R}^{n+1}_-)$ be a parabolically convex solution of (1.3) that further satisfies (1.4), normalizations (2.1) and (2.2). Then for some positive constant $C = C(n, m_1, m_2)$,*

$$|a_H| \le C, \quad |a_H^{-1}| \le C. \tag{3.12}$$

Moreover,

$$\sup_{\mathbb{R}^{n+1}_-} |D^2 u| \le C. \tag{3.13}$$

Proof Let us define

$$\Gamma_H(x,t) = \left(\frac{1}{R} a_H x, \frac{t}{R^2}\right), \quad \text{and} \quad \Gamma_H(Q_H) = Q_H^*.$$

Consider

$$w(y,s) := \frac{1}{R^2} u(\Gamma_H^{-1}(y,s)) = \frac{1}{R^2} u(R a_H^{-1} y, R^2 s), \quad (y,s) \in Q_H^*.$$

By (1.3) and $\det a_H = 1$, we have

$$-\, w_t \det D^2 w = 1 \quad \text{on } Q_H^*.$$

It follows from Proposition 3.1 and (3.11) that

$$C^{-1} \le w = \frac{H}{R^2} \le C \quad \text{on } \partial_p Q_H^*,$$

and

$$B_{\frac{\varepsilon_0'}{C}}(0) \times [-\varepsilon_1' C^{-1}, 0] \subset Q_H^* \subset B_2(0) \times [-\varepsilon_2' C, 0]. \tag{3.14}$$

By (3.14) and the interior second derivative estimates of Pogorelov (see [24]),

$$|D^2 w| \le C, \quad in\ B_{\frac{\varepsilon_0'}{2C}}(0) \times \left[-\frac{\varepsilon_1'}{2C}, 0\right], \tag{3.15}$$

in particular,

$$|D^2 w(0,0)| \le C.$$

Since

$$D^2 w(0,0) = (a_H^{-1})^T D^2 u(0,0)(a_H^{-1}) = (a_H^{-1})^T (a_H^{-1}),$$

we get

$$|a_H^{-1}| \le C.$$

Since $\det(a_H^{-1}) = \det a_H = 1$, we then have

$$|a_H| \le C.$$

Estimate (3.12) is established.

By (3.15) and (3.12),

$$|D^2u| \le C \quad in\ B_{\frac{\varepsilon_0' R}{2C}}(0) \times \left[-\frac{\varepsilon_1' R^2}{2C}, 0\right],$$

where $C = C(n, m_1, m_2)$. Since R can be arbitrary large (as can H), estimate (3.13) follows from the above. □

Theorem 1.1 can be deduced from (3.13) and the interior estimates of Evans and Krylov as follows:

Proof of Theorem 1.1 By (3.13), we have

$$|u(x,t)| \le C(|x|^2 - t) \quad in \quad \mathbb{R}^{n+1}_-. \tag{3.16}$$

For $(\bar{x}, \bar{t}) \in \mathbb{R}^{n+1}_-$, we will show that $D^2u(\bar{x}, \bar{t}) = D^2u(0,0)$ and consequently by (1.3),

$$u_t(\bar{x}, \bar{t}) = -\frac{1}{\det D^2u(\bar{x}, \bar{t})} = -\frac{1}{\det D^2u(0,0)} = u_t(0,0).$$

Since $(\bar{x}, \bar{t})$ is arbitrary, u must have the form $u(x,t) = C_1 t + p(x)$, where $C_1 < 0$ is a constant and p is a convex quadratic polynomial on x. Theorem 1.1 is established.

For $R > 1$, $R > 2|\bar{x}|$ and $R^2 > -2\bar{t}$, we consider

$$w(y,s) = \frac{1}{R^2} u(Ry, R^2 s), \quad (y,s) \in B_1(0) \times (-1, 0].$$

By (1.3), (3.16) and (3.13),

$$-w_t \det D^2 w = 1, \quad |w| + |D^2 w| \le C \quad \text{in } B_1(0) \times (-1, 0].$$

It follows from the interior estimates of Evans and Krylov that for some $\alpha \in (0,1)$ and C (independent of R and H),

$$|D^2w(y,s) - D^2w(0,0)| \le C(|y|^2 + |s|)^{\frac{\alpha}{2}}, \quad (y,s) \in B_{\frac{1}{2}}(0) \times \left(-\frac{1}{2}, 0\right].$$

In particular,

$$\left| D^2 w\left(\frac{\bar{x}}{R}, \frac{\bar{t}}{R^2}\right) - D^2w(0,0)\right| \le C\left(\left(\frac{|\bar{x}|}{R}\right)^2 + \frac{|\bar{t}|}{R^2}\right)^{\frac{\alpha}{2}},$$

i.e.,

$$|D^2u(\bar{x}, \bar{t}) - D^2u(0,0)| \le \frac{C}{R^\alpha}(|\bar{x}|^2 + |\bar{t}|)^{\frac{\alpha}{2}}.$$

Sending $R \to +\infty$, we have

$$D^2u(\bar{x}, \bar{t}) = D^2u(0,0).$$

□

4 Proof of Theorem 1.2

Let $T_H(x) = a_H x + b_H$ be an affine transformation satisfying (3.3), (3.4) and (3.5), and let

$$v(y,s) = \frac{1}{R^2} u(\Gamma_H^{-1}(y,s)) = \frac{1}{R^2} u(Ra_H^{-1} y, R^2 s), \quad (y,s) \in Q_H^*. \tag{4.1}$$

By (3.11),

$$B_{\frac{\varepsilon_0}{C}}(0) \times \left[-\frac{\varepsilon_1}{C}, 0\right] \subset Q_H^* \subset B_2(0) \times [-C\varepsilon_2, 0].$$

Clearly

$$-v_s \det D^2 v = f(\Gamma_H^{-1}(y,s)) = f(Ra_H^{-1} y, R^2 s) \quad \text{in } Q_H^*.$$

By Proposition 3.1,

$$v = \frac{H}{R^2} \in (C^{-1}, C) \quad \text{on } \partial_p Q_H^*. \tag{4.2}$$

By [24], there exists a unique parabolically convex solution $\overline{v} \in C^0(\overline{Q_H^*}) \bigcap C^\infty(Q_H^*)$ of

$$\begin{cases} -\overline{v}_s \det D^2 \overline{v} = 1 & \text{in } Q_H^*, \\ \overline{v} = \frac{H}{R^2} \in (C^{-1}, C) & \text{on } \partial_p Q_H^*, \\ -C \le \overline{v}_s \le -C^{-1} & \text{in } Q_H^*. \end{cases}$$

And for every $\delta > 0$, there exists some positive constant $C = C(\delta)$ such that for all $(y,s) \in Q_H^*$ and $dist_p((y,s), \partial_p Q_H^*) \ge \delta$, we have

$$C^{-1} I \le D^2 \overline{v}(y,s) \le CI, \quad |D^3 \overline{v}(y,s)| \le C, \quad |D\overline{v}_s(y,s)| \le C \tag{4.3}$$

Lemma 4.1 *For some positive constant $\widetilde{C}$ depending only on n and f, we have*

$$|v - \overline{v}| \le \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}} \quad \textit{in } Q_H^*. \tag{4.4}$$

In fact, $\widetilde{C} = C(n) \|f - 1\|_{L^1(f \neq 1)}^{\frac{1}{n+1}}$.

Proof By replacing u in Proposition 2.1 ([22]) with $-u$, we have that

$$-\min_{Q_H^*}(v - \overline{v}) \le C(n) \left(\int_{S_1^+} -(v - \overline{v})_s \det D^2 (v - \overline{v}) dy ds \right)^{\frac{1}{n+1}},$$

where

$$S_1^+ = \{(y,s) \in Q_H^* : (v - \overline{v})_s < 0, D^2(v - \overline{v}) > 0\}.$$

On S_1^+,

$$\det D^2(v - \overline{v}) \le \det D^2 v - \det D^2 \overline{v} \le \det D^2 v,$$

so we have

$$\begin{aligned} -(v - \overline{v})_s \det D^2(v - \overline{v}) &\le -(v - \overline{v})_s \det D^2 v \\ &= -v_s \det D^2 v + \overline{v}_s \det D^2 v \\ &\le -v_s \det D^2 v + \overline{v}_s \det D^2 \overline{v} \\ &= f(Ra_H^{-1} y, R^2 s) - 1, \end{aligned}$$

It follows that

$$
\begin{aligned}
-\min_{\overline{Q_H^*}}(v-\overline{v}) &\le C(n)\Big(\int_{S_1^+}(f(Ra_H^{-1}y, R^2 s)-1)dyds\Big)^{\frac{1}{n+1}}\\
&= C(n)\Big(\int_{\Gamma_H^{-1}(S_1^+)}(f(x,t)-1)\frac{\det a_H}{R^{n+2}}dxdt\Big)^{\frac{1}{n+1}}\\
&= \frac{C(n)}{R^{\frac{n+2}{n+1}}}\|f-1\|_{L^1(f>1)}^{\frac{1}{n+1}}\\
&\le \frac{\widetilde{C}(n,f)}{R^{\frac{n+2}{n+1}}}.
\end{aligned}
$$

Similarly, we can show that

$$
-\min_{\overline{Q_H^*}}(\overline{v}-v) \le \frac{C(n)}{R^{\frac{n+2}{n+1}}}\|1-f\|_{L^1(f<1)}^{\frac{1}{n+1}} \le \frac{\widetilde{C}(n,f)}{R^{\frac{n+2}{n+1}}}.
$$

Lemma 4.1 is established. □

Let $(\overline{y}, 0)$ be the unique minimum point of $\overline{v}$ in $\overline{Q_H^*}$. For $\overline{v}(\overline{y}, 0) < \widetilde{H} \le H$, let

$$
\begin{aligned}
S_{\widetilde{H}}(0,0) &= \left\{(y,s)\in\mathbb{R}_-^{n+1} : \frac{1}{2}y^T D^2\overline{v}(\overline{y},0)y + \overline{v}_s(\overline{y},0)s = \widetilde{H}\right\},\\
E_{\widetilde{H}}(0,0) &= \left\{(y,s)\in\mathbb{R}_-^{n+1} : \frac{1}{2}y^T D^2\overline{v}(\overline{y},0)y + \overline{v}_s(\overline{y},0)s < \widetilde{H}\right\},\\
S_{\widetilde{H}}(\overline{y},0) &= \left\{(y,s)\in\mathbb{R}_-^{n+1} : \frac{1}{2}(y-\overline{y})^T D^2\overline{v}(\overline{y},0)(y-\overline{y}) + \overline{v}_s(\overline{y},0)s = \widetilde{H}\right\},\\
E_{\widetilde{H}}(\overline{y},0) &= \left\{(y,s)\in\mathbb{R}_-^{n+1} : \frac{1}{2}(y-\overline{y})^T D^2\overline{v}(\overline{y},0)(y-\overline{y}) + \overline{v}_s(\overline{y},0)s < \widetilde{H}\right\}.
\end{aligned}
$$

We also denote that

$$
\begin{aligned}
mE_{\widetilde{H}}(0,0) &= \left\{(y,s) : \frac{1}{2}y^T D^2\overline{v}(\overline{y},0)y + \overline{v}_s(\overline{y},0)s < m^2\widetilde{H}\right\}, m\in\mathbb{R}^+,\\
mE_{\widetilde{H}}(\overline{y},0) &= \left\{(y,s) : \frac{1}{2}(y-\overline{y})^T D^2\overline{v}(\overline{y},0)(y-\overline{y}) + \overline{v}_s(\overline{y},0)s < m^2\widetilde{H}\right\}, m\in\mathbb{R}^+,
\end{aligned}
$$

and

$$
mQ_H = \{(y', s') = (my, m^2 t) : (y,s)\in Q_H\}, m\in\mathbb{R}^+.
$$

Proposition 4.2 *There exist $\overline{k}$ and $\overline{C}$, depending only on n and f, such that for $\epsilon = \frac{1}{3}$, $H = 2^{(1+\epsilon)k}$ and $2^{k-1} \le H' \le 2^k$, we have*

$$
\left(\frac{H'}{R^2} - \overline{C}2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(0,0) \subset \Gamma_H(Q_{H'}) \subset \left(\frac{H'}{R^2} + \overline{C}2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(0,0), \quad \forall k \ge \overline{k}. \tag{4.5}
$$

Proof Clearly, it follows from (3.8) and (4.1) that

$$
C^{-1}2^{-\epsilon k} \le \frac{H'}{R^2} \le C2^{-\epsilon k}, \quad C^{-1}2^{\frac{(1+\epsilon)k}{2}} \le R \le C2^{\frac{(1+\epsilon)k}{2}},
$$

and

$$
\left\{v < \frac{H'}{R^2}\right\} := \left\{(y,s) : v(y,s) < \frac{H'}{R^2}\right\} = \Gamma_H(Q_{H'}) \subset Q_H^*.
$$

By Lemma 4.1,

$$|v-\overline{v}| \le \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}} \le \widetilde{C}C2^{-\frac{1+\epsilon}{2}k\left(1+\frac{1}{n+1}\right)} \quad \text{in } Q_H^*.$$

Since

$$\frac{H'}{R^2} \gg \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}, \quad as\ R \to \infty,$$

the level surface of v can be well approximated by the level surface of $\overline{v}$:

$$\left\{\overline{v} < \frac{H'}{R^2} - \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right\} \subset \left\{v < \frac{H'}{R^2}\right\} \subset \left\{\overline{v} < \frac{H'}{R^2} + \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right\}.$$

By Lemma 4.1, the fact $v \ge 0$ and $v(0,0) = 0$, we have

$$-\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}} \le v(\overline{y},0) - \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}} \le \overline{v}(\overline{y},0) \le \overline{v}(0,0) \le v(0,0) + \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}} = \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}.$$

Therefore by Lemmas 2.3 and (4.3),

$$|\overline{v}(y,s) - \overline{v}(\overline{y},0) - \overline{v}_s(\overline{y},0)s - \frac{1}{2}(y-\overline{y})^T D^2\overline{v}(\overline{y},0)(y-\overline{y})| \le C(|y-\overline{y}|^2 + |s|)^{\frac{3}{2}},$$

$dist_p((y,s),(\overline{y},0)) < \frac{1}{C}$ and

$$2C^{-1}I \le D^2\overline{v}(\overline{y},0) \le 2CI.$$

On one hand, we take a positive constant C_1 to be determined. For $(y,s) \in \left(\frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(\overline{y},0)$, we have

$$\begin{aligned}
&\overline{v}_s(\overline{y},0)s + \frac{1}{2}(y-\overline{y})^T D^2\overline{v}(\overline{y},0)(y-\overline{y}) < \frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2}},\\
&\frac{1}{C}|s| + \frac{1}{2C}|y-\overline{y}|^2 < \frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2}},\\
&|y-\overline{y}|^2 + |s| < C\left(\frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2}}\right).
\end{aligned}$$

We can take $\overline{k}_1$ satisfying for $k \ge \overline{k}_1$, then

$$|y-\overline{y}|^2 + |s| < C\left(\frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2}}\right) \le \frac{1}{C^2}.$$

Thus,

$$
\begin{aligned}
\overline{v}(y,s) &\le \overline{v}(\overline{y},0)+\overline{v}_s(\overline{y},0)s+\frac{1}{2}(y-\overline{y})^T D^2\overline{v}(\overline{y},0)(y-\overline{y})+C(|y-\overline{y}|^2+|s|)^{\frac{3}{2}}\\
&\le \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2}}+C^{\frac{5}{2}}\left(\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{3}{2}}\\
&\le \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2}}+C^{\frac{5}{2}}\left(\frac{H'}{R^2}\right)^{\frac{3}{2}}\\
&\le \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2}}+C^4 2^{-\frac{3}{2}\epsilon k}\\
&= \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+(C^4-C_1)2^{-\frac{3\epsilon k}{2}}.
\end{aligned}
$$

We can take $C_1 > C^4$ satisfying $\frac{2\widetilde{C}C}{C_1-C^4} < 1$, then

$$
2\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}} \le 2\widetilde{C}C2^{-\frac{(1+\epsilon)(n+2)k}{2(n+1)}} < (C_1-C^4)2^{-\frac{3\epsilon k}{2}}.
$$

For $k \ge \overline{k}_1$, we can obtain

$$
\overline{v}(y,s) \le \frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+(C^4-C_1)2^{-\frac{3\epsilon k}{2}} < \frac{H'}{R^2}-\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}.
$$

In conclusion, we have

$$
\left(\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(\overline{y},0) \subset \left\{\overline{v} < \frac{H'}{R^2}-\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right\}, \quad \forall k \ge \overline{k}_1.
$$

On the other hand, we take a positive constant C_2 to be determined. In order to prove

$$
\left\{\overline{v} < \frac{H'}{R^2}+\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right\} \subset \left(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(\overline{y},0),
$$

using the fact

$$
(\overline{y},0) \in \left\{\overline{v} < \frac{H'}{R^2}+\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right\} \cap \left(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(\overline{y},0),
$$

we only need to prove

$$
\left(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} S_1(\overline{y},0) \subset \left\{\overline{v} < \frac{H'}{R^2}+\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right\}^c.
$$

For $(y,s) \in \left(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} S_1(\overline{y},0)$, then we get

$$
\begin{aligned}
&\overline{v}_s(\overline{y},0)s+\frac{1}{2}(y-\overline{y})^T D^2\overline{v}(\overline{y},0)(y-\overline{y}) = \frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}},\\
&\frac{1}{C}|s|+\frac{1}{2C}|y-\overline{y}|^2 < \frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}},\\
&|y-\overline{y}|^2+|s| < 2C\left(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}\right).
\end{aligned}
$$

Taking $\overline{k}_2$ satisfying for $k \geq \overline{k}_2$, we obtain

$$|y-\overline{y}|^2+|s| < C\left(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}\right) \leq \frac{1}{C^2}.$$

Thus,

$$\begin{aligned}
\overline{v}(y,s) &\geq \overline{v}(\overline{y},0)+\overline{v}_s(\overline{y},0)s+\frac{1}{2}(y-\overline{y})^T D^2\overline{v}(\overline{y},0)(y-\overline{y})-C(|y-\overline{y}|^2+|s|)^{\frac{3}{2}}\\
&\geq -\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}-C^{\frac{5}{2}}\left(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{3}{2}}\\
&\geq -\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}-C^{\frac{5}{2}}\left(2\frac{H'}{R^2}\right)^{\frac{3}{2}}\\
&\geq -\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}-C^4 2^{\frac{3}{2}} 2^{-\frac{3}{2}\epsilon k}\\
&= -\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+\left(C_2-2^{\frac{3}{2}}C^4\right)2^{-\frac{3\epsilon k}{2}}.
\end{aligned}$$

We can take $C_2 > 2^{\frac{3}{2}}C^4$ satisfying $\frac{C_2-2^{\frac{3}{2}}C^4}{2\widetilde{C}C^{\frac{n+2}{n+1}}} > 1$, and then

$$2\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}} \leq 2\widetilde{C}C^{\frac{n+2}{n+1}}2^{-\frac{(1+\epsilon)(n+2)k}{2(n+1)}} < \left(C_2-2^{\frac{3}{2}}C^4\right)2^{-\frac{3\epsilon k}{2}}.$$

For $k \geq \overline{k}_2$, we obtain

$$\overline{v}(y,s) \geq -\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\frac{H'}{R^2}+\left(C_2-2^{\frac{3}{2}}C^4\right)2^{-\frac{3\epsilon k}{2}} > \frac{H'}{R^2}+\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}.$$

In conclusion, we have

$$\left\{\overline{v} < \frac{H'}{R^2}+\frac{\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right\} \subset \left(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(\overline{y},0), \quad \forall k \geq \overline{k}_2.$$

Therefore, taking $C_3 > \max\{C_1, C_2\}$ and $\overline{k} = \max\{\overline{k}_1, \overline{k}_2\}$, we see

$$\left(\frac{H'}{R^2}-C_3 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(\overline{y},0) \subset \left\{v < \frac{H'}{R^2}\right\} \subset \left(\frac{H'}{R^2}+C_3 2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(\overline{y},0) \quad \forall k \geq \overline{k}. \tag{4.6}$$

Finally, we want to obtain (4.5). We first show that

$$\partial_p(Q^*_{\widetilde{H}+\overline{v}(\overline{y},0)}(\overline{v})) \subset N_{\delta_1}(S_{\widetilde{H}}(\overline{y},0)), \quad 0 < \widetilde{H} \leq \frac{H}{R^2}-\overline{v}(\overline{y},0), \quad \delta_1 \leq C\widetilde{H}^{\frac{1}{2}}, \tag{4.7}$$

and neighborhood N is measured by parabolic distance

$$dist_p[(y_1,s_1),(y_2,s_2)] := (|y_1-y_2|^2+|s_1-s_2|)^{\frac{1}{2}}.$$

In fact, for $(y, s) \in \partial_p(Q^*_{\widetilde{H}+\overline{v}(\overline{y},0)}(\overline{v}))$, by the Mean Theorem, Lemma 2.3 and (4.3), we have

$$\begin{aligned}\widetilde{H} &= \overline{v}(y, s) - \overline{v}(\overline{y}, 0)\\ &= \overline{v}(y, s) - \overline{v}(y, 0) + \overline{v}(y, 0) - \overline{v}(\overline{y}, 0)\\ &= \overline{v}_s(y, s')s + \frac{1}{2}(y - \overline{y})^T D^2\overline{v}(y', 0)(y - \overline{y})\\ &\geq \frac{1}{2C}(|s| + |y - \overline{y}|^2),\end{aligned}$$

where $(y', s') \in Q^*_{\widetilde{H}+\overline{v}(\overline{y},0)}(\overline{v})$. Writing

$$\begin{aligned}\widetilde{H} &= \overline{v}(y, s) - \overline{v}(\overline{y}, 0)\\ &= \overline{v}_s(\overline{y}, 0)s + (\overline{v}_s(y, s') - \overline{v}_s(\overline{y}, 0))s + \frac{1}{2}(y - \overline{y})^T D^2\overline{v}(\overline{y}, 0)(y - \overline{y})\\ &\quad + \frac{1}{2}(y - \overline{y})^T (D^2\overline{v}(y', 0) - D^2\overline{v}(\overline{y}, 0))(y - \overline{y}),\end{aligned}$$

for $(y, s) \in \partial_p(Q^*_{\widetilde{H}+\overline{v}(\overline{y},0)}(\overline{v}))$, it follows

$$\begin{aligned}&\left|\widetilde{H} - \overline{v}_s(\overline{y}, 0)s - \frac{1}{2}(y - \overline{y})^T D^2\overline{v}(\overline{y}, 0)(y - \overline{y})\right|\\ &= \left|(\overline{v}_s(y, s') - \overline{v}_s(\overline{y}, 0))s + \frac{1}{2}(y - \overline{y})^T (D^2\overline{v}(y', 0) - D^2\overline{v}(\overline{y}, 0))(y - \overline{y})\right|\\ &\leq C|s| + C|y - \overline{y}|^2\\ &\leq C\widetilde{H}.\end{aligned}$$

For any $(y, s) \in \partial_p(Q^*_{\widetilde{H}+\overline{v}(\overline{y},0)}(\overline{v}))$ and any $(\widetilde{y}, \widetilde{s}) \in S_{\widetilde{H}}(\overline{y}, 0)$, by the above inequality, we show

$$\begin{aligned}&\left|\overline{v}_s(\overline{y}, 0)\widetilde{s} + \frac{1}{2}(\widetilde{y} - \overline{y})^T D^2\overline{v}(\overline{y}, 0)(\widetilde{y} - \overline{y}) - \overline{v}_s(\overline{y}, 0)s - \frac{1}{2}(y - \overline{y})^T D^2\overline{v}(\overline{y}, 0)(y - \overline{y})\right|\\ &\quad \leq C\widetilde{H}.\end{aligned}$$

Taking $\widetilde{y}, \overline{y}, y$ on the same line l with $\widetilde{y}$ and y on the same side of the line l with respect to $\overline{y}$ (rotating the coordinates again so that l is parallel to some axis), we have

$$||\widetilde{y} - \overline{y}|^2 - |y - \overline{y}|^2| \geq |y - \widetilde{y}|^2.$$

Then for $s = \widetilde{s}$, we get

$$\frac{1}{2C}||\widetilde{y} - \overline{y}|^2 - |y - \overline{y}|^2| \leq C\widetilde{H}.$$

In fact, there exists an orthogonal matrix O such that $D^2\overline{v}(\overline{y}, 0) = O^T diag\{\lambda_1, \ldots, \lambda_n\}O$, and the length of a vector in Euclidean space is invariant under orthogonal transformation. Therefore, we get

$$|y - \widetilde{y}|^2 \leq C\widetilde{H}.$$

Similarly, for $y = \widetilde{y}$,

$$|\overline{v}_s(\overline{y}, 0)\widetilde{s} - \overline{v}_s(\overline{y}, 0)s| \leq C\widetilde{H},$$

So we get

$$|s - \widetilde{s}| \leq C\widetilde{H}.$$

This completes the proof of (4.7).

Next we estimate the distance between (0, 0) and $(\overline{y}, 0)$. By Lemma 4.1, we have

$$\begin{aligned} 0 &\le \overline{v}(0,0) - \overline{v}(\overline{y},0) \\ &= (\overline{v}(0,0) - v(0,0)) + (v(0,0) - v(\overline{y},0)) + (v(\overline{y},0) - \overline{v}(\overline{y},0)) \\ &\le \frac{2\widetilde{C}}{R^{\frac{n+2}{n+1}}}, \end{aligned}$$

so $(0,0) \in Q^*_{\frac{2\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\overline{v}(\overline{y},0)}(\overline{v})$, and by (4.7) (taking $\widetilde{H} = \frac{2\widetilde{C}}{R^{\frac{n+2}{n+1}}}$), we have

$$\partial\left(Q^*_{\frac{2\widetilde{C}}{R^{\frac{n+2}{n+1}}}+\overline{v}(\overline{y},0)}(\overline{v})\right) \subset N_{\delta_1}\left(S_{\frac{2\widetilde{C}}{R^{\frac{n+2}{n+1}}}}(\overline{y},0)\right), \quad \delta_1 \le C\left(\frac{2\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right)^{1/2}.$$

Thus we get

$$dist_p((0,0),(\overline{y},0)) \le C\left(\frac{2\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right)^{1/2}.$$

So by (4.6), we have

$$\begin{aligned} &\left(\frac{H'}{R^2} - C_3 2^{-\frac{3\epsilon k}{2}} - C^2\frac{2\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right)^{\frac{1}{2}} E_1(0,0) \subset \left\{v < \frac{H'}{R^2}\right\} \\ &\subset \left(\frac{H'}{R^2} + C_3 2^{-\frac{3\epsilon k}{2}} + C^2\frac{2\widetilde{C}}{R^{\frac{n+2}{n+1}}}\right)^{\frac{1}{2}} E_1(0,0) \quad \forall k \ge \overline{k}. \end{aligned}$$

Since $2^{-\frac{3\epsilon k}{2}} \gg \frac{1}{R^{\frac{n+2}{n+1}}}$, then we can obtain (4.5) by taking $\overline{C} = 2C^2\widetilde{C} + C_3$. □

Let $\widetilde{E}$ denote the set $\{(y,s) \in \mathbb{R}^{n+1}_- : \frac{1}{2}|y|^2 - s < 1\}$, then we have the following proposition.

Proposition 4.3 *There exist positive constants $\widehat{k}$, $\widehat{C}$, some real invertible upper-triangular matrix $\{T_k\}_{k\ge\widehat{k}}$ and negative number $\{\tau_k\}_{k\ge\widehat{k}}$ such that*

$$-\tau_k \det T_k^T T_k = 1, \quad \|T_k T_{k-1}^{-1} - I\| \le \widehat{C}2^{-\frac{\epsilon k}{4}}, \quad |\tau_k \tau_{k-1}^{-1} - 1| \le \widehat{C}2^{-\frac{\epsilon k}{4}}, \tag{4.8}$$

and

$$\left(1 - \widehat{C}2^{-\frac{\epsilon k}{2}}\right)\sqrt{H'}\widetilde{E} \subset \Sigma_k(Q_{H'}) \subset \left(1 + \widehat{C}2^{-\frac{\epsilon k}{2}}\right)\sqrt{H'}\widetilde{E}, \quad \forall 2^{k-1} \le H' \le 2^k, \tag{4.9}$$

where $\Sigma_k = (T_k, -\tau_k)$. Consequently, for some invertible T and τ,

$$-\tau \det T^T T = 1, \quad \|T_k - T\| \le \widehat{C}2^{-\frac{\epsilon k}{4}}, \quad |\tau_k - \tau| \le \widehat{C}2^{-\frac{\epsilon k}{4}}. \tag{4.10}$$

Proof Let $H = 2^{(1+\epsilon)k}$ and $2^{k-1} \le H' \le 2^k$. By Proposition 4.2, there exist some positive constants $\overline{C}$ and $\overline{k}$ depending only on n and f such that

$$\left(\frac{H'}{R^2} - \overline{C}2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(0,0) \subset \Gamma_H(Q_{H'}) \subset \left(\frac{H'}{R^2} + \overline{C}2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_1(0,0), \quad \forall k \ge \overline{k}.$$

Then we have

$$\left(H' - \overline{C}2^{-\frac{3\epsilon k}{2}}R^2\right)^{\frac{1}{2}} E_1(0,0) \subset (a_H, id)(Q_{H'}) \subset \left(H' + \overline{C}2^{-\frac{3\epsilon k}{2}}R^2\right)^{\frac{1}{2}} E_1(0,0),$$

$$\left(1 - \overline{C}2^{-\frac{3\epsilon k}{2}}\frac{R^2}{H'}\right)^{\frac{1}{2}} \sqrt{H'}E_1(0,0) \subset (a_H, id)(Q_{H'})$$
$$\subset \left(1 + \overline{C}2^{-\frac{3\epsilon k}{2}}\frac{R^2}{H'}\right)^{\frac{1}{2}} \sqrt{H'}E_1(0,0).$$

Since

$$C^{-1}2^{-\epsilon k} \le \frac{H'}{R^2} \le C2^{-\epsilon k},$$

we can get

$$\left(1 - \overline{C}C2^{-\frac{\epsilon k}{2}}\right)^{\frac{1}{2}} \sqrt{H'}E_1(0,0) \subset (a_H, id)(Q_{H'}) \subset \left(1 + \overline{C}C2^{-\frac{\epsilon k}{2}}\right)^{\frac{1}{2}} \sqrt{H'}E_1(0,0).$$

On one hand, we take $\overline{C}_1 > \frac{\overline{C}C}{2}$, $\overline{k}_5$ satisfying when $k \ge \overline{k}_5$, $2^{\frac{k\epsilon}{2}} \ge \frac{\overline{C}_1^2}{2\overline{C}_1 - C\overline{C}}$. If $k \ge \overline{k}_6 := \max\{\overline{k}_5, \overline{k}\}$, we have

$$\begin{aligned}
\overline{C}_1^2 &\le 22^{\frac{k\epsilon}{2}}\overline{C}_1 - 2^{\frac{k\epsilon}{2}}C\overline{C},\\
2^{-\epsilon k}\overline{C}_1^2 &\le 22^{-\frac{k\epsilon}{2}}\overline{C}_1 - 2^{-\frac{k\epsilon}{2}}C\overline{C},\\
2^{-\epsilon k}\overline{C}_1^2 - 22^{-\frac{k\epsilon}{2}}\overline{C}_1 &\le -2^{-\frac{k\epsilon}{2}}C\overline{C},\\
2^{-\epsilon k}\overline{C}_1^2 - 22^{-\frac{k\epsilon}{2}}\overline{C}_1 + 1 &\le 1 - 2^{-\frac{k\epsilon}{2}}C\overline{C},\\
\left(1 - \overline{C}_1 2^{-\frac{k\epsilon}{2}}\right)^2 &\le 1 - 2^{-\frac{k\epsilon}{2}}C\overline{C}.
\end{aligned}$$

Therefore,

$$\left(1 - \overline{C}_1 2^{-\frac{k\epsilon}{2}}\right)\sqrt{H'}E_1(0,0) \subset (a_H, id)(Q_{H'}), \quad k \ge \overline{k}_6.$$

On the other hand, if taking $\overline{C}_2 > \frac{\overline{C}C}{2}$, then for any $k \ge \overline{k}$, we have

$$\left(1 + \overline{C}C2^{-\frac{k\epsilon}{2}}\right)^{\frac{1}{2}} \le \left(1 + \overline{C}_2 2^{-\frac{k\epsilon}{2}}\right).$$

So we show

$$(a_H, id)(Q_{H'}) \subset \left(1 + \overline{C}_2 2^{-\frac{k\epsilon}{2}}\right)\sqrt{H'}E_1(0,0), \quad k \ge \overline{k}.$$

In conclusion, taking $\widehat{C} > \frac{\overline{C}C}{2}$, $\widehat{k} = \overline{k}_6$, we have

$$\left(1 - \widehat{C}2^{-\frac{k\epsilon}{2}}\right)\sqrt{H'}E_1(0,0) \subset (a_H, id)(Q_{H'}) \subset \left(1 + \widehat{C}2^{-\frac{k\epsilon}{2}}\right)\sqrt{H'}E_1(0,0), \quad k \ge \widehat{k}. \tag{4.11}$$

Let Q be the real symmetric positive definite matrix satisfying $Q^2 = Q^T Q = D^2\overline{v}(\overline{y}, 0)$ and O be an orthogonal matrix such that

$$T_k := OQa_H \quad \text{is the upper-triangular.}$$

And we also define $\tau_k = \overline{v}_s(\overline{y}, 0)$ and $\Sigma_k = (T_k, -\tau_k)$. Clearly,

$$-\tau_k \det T_k^T T_k = -\overline{v}_s(\overline{y}, 0)(\det a_H)^2 \det D^2\overline{v}(\overline{y}, 0) = 1.$$

Now we claim that $\widetilde{E} = (OQ, -\tau_k)E_1(0,0)$. $\forall (y,s) \in E_1(0,0)$, $(x,t) = (OQy, -\tau_k s)$, $x^T x = y^T Q^T O^T O Q y = y^T D^2\overline{v}(\overline{y},0)y$, $t = -\tau_k s = -\overline{v}_s(\overline{y},0)s$. Recall that

$$\frac{1}{2}y^T D^2\overline{v}(\overline{y},0)y + \overline{v}_s(\overline{y},0)s = 1,$$

so $(x,t) \in \widetilde{E}$, and vice versa. From (4.11), we have

$$\left(1-\widehat{C}2^{-\frac{k\epsilon}{2}}\right)\sqrt{H'}\widetilde{E} \subset \Sigma_k(Q_{H'}) \subset \left(1+\widehat{C}2^{-\frac{k\epsilon}{2}}\right)\sqrt{H'}\widetilde{E}, \quad k \geq \widehat{k}.$$

If taking $H = 2^{(1+\epsilon)k}$ and $H' = 2^{k-1}$, we can obtain

$$\left(1-\widehat{C}2^{-\frac{k\epsilon}{2}}\right)\sqrt{2^{k-1}}\widetilde{E} \subset \Sigma_k(Q_{2^{k-1}}) \subset \left(1+\widehat{C}2^{-\frac{k\epsilon}{2}}\right)\sqrt{2^{k-1}}\widetilde{E}, \tag{4.12}$$

and if taking $H = 2^{(1+\epsilon)(k-1)}$ and $H' = 2^{k-1}$, we can get

$$\left(1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}\right)\sqrt{2^{k-1}}\widetilde{E} \subset \Sigma_{k-1}(Q_{2^{k-1}}) \subset \left(1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}\right)\sqrt{2^{k-1}}\widetilde{E}.$$

Thus we obtain

$$\left(1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}\right)\sqrt{2^{k-1}}\Sigma_{k-1}^{-1}\widetilde{E} \subset Q_{2^{k-1}} \subset \left(1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}\right)\sqrt{2^{k-1}}\Sigma_{k-1}^{-1}\widetilde{E}, \tag{4.13}$$

$$\begin{aligned}&\left(1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}\right)\sqrt{2^{k-1}}\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E} \subset \Sigma_k(Q_{2^{k-1}})\\ &\subset \left(1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}\right)\sqrt{2^{k-1}}\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E}.\end{aligned} \tag{4.14}$$

On one hand, from the left hand of (4.14) and the right hand of (4.12), we see

$$\left(1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}\right)\sqrt{2^{k-1}}\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E} \subset \left(1+\widehat{C}2^{-\frac{k\epsilon}{2}}\right)\sqrt{2^{k-1}}\widetilde{E},$$

thus we have

$$\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E} \subset \frac{1+\widehat{C}2^{-\frac{k\epsilon}{2}}}{1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}}\widetilde{E} = \left(1+\frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}+\widehat{C}2^{-\frac{k\epsilon}{2}}}{1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}}\right)\widetilde{E}.$$

Since

$$\lim_{k\to+\infty} 2^{\frac{\epsilon k}{2}}\frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}+\widehat{C}2^{-\frac{k\epsilon}{2}}}{1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}} = \lim_{k\to+\infty}\frac{\widehat{C}2^{\frac{\epsilon}{2}}+\widehat{C}}{1-\widehat{C}2^{-\frac{\epsilon(k-1)}{2}}} = \widehat{C}2^{\frac{\epsilon}{2}}+\widehat{C},$$

by taking k sufficiently large, we can obtain

$$\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E} \subset \left(1+\frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}+\widehat{C}2^{-\frac{k\epsilon}{2}}}{1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}}\right)\widetilde{E} \subset \left(1+\widehat{C}2^{-\frac{\epsilon k}{2}}\right)\widetilde{E}.$$

On the other hand, from the left hand of (4.12) and the right hand of (4.14), we get

$$\left(1-\widehat{C}2^{-\frac{k\epsilon}{2}}\right)\sqrt{2^{k-1}}\widetilde{E} \subset \left(1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}\right)\sqrt{2^{k-1}}\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E},$$

thus we obtain

$$\left(1-\frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}+\widehat{C}2^{-\frac{k\epsilon}{2}}}{1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}}\right)\widetilde{E} = \frac{1-\widehat{C}2^{-\frac{k\epsilon}{2}}}{1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}}\widetilde{E} \subset \Sigma_k\Sigma_{k-1}^{-1}\widetilde{E},$$

Since

$$\lim_{k\to+\infty} 2^{\frac{\epsilon k}{2}} \frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2}} + \widehat{C}2^{-\frac{k\epsilon}{2}}}{1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}} = \lim_{k\to+\infty} \frac{\widehat{C}+\widehat{C}2^{\frac{\epsilon}{2}}}{1+\widehat{C}2^{-\frac{\epsilon(k-1)}{2}}} = \widehat{C}+\widehat{C}2^{\frac{\epsilon}{2}},$$

by taking k sufficiently large, we can show

$$\left(1-\widehat{C}2^{-\frac{\epsilon k}{2}}\right)\widetilde{E} \subset \left(1-\frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2}} + \widehat{C}2^{-\frac{k\epsilon}{2}}}{1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2}}}\right)\widetilde{E} \subset \Sigma_k\Sigma_{k-1}^{-1}\widetilde{E}.$$

So we have

$$\left(1-\widehat{C}2^{-\frac{\epsilon k}{2}}\right)\widetilde{E} \subset \Sigma_k\Sigma_{k-1}^{-1}\widetilde{E} \subset \left(1+\widehat{C}2^{-\frac{\epsilon k}{2}}\right)\widetilde{E}, \quad k \ge \widehat{k}.$$

Since $\Sigma_k\Sigma_{k-1}^{-1}$ is still upper-triangular, we apply Lemma 2.1 (with $U = \Sigma_k\Sigma_{k-1}^{-1}$) to obtain that

$$\|\Sigma_k\Sigma_{k-1}^{-1} - I\| \le C(n)\widehat{C}2^{-\frac{\epsilon k}{4}}, \quad k \ge \widehat{k}.$$

Estimates (4.8) and (4.9) have been established. The existence of T, τ and (4.10) follow by elementary consideration. □

From Proposition 4.3, we can define

$$\Sigma = (T, -\tau),$$

and let $w = u \circ \Sigma^{-1}$, then we have

$$-w_s \det D^2 w = 1, \quad \text{in } \mathbb{R}^{n+1}_- \setminus \Sigma(Q_H),$$

in fact, $w_s = -\frac{u_t}{\tau}$, $\det D^2 w = (\det T^{-1})^2 \det D^2 u$,

$$-w_s \det D^2 w = \frac{1}{\tau}\frac{1}{(\det T)^2} u_t \det D^2 u = 1$$

from (4.10). Since $\{(y,s) : w(y,s) < H'\} = \Sigma(Q_{H'})$, (3.11) and

$$\frac{Q_{H'}}{\sqrt{H'}} = \left(diag\left\{\frac{1}{\sqrt{H'}}, \frac{1}{\sqrt{H'}}, \ldots, \frac{1}{\sqrt{H'}}\right\}, \frac{1}{H'}\right) Q_{H'},$$

then we can deduce from (4.9) and (4.10) that on one hand

$$\Sigma(Q_{H'}) - \Sigma_k(Q_{H'}) \subset \widehat{C}2^{-\frac{\epsilon k}{4}}\sqrt{H'}\widetilde{E},$$
$$\Sigma(Q_{H'}) \subset \left(1+2\widehat{C}2^{-\frac{\epsilon k}{4}}\right)\sqrt{H'}\widetilde{E},$$

on the other hand

$$\Sigma_k(Q_{H'}) - \Sigma(Q_{H'}) \subset \widehat{C}2^{-\frac{\epsilon k}{4}}\sqrt{H'}\widetilde{E},$$
$$\left(1-2\widehat{C}2^{-\frac{\epsilon k}{4}}\right)\sqrt{H'}\widetilde{E} \subset \Sigma(Q_{H'}).$$

In particular, taking $H' = 2^k$, then we get

$$\left(1-2\widehat{C}(H')^{-\frac{\epsilon}{4}}\right)\sqrt{H'}\widetilde{E} \subset \{(y,s) : w(y,s) < H'\}$$
$$\subset \left(1+2\widehat{C}(H')^{-\frac{\epsilon}{4}}\right)\sqrt{H'}\widetilde{E}, \quad \forall H' \ge 2^{\widehat{k}}.$$

So we have

$$\left(1-2\widehat{C}(w(y,s))^{-\frac{\epsilon}{4}}\right)^2 w(y,s) < -s + \frac{1}{2}|y|^2 < \left(1+2\widehat{C}(w(y,s))^{-\frac{\epsilon}{4}}\right)^2 w(y,s),$$

on one hand, we see

$$
\begin{aligned}
&-s+\frac{1}{2}|y|^2 < \left(1+2\widehat{C}(w(y,s))^{-\frac{\epsilon}{4}}\right)^2 w(y,s),\\
&-s+\frac{1}{2}|y|^2 < w(y,s)+4\widehat{C}(w(y,s))^{1-\frac{\epsilon}{4}}+4\widehat{C}^2(w(y,s))^{1-\frac{\epsilon}{2}},\\
&-s+\frac{1}{2}|y|^2 < w(y,s)+(4\widehat{C}+4\widehat{C}^2)(w(y,s))^{1-\frac{\epsilon}{4}},\\
&w(y,s)-\left(-s+\frac{1}{2}|y|^2\right) > -(4\widehat{C}+4\widehat{C}^2)(w(y,s))^{1-\frac{\epsilon}{4}},
\end{aligned}
$$

on the other hand, we show

$$
\begin{aligned}
&\left(1-2\widehat{C}(w(y,s))^{-\frac{\epsilon}{4}}\right)^2 w(y,s) < -s+\frac{1}{2}|y|^2,\\
&w(y,s)-4\widehat{C}(w(y,s))^{1-\frac{\epsilon}{4}}+4\widehat{C}^2(w(y,s))^{1-\frac{\epsilon}{2}} < -s+\frac{1}{2}|y|^2,\\
&w(y,s)-\left(-s+\frac{1}{2}|y|^2\right) < 4\widehat{C}(w(y,s))^{1-\frac{\epsilon}{4}}-4\widehat{C}^2(w(y,s))^{1-\frac{\epsilon}{2}},\\
&w(y,s)-\left(-s+\frac{1}{2}|y|^2\right) < 4\widehat{C}(w(y,s))^{1-\frac{\epsilon}{4}}.
\end{aligned}
$$

Thus we obtain

$$
\left|w(y,s)-\left(-s+\frac{1}{2}|y|^2\right)\right| < \widehat{\widehat{C}}(w(y,s))^{1-\frac{\epsilon}{4}}.
$$

Consequently, by the fact $C^{-1}w(y,s)\le |y|^2+|s|$, we get

$$
\left|w(y,s)-\left(-s+\frac{1}{2}|y|^2\right)\right| \le C(|y|^2+|s|)^{\frac{2-\epsilon/2}{2}}, \quad \sqrt{|y|^2+|s|}\ge 2^{\overline{k}}. \tag{4.15}
$$

Proposition 4.4 *Let $g\in C^\infty(\mathbb{R}^{n+1}_-\setminus\widetilde{E})$ satisfy*

$$
(1-g_s)\det(I+D^2g)=1,\ I+D^2g>0,\ -m_5<g_s-1<-m_6,\ (y,s)\in\mathbb{R}^{n+1}_-\setminus\widetilde{E},
$$

where m_5, m_6 are two positive constants, and for some constants $\beta>0$ and $\gamma>-2$,

$$
|g(y,s)|\le\frac{\beta}{(\sqrt{|y|^2/2+|s|})^\gamma},\quad (y,s)\in\mathbb{R}^{n+1}_-\setminus\widetilde{E},
$$

Then there exist some constant $r=r(n,\beta,\gamma)\ge 1$ such that

$$
|D_y^iD_s^jg(y,s)|\le\frac{C}{(\sqrt{|y|^2/2+|s|})^{\gamma+k}},\quad i+2j=k,\quad \frac{|y|^2}{2}-s>r,\ \ k=1,2,3\ldots.
$$

where C depends only on n, k, β and γ.

Proof Let

$$
\eta(y,s):=-s+\frac{|y|^2}{2}+g(y,s).
$$

For $\left(\frac{|y|^2}{2}+|s|\right)^{\frac{1}{2}}=R>2$, let

$$
\eta_R(z,\iota):=\left(\frac{4}{R}\right)^2\eta\left(y+\frac{R}{4}z,s+\frac{R^2}{16}\iota\right),\quad \left(\frac{|z|^2}{2}+|\iota|\right)^{\frac{1}{2}}\le\frac{3}{2}.
$$

We have to show that η_R is well defined when $\left(\frac{|z|^2}{2}+|\iota|\right)^{\frac{1}{2}} \le \frac{3}{2}$.

$$
\begin{aligned}
\frac{\left|y+\frac{R}{4}z\right|^2}{2}-\left(s+\frac{R^2}{16}\iota\right) &= \frac{1}{2}\left(|y|^2+\frac{R}{2}y^Tz+\frac{R^2}{16}|z|^2\right)-\left(s+\frac{R^2}{16}\iota\right)\\
&\ge \frac{1}{2}|y|^2-s+\frac{R}{4}y^Tz\\
&\ge R^2-\frac{R}{4}|y||z|\\
&\ge R^2-\frac{R}{4}(\sqrt{2}R)\left(\frac{3}{\sqrt{2}}\right)\\
&= \frac{R^2}{4}\\
&> 1.
\end{aligned}
$$

Then let

$$
\begin{aligned}
g_R(z,\iota) :&= \left(\frac{4}{R}\right)^2 g\left(y+\frac{R}{4}z, s+\frac{R^2}{16}\iota\right)\\
&= \left(\frac{4}{R}\right)^2\left[\eta\left(y+\frac{R}{4}z, s+\frac{R^2}{16}\iota\right)+s+\frac{R^2}{16}\iota-\frac{\left|y+\frac{R}{4}z\right|^2}{2}\right]\\
&= \eta_R(z,\iota)+\iota+\frac{16}{R^2}s-8\left|\frac{y}{R}+\frac{z}{4}\right|^2, \quad \left(\frac{|z|^2}{2}+|\iota|\right)^{\frac{1}{2}} \le \frac{3}{2}.
\end{aligned}
$$

By the decay hypothesis on g, on one hand, we have

$$
\begin{aligned}
\eta_R(z,\iota) &\le 8\left|\frac{y}{R}+\frac{z}{4}\right|^2-\left(\iota+\frac{16}{R^2}s\right)+\left(\frac{4}{R}\right)^2\frac{\beta}{\left(\frac{\left|y+\frac{R}{4}z\right|^2}{2}+\left|s+\frac{R^2}{16}\iota\right|\right)^{\frac{\gamma}{2}}}\\
&\le 8\left(\frac{|y|}{R}+\frac{|z|}{4}\right)^2+\frac{9}{4}+16+\frac{16\beta}{R^2}\frac{1}{\left(\frac{25}{64}R^2\right)^{\frac{\gamma}{2}}}\\
&\le \frac{97}{2}+\frac{16\beta 8^\gamma}{5^\gamma R^{2+\gamma}}.
\end{aligned}
$$

Taking r_1 satisfying $\frac{16\beta 8^\gamma}{5^\gamma r_1^{2+\gamma}}=1$, for $R \ge \max\{r_1, 8\}$, we have

$$
\eta_R(z,\iota) \le \frac{99}{2}.
$$

On the other hand, we have

$$
\begin{aligned}
\eta_R(z,\iota) &\geq 8\left|\frac{y}{R}+\frac{z}{4}\right|^2-\left(\iota+\frac{16}{R^2}s\right)-\left(\frac{4}{R}\right)^2\frac{\beta}{\left(\left|y+\frac{R}{4}z\right|^2+\left|s+\frac{R^2}{16}\iota\right|\right)^{\frac{\gamma}{2}}}\\
&\geq 8\left(\frac{|y|}{R}-\frac{|z|}{4}\right)^2+16-\frac{8|y|^2}{R^2}-\frac{16\beta}{R^2}\frac{1}{\left(\frac{25}{64}R^2\right)^{\frac{\gamma}{2}}}\\
&=8\left(\frac{|y|^2}{R^2}-\frac{|yz|}{2R}+\frac{z^2}{16}\right)+16-\frac{8|y|^2}{R^2}-\frac{16\beta}{R^2}\frac{1}{\left(\frac{25}{64}R^2\right)^{\frac{\gamma}{2}}}\\
&\geq -\frac{4|yz|}{R}+16-\frac{16\beta 8^\gamma}{5^\gamma R^{2+\gamma}}\\
&\geq -\frac{4|y|}{R}\cdot\frac{3\sqrt{2}}{2}+16-\frac{16\beta 8^\gamma}{5^\gamma R^{2+\gamma}}\\
&\geq -12+16-\frac{16\beta 8^\gamma}{5^\gamma R^{2+\gamma}}\\
&=4-\frac{16\beta 8^\gamma}{5^\gamma R^{2+\gamma}}.
\end{aligned}
$$

then taking $R\geq\max\{r_1,8\}$, we have

$$\eta_R(z,\iota)\geq 3.$$

In conclusion, there exists some $r=r(n,\beta,\gamma)=\max\{r_1,8\}\geq 1$ such that for $(\frac{|y|^2}{2}+|s|)^{\frac{1}{2}}=R\geq r$,

$$3\leq\eta_R(z,\iota)\leq\frac{99}{2},\quad\left(\frac{|z|^2}{2}+|\iota|\right)^{\frac{1}{2}}\leq\frac{3}{2}.$$

Since η_R satisfies

$$-\eta_{R\iota}\det D^2\eta_R=1,\ D^2\eta_R>0,\ -m_5<\eta_{R\iota}<-m_6,\ \left(\frac{|z|^2}{2}+|\iota|\right)^{\frac{1}{2}}\leq\frac{3}{2},$$

by the estimates of Pogorelov, Evans-Krylov and regularity theory of the parabolic equation, we have

$$\|\eta_R\|_{C^{k,k/2}(\widetilde{E})}\leq C,\ C^{-1}I\leq D^2\eta_R\leq CI\ \ in\ \widetilde{E}.$$

Here and in the following, $C\geq 1$ denotes some constant depending only on n unless otherwise stated.

$$\|g_R\|_{C^{k,k/2}(\widetilde{E})}\leq C,\ C^{-1}I\leq(I+D^2g_R)\leq CI\ \ in\ \widetilde{E},\ k\geq 2. \tag{4.16}$$

Clearly, g_R satisfies

$$\widehat{a}_1 g_{R\iota}+\widehat{a}_{ij}D_{ij}g_R=0,\quad\left(\frac{|z|^2}{2}+|\iota|\right)^{\frac{1}{2}}<\frac{3}{2},$$

where

$$
\begin{aligned}
\widehat{a}_1(z,\iota)=\int_0^1 F_1(-1+\theta g_{R\iota},I+\theta D^2g_R)d\theta,\ \widehat{a}_{ij}(z,\iota)\\
=\int_0^1 F_{ij}(-1+\theta g_{R\iota},I+\theta D^2g_R)d\theta
\end{aligned}
$$

satisfies, in view of (4.16), that

$$\|\widehat{a}_1\|_{C^{k,k/2}(\widetilde{E})} \le C,\ C^{-1} \le -\widehat{a}_1 \le C \ \ in\ \widetilde{E},$$

and

$$\|\widehat{a}_{ij}\|_{C^{k,k/2}(\widetilde{E})} \le C,\ C^{-1}I \le (\widehat{a}_{ij}) \le CI \ \ in\ \widetilde{E}.$$

Here we use the notation

$$F(a, M) := -a \det M.$$

By interior estimate of parabolic equations, we have

$$\begin{aligned} |D_z^i D_\iota^j g_R(0,0)| &\le C\|g_R\|_{L^\infty(\widetilde{E})} \\ &\le \frac{16}{R^2}\frac{\beta}{\left(\frac{\left|y+\frac{R}{4}z\right|^2}{2}+\left|s+\frac{R^2}{16}\iota\right|\right)^{\frac{\gamma}{2}}} \\ &\le \frac{C(n,k,\beta,\gamma)}{R^{2+\gamma}}, \quad i+2j=k. \end{aligned}$$

It follows that

$$|D_y^i D_s^j g(y,s)| \le \frac{C(n,k,\beta,\gamma)}{(\sqrt{|y|^2/2+|s|})^{\gamma+k}}, \quad i+2j=k,\ k=1,2,\dots.$$

□

Proposition 4.5 *There exist $\widetilde{b} \in \mathbb{R}^n$, $\widetilde{c} \in \mathbb{R}$ and some positive constant C such that*

$$\left|w(y,s)+s-\frac{|y|^2}{2}-\widetilde{b}^T y-\widetilde{c}\right| \le \frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}}, \quad \forall\ (y,s) \in \mathbb{R}^{n+1}_- \setminus \Sigma(Q_H).$$

Proof Let

$$\widehat{E}(y,s) := w(y,s) - \left(-s+\frac{|y|^2}{2}\right), \quad (y,s) \in \mathbb{R}^{n+1}_- \setminus \Sigma(Q_H),$$

and by (4.15) and Proposition 4.4

$$|D^2\widehat{E}(y,s)| + |\widehat{E}_s(y,s)| \le \frac{C}{(|y|^2+|s|)^{\frac{\epsilon}{2}}}, \tag{4.17}$$

and

$$|D^3\widehat{E}(y,s)| + |D_m\widehat{E}_s(y,s)| \le \frac{C}{(|y|^2+|s|)^{\frac{\epsilon+1}{2}}}. \tag{4.18}$$

It follows that

$$\widehat{a}_1\widehat{E}_s + \widehat{a}_{ij}D_{ij}\widehat{E} = 0 \ \text{ in } \mathbb{R}^{n+1}_- \setminus \Sigma(Q_H), \tag{4.19}$$

where

$$\begin{aligned} \widehat{a}_1(y,s) &= \int_0^1 F_1(-1+\theta\widehat{E}_s, I+\theta D^2\widehat{E})d\theta, \\ \widehat{a}_{ij}(y,s) &= \int_0^1 F_{ij}(-1+\theta\widehat{E}_s, I+\theta D^2\widehat{E})d\theta. \end{aligned}$$

Next we adopt the notation

$$\widetilde{F}(a, M) = (a \det M)^{\frac{1}{n}},$$

A is an $n \times n$ symmetric positive definite matrix and $a > 0$. It is well-known that in the open set of $n \times n$ symmetric positive definite matrices, $\det^{\frac{1}{n}}$ is concave. If

$$\widetilde{F}(a(y, s), M(y, s)) = 1,$$

then by differentiating it, we have

$$\widetilde{F}_a a_s + \widetilde{F}_{ij} M_{ijs} = 0,$$

and

$$\widetilde{F}_a a_k + \widetilde{F}_{ij} M_{ijk} = 0, \quad k = 1, 2, \dots, n.$$

We differentiate the above equation again, and get

$$\widetilde{F}_{aa} a_l a_k + \widetilde{F}_a a_{kl} + \widetilde{F}_{ij,\alpha\beta} M_{\alpha\beta l} M_{ijk} + \widetilde{F}_{ij} M_{ijkl} + 2\widetilde{F}_{ija} a_k M_{ijk} = 0, \quad k, l = 1, 2, \dots, n.$$

By the concavity of $\widetilde{F}$ and let $e \in \mathbb{R}^n$ be a unit vector, we have

$$\widetilde{F}_{aa}(D_e a)^2 + \widetilde{F}_a(D_{ee} a) + \widetilde{F}_{ij}(e^T Me)_{ij} \geq 0,$$

where $\widetilde{F}_a = \frac{1}{na}$, $\widetilde{F}_{aa} = -\frac{1}{na^2}$, $\widetilde{F}_{ij} = \frac{1}{n} M^{ij}$ and $(M^{ij}) = (M_{ij})^{-1}$. Thus we show

$$-\frac{1}{a^2}(D_e a)^2 + \frac{1}{a}(D_{ee} a) + M^{ij}(e^T Me)_{ij} \geq 0.$$

Let $a = -w_s = -\widehat{E}_s + 1$, $M = D^2 w = D^2 \widehat{E} + I$, we have

$$-\frac{1}{1 - \widehat{E}_s}(\widehat{E}_s)_s + ((D^2 \widehat{E} + I)^{-1})_{ij}(\widehat{E}_s)_{ij} = 0,$$
$$-\frac{1}{1 - \widehat{E}_s}(D_m \widehat{E})_s + ((D^2 \widehat{E} + I)^{-1})_{ij}(D_m \widehat{E})_{ij} = 0, \ m = 1, 2, \dots, n,$$

and

$$-\frac{1}{1 - \widehat{E}_s}(D_{ee} \widehat{E})_s + ((D^2 \widehat{E} + I)^{-1})_{ij}(D_{ee} \widehat{E})_{ij} \geq \frac{(D_e(-\widehat{E}_s))^2}{(1 - \widehat{E}_s)^2} \geq 0.$$

Taking $B_1 = \frac{1}{1 - \widehat{E}_s}$ and $B_{ij} = ((D^2 \widehat{E} + I)^{-1})_{ij}$, we have

$$-B_1(\widehat{E}_s)_s + B_{ij}(\widehat{E}_s)_{ij} = 0,$$
$$-B_1(D_m \widehat{E})_s + B_{ij}(D_m \widehat{E})_{ij} = 0, \ m = 1, 2, \dots, n,$$

and

$$-B_1(D_{ee} \widehat{E})_s + B_{ij}(D_{ee} \widehat{E})_{ij} \geq \frac{(D_e(-\widehat{E}_s))^2}{(1 - \widehat{E}_s)^2} \geq 0.$$

We claim that

$$|B_{ij} - \delta_{ij}| \leq \frac{C}{(|y|^2 + |s|)^{\frac{\epsilon}{2}}}, \quad |B_1 - 1| \leq \frac{C}{(|y|^2 + |s|)^{\frac{\epsilon}{2}}}.$$

Indeed, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of $D^2\widehat{E}$, then on one hand,

$$\begin{aligned}\left|\frac{1}{1-\widehat{E}_s}-1\right| &= |\det(D^2\widehat{E}+I)-1| \\ &= |1+\lambda_1+\lambda_2+\cdots+\lambda_n+\lambda_1\lambda_2+\cdots+\lambda_1\lambda_2\cdots\lambda_n-1| \\ &= |\lambda_1+\lambda_2+\cdots+\lambda_n+\lambda_1\lambda_2+\cdots+\lambda_1\lambda_2\cdots\lambda_n| \\ &\leq \frac{C}{(|y|^2+|s|)^{\frac{\epsilon}{2}}},\end{aligned}$$

on the other hand,

$$\begin{aligned}\left|\frac{1}{1+\lambda_i}-1\right| &= \frac{|\lambda_i|}{|1+\lambda_i|} \\ &\leq \frac{|\lambda_i|}{1-|\lambda_i|} \\ &\leq \frac{C}{(|y|^2+|s|)^{\frac{\epsilon}{2}}},\end{aligned}$$

this completes the proof of the claim. So we have

$$-(\widehat{E}_s)_s+\widetilde{B_{ij}}(\widehat{E}_s)_{ij}=0, \tag{4.20}$$

$$-(D_m\widehat{E})_s+\widetilde{B_{ij}}(D_m\widehat{E})_{ij}=0,\ m=1,2,\ldots,n, \tag{4.21}$$

and

$$-(D_{ee}\widehat{E})_s+\widetilde{B_{ij}}(D_{ee}\widehat{E})_{ij}\geq 0, \tag{4.22}$$

where $\widetilde{B_{ij}}=\frac{B_{ij}}{B_1}$ and

$$\begin{aligned}|\widetilde{B_{ij}}-\delta_{ij}| &= \frac{|B_{ij}-\delta_{ij}+(1-B_1)\delta_{ij}|}{|B_1|} \\ &\leq \frac{|B_{ij}-\delta_{ij}|+|1-B_1|}{|B_1|} \\ &\leq \frac{\frac{2C}{(|y^2|+|s|)^{\epsilon/2}}}{1-\frac{C}{(|y^2|+|s|)^{\epsilon/2}}} \\ &\leq \frac{\frac{2C}{(|y^2|+|s|)^{\epsilon/2}}}{1-\frac{1}{2}} \\ &= \frac{4C}{(|y^2|+|s|)^{\epsilon/2}}.\end{aligned}$$

By Lemma 2.4, for such coefficients, there exists a positive solution $G(y,s)$ to

$$-G_s+\widetilde{B_{ij}}G_{ij}=-e^s\left[\left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}+n(n-2)\left(\frac{1}{1+|y|^2}\right)^{\frac{n+2}{2}}\right]\leq 0 \tag{4.23}$$

satisfying

$$0\leq G(y,s)\leq\frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}}.$$

Similarly, there exists a negative solution $G'(y,s)$ to

$$-G'_s + \widetilde{B_{ij}}G'_{ij} = e^s\left[\left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}} + n(n-2)\left(\frac{1}{1+|y|^2}\right)^{\frac{n+2}{2}}\right] \geq 0 \tag{4.24}$$

satisfying

$$0 \leq -G'(y,s) \leq C\frac{e^s}{(1+|y|^2)^{\frac{n-2}{2}}}.$$

By (4.20), (4.23) and the maximum principle, we have

$$-\widehat{E}_s(y,s) \leq CG(y,s) \leq Ce^s\left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}. \tag{4.25}$$

By (4.22), (4.23) and the maximum principle, we also get

$$D_{ee}\widehat{E}(y,s) \leq CG(y,s) \leq Ce^s\left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}. \tag{4.26}$$

This means the largest eigenvalue of $D^2\widehat{E}(y,s)$ is bounded from above by $\frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}}$. Denoting $N(y,s)$, $N'(y,s)$ by

$$N(y,s) = \begin{pmatrix} \widehat{a}_{ij}(y,s) & 0 \\ 0 & -\widehat{a}_1(y,s) \end{pmatrix},$$

$$N'(y,s) = \begin{pmatrix} D_{ij}\widehat{E}(y,s) & 0 \\ 0 & -\widehat{E}_s(y,s) \end{pmatrix},$$

we can regard (4.19) as

$$tr(N(y,s)N'(y,s)) = 0.$$

By (4.25) and (4.26), the least eigenvalue of $D^2\widehat{E}(y,s)$ is bounded below by a negative constant multiple of the largest eigenvalue of $N'(y,s)$. Thus we have

$$|D^2\widehat{E}(y,s)| \leq \frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}}, \tag{4.27}$$

and

$$|\widehat{E}_s(y,s)| \leq \frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}}. \tag{4.28}$$

Now we claim that for $1 \leq m \leq n$, there exists $\widetilde{b}_m \in \mathbb{R}$ satisfying $\lim_{|y|^2-s\to\infty} D_m\widehat{E} = \widetilde{b}_m$. The idea of the proof comes from Elmar Schrohe and the first author. We divide the proof into four steps:

Step 1

$$|D_m\widehat{E}_s(y,s)| \leq \frac{C}{(1+|y|^2-s)^{\frac{n-1}{2}}}. \tag{4.29}$$

Since $\widehat{E}_s$ is a solution to (4.20), (4.29) is the gradient estimates for $\widehat{E}_s$. From the definition of $\widetilde{B_{ij}}$, (4.18), (4.27) and (4.28), it is easy to see that

$$|D_m\widetilde{B_{ij}}(y,s)| \leq \frac{C}{(1+|y|^2-s)^{\frac{1+\epsilon}{2}}}. \tag{4.30}$$

Let v be a solution to

$$-\, v_s + \widetilde{B_{ij}} D_{ij} v = 0 \ \text{ in } \mathbb{R}^{n+1}_- \setminus \Sigma(Q_H). \tag{4.31}$$

For $\frac{|y|^2}{2} - s = R^2 > 4$, let

$$v_R(z, \iota) = \frac{16}{R^2} v\left(y + \frac{R}{4}z, s + \frac{R^2}{16}\iota\right) \ \text{ in } E_{9/4}, \tag{4.32}$$

where $E_{9/4} := \{(z, \iota) : \frac{|z|^2}{2} - \iota < 9/4\}$. Since

$$\begin{aligned}
\frac{\left|y + \frac{R}{4}z\right|^2}{2} - \left(s + \frac{R^2}{16}\iota\right) &= \frac{1}{2}\left(|y|^2 + \frac{R}{2}y^T z + \frac{R^2}{16}|z|^2\right) - \left(s + \frac{R^2}{16}\iota\right) \\
&\geq \frac{1}{2}|y|^2 - s + \frac{R}{4}y^T z \\
&\geq R^2 - \frac{R}{4}|y||z| \\
&\geq R^2 - \frac{R}{4}(\sqrt{2}R)\left(\frac{3}{\sqrt{2}}\right) \\
&= \frac{R^2}{4} \\
&> 1,
\end{aligned}$$

we see that v_R is well defined when $(z, \iota) \in E_{9/4}$. Clearly, v_R satisfies

$$-\,(v_R)_\iota + (\widetilde{B}_R)_{ij} D_{ij} v_R = 0, \tag{4.33}$$

where $(\widetilde{B}_R)_{ij}(z, \iota) := \widetilde{B}_{ij}(y + \frac{R}{4}z, s + \frac{R^2}{16}\iota)$, and

$$|D_m(\widetilde{B}_R)_{ij}| \leq \frac{R}{4}\frac{C}{R^{1+\epsilon}} = \frac{C}{R^\epsilon}$$

From the gradient estimate for uniformly parabolic differential equation (4.33), we get

$$\sup_{E_{1/4}} |D_m v_R| \leq C_1 \sup_{E_{9/4}} |v_R|, \tag{4.34}$$

where C_1 depends only on n, the decay of $|D_m(\widetilde{B}_R)_{ij}|$ and the parabolic constants of (4.33). In fact, we use an idea that goes back to Bernstein, and choose a function

$$\eta_R(z, \iota) := \left(\frac{9}{4} - \frac{|z|^2}{2} + \iota\right)^2 \sum_{m=1}^{n} (D_m v_R)^2 + C_1^2 v_R^2.$$

After a simple computation and an application of Cauchy inequality, we can find C_1 depending only on n, the decay of $|D_m(\widetilde{B}_R)_{ij}|$ and the parabolic constants of (4.33), such that

$$-\,(\eta_R)_\iota + (\widetilde{B}_R)_{ij} D_{ij} \eta_R \geq 0 \ \ in \ \ E_{9/4}.$$

Then we can apply the maximum principle and obtain (4.34). Particularly, from (4.34), we see that

$$|D_m v_R(0, 0)| \leq C_1 \sup_{E_{9,4}} |v_R|,$$

that is,

$$|D_m v(y,s)| = \frac{R}{4}|D_m v_R(0,0)| \le CR \sup_{E_{9/4}} |v_R| \le \frac{C}{R} \sup_{(y,s)+\frac{R}{4}E_{9/4}} |v|.$$

Setting $v(y,s) = \widehat{E}_s(y,s)$, we get (4.29).

Step 2 $|D_m \widehat{E}| \le C \ln(1+|y|^2 - s)$ for $n=3$, or $|D_m\widehat{E}| \le C$ for $n \ge 4$. Indeed, for $n=3$ and fixed $a>0$ and $b<0$, we set $F(t) = D_m\widehat{E}(t(y_1,y_2,y_3,s)+(1-t)(a,a,a,b))$, from (4.29), then have

$$\begin{aligned}
&|D_m\widehat{E}(y_1,y_2,y_3,s) - D_m\widehat{E}(a,a,a,b)| = |F(1)-F(0)| = \left|\int_0^1 \frac{dF}{dt}dt\right| \\
&= \left|\sum_{i=1}^3 \left(\int_0^1 D_{mi}\widehat{E}dt\right)(y_i - a) + \left(\int_0^1 D_m\widehat{E}_s dt\right)(s-b)\right| \\
&\le \sum_{i=1}^3 \left(\int_0^1 \frac{Ce^{ts+(1-t)b}}{\left(1+\sum_{j=1}^3 |a+t(y_j-a)|^2\right)^{\frac{1}{2}}}dt\right)|y_i - a| \\
&\quad + \left(\int_0^1 \frac{C}{1+\sum_{j=1}^3 |a+t(y_j-a)|^2 + |b+t(s-b)|}dt\right)|s-b| \\
&\le C\ln(1+|y|^2 - s)
\end{aligned}$$

Similarly, we obtain, for $n \ge 4$,

$$|D_m\widehat{E}(y,s) - D_m\widehat{E}(a,a,\dots,a,b)| \le C.$$

Step 3 $D_m\widehat{E}(y,s) \equiv D_m\widehat{E}(0,\tau_0)$, $(y,s) \in \mathbb{R}^n \times (-\infty,\tau_0]$, $\tau_0 < 0$. It is easy to see that $D_m\widehat{E}$ satisfies (4.21) in $\mathbb{R}^n \times (-\infty,\tau_0]$, since (4.21) holds outside a compact set in $\mathbb{R}^{n+1}_-$. Let v be a solution to

$$-\ v_s + \widetilde{B_{ij}} D_{ij} v = 0 \text{ in } \mathbb{R}^n \times (-\infty,\tau_0], \tag{4.35}$$

where $\widetilde{B_{ij}} \in C^\infty(\mathbb{R}^n \times (-\infty,\tau_0])$ satisfies (4.30). For any $(y_0,s_0) \in \mathbb{R}^n \times (-\infty,\tau_0]$, we apply interior gradient estimates for v in $E_R = \{(y,s) | \frac{|y-y_0|^2}{2} - (s-s_0) < R^2 \text{ and } s \le s_0\}$, and have

$$|D_m v|_{L^\infty(E_{R/2})} \le \frac{C_2 |v|_{L^\infty(E_R)}}{R^2}, \tag{4.36}$$

where C_2 depends only on n, the decay of $|D_m(\widetilde{B}_R)_{ij}|$ and the parabolic constants of (4.35). In fact, we choose $\eta(y,s) = (R^2 - \frac{|y-y_0|^2}{2} + (s-s_0))^2 \sum_{m=1}^n (D_m v)^2 + C_2^2 v^2$, such that $-\eta_s + \widetilde{B}_{ij} D_{ij}\eta \ge 0$ in E_R. Then applying the maximum principle, we get (4.36). In particular,

$$|Dv(y_0,s_0)| \le \frac{C_2|v|_{L^\infty(E_R)}}{R^2}. \tag{4.37}$$

If $|v|_{L^\infty(E_R)} \le C\ln(1+R^2)$, or $|v|_{L^\infty(\mathbb{R}^{n+1}_- \setminus \Sigma(Q_H))} \le C$, we send $R \to \infty$, then get $|Dv(y_0,s_0)| = 0$, and then $v(y_0,s_0) = v(0,s_0)$, $D^2 v(y_0,s_0) = 0$. From (4.35), we conclude $v_s(y_0,s_0) = 0$ and $v(y_0,s_0) = v(0,s_0) = v(0,\tau_0)$ for $(y_0,s_0) \in \mathbb{R}^n \times (-\infty,\tau_0]$. Now Setting $v = D_m\widehat{E}$, by the estimate of $|D_m\widehat{E}|$ from Step 2, we finish the proof of Step 3.

Step 4 $\lim_{|y|\to\infty} D_m\widehat{E}(y,s) = \lim_{|y|\to\infty} D_m\widehat{E}(y,\tau_0) = D_m\widehat{E}(0,\tau_0)$, $(y,s)\in\mathbb{R}^n\times[\tau_0,0]\backslash\Sigma(Q_H)$. Indeed, for fixed $y_0\in\mathbb{R}^n$, from (4.29), we have

$$|D_m\widehat{E}(y_0,s)-D_m\widehat{E}(y_0,\tau_0)|\le\int_{\tau_0}^{s}|D_m\widehat{E}_s(y_0,s)|ds\le\frac{C|\tau_0|}{(1+|y_0|^2)^{\frac{n-1}{2}}}. \tag{4.38}$$

Let $|y_0|\to\infty$, we get

$$\lim_{|y|\to\infty} D_m\widehat{E}(y,s)=\lim_{|y|\to\infty} D_m\widehat{E}(y,\tau_0).$$

From above four steps, we finally obtain the claim.

Next we will prove that $\lim_{|y|^2-s\to\infty}(\widehat{E}(y,s)-\widetilde{b}^T y)=\widetilde{c}$, following the same line of the proof of the existence of $\widetilde{b}_m$. Since $D_m\widehat{E}(y,s)-\widetilde{b}_m$ satisfies

$$-(D_m\widehat{E}-\widetilde{b}_m)_s+\widetilde{B_{ij}}(D_m\widehat{E}-\widetilde{b}_m)_{ij}=0$$

from (4.21). Combining (4.23), (4.24) and maximum principle, we have

$$|D\widehat{E}(y,s)-\widetilde{b}|\le\frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}},$$

where $\widetilde{b}=(\widetilde{b}_1,\widetilde{b}_2,\ldots,\widetilde{b}_n)$.

Let $\widetilde{E}(y,s):=\widehat{E}(y,s)-\widetilde{b}^T y$. From (4.19), we have that $\widetilde{E}$ satisfies

$$-\widetilde{E}_s+A_{ij}D_{ij}\widetilde{E}=0, \tag{4.39}$$

where $A_{ij}:=\frac{\widehat{a}_{ij}}{-\widehat{a}_1}$. Recall that $\widehat{a}_1(y,s)=\int_0^1 F_1(-1+\theta\widehat{E}_s, I+\theta D^2\widehat{E})d\theta$, $\widehat{a}_{ij}(y,s)=\int_0^1 F_{ij}(-1+\theta\widehat{E}_s, I+\theta D^2\widehat{E})d\theta$, and $F(a,M)=-a\det M$. By (4.17) and (4.18), we get

$$|A_{ij}-\delta_{ij}|\le\frac{C}{(|y|^2-s)^{\frac{\epsilon}{2}}}, \tag{4.40}$$

and

$$|D_mA_{ij}|\le\frac{C}{(|y|^2-s)^{\frac{1+\epsilon}{2}}}. \tag{4.41}$$

It follows from Lemma 2.4 that there exists a positive solution $g(y,s)$ to

$$-g_s+A_{ij}g_{ij}=-e^s\left[\left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}+n(n-2)\left(\frac{1}{1+|y|^2}\right)^{\frac{n+2}{2}}\right]\le 0 \tag{4.42}$$

satisfying

$$0\le g(y,s)\le\frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}}.$$

Similarly, there exists a negative solution $g'(y,s)$ to

$$-g'_s+A_{ij}g'_{ij}=e^s\left[\left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}+n(n-2)\left(\frac{1}{1+|y|^2}\right)^{\frac{n+2}{2}}\right]\ge 0 \tag{4.43}$$

satisfying

$$0\le -g'(y,s)\le\frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}}.$$

Since

$$|\widetilde{E}_s(y,s)| = |\widehat{E}_s(y,s)| \le \frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}},$$

we skip the Step 1 in the proof of existence of $\widetilde{b}_m$, along the same line of other Steps, then see that there exists $\widetilde{c} \in \mathbb{R}$ such that

$$\lim_{|y|^2 - s \to \infty} \widetilde{E}(y,s) = \widetilde{c}.$$

It follows that $\widetilde{E}(y,s) - \widetilde{c}$ is a solution to (4.39). Then from (4.42), (4.43) and maximum principle, we obtain

$$|\widetilde{E}(y,s) - \widetilde{c}| \le \frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}}, \quad \forall (y,s) \in \mathbb{R}^{n+1}_{-} \setminus \Sigma(Q_H),$$

that is,

$$|\widehat{E}(y,s) - \widetilde{b}^T y - \widetilde{c}| \le \frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}}, \quad \forall (y,s) \in \mathbb{R}^{n+1}_{-} \setminus \Sigma(Q_H).$$

□

Proof of Theorem 1.2 Recall that w is a solution to $-w_s \det D^2 w = 1$ in $\mathbb{R}^{n+1}_{-} \setminus \Sigma(Q_H)$. Let

$$\breve{E}(y,s) := w(y,s) - \left(-s + \frac{|y|^2}{2} + \widetilde{b}^T y - \widetilde{c}\right).$$

with

$$|\breve{E}(y,s)| \le \frac{Ce^s}{(1+|y|^2)^{\frac{n-2}{2}}} \le \frac{C}{(|y|^2/2 - s)^{\frac{n-2}{2}}}.$$

From Proposition 4.4, we see that

$$|D_y^i D_t^j \breve{E}(y,s)| \le \frac{C}{(|y|^2/2 - s)^{\frac{n-2+k}{2}}}, \ i + 2j = k, \ \forall\, k \ge 1.$$

Since $w(y,s) = u(T^{-1}y, \frac{s}{-\tau})$, $x = T^{-1}y$ and $t = \frac{s}{-\tau}$, we have

$$|u(x,t) - \tau t - \frac{x^T T^T T x}{2} - \widetilde{b}^T T x - \widetilde{c}| \le \frac{Ce^{-\tau t}}{(1+|x|^2)^{\frac{n-2}{2}}},$$

and

$$|D_x^i D_t^j (u(x,t) - \tau t - \frac{x^T T^T T x}{2} - \widetilde{b}^T T x - \widetilde{c})| \le \frac{C}{(|y|^2/2 - s)^{\frac{n-2+k}{2}}}, \ i + 2j = k, \ \forall\, k \ge 1.$$

If taking $A = T^T T$, $b = T^T \widetilde{b}$, $c = \widetilde{c}$, then we complete the proof of Theorem 1.2. □

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