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A Calabi theorem for solutions to the parabolic Monge–Ampère equation with periodic data ☆

Un théorème de Calabi pour les solutions de l'équation de Monge–Ampère avec données periodiques

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Abstract

We classify all solutions to

$$-u_t \det D^2 u = f(x) \text{ in } \mathbb{R}^{n+1}_-,$$

where $f \in C^\alpha(\mathbb{R}^n)$ is a positive periodic function in x. More precisely, if u is a parabolically convex solution to above equation, then u is the sum of a convex quadratic polynomial in x, a periodic function in x and a linear function of t. It can be viewed as a generalization of the work of Gutiérrez and Huang in 1998. And along the line of approach in this paper, we can treat other parabolic Monge–Ampère equations.

Résumé

Nous classifions toutes les solutions à

$$-u_t \det D^2 u = f(x) \text{ in } \mathbb{R}^{n+1}_-,$$

où $f \in C^\alpha(\mathbb{R}^n)$ est une fonction périodique positive en x. Plus précisément, si u est une solution paraboliquement convexe de l'équation ci-dessus, alors u est la somme d'un polynôme quadratique convexe en x, une fonction périodique en x et une fonction linéaire de t. Cela peut être considéré comme une généralisation du travail de Gutiérrez et Huang en 1998. Et le long de la ligne d'approche dans cet article, nous pouvons traiter d'autres équations paraboliques Monge–Ampère.

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1. Introduction

A celebrated result of Jörgens ($n = 2$ [15]), Calabi ($n \le 5$ [7]) and Pogorelov ($n \ge 2$ [23]) states that any classical convex solutions to the Monge–Ampère equation

$$\det D^2 u = 1 \text{ in } \mathbb{R}^n \tag{1}$$

must be a quadratic polynomial. A simpler and more analytical proof was given by S.Y. Cheng and S.T. Yau [8]. J. Jost and Y.L. Xin showed a quite different proof in [16]. L. Caffarelli [3] extended above result for classical solutions to viscosity solutions. L. Caffarelli and Y.Y. Li [5] considered

$$\det D^2 u = f \text{ in } \mathbb{R}^n, \tag{2}$$

where f is a positive continuous function and is not equal to 1 only on a bounded set. They proved that for $n \ge 3$, the convex viscosity solution u is very close to quadratic polynomial at infinity. More precisely, for $n \ge 3$, there exist $c \in \mathbb{R}$, $b \in \mathbb{R}^n$ and an $n \times n$ symmetric positive definite matrix A with $\det A = 1$, such that

$$\limsup_{|x|\to\infty} |x|^{n-2} |u(x) - (\frac{1}{2} x^T A x + b \cdot x + c)| < \infty.$$

In a subsequent work [6], L. Caffarelli and Y.Y. Li proved that if f is periodic, then u must be the sum of a quadratic polynomial and a periodic function. To be concrete, for $n \ge 2$, there exist $b \in \mathbb{R}^n$ and a symmetric positive definite $n \times n$ matrix A with $\det A = \fint_{\prod_{1 \le i \le n}[0,a_i]} f$, such that $v := u - [\frac{1}{2} x^T A x + b^T x]$ is a_i-periodic in ith variable, i.e., $v(x + a_i e_i) = v(x)$, $\forall\, x \in \mathbb{R}^n$, $1 \le i \le n$. In recent paper [24], E. Teixeira and L. Zhang obtained that if $f \in C^{1,\alpha}(\mathbb{R}^n)$ is asymptotically close to a periodic function, then the difference between u and a parabola is asymptotically close to a periodic function at infinity, for $n \ge 3$.

Above famous Jörgens, Calabi and Pogorelov theorem was extended by C.E. Gutiérrez and Q. Huang [11] to solutions of the following parabolic Monge–Ampère equation

$$-u_t \det D^2 u = 1, \tag{3}$$

where $u = u(x,t)$ is parabolically convex, i.e., u is convex in x and nonincreasing in t, and $D^2 u$ denotes the Hessian of u with respect to the variable x. They got

Theorem 1.1. *Let $u \in C^{4,2}(\mathbb{R}^{n+1}_-)$ be a parabolically convex solution to the parabolic Monge–Ampère equation (3) in $\mathbb{R}^{n+1}_- := \mathbb{R}^n \times (-\infty, 0]$, such that there exist positive constants m_1 and m_2 with*

$$-m_1 \le u_t(x,t) \le -m_2, \quad \forall (x,t) \in \mathbb{R}^{n+1}_-. \tag{4}$$

Then u must have the form $u(x,t) = C_1 t + p(x)$, where $C_1 < 0$ is a constant and p is a convex quadratic polynomial on x.

and they gave an example to show that viscosity solutions to (3) may not be of the form given by above theorem. Recently, J. Bao and J. Xiong [1] extended this theorem to general parabolic Monge–Ampère equations.

This type of parabolic Monge–Ampère operator was first introduced by N.V. Krylov [17]. Owing to its importance in stochastic theory, he further considered it in [18–20]. This operator is relevant in the study of deformation of a surface by Gauss–Kronecker curvature [9]. Indeed, K. Tso [26] solved this problem by noting that the support function to the surface that is deforming satisfies an initial value problem involving that parabolic operator. And the operator plays an important role in a maximum principle for parabolic equations [25].

Solutions of elliptic Monge–Ampère equations with periodic right-hand side appear in several contexts of geometry and applied mathematics: when lifting the equation from a Hessian manifold, in problems of optimal transportation, vorticity arrays, homogenization, etc. And the solutions to some kind of parabolic Monge–Ampère equations with the same periodic right-hand side can be considered as a flow of above problems.

In the present paper we extend the Liouville theorem of L. Caffarelli and Y.Y. Li [6] to this parabolic Monge–Ampère equation:

$$-u_t \det D^2 u = f(x), \quad \text{in } \mathbb{R}^{n+1}_-, \tag{5}$$

where f is a positive periodic function, i.e.,

$$f(x + a_i e_i) = f(x) > 0, \quad \forall x \in \mathbb{R}^n, \quad 1 \le i \le n, \tag{6}$$

where $e_1 = (1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 0, 1)$. And assuming that

$$-\infty < -m_1 \le u_t \le -m_2 < 0, \tag{7}$$

then we obtain

Theorem 1.2. *Let $f \in C^\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$, satisfy (6), and let $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ be a parabolically convex solution to (5) satisfying (7). Then there exist $\tau < 0$, $b \in \mathbb{R}^n$ and a symmetric positive definite $n \times n$ matrix A with $-\tau \det A = \fint_{\Pi_{1\le i\le n}[0,a_i]} f$, such that*

$$v(x) := u(x,t) - [\tau t + \frac{1}{2} x^T A x + b \cdot x]$$

is a_i periodic in the ith variable, i.e.,

$$v(x + a_i e_i) = v(x), \quad x \in \mathbb{R}^n, \quad 1 \le i \le n.$$

Next we give some remarks on above theorem.

Remark 1.3. The theorem of Jörgens, Calabi, and Pogorelov for (3) is an easy consequence of the above theorem.

Remark 1.4. Because of the affine invariance, we only need to establish Theorem 1.2 for $a_i = 1$ $\forall i$ and for f satisfying in addition

$$\int_{[0,1]^n} f = 1 \tag{8}$$

Remark 1.5. From the regularity theorem obtained by the first author [30], we are able to get the above theorem under the weaker condition $f \in VMO^\psi(\mathbb{R}^n)$.

In the paper we work on the parabolic Monge–Ampère equation (5), but our methods can be applied to other parabolic Monge–Ampère equations, such as

$$u_t = (\det D^2 u)^{\frac{1}{n}} + f(x), \tag{9}$$

$$u_t = \log \det D^2 u + f(x). \tag{10}$$

Taking (10) for example, we have

Corollary 1.6. *Let $f \in C^\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$, satisfy (6), and let $u \in C^{2,1}(\mathbb{R}^{n+1}_-)$ be a convex solution to (10) satisfying*

$$\hat{m} \le u_t \le \hat{M} \tag{11}$$

Then there exist $\tau \in \mathbb{R}$, $b \in \mathbb{R}^n$ and a symmetric positive definite $n \times n$ matrix A with $\tau - \log \det A = \fint_{\Pi_{1\le i\le n}[0,a_i]} f$, such that

$$v(x) := u(x,t) - [\tau t + \frac{1}{2}x^T Ax + b \cdot x]$$

is a_i periodic in the ith variable.

Proof. Let

$$\overline{u}(x,t) = u(x,t) - (1+\hat{M})t.$$

Then $\overline{u} \in C^{2,1}(\mathbb{R}^{n+1}_-)$ is a solution to

$$\overline{u}_t = \log\det D^2\overline{u} + \overline{f},$$

where $\hat{m} - 1 - \hat{M} \le \overline{u}_t \le -1$, $\overline{f} = f - (1+\hat{M})$. Then

$$\frac{1}{C} \le \det D^2\overline{u} = \exp(\overline{u}_t - \overline{f}) \le C$$

where $C > 0$ depends on $\hat{m}$, $\hat{M}$ and $\min_{\mathbb{R}^n} f$ and $\max_{\mathbb{R}^n} f$. Therefore we get the density of parabolic Monge–Ampère measure associated to $\overline{u}$, $-\overline{u}_t \det D^2\overline{u}$, is bounded away from 0 and ∞. Now following almost the same line of the proof of above theorem, we get the corollary. □

The existence and uniqueness (modulo constants) of solutions to periodic elliptic Monge–Ampère equations were studied by Y.Y. Li.

Theorem 1.7. *([22]) Let $\mathbb{T}^n$ be a flat torus, $f \in C^\alpha(\mathbb{T}^n)$ be a positive function, and let A be a symmetric positive definite $n \times n$ matrix satisfying*

$$\det A = \fint_{\mathbb{T}^n} f. \tag{12}$$

Then there exists a function $v \in C^{2,\alpha}(\mathbb{T}^n)$ satisfying

$$\det(A + D^2v) = f, \quad on\ \mathbb{T}^n, \tag{13}$$

$$A + D^2v > 0, \quad on\ \mathbb{T}^n. \tag{14}$$

Moreover, condition (12) is necessary for the solvability of (13), and solutions of (13) and (14) are unique up to addition of constants.

Remark 1.8. Considering

$$-\tilde{v}_t \det(A + D^2\tilde{v}) = f, \quad \text{on } \mathbb{T}^n \times (-\infty, 0], \tag{15}$$

with $A + D^2\tilde{v} > 0$ on $\mathbb{T}^n \times (-\infty, 0]$ and $\det A = \fint_{\mathbb{T}^n} f$, we may easily find a solution to above equation. In fact, $\tilde{v} = -t + v(x)$, and $v(x)$ satisfies $\det(A + D^2v) = f$ on $\mathbb{T}^n$ with $A + D^2v > 0$ on $\mathbb{T}^n$.

In our proof of Theorem 1.2, we need a homogenization type estimate. It states that a solution w of the parabolic Monge–Ampère equation with periodic right-hand side differs from the corresponding solution $\overline{w}$, with constant right-hand side, a power of the diameter of the lattice. Let $Q^* \subset \mathbb{R}^{n+1}_-$ be a bowl-shaped domain satisfying

$$B_{\varepsilon_0}(0) \times [-\varepsilon_1, 0] \subset Q^* \subset B_2 \times [-\varepsilon_2, 0], \tag{16}$$

where ε_0, ε_1 and ε_2 depending only on n, m_1 and m_2. And let $\overline{w} \in C^0(\overline{Q^*}) \cap C^\infty(Q^*)$ denote the parabolically convex solution of

$$\begin{cases} -\overline{w}_t \det D^2\overline{w} = 1 & \text{in } Q^*, \\ \overline{w} = 0 & \text{on } \partial_p Q^*, \\ -C \le \overline{w}_t \le -C^{-1} & \text{in } Q^*, \end{cases}$$

see [28].

Let $\tilde{\epsilon_1}, \cdots, \tilde{\epsilon_n}$ be n linearly independent vectors in $\mathbb{R}^n$, and let $g \in C^0(\mathbb{R}^n)$ be a positive function satisfying

$$g(x+\tilde{\epsilon_i}) = g(x), \quad \forall x \in \mathbb{R}^n, \quad 1 \leq i \leq n, \tag{17}$$

$$\fint_{\Omega_i} g = 1, \tag{18}$$

where $\Omega_i = \{x \in \mathbb{R}^n : x = \sum_{i=1}^n t_i \tilde{\epsilon_i}, 0 \leq t_i \leq 1\}$ in the fundamental domain for the periodicity.

Considering

$$\begin{cases} -w_t \det D^2 w = g, & \text{in } Q^*, \\ w = 0 & \text{on } \partial_p Q^*, \end{cases} \tag{19}$$

then we give an estimate to the L^∞ norm of $|w - \overline{w}|$ on $\overline{Q^*}$:

Theorem 1.9. *Let $\tilde{\epsilon_1}, \tilde{\epsilon_2}, \cdots, \tilde{\epsilon_n} \in \mathbb{R}^n$ and $Q^* \subset \mathbb{R}^{n+1}_-$ be as above, $g \in C^0(\mathbb{R}^n)$ be a positive function satisfying (17) and (18), and let $w \in C^2(Q^*) \cap C^0(\overline{Q^*})$ be the parabolically convex solution of (19). Then we have*

$$\|w - \overline{w}\|_{L^\infty(Q^*)} \leq C \sum_{i=1}^n |\tilde{\epsilon_i}|^\beta, \tag{20}$$

for some constants β and C, depending only on n and the upper bound of g.

Remark 1.10. We have estimate (20) with the constant C independent of the smoothness of g, then g can be approximated by smooth g_j.

Next we give the local maximum principle for sub-solution of linearized parabolic Monge–Ampère equation:

Theorem 1.11. *Let Q^* be a bowl-shaped domain in $\mathbb{R}^{n+1}$ satisfying (16), and let $\phi \in C^{2,1}(\overline{Q^*})$ be a parabolically convex function satisfying, for some constants λ and Λ,*

$$\begin{cases} 0 < \lambda \leq -\phi_t \det D^2\phi \leq \Lambda < \infty, & \text{in } Q^*, \\ \phi = 0, & \text{on } \partial_p Q^*, \\ -m_1 \leq \phi_t(x,t) \leq -m_2, & \text{in } Q^*. \end{cases} \tag{21}$$

Assume that $\omega \in C^{2,1}(Q^*)$ satisfies

$$L_\phi \omega = \frac{\omega_t}{\phi_t} + trace\,((D^2\phi(x,t))^{-1} D^2\omega) \geq 0, \quad \omega \geq 0, \quad in\ Q^*.$$

Then for any $r > s > 0$,

$$\max_{X \in Q^*, dist(X, \partial_p Q^*) > r} \omega \leq C \int_{X \in Q^*, dist(X, \partial_p Q^*) > s} \omega,$$

where $X = (x,t)$, C depends only on n, λ, Λ, m_1, m_2, r and s.

This theorem can be viewed as an affine invariant counterpart of the classical local maximum principle for heat equation, parabolic version of Caffarelli and Gutiérrezs' [4] local maximum principle for linearized elliptic Monge–Ampère equation, and an extension of Huang's [14] local maximum principle for linearized parabolic Monge–Ampère equation to general $\phi(x,t)$. And we should note that the theorem is valid for other linearized parabolic Monge–Ampère equations, (9) and (10), once the density of parabolic Monge–Ampère measure associated to ϕ is bounded away from 0 and ∞.

Our paper is organized as follows. In Section 2, we list some preliminary facts. Theorem 1.9 is established in Section 3. We give a proof of our main theorem, Theorem 1.2, in Section 4. In the last section, the local maximum principle (Theorem 1.11) is obtained.

2. Preliminary results

In this section, we list some results that are used in the text.

First we recall some notations about the sections of parabolically convex functions. Let $Q \subset \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$, then we define

$$Q(t) = \{x : (x,t) \in Q\}. \tag{22}$$

If Q be a bounded set and $\tilde{t} = \inf\{t : Q(t) \neq \emptyset\}$. The parabolic boundary of the bounded domain Q is defined by

$$\partial_p Q = (\overline{Q(\tilde{t})}) \cup \bigcup_{t \in \mathbb{R}} (\partial Q(t) \times \{t\}),$$

where $\overline{Q(\tilde{t})}$ denotes the closure of $Q(\tilde{t})$ and $\partial Q(t)$ denotes the boundary of $Q(t)$. We say that Q is a bowl-shaped domain if $Q(t)$ is convex for each t and $Q(t_1) \subset Q(t_2)$ for $t_1 \leq t_2$. A function $\phi(x,t)$ is parabolically convex in Q if it is convex in x and nonincreasing in t. Given $X_0 = (x_0, t_0) \in Q$, ℓ_{X_0} is a supporting affine function, or supporting hyperplane for $\phi(\cdot, t_0)$ at $x = x_0$, if $\ell_{X_0} = \phi(x_0, t_0) + p \cdot (x - x_0)$ and $\phi(x, t_0) \geq \ell_{X_0}(x)$ for all $x \in Q(t_0)$. When $\phi \in C^1(Q)$, we have $p = D\phi(x_0, t_0)$.

Given $h > 0$, we define

$$Q_\phi(X_0, h) = \{(x,t) : \phi(x,t) \leq \ell_{X_0}(x) + h \text{ and } t \leq t_0\}, \tag{23}$$

and

$$S_\phi(x_0|t_0, h) = \{x : \phi(x, t_0) \leq \ell_{X_0}(x) + h\}. \tag{24}$$

We can always normalize u so that

$$u(0,0) = 0, \quad u(x,t) \geq 0 \quad \text{in } \mathbb{R}^{n+1}_-.$$

Let

$$Q_H = \{(x,t) \in \mathbb{R}^{n+1}_- : u(x,t) < H\}.$$

In fact, Q_H is $Q_u((0,0), H)$. Denote $v(x) = u(x,0)$. Let

$$\Omega_H = \{x \in \mathbb{R}^n : v(x) < H\}.$$

Indeed, Ω_H is $S_u(0|0, H)$. By a normalization lemma of John-Cordoba and Gallegos, there exists some affine transformation

$$T_H(x) = a_H x + b_H$$

with $\det a_H = 1$ such that $B_{\alpha_n R}(0) \subset T_H(\Omega_H) \subset B_R(0)$, where $\alpha_n = n^{-\frac{3}{2}}$.

Let

$$v_H(y) = \frac{1}{R^2} v(a_H^{-1}(Ry)), \quad y \in O_H := \frac{1}{R} a_H(\Omega_H). \tag{25}$$

From Proposition 2.12 in [5], $B_{1/C}(0) \subset O_H \subset B_2(0)$.

It is clear that $v_H(0) = \frac{1}{R^2} v(0) = 0$

$$\det D^2 v_H(y) = \frac{f(a_H^{-1}(Ry))}{-(u_H(y,0))_t}, \quad y \in O_H,$$

and

$$v_H|_{\partial O_H} = \frac{H}{R^2} \in (\frac{1}{C}, \ C). \tag{26}$$

Then, by the convexity of v_H, $0 \leq v_H \leq C$ in O_H.

Lemma 2.1. *(Lemma 2.9 in [5]) For $\lambda > 0$ and $r \geq 2$, let $v \in C^2((-3,3)^{n-1} \times (-r,r))$ satisfy*

$$D^2 v > 0, \quad \det D^2 v \geq \lambda, \quad in\ (-3,3)^{n-1} \times (-r,r),$$

and

$$0 \leq v \leq 1 \quad in\ (-2,2)^n.$$

Then, for some positive constant $C = C(n) > 0$,

$$\max_{|y_n| \leq r} v(0', y_n)^n \geq (\frac{r\lambda}{C} - 1).$$

Let

$$E = \{k_1 e_1 + \cdots + k_n e_n; \quad k_1, \cdots, k_n \text{ are integers}, \quad k_1^2 + \cdots + k_n^2 > 0\}.$$

For $e \in E$, let

$$\tilde{e} = \frac{1}{R} a_H(e). \tag{27}$$

Lemma 2.2. *For some positive constants $\alpha \in (-1,1)$ and C, depending only on n, m_1, m_2, $\max_{\mathbb{R}^n} f$ and $\min_{\mathbb{R}^n} f$,*

$$|\tilde{e}| \leq \frac{C}{R^{1+\alpha}} |e|, \quad e \in E. \tag{28}$$

Proof. For any $y \in \partial O_H$, we have, by [2],

$$v_H(y) \leq C v_H(\frac{y}{2}),$$

where $C > 2$ depends on n, m_1, m_2, $\max_{\mathbb{R}^n} f$ and $\min_{\mathbb{R}^n} f$. Then we deduce

$$v_H(y) \leq C^k v_H(\frac{y}{2^k})$$

for all $y \in \partial O_H$. Scaling back, the above inequality implies that for any $x \in \mathbb{R}^n$ satisfying $|x| > 1$,

$$v(x) \leq C^k v(\frac{x}{2^k}),$$

where k is an integer such that $2^{k-1} < |x| \leq 2^k$. Choosing $\alpha' > 0$ such that $C = 2^{1+\alpha'}$, we have

$$v(x) \leq 2^{k(1+\alpha')} v(\frac{x}{2^k}) \leq C|x|^{1+\alpha'},$$

where α' depends on n, m_1, m_2, $\max_{\mathbb{R}^n} f$ and $\min_{\mathbb{R}^n} f$.

For $\lambda e \in \partial\Omega_H$, we get

$$H = v(\lambda e) \leq C|\lambda e|^{1+\alpha'} \tag{29}$$

from above inequality. Then (26) and (29) imply that

$$\frac{1}{|\lambda|} \leq \frac{C}{R^{2/(1+\alpha')}} |e|. \tag{30}$$

On the other hand, since $\frac{1}{R} a_H(\lambda e) \in \partial O_H \subset B_2$, we have

$$|\lambda||\tilde{e}| = |\frac{1}{R} a_H(\lambda e)| \leq 4,$$

i.e.,

$$|\tilde{e}| \leq \frac{4}{|\lambda|} \leq \frac{C}{R^{1+\alpha}} |e|, \quad \forall e \in E,$$

from (30), where $\alpha = \frac{1-\alpha'}{1+\alpha'} \in (-1,1)$. $\square$

Let

$$(y,s) := \Gamma_H(x,t) = (\frac{a_H x}{R}, \frac{t}{R^2}), \quad (y,s) \in Q_H^* := \Gamma_H(Q_H),$$

and

$$w(y,s) = \frac{1}{R^2} u(\Gamma_H^{-1}(y,s)) = \frac{1}{R^2} u(Ra_H^{-1} y, R^2 s), \quad (y,s) \in Q_H^*. \tag{31}$$

Clearly

$$-w_s \det D^2 w = f(Ra_H^{-1} y) := g(y) \quad \text{in } Q_H^*.$$

By Proposition 3.1 in [29],

$$w = \frac{H}{R^2} \in (C^{-1}, C) \quad \text{on } \partial_p Q_H^*. \tag{32}$$

From Proposition 3.2 in [29],

$$B_{\varepsilon_0}(0) \times [-\varepsilon_1, 0] \subset Q_H^* \subset B_2(0) \times [-\varepsilon_2, 0].$$

By [28], there exists a unique parabolically convex solution $\overline{w} \in C^0(\overline{Q_H^*}) \bigcap C^\infty(Q_H^*)$ of

$$\begin{cases} -\overline{w}_s \det D^2 \overline{w} = 1 \quad \text{in } Q_H^*, \\ \overline{w} = \frac{H}{R^2} \in (C^{-1}, C) \quad \text{on } \partial_p Q_H^*, \\ -C \le \overline{w}_s \le -C^{-1} \quad \text{in } Q_H^*. \end{cases}$$

And for every $\delta > 0$, there exists some positive constant $C = C(\delta)$ such that for all $(y,s) \in Q_H^*$ and $dist_p((y,s), \partial_p Q_H^*) \ge \delta$, we have

$$C^{-1} I \le D^2 \overline{w}(y,s) \le CI, \quad |D^3 \overline{w}(y,s)| \le C. \tag{33}$$

Lemma 2.3. *([28]) Let $Q^* \subset \mathbb{R}^{n+1}$ be a bowl-shaped domain satisfying (16), and let $\overline{w} \in C^{2,1}(Q^*) \cap C^0(\overline{Q^*})$ be a parabolically convex solution of*

$$\begin{cases} -\overline{w}_t \det D^2 \overline{w} = 1, \quad \textit{in } Q^*, \\ \overline{w} = 0, \quad \textit{on } \partial_p Q^*. \end{cases} \tag{34}$$

Then for some positive constants C_k and β_k, depending only on n and k,

$$|D^k \overline{w}(X)| \le C_k dist(X, \partial_p Q^*)^{-\beta_k}, \quad X \in Q^*, \quad k = 1, 2, \cdots. \tag{35}$$

3. Proof of Theorem 1.9

In this section we prove Theorem 1.9.

Proof of Theorem 1.9. Throughout the proof, and unless otherwise stated, $\mu_i \in (0,1)$ and $C_i > 1$ denote various positive constants depending only on n and the upper bound of g. Let

$$m = \max_{\overline{Q^*}} |w - \overline{w}|. \tag{36}$$

By a barrier function argument [28],

$$-C_1 dist(X, \partial_p Q^*)^{\beta_1} \le w \le 0, \tag{37}$$

and

$$-C_1 dist(X, \partial_p Q^*)^{\beta_1} \le \overline{w} \le 0. \tag{38}$$

Particularly $m \le C_1$.

We will only treat the case

$$m = \max_{\overline{Q^*}}(w - \overline{w}) > 0,$$

since the other case can be settled similarly.

Let $\overline{X} := (\bar{x}, \bar{t}) \in Q^*$ be a maximum point of $w - \overline{w}$: $m = w(\overline{X}) - \overline{w}(\overline{X})$. By $w \leq 0$ and (38),

$$dist(\overline{X}, \partial_p Q^*) \geq \mu_1 m^{1/\beta_1}. \tag{39}$$

Let

$$u(x,t) = w(x,t) + \frac{m}{12^2}|x - \overline{x}|^2 + \frac{m}{9\varepsilon_2}|t - \overline{t}|. \tag{40}$$

Then $(u - \overline{w})(\overline{x}, \overline{t}) = m$. On the other hand, since

$$|u - w| = |\frac{m}{12^2}|x - \overline{x}|^2 + \frac{m}{9\varepsilon_2}|t - \overline{t}|| \leq \frac{2m}{9}, \quad \text{in } \overline{Q^*}, \tag{41}$$

we have

$$u - \overline{w} \leq \frac{2m}{9}, \quad \text{on } \partial_p Q^*$$

So for some interior point $\tilde{X} := (\tilde{x}, \tilde{t}) \in Q^*$,

$$(u - \overline{w})(\tilde{x}, \tilde{t}) = \max_{\overline{Q^*}}(u - \overline{w}) \geq m. \tag{42}$$

From (41) and (42),

$$(w - \overline{w})(\tilde{x}, \tilde{t}) = [(u - \overline{w}) - (u - w)](\tilde{x}, \tilde{t}) \geq m - \frac{2m}{9} = \frac{7m}{9}. \tag{43}$$

It follows, by (37) and (38), that

$$dist(\tilde{X}, \partial_p Q) \geq \mu_1 m^{1/\beta_1}, \quad \text{in } Q^*, \tag{44}$$

where the values of μ_1 is smaller than previous one.

Let $\xi \in C^\infty(\mathbb{R}^n)$ be the unique solution of

$$\det(D^2[\frac{1}{2}x^T D^2\overline{w}(\tilde{x}, \tilde{t})x + \xi(x)]) = \frac{g(x)}{m_1}, \quad \text{in } \mathbb{R}^n, \tag{45}$$

satisfying

$$D^2[\frac{1}{2}x^T D^2\overline{w}(\tilde{x}, \tilde{t})x + \xi(x)]) > 0, \quad x \in \mathbb{R}^n, \tag{46}$$

$$\xi(x + \tilde{\epsilon_i}) = \xi(x), \quad x \in \mathbb{R}^n, \quad 1 \leq i \leq n, \tag{47}$$

and

$$\int_{\Omega_i} \xi = 0. \tag{48}$$

The existence and uniqueness of ξ follows from Theorem 2.2 in [22].

Now we claim that

$$\|\xi\|_{L^\infty(\mathbb{R}^n)} \leq 2C_2 \sum_{i=1}^n |\tilde{\epsilon_i}|^2 \mu_1^{-\beta_2} m^{-\beta_2/\beta_1}. \tag{49}$$

In fact, let $\varphi(x) = \frac{1}{2}x^T D^2\overline{w}(\tilde{x}, \tilde{t})x + \xi(x)$, $x \in \mathbb{R}^n$, and for any fixed $y \in \mathbb{R}^n$ and $1 \leq i \leq n$, let $h(t) = \xi(y + t\tilde{\epsilon_i})$, $t \in \mathbb{R}$. Since $D^2\varphi > 0$ in $\mathbb{R}^n$, we have $\frac{d^2}{dt^2}\varphi(y + t\tilde{\epsilon_i}) > 0$ for $t \in \mathbb{R}$. Since

$$\frac{d^2}{dt^2}\varphi(y+t\tilde{\epsilon}_i) = \tilde{\epsilon}_i^T D^2\overline{w}(\tilde{x},\tilde{t})\tilde{\epsilon}_i + h''(t) \geq 0,$$

we then get, from (35) and (44),

$$h''(t) \geq -\tilde{\epsilon}_i^T D^2\overline{w}(\tilde{x},\tilde{t})\tilde{\epsilon}_i \geq -|\tilde{\epsilon}_i|^2 \|D^2\overline{w}(\tilde{x},\tilde{t})\| \geq -C_2|\tilde{\epsilon}_i|^2\mu_1^{-\beta_2}m^{-\beta_2/\beta_1}.$$

Since h is a periodic function of period 1, we can let $\overline{t} \in [-1,0]$ be a point where $h'=0$. For all $0<t<s<1$, we have, by the above lower bound of h'' and (35), that

$$h(s)-h(t) = \int_t^s h'(\tau_1)d\tau_1 = \int_t^s \int_{\overline{t}}^{\tau_1} h''(\tau_2)d\tau_2 d\tau_1 \geq -2C_2|\tilde{\epsilon}_i|^2\mu_1^{-\beta_2}m^{-\beta_2/\beta_1}.$$

So we have

$$\|\xi\|_{L^\infty(\mathbb{R}^n)} \leq osc_{\mathbb{R}^n} h \leq 2C_2\sum_{i=1}^n |\tilde{\epsilon}_i|^2\mu_1^{-\beta_2}m^{-\beta_2/\beta_1}.$$

Since $(\tilde{x},\tilde{t})$ is an interior maximum point of $u-\overline{w}$, we have

$$D^2(u-\overline{w})(\tilde{x},\tilde{t}) \leq 0, \tag{50}$$

that is,

$$0 < D^2w(\tilde{x},\tilde{t}) = D^2(u - \frac{m}{12^2}|x-\overline{x}|^2 - \frac{m}{9\varepsilon_2}|t-\overline{t}|)(\tilde{x},\tilde{t}) \leq D^2\overline{w}(\tilde{x},\tilde{t}) - \frac{2m}{12^2}I. \tag{51}$$

Let

$$v(x,t) = \overline{w}(x,t) + \xi(x) - \frac{m}{12^2}|x-\overline{x}|^2 + \frac{m}{24^2}|x-\tilde{x}|^2 - \frac{m}{9\varepsilon_2}|t-\overline{t}| + \frac{m}{18\varepsilon_2}|t-\tilde{t}|. \tag{52}$$

Then

$$w(x,t)-v(x,t) = u(x,t) - (\overline{w}(x,t) + \xi(x) + \frac{m}{24^2}|x-\tilde{x}|^2 + \frac{m}{18\varepsilon_2}|t-\tilde{t}|) \tag{53}$$

From (35) and (44) we can find β_3 and C_3 such that

$$|D^3\overline{w}(x,t)| \leq C_3m^{-\beta_3}, \quad |D^2\overline{w}_t(x,t)| \leq C_3m^{-\beta_3}, \quad \forall (x,t) \in B_{m^{\beta_3}/C_3}(\tilde{x},\tilde{t}) \cap Q^*.$$

Thus we can find larger β_4 and C_4 such that

$$\begin{aligned}
&B_{m^{\beta_4}/C_4}(\tilde{x},\tilde{t}) \subset B_{m^{\beta_3}/C_3}(\tilde{x},\tilde{t}), \quad \beta_4-\beta_3=1,\\
&D^2v(x,t) = D^2\overline{w}(x,t) + D^2\xi(x) - \frac{m}{96}I\\
&\qquad \leq D^2\overline{w}(\tilde{x},\tilde{t}) + n^2C_3m^{-\beta_3}(|x-\tilde{x}|+|t-\tilde{t}|)I + D^2\xi(x) - \frac{m}{96}I\\
&\qquad \leq D^2\overline{w}(\tilde{x},\tilde{t}) + \frac{2n^2C_3}{C_4m^{\beta_3-\beta_4}}I - \frac{m}{96}I + D^2\xi(x)\\
&\qquad < D^2\overline{w}(\tilde{x},\tilde{t}) + D^2\xi(x), \quad \forall (x,t) \in B_{m^{\beta_4}/C_4}(\tilde{x},\tilde{t}).
\end{aligned}$$

Then we get

$$\det D^2v(x,t) < \det(D^2\overline{w}(\tilde{x},\tilde{t}) + D^2\xi(x)) = \frac{g(x)}{m_1} = -\frac{w_t}{m_1}\det D^2w(x,t) \leq \det D^2w(x,t), \tag{54}$$

for all $(x,t) \in B_{m^{\beta_4}/C_4}(\tilde{x},\tilde{t})$ with $D^2v(x,t) \geq 0$.

Now (53) at $(\tilde{x},\tilde{t})$ implies that

$$(w-v)(\tilde{x},\tilde{t}) = (u-\overline{w})(\tilde{x},\tilde{t}) - \xi(\tilde{x}) \geq (u-\overline{w})(\tilde{x},\tilde{t}) - 2C_2\sum_{i=1}^n |\tilde{\epsilon}_i|^2\mu_1^{-\beta_2}m^{-\beta_2/\beta_1}.$$

Since $(u-\overline{w})(\tilde{x},\tilde{t})$ is the maximum value of $u-\overline{w}$, we have, for all $(x,\tilde{t})\in\partial B_{m^{\beta_4}/C_4}(\tilde{x},\tilde{t})$, that

$$(w-v)(x,\tilde{t})=(u-\overline{w})(x,\tilde{t})-\xi(x)-\frac{m}{24^2}|x-\tilde{x}|^2\le(u-\overline{w})(\tilde{x},\tilde{t})+2C_2\sum_{i=1}^{n}|\tilde{\epsilon_i}|^2\mu_1^{-\beta_2}m^{-\beta_2/\beta_1}-\frac{m^{1+2\beta_4}}{(24C_4)^2}.$$

If

$$4C_2\sum_{i=1}^{n}|\tilde{\epsilon_i}|^2\mu_1^{-\beta_2}m^{-\beta_2/\beta_1}\ge\frac{m^{1+2\beta_4}}{(24C_4)^2},$$

we have done, that is,

$$m\le(2304C_4^2C_2\mu_1^{-\beta_2}\sum_{i=1}^{n}|\tilde{\epsilon_i}|^2)^{\frac{1}{1+2\beta_4+\frac{\beta_2}{\beta_1}}}.$$

Otherwise,

$$(w-v)(x,\tilde{t})<(w-v)(\tilde{x},\tilde{t}),\quad\forall(x,\tilde{t})\in\partial B_{m^{\beta_4}/C_4}(\tilde{x},\tilde{t}).$$

Let $x_1\in B_{m^{\beta_4}/C_4}(\tilde{x},\tilde{t})$ be an interior maximum point $(w-v)(x,\tilde{t})$, then $D^2v(x_1,\tilde{t})\ge D^2w(x_1,\tilde{t})>0$ and $\det D^2v(x_1,\tilde{t})\ge\det D^2w(x_1,\tilde{t})$. This contradicts (54). Theorem 1.9 is established. □

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. We divide it into two steps.

Step 1. Modulo an affine transformation $(AT(n)\times AT(1))$, the behavior of u at infinity is $-t+\frac{1}{2}|x|^2$, where $AT(n)$ denotes the group of all invertible affine transformations on $\mathbb{R}^n$:

Proposition 4.1. *There exist some $\tau\in\mathbb{R}_-$, and some $n\times n$ symmetric positive definite matrix A with $-\tau\det A=1$, and some positive constants $0<\varepsilon<1$ and $C>1$, such that*

$$|u(x,t)-(\tau t+\frac{1}{2}x^TAx)|\le C(\sqrt{|x|^2+|t|})^{2-\varepsilon},\quad |x|^2+|t|\ge 1. \tag{55}$$

Owing to Lemma 2.2 and Theorem 1.9, we have

$$\|w-\overline{w}\|_{L^\infty(Q_H^*)}\le C\sum_{i=1}^{n}|\tilde{\epsilon_i}|^\beta=\frac{\widetilde{C}}{R^\theta}, \tag{56}$$

where $\theta=\min\{1,(1+\alpha)\beta\}$.

Let $(\overline{y},0)$ be the unique minimum point of $\overline{w}$ in $\overline{Q_H^*}$. For $\overline{w}(\overline{y},0)<\widetilde{H}\le H$, let

$$S_{\widetilde{H}}(0,0)=\{(y,s)\in\mathbb{R}_-^{n+1}:\frac{1}{2}y^TD^2\overline{w}(\overline{y},0)y+\overline{w}_s(\overline{y},0)s=\widetilde{H}\},$$
$$E_{\widetilde{H}}(0,0)=\{(y,s)\in\mathbb{R}_-^{n+1}:\frac{1}{2}y^TD^2\overline{w}(\overline{y},0)y+\overline{w}_s(\overline{y},0)s<\widetilde{H}\},$$
$$S_{\widetilde{H}}(\overline{y},0)=\{(y,s)\in\mathbb{R}_-^{n+1}:\frac{1}{2}(y-\overline{y})^TD^2\overline{w}(\overline{y},0)(y-\overline{y})+\overline{w}_s(\overline{y},0)s=\widetilde{H}\},$$
$$E_{\widetilde{H}}(\overline{y},0)=\{(y,s)\in\mathbb{R}_-^{n+1}:\frac{1}{2}(y-\overline{y})^TD^2\overline{w}(\overline{y},0)(y-\overline{y})+\overline{w}_s(\overline{y},0)s<\widetilde{H}\}.$$

We also denote that

$$mE_{\widetilde{H}}(0,0)=\{(y,s):\frac{1}{2}y^TD^2\overline{w}(\overline{y},0)y+\overline{w}_s(\overline{y},0)s<m^2\widetilde{H}\},m\in\mathbb{R}^+,$$
$$mE_{\widetilde{H}}(\overline{y},0)=\{(y,s):\frac{1}{2}(y-\overline{y})^TD^2\overline{w}(\overline{y},0)(y-\overline{y})+\overline{w}_s(\overline{y},0)s<m^2\widetilde{H}\},m\in\mathbb{R}^+,$$

and

$$mQ_H = \{(y', s') = (my, m^2t) : (y, s) \in Q_H\}, m \in \mathbb{R}^+.$$

Proposition 4.2. *There exist $\overline{k}$ and $\overline{C}$, depending only on n and f, such that for $\epsilon = \frac{\theta}{3}$, $H = 2^{(1+\epsilon)k/\theta}$ and $2^{(k-1)/\theta} \leq H' \leq 2^{k/\theta}$, we have*

$$(\frac{H'}{R^2} - \overline{C}2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}} E_1(0,0) \subset \Gamma_H(Q_{H'}) \subset (\frac{H'}{R^2} + \overline{C}2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}} E_1(0,0), \quad \forall k \geq \overline{k}. \tag{57}$$

Proof. Clearly, it follows from Proposition 3.1 in [29] and (31) that

$$C^{-1}2^{-\epsilon k/\theta} \leq \frac{H'}{R^2} \leq C2^{-\epsilon k/\theta}, \quad C^{-1}2^{\frac{(1+\epsilon)k}{2\theta}} \leq R \leq C2^{\frac{(1+\epsilon)k}{2\theta}},$$

and

$$\{w < \frac{H'}{R^2}\} := \{(y, s) : w(y, s) < \frac{H'}{R^2}\} = \Gamma_H(Q_{H'}) \subset Q_H^*.$$

By (56),

$$|w - \overline{w}| \leq \frac{\widetilde{C}}{R^\theta} \leq \widetilde{C}C2^{-\frac{1+\epsilon}{2}k} \quad \text{in } Q_H^*.$$

Since

$$\frac{H'}{R^2} \gg \frac{\widetilde{C}}{R^\theta}, \quad \text{as } R \to \infty,$$

the level surface of w can be well approximated by the level surface of $\overline{w}$:

$$\{\overline{w} < \frac{H'}{R^2} - \frac{\widetilde{C}}{R^\theta}\} \subset \{w < \frac{H'}{R^2}\} \subset \{\overline{w} < \frac{H'}{R^2} + \frac{\widetilde{C}}{R^\theta}\}.$$

By (56), the fact $w \geq 0$ and $w(0,0) = 0$, we have

$$-\frac{\widetilde{C}}{R^\theta} \leq w(\overline{y}, 0) - \frac{\widetilde{C}}{R^\theta} \leq \overline{w}(\overline{y}, 0) \leq \overline{w}(0,0) \leq w(0,0) + \frac{\widetilde{C}}{R^\theta} = \frac{\widetilde{C}}{R^\theta}.$$

Therefore by (33) and Lemma 2.3 in [29],

$$|\overline{w}(y, s) - \overline{w}(\overline{y}, 0) - \overline{w}_s(\overline{y}, 0)s - \frac{1}{2}(y - \overline{y})^T D^2\overline{w}(\overline{y}, 0)(y - \overline{y})| \leq C(|y - \overline{y}|^2 + |s|)^{\frac{3}{2}},$$

$dist_p((y, s), (\overline{y}, 0)) < \frac{1}{C}$ and

$$2C^{-1}I \leq D^2\overline{w}(\overline{y}, 0) \leq 2CI.$$

On one hand, we take a positive constant C_1 to be determined. For $(y, s) \in (\frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}} E_1(\overline{y}, 0)$, then

$$\begin{aligned} \overline{w}_s(\overline{y}, 0)s + \frac{1}{2}(y - \overline{y})^T D^2\overline{w}(\overline{y}, 0)(y - \overline{y}) &< \frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2\theta}}, \\ \frac{1}{C}|s| + \frac{1}{C}|y - \overline{y}|^2 &< \frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2\theta}}, \\ |y - \overline{y}|^2 + |s| &< C(\frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2\theta}}). \end{aligned}$$

We can take $\overline{k}_1$ satisfying for $k \geq \overline{k}_1$,

$$|y - \overline{y}|^2 + |s| < C(\frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2\theta}}) \leq \frac{1}{C^2}.$$

Thus,

$$\begin{aligned}
\overline{w}(y,s) &\le \overline{w}(\overline{y},0)+\overline{w}_s(\overline{y},0)s+\frac{1}{2}(y-\overline{y})^T D^2\overline{w}(\overline{y},0)(y-\overline{y})+C(|y-\overline{y}|^2+|s|)^{\frac{3}{2}}\\
&\le \frac{\widetilde{C}}{R^\theta}+\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2\theta}}+C^{\frac{5}{2}}(\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{3}{2}}\\
&\le \frac{\widetilde{C}}{R^\theta}+\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2\theta}}+C^{\frac{5}{2}}(\frac{H'}{R^2})^{\frac{3}{2}}\\
&\le \frac{\widetilde{C}}{R^\theta}+\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2\theta}}+C^4 2^{-\frac{3}{2\theta}\epsilon k}\\
&= \frac{\widetilde{C}}{R^\theta}+\frac{H'}{R^2}+(C^4-C_1)2^{-\frac{3\epsilon k}{2\theta}}.
\end{aligned}$$

We can take $C_1 > C^4$ satisfying $\frac{2\widetilde{C}C}{C_1-C^4}<1$, then

$$2\frac{\widetilde{C}}{R^\theta}\le 2\widetilde{C}C2^{-\frac{(1+\epsilon)k}{2}}<(C_1-C^4)2^{-\frac{3\epsilon k}{2\theta}}.$$

For $k\ge \overline{k}_1$, we can obtain

$$\overline{w}(y,s)\le \frac{\widetilde{C}}{R^\theta}+\frac{H'}{R^2}+(C^4-C_1)2^{-\frac{3\epsilon k}{2\theta}}<\frac{H'}{R^2}-\frac{\widetilde{C}}{R^\theta}.$$

In conclusion, we have

$$(\frac{H'}{R^2}-C_1 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}}E_1(\overline{y},0)\subset\{\overline{w}<\frac{H'}{R^2}-\frac{\widetilde{C}}{R^\theta}\},\quad \forall k\ge \overline{k}_1.$$

On the other hand, we take a positive constant C_2 to be determined. In order to prove

$$\{\overline{w}<\frac{H'}{R^2}+\frac{\widetilde{C}}{R^\theta}\}\subset(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}}E_1(\overline{y},0),$$

using the fact

$$(\overline{y},0)\in\{\overline{w}<\frac{H'}{R^2}+\frac{\widetilde{C}}{R^\theta}\}\cap(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}}E_1(\overline{y},0),$$

we only need to prove

$$(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}}S_1(\overline{y},0)\subset\{\overline{w}<\frac{H'}{R^2}+\frac{\widetilde{C}}{R^\theta}\}^c.$$

For $(y,s)\in(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}}S_1(\overline{y},0)$, then

$$\begin{aligned}
\overline{w}_s(\overline{y},0)s+\frac{1}{2}(y-\overline{y})^T D^2\overline{w}(\overline{y},0)(y-\overline{y}) &= \frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}},\\
\frac{1}{C}|s|+\frac{1}{C}|y-\overline{y}|^2 &< \frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}},\\
|y-\overline{y}|^2+|s| &< C(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}}).
\end{aligned}$$

Taking $\overline{k}_2$ satisfying for $k\ge \overline{k}_2$,

$$|y-\overline{y}|^2+|s|<C(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}})\le\frac{1}{C^2}.$$

Thus,

$$\begin{aligned}
\overline{w}(y,s) &\geq \overline{w}(\overline{y},0)+\overline{w}_s(\overline{y},0)s+\frac{1}{2}(y-\overline{y})^T D^2\overline{w}(\overline{y},0)(y-\overline{y})-C(|y-\overline{y}|^2+|s|)^{\frac{3}{2}}\\
&\geq -\frac{\widetilde{C}}{R^\theta}+\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}}-C^{\frac{5}{2}}(\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{3}{2}}\\
&\geq -\frac{\widetilde{C}}{R^\theta}+\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}}-C^{\frac{5}{2}}(2\frac{H'}{R^2})^{\frac{3}{2}}\\
&\geq -\frac{\widetilde{C}}{R^\theta}+\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}}-C^4 2^{\frac{3}{2}}2^{-\frac{3}{2\theta}\epsilon k}\\
&= -\frac{\widetilde{C}}{R^\theta}+\frac{H'}{R^2}+(C_2-2^{\frac{3}{2}}C^4)2^{-\frac{3\epsilon k}{2\theta}}.
\end{aligned}$$

We can take $C_2 > 2^{\frac{3}{2}}C^4$ satisfying $\frac{C_2-2^{\frac{3}{2}}C^4}{2\widetilde{C}C^\theta} > 1$, then

$$2\frac{\widetilde{C}}{R^\theta} \leq 2\widetilde{C}C^\theta 2^{-\frac{(1+\epsilon)k}{2}} < (C_2-2^{\frac{3}{2}}C^4)2^{-\frac{3\epsilon k}{2\theta}}.$$

For $k \geq \overline{k}_2$, we can obtain

$$\overline{w}(y,s) \geq -\frac{\widetilde{C}}{R^\theta}+\frac{H'}{R^2}+(C_2-2^{\frac{3}{2}}C^4)2^{-\frac{3\epsilon k}{2\theta}} > \frac{H'}{R^2}+\frac{\widetilde{C}}{R^\theta}.$$

In conclusion, we have

$$\{\overline{w} < \frac{H'}{R^2}+\frac{\widetilde{C}}{R^\theta}\} \subset (\frac{H'}{R^2}+C_2 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}}E_1(\overline{y},0), \quad \forall k \geq \overline{k}_2.$$

Therefore, if we take $C_3 > \max\{C_1, C_2\}$ and $\overline{k} = \max\{\overline{k}_1, \overline{k}_2\}$, then

$$(\frac{H'}{R^2}-C_3 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}}E_1(\overline{y},0) \subset \{w < \frac{H'}{R^2}\} \subset (\frac{H'}{R^2}+C_3 2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}}E_1(\overline{y},0) \quad \forall k \geq \overline{k}. \tag{58}$$

Finally, we want to obtain (57). We first show that

$$\partial_p(Q^*_{\widetilde{H}+\overline{w}(\overline{y},0)}(\overline{w})) \subset N_{\delta_1}(S_{\widetilde{H}}(\overline{y},0)), \quad 0 < \widetilde{H} \leq \frac{H}{R^2}-\overline{w}(\overline{y},0), \quad \delta_1 \leq C\widetilde{H}^{\frac{1}{2}}, \tag{59}$$

and neighborhood N is measured by parabolic distance

$$dist_p[(y_1,s_1),(y_2,s_2)] := (|y_1-y_2|^2+|s_1-s_2|)^{\frac{1}{2}}.$$

In fact, for $(y,s) \in \partial_p(Q^*_{\widetilde{H}+\overline{w}(\overline{y},0)}(\overline{w}))$, by the mean value theorem, (33) and Lemma 2.3 in [29], we have

$$\begin{aligned}
\widetilde{H} &= \overline{w}(y,s)-\overline{w}(\overline{y},0)\\
&= \overline{w}(y,s)-\overline{w}(y,0)+\overline{w}(y,0)-\overline{w}(\overline{y},0)\\
&= \overline{w}_s(y,s')s+\frac{1}{2}(y-\overline{y})^T D^2\overline{w}(y',0)(y-\overline{y})\\
&\geq \frac{1}{2C}(|s|+|y-\overline{y}|^2),
\end{aligned}$$

where $(y',s') \in Q^*_{\widetilde{H}+\overline{w}(\overline{y},0)}(\overline{w})$. Writing

$$\begin{aligned}
\widetilde{H} &= \overline{w}(y,s)-\overline{w}(\overline{y},0)\\
&= \overline{w}_s(\overline{y},0)s+(\overline{w}_s(y,s')-\overline{w}_s(\overline{y},0))s+\frac{1}{2}(y-\overline{y})^T D^2\overline{w}(\overline{y},0)(y-\overline{y})\\
&\quad +\frac{1}{2}(y-\overline{y})^T(D^2\overline{w}(y',0)-D^2\overline{w}(\overline{y},0))(y-\overline{y}),
\end{aligned}$$

for $(y,s)\in\partial_p(Q^*_{\widetilde{H}+\overline{w}(\overline{y},0)}(\overline{w}))$, then

$$\begin{aligned}
&|\widetilde{H}-\overline{w}_s(\overline{y},0)s-\frac{1}{2}(y-\overline{y})^T D^2\overline{w}(\overline{y},0)(y-\overline{y})|\\
&=|(\overline{w}_s(y,s')-\overline{w}_s(\overline{y},0))s+\frac{1}{2}(y-\overline{y})^T(D^2\overline{w}(y',0)-D^2\overline{w}(\overline{y},0))(y-\overline{y})|\\
&\le C|s|+C|y-\overline{y}|^2\\
&\le C\widetilde{H}.
\end{aligned}$$

For any $(y,s)\in\partial_p(Q^*_{\widetilde{H}+\overline{w}(\overline{y},0)}(\overline{w}))$ and any $(\widetilde{y},\widetilde{s})\in S_{\widetilde{H}}(\overline{y},0)$, by the above inequality, we have

$$|\overline{w}_s(\overline{y},0)\widetilde{s}+\frac{1}{2}(\widetilde{y}-\overline{y})^T D^2\overline{w}(\overline{y},0)(\widetilde{y}-\overline{y})-\overline{w}_s(\overline{y},0)s-\frac{1}{2}(y-\overline{y})^T D^2\overline{w}(\overline{y},0)(y-\overline{y})|\le C\widetilde{H}.$$

Taking $\widetilde{y},\overline{y},y$ on the same line l with $\widetilde{y}$ and y on the same side of the line l with respect to $\overline{y}$ (rotating the coordinates again so that l is parallel to some axis), we have

$$||\widetilde{y}-\overline{y}|^2-|y-\overline{y}|^2|\ge|y-\widetilde{y}|^2.$$

Then for $s=\widetilde{s}$, we get

$$\frac{1}{2C}||\widetilde{y}-\overline{y}|^2-|y-\overline{y}|^2|\le C\widetilde{H}.$$

In fact, there exists an orthogonal matrix O such that $D^2\overline{w}(\overline{y},0)=O^T diag\{\lambda_1,\cdots,\lambda_n\}O$, and the length of a vector in Euclidean space is invariant in orthogonal transformation. Therefore, we get

$$|y-\widetilde{y}|^2\le C\widetilde{H}.$$

Similarly, for $y=\widetilde{y}$,

$$|\overline{w}_s(\overline{y},0)\widetilde{s}-\overline{w}_s(\overline{y},0)s|\le C\widetilde{H}.$$

So we get

$$|s-\widetilde{s}|\le C\widetilde{H}.$$

This completes the proof of (59).

Next we estimate the distance between $(0,0)$ and $(\overline{y},0)$. By (56), we have

$$\begin{aligned}
0&\le\overline{w}(0,0)-\overline{w}(\overline{y},0)\\
&=(\overline{w}(0,0)-w(0,0))+(w(0,0)-w(\overline{y},0))+(w(\overline{y},0)-\overline{w}(\overline{y},0))\\
&\le\frac{2\widetilde{C}}{R^\theta},
\end{aligned}$$

so $(0,0)\in Q^*_{\frac{2\widetilde{C}}{R^\theta}+\overline{w}(\overline{y},0)}(\overline{w})$, and by (59) (taking $\widetilde{H}=\frac{2\widetilde{C}}{R^\theta}$), we have

$$\partial_p(Q^*_{\frac{2\widetilde{C}}{R^\theta}+\overline{w}(\overline{y},0)}(\overline{w}))\subset N_{\delta_1}(S_{\frac{2\widetilde{C}}{R^\theta}}(\overline{y},0)),\qquad \delta_1\le C(\frac{2\widetilde{C}}{R^\theta})^{1/2},$$

thus we get

$$dist_p((0,0),(\overline{y},0))\le C(\frac{2\widetilde{C}}{R^\theta})^{1/2}.$$

So by (58), we have

$$(\frac{H'}{R^2}-C_3 2^{-\frac{3\epsilon k}{2\theta}}-C^2\frac{2\widetilde{C}}{R^\theta})^{\frac{1}{2}}E_1(0,0)\subset\{w<\frac{H'}{R^2}\}\subset(\frac{H'}{R^2}+C_3 2^{-\frac{3\epsilon k}{2\theta}}+C^2\frac{2\widetilde{C}}{R^\theta})^{\frac{1}{2}}E_1(0,0)\quad\forall k\ge\overline{k}.$$

Since $2^{-\frac{3\epsilon k}{2\theta}}\gg\frac{1}{R^\theta}$ and let $\overline{C}=2C^2\widetilde{C}+C_3$, then we can obtain (57). $\square$

Let $\widetilde{E}$ denote the set $\{(y,s)\in\mathbb{R}^{n+1}_{-}:\frac{1}{2}|y|^2-s<1\}$, then we have the following proposition.

Proposition 4.3. *There exist positive constants $\widehat{k}$, $\widehat{C}$, some real invertible upper-triangular matrices $\{T_k\}_{k\geq\widehat{k}}$ and negative number $\{\tau_k\}_{k\geq\widehat{k}}$ such that*

$$-\tau_k\det T_k^T T_k=1,\quad \|T_kT_{k-1}^{-1}-I\|\leq\widehat{C}2^{-\frac{\epsilon k}{4\theta}},\quad |\tau_k\tau_{k-1}^{-1}-1|\leq\widehat{C}2^{-\frac{\epsilon k}{4\theta}},\tag{60}$$

and

$$(1-\widehat{C}2^{-\frac{\epsilon k}{2\theta}})\sqrt{H'}\widetilde{E}\subset\Sigma_k(Q_{H'})\subset(1+\widehat{C}2^{-\frac{\epsilon k}{2\theta}})\sqrt{H'}\widetilde{E},\quad \forall 2^{(k-1)/\theta}\leq H'\leq 2^{k/\theta},\tag{61}$$

where $\Sigma_k=(T_k,-\tau_k)$. Consequently, for some invertible T and τ,

$$-\tau\det T^TT=1,\quad \|T_k-T\|\leq\widehat{C}2^{-\frac{\epsilon k}{2\theta}},\quad |\tau_k-\tau|\leq\widehat{C}2^{-\frac{\epsilon k}{2\theta}}.\tag{62}$$

Proof. Let $H=2^{(1+\epsilon)k/\theta}$ and $2^{(k-1)/\theta}\leq H'\leq 2^{k/\theta}$. By Proposition 4.2, there exist some positive constants $\overline{C}$ and $\overline{k}$ depending only on n and f such that

$$(\frac{H'}{R^2}-\overline{C}2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}}E_1(0,0)\subset\Gamma_H(Q_{H'})\subset(\frac{H'}{R^2}+\overline{C}2^{-\frac{3\epsilon k}{2\theta}})^{\frac{1}{2}}E_1(0,0),\quad \forall k\geq\overline{k}.$$

Then

$$(H'-\overline{C}2^{-\frac{3\epsilon k}{2\theta}}R^2)^{\frac{1}{2}}E_1(0,0)\subset(a_H,id)(Q_{H'})\subset(H'+\overline{C}2^{-\frac{3\epsilon k}{2\theta}}R^2)^{\frac{1}{2}}E_1(0,0),$$
$$(1-\overline{C}2^{-\frac{3\epsilon k}{2\theta}}\frac{R^2}{H'})^{\frac{1}{2}}\sqrt{H'}E_1(0,0)\subset(a_H,id)(Q_{H'})\subset(1+\overline{C}2^{-\frac{3\epsilon k}{2\theta}}\frac{R^2}{H'})^{\frac{1}{2}}\sqrt{H'}E_1(0,0).$$

Since

$$C^{-1}2^{-\epsilon k/\theta}\leq\frac{H'}{R^2}\leq C2^{-\epsilon k/\theta},$$

we can get

$$(1-\overline{C}C2^{-\frac{\epsilon k}{2\theta}})^{\frac{1}{2}}\sqrt{H'}E_1(0,0)\subset(a_H,id)(Q_{H'})\subset(1+\overline{C}C2^{-\frac{\epsilon k}{2\theta}})^{\frac{1}{2}}\sqrt{H'}E_1(0,0).$$

On one hand, we take $\overline{C}_1>\frac{\overline{C}C}{2}$, $\overline{k}_5$ satisfying when $k\geq\overline{k}_5$, $2^{\frac{k\epsilon}{2\theta}}\geq\frac{\overline{C}_1^2}{2\overline{C}_1-C\overline{C}}$, and $\overline{k}_6=\max\{\overline{k}_5,\overline{k}\}$, then if $k\geq\overline{k}_6$, we have

$$\begin{aligned}\overline{C}_1^2&\leq 22^{\frac{k\epsilon}{2\theta}}\overline{C}_1-2^{\frac{k\epsilon}{2\theta}}C\overline{C},\\ 2^{-\epsilon k/\theta}\overline{C}_1^2&\leq 22^{-\frac{k\epsilon}{2\theta}}\overline{C}_1-2^{-\frac{k\epsilon}{2\theta}}C\overline{C},\\ 2^{-\epsilon k/\theta}\overline{C}_1^2-22^{-\frac{k\epsilon}{2\theta}}\overline{C}_1&\leq-2^{-\frac{k\epsilon}{2\theta}}C\overline{C},\\ 2^{-\epsilon k/\theta}\overline{C}_1^2-22^{-\frac{k\epsilon}{2\theta}}\overline{C}_1+1&\leq 1-2^{-\frac{k\epsilon}{2\theta}}C\overline{C},\\ (1-\overline{C}_12^{-\frac{k\epsilon}{2\theta}})^2&\leq 1-2^{-\frac{k\epsilon}{2\theta}}C\overline{C}.\end{aligned}$$

Therefore,

$$(1-\overline{C}_12^{-\frac{k\epsilon}{2\theta}})\sqrt{H'}E_1(0,0)\subset(a_H,id)(Q_{H'}),\quad k\geq\overline{k}_6.$$

On the other hand, if we also take $\overline{C}_2>\frac{\overline{C}C}{2}$, then for any $k\geq\overline{k}$, we have

$$(1+\overline{C}C2^{-\frac{k\epsilon}{2\theta}})^{\frac{1}{2}}\leq(1+\overline{C}_22^{-\frac{k\epsilon}{2\theta}}).$$

So

$$(a_H,id)(Q_{H'})\subset(1+\overline{C}_22^{-\frac{k\epsilon}{2\theta}})\sqrt{H'}E_1(0,0),\quad k\geq\overline{k}.$$

In conclusion, if we take $\widehat{C} > \frac{\overline{C}C}{2}$, $\widehat{k} = \overline{k}_6$, then

$$(1-\widehat{C}2^{-\frac{k\epsilon}{2\theta}})\sqrt{H'}E_1(0,0) \subset (a_H, id)(Q_{H'}) \subset (1+\widehat{C}2^{-\frac{k\epsilon}{2\theta}})\sqrt{H'}E_1(0,0), \quad k \ge \widehat{k}. \tag{63}$$

Let Q be the real symmetric positive definite matrix satisfying $Q^2 = Q^T Q = D^2\overline{w}(\overline{y}, 0)$ and O be an orthogonal matrix such that

$$T_k := OQa_H \quad \text{is the upper-triangular.}$$

And we also define $\tau_k = \overline{w}_s(\overline{y}, 0)$ and $\Sigma_k = (T_k, -\tau_k)$. Clearly,

$$-\tau_k \det T_k^T T_k = -\overline{w}_s(\overline{y}, 0)(\det a_H)^2 \det D^2\overline{w}(\overline{y}, 0) = 1.$$

Now we claim that $\widetilde{E} = (OQ, -\tau_k)E_1(0,0)$. $\forall (y, s) \in E_1(0,0)$, $(x, t) = (OQy, -\tau_k s)$, $x^T x = y^T Q^T O^T OQy = y^T D^2\overline{w}(\overline{y}, 0)y$, $t = -\tau_k s = -\overline{w}_s(\overline{y}, 0)s$. Recall that

$$\frac{1}{2}y^T D^2\overline{w}(\overline{y}, 0)y + \overline{w}_s(\overline{y}, 0)s = 1,$$

so $(x, t) \in \widetilde{E}$, and vice versa. From (63), we have

$$(1-\widehat{C}2^{-\frac{k\epsilon}{2\theta}})\sqrt{H'}\widetilde{E} \subset \Sigma_k(Q_{H'}) \subset (1+\widehat{C}2^{-\frac{k\epsilon}{2\theta}})\sqrt{H'}\widetilde{E}, \quad k \ge \widehat{k}.$$

If we take $H = 2^{(1+\epsilon)k/\theta}$ and $H' = 2^{(k-1)/\theta}$, we can obtain

$$(1-\widehat{C}2^{-\frac{k\epsilon}{2\theta}})\sqrt{2^{k-1}}\widetilde{E} \subset \Sigma_k(Q_{2^{k-1}}) \subset (1+\widehat{C}2^{-\frac{k\epsilon}{2\theta}})\sqrt{2^{k-1}}\widetilde{E}, \tag{64}$$

and if we take $H = 2^{(1+\epsilon)(k-1)/\theta}$ and $H' = 2^{(k-1)/\theta}$, we can get

$$(1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}})\sqrt{2^{k-1}}\widetilde{E} \subset \Sigma_{k-1}(Q_{2^{k-1}}) \subset (1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}})\sqrt{2^{k-1}}\widetilde{E},$$

then

$$(1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}})\sqrt{2^{k-1}}\Sigma_{k-1}^{-1}\widetilde{E} \subset Q_{2^{k-1}} \subset (1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}})\sqrt{2^{k-1}}\Sigma_{k-1}^{-1}\widetilde{E}, \tag{65}$$

$$(1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}})\sqrt{2^{k-1}}\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E} \subset \Sigma_k(Q_{2^{k-1}}) \subset (1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}})\sqrt{2^{k-1}}\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E}. \tag{66}$$

From the left hand of (66) and the right hand of (64), we see

$$(1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}})\sqrt{2^{k-1}}\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E} \subset (1+\widehat{C}2^{-\frac{k\epsilon}{2\theta}})\sqrt{2^{k-1}}\widetilde{E},$$

thus

$$\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E} \subset \frac{1+\widehat{C}2^{-\frac{k\epsilon}{2\theta}}}{1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}}}\widetilde{E} = (1 + \frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}} + \widehat{C}2^{-\frac{k\epsilon}{2\theta}}}{1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}}})\widetilde{E}.$$

Since

$$\lim_{k\to+\infty} 2^{\frac{\epsilon k}{2\theta}} \frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}} + \widehat{C}2^{-\frac{k\epsilon}{2\theta}}}{1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}}} = \lim_{k\to+\infty} \frac{\widehat{C}2^{\frac{\epsilon}{2\theta}} + \widehat{C}}{1-\widehat{C}2^{-\frac{\epsilon(k-1)}{2\theta}}} = \widehat{C}2^{\frac{\epsilon}{2\theta}} + \widehat{C},$$

by taking k sufficiently large, we can obtain

$$\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E} \subset (1 + \frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}} + \widehat{C}2^{-\frac{k\epsilon}{2\theta}}}{1-\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}}})\widetilde{E} \subset (1+\widehat{C}2^{-\frac{\epsilon k}{2\theta}})\widetilde{E}.$$

At the same time, from the left hand of (64) and the right hand of (66), we get

$$(1-\widehat{C}2^{-\frac{k\epsilon}{2\theta}})\sqrt{2^{k-1}}\widetilde{E} \subset (1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}})\sqrt{2^{k-1}}\Sigma_k\Sigma_{k-1}^{-1}\widetilde{E},$$

thus

$$(1 - \frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}} + \widehat{C}2^{-\frac{k\epsilon}{2\theta}}}{1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}}})\widetilde{E} = \frac{1-\widehat{C}2^{-\frac{k\epsilon}{2\theta}}}{1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}}}\widetilde{E} \subset \Sigma_k\Sigma_{k-1}^{-1}\widetilde{E}.$$

Since

$$\lim_{k\to+\infty} 2^{\frac{\epsilon k}{2\theta}} \frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}} + \widehat{C}2^{-\frac{k\epsilon}{2\theta}}}{1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}}} = \lim_{k\to+\infty} \frac{\widehat{C}+\widehat{C}2^{\frac{\epsilon}{2\theta}}}{1+\widehat{C}2^{-\frac{\epsilon(k-1)}{2\theta}}} = \widehat{C}+\widehat{C}2^{\frac{\epsilon}{2\theta}},$$

by taking k sufficiently large, we can obtain

$$(1-\widehat{C}2^{-\frac{\epsilon k}{2\theta}})\widetilde{E} \subset (1 - \frac{\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}} + \widehat{C}2^{-\frac{k\epsilon}{2\theta}}}{1+\widehat{C}2^{-\frac{(k-1)\epsilon}{2\theta}}})\widetilde{E} \subset \Sigma_k \Sigma_{k-1}^{-1}\widetilde{E}.$$

So we have

$$(1-\widehat{C}2^{-\frac{\epsilon k}{2\theta}})\widetilde{E} \subset \Sigma_k \Sigma_{k-1}^{-1}\widetilde{E} \subset (1+\widehat{C}2^{-\frac{\epsilon k}{2\theta}})\widetilde{E}, \quad k \geq \widehat{k}.$$

Since $\Sigma_k \Sigma_{k-1}^{-1}$ is still upper-triangular, we apply Lemma 2.1 in [29] (with $U = \Sigma_k \Sigma_{k-1}^{-1}$) to obtain that

$$\|\Sigma_k \Sigma_{k-1}^{-1} - I\| \leq C(n)\widehat{C}2^{-\frac{\epsilon k}{2\theta}}, \quad k \geq \widehat{k}.$$

Estimate (60) and (61) have been established. The existence of T, τ and (62) follow by an elementary consideration. □

Proof of Proposition 4.1. From Proposition 4.3, we can define

$$\Sigma = (T, -\tau),$$

and let $\hat{w} = u \circ \Sigma^{-1}$, then

$$-\hat{w}_s \det D^2 \hat{w} = 1, \quad \text{in } \mathbb{R}^{n+1}_- \setminus \Sigma(Q_H),$$

in fact, $\hat{w}_s = -\frac{u_t}{\tau}$, $\det D^2\hat{w} = (\det T^{-1})^2 \det D^2 u$,

$$-\hat{w}_s \det D^2 \hat{w} = \frac{1}{\tau}\frac{1}{(\det T)^2} u_t \det D^2 u = 1$$

from (62). Since $\{(y,s) : \hat{w}(y,s) < H'\} = \Sigma(Q_{H'})$ and

$$\frac{Q_{H'}}{\sqrt{H'}} = (diag\{\frac{1}{\sqrt{H'}}, \frac{1}{\sqrt{H'}}, \cdots, \frac{1}{\sqrt{H'}}\}, \frac{1}{H'})Q_{H'},$$

then we can deduce from (61) and (62) that

$$\begin{aligned}
&\Sigma(Q_{H'}) - \Sigma_k(Q_{H'}) \subset \widehat{C}2^{-\frac{\epsilon k}{2\theta}}\sqrt{H'}\widetilde{E},\\
&\qquad \Sigma(Q_{H'}) \subset (1+2\widehat{C}2^{-\frac{\epsilon k}{2\theta}})\sqrt{H'}\widetilde{E},
\end{aligned}$$

and

$$\begin{aligned}
&\Sigma_k(Q_{H'}) - \Sigma(Q_{H'}) \subset \widehat{C}2^{-\frac{\epsilon k}{2\theta}}\sqrt{H'}\widetilde{E},\\
&\qquad (1-2\widehat{C}2^{-\frac{\epsilon k}{2\theta}})\sqrt{H'}\widetilde{E} \subset \Sigma(Q_{H'}).
\end{aligned}$$

In particular, if we take $H' = 2^{k/\theta}$, then

$$(1-2\widehat{C}(H')^{-\frac{\epsilon}{2}})\sqrt{H'}\widetilde{E} \subset \{(y,s) : \hat{w}(y,s) < H'\} \subset (1+2\widehat{C}(H')^{-\frac{\epsilon}{2}})\sqrt{H'}\widetilde{E}, \quad \forall H' \geq 2^{\widehat{k}}.$$

So we have

$$(1-2\widehat{C}(\hat{w}(y,s))^{-\frac{\epsilon}{2}})^2\hat{w}(y,s) < -s + \frac{1}{2}|y|^2 < (1+2\widehat{C}(\hat{w}(y,s))^{-\frac{\epsilon}{2}})^2\hat{w}(y,s).$$

On one hand, we see

$$-s+\frac{1}{2}|y|^2<(1+2\widehat{C}(\hat{w}(y,s))^{-\frac{\epsilon}{2}})^2\hat{w}(y,s),$$
$$-s+\frac{1}{2}|y|^2<\hat{w}(y,s)+4\widehat{C}(\hat{w}(y,s))^{1-\frac{\epsilon}{2}}+4\widehat{C}^2(\hat{w}(y,s))^{1-\epsilon},$$
$$-s+\frac{1}{2}|y|^2<\hat{w}(y,s)+(4\widehat{C}+4\widehat{C}^2)(\hat{w}(y,s))^{1-\frac{\epsilon}{2}},$$
$$\hat{w}(y,s)-(-s+\frac{1}{2}|y|^2)>-(4\widehat{C}+4\widehat{C}^2)(\hat{w}(y,s))^{1-\frac{\epsilon}{2}}.$$

Meanwhile we show

$$(1-2\widehat{C}(\hat{w}(y,s))^{-\frac{\epsilon}{2}})^2\hat{w}(y,s)<-s+\frac{1}{2}|y|^2,$$
$$\hat{w}(y,s)-4\widehat{C}(\hat{w}(y,s))^{1-\frac{\epsilon}{2}}+4\widehat{C}^2(\hat{w}(y,s))^{1-\epsilon}<-s+\frac{1}{2}|y|^2,$$
$$\hat{w}(y,s)-(-s+\frac{1}{2}|y|^2)<4\widehat{C}(\hat{w}(y,s))^{1-\frac{\epsilon}{2}}-4\widehat{C}^2(\hat{w}(y,s))^{1-\epsilon},$$
$$\hat{w}(y,s)-(-s+\frac{1}{2}|y|^2)<4\widehat{C}(\hat{w}(y,s))^{1-\frac{\epsilon}{2}}$$

Combining the above inequalities, we get

$$|\hat{w}(y,s)-(-s+\frac{1}{2}|y|^2)|<\widehat{\widehat{C}}(\hat{w}(y,s))^{1-\frac{\epsilon}{2}}.$$

Consequently, by the fact $C^{-1}\hat{w}(y,s)\le|y|^2+|s|$, we get

$$|\hat{w}(y,s)-(-s+\frac{1}{2}|y|^2)|\le C(|y|^2+|s|)^{\frac{2-\epsilon}{2}},\quad \sqrt{|y|^2+|s|}\ge 2^{\overline{k}}. \tag{67}$$

Note that $\hat{w}(y,s)=u(T^{-1}y,\frac{s}{-\tau})$. Then we have

$$|u(x,t)-(\tau t+\frac{1}{2}x^TT^TTx)|\le C(\sqrt{|x|^2+|t|})^{2-\varepsilon},\quad |x|^2+|t|\ge 2^{2\overline{k}}.$$

Taking $A=T^TT$, we complete the proof. $\square$

One consequence of Proposition 4.1 is that for some positive constant C,

$$\|a_H\|,\quad \|a_H^{-1}\|\le C,\quad \forall H\ge 1.$$

Let $F(-u_t,D^2u)=(-u_t\det D^2u)^{\frac{1}{n+1}}$.

Lemma 4.4. *Let f satisfies (6) (with $a_i=1$), and let u satisfy (5). Then for every $e\in E$,*

$$\frac{(u(x+e,t)+u(x-e,t)-2u(x,t))_t}{u_t(x,t)}+u^{ij}D_{ij}(u(x+e,t)+u(x-e,t)-2u(x,t))\ge 0,\quad \text{in } \mathbb{R}^{n+1}_-,$$

where (u^{ij}) is the inverse of (u_{ij}).

Proof. By the concavity of F, the equation of u, and the periodicity of f, we have

$$\begin{aligned}F(-w_t,D^2w)&\ge\frac{1}{2}[F(-u_t(x+e,t),D^2u(x+e,t))+F(-u_t(x-e,t),D^2u(x-e,t))]\\&=\frac{1}{2}[f(x+e)+f(x-e)]=f(x),\end{aligned}$$

where $w(x,t)=\frac{1}{2}(u(x+e,t)+u(x-e,t))$.

On the other hand, from the concavity of F and the equation of u,

$$F(-w_t, D^2w) \le F(-u_t, D^2u) - F_a(w-u)_t + F_{ij}D_{ij}(w-u) = f - F_a(w-u)_t + F_{ij}D_{ij}(w-u). \tag{68}$$

So we have

$$\frac{(u(x+e,t)+u(x-e,t)-2u(x,t))_t}{u_t(x,t)} + u^{ij}D_{ij}(u(x+e,t)+u(x-e,t)-2u(x,t)) \ge 0. \quad \square$$

Step 2: L^∞ estimate of the Hessian of u.

Proposition 4.5. *There exists some positive constant C such that*

$$\frac{I}{C} \le D^2u(x,t) \le CI, \quad \forall (x,t) \in \mathbb{R}^{n+1}_-. \tag{69}$$

For nonzero $e \in \mathbb{R}^n$, we introduce a notation of the second incremental quotient:

$$\triangle_e^2 u(x,t) = \frac{u(x+e,t)+u(x-e,t)-2u(x,t)}{|e|^2}.$$

The following lemma is a consequence of Theorem 1.11, a result of authors on the linearization of the parabolic Monge–Ampère equation, which will be proved in Section 5.

Lemma 4.6. *For $r > 0$ and $e \in E$, there exists H_0, depending on n, r and $|e|$, such that for all $H \ge H_0$,*

$$\int\limits_{Y \in Q_H^*,\ dist(Y,\partial_p Q_H^*)>r} \triangle_{\tilde{e}}^2 u_H \le C, \tag{70}$$

and

$$0 < \triangle_{\tilde{e}}^2 u_H(Y) \le C, \quad \forall Y \in Q_H^*,\ dist(Y,\partial_p Q_H^*) > r, \tag{71}$$

where C depends only on n, r, $\max_{\mathbb{R}^n} f$, $\min_{\mathbb{R}^n} f$, m_1 and m_2.

Proof. Let $e \in E$, $\triangle_{\tilde{e}}^2 u_H$ is positive since u is strictly convex. By Lemma 2.2, $|\tilde{e}| \to 0$ as $H \to \infty$ ($H \approx R^2$). So there exists H_0 such that for $H \ge H_0$, $|\tilde{e}| \le \frac{r}{8}$. Let L be a line parallel to $\tilde{e}$, we have, by Lemma A.1 in Appendix A in [6], that

$$\int\limits_{L \cap \{Y \in Q_H^*,\ dist(Y,\partial_p Q_H^*)>r\}} \triangle_{\tilde{e}}^2 u_H \le C, \tag{72}$$

where C depends on n, r, $\max_{\mathbb{R}^n} f$, $\min_{\mathbb{R}^n} f$, m_1 and m_2, not depends on H. Integrating the above over all such lines, we could get (70).

By Lemma 4.4, $w := \triangle_{\tilde{e}}^2 u_H$ satisfies

$$\frac{w_s(Y)}{(u_H(Y))_s} + u_H^{ij}(Y)D_{ij}w(Y) \ge 0, \quad Y \in Q_H^*,\ dist(Y,\partial_p Q_H^*) > \frac{r}{2}. \tag{73}$$

Combining (70) and Theorem 1.11, we obtain (71). $\square$

Lemma 4.7.

$$\gamma := \sup_{e \in E} \sup_{(x,t)\in\mathbb{R}^{n+1}_-} \triangle_e^2 u(x,t) < \infty.$$

Proof. For $e \in E$ and $(x,t) \in \mathbb{R}^{n+1}_-$, let $Y = (y,s) = (\frac{a_H(x)}{R}, \frac{t}{R^2})$. Taking H large so that $(x,t) \in Q_{H/2}$, by Lemma 2.3 in [29], we have

$$dist(Y, \partial_p Q^*_H) \geq \frac{1}{C},$$

for some C depending only on n, $\min_{\mathbb{R}^n} f$, $\max_{\mathbb{R}^n} f$, m_1 and m_2. Then from (71), we see

$$\begin{aligned}
\triangle^2_e u(x,t) &= \frac{u(x+e,t)+u(x-e,t)-2u(x,t)}{\|e\|^2} \\
&= \frac{\|a_H(e)\|^2}{\|e\|^2}\frac{[u(a_H^{-1}(Ry+R\tilde{e}), R^2s)+u(a_H^{-1}(Ry-R\tilde{e}), R^2s)-2u(a_H^{-1}Ry, R^2s)]}{\|a_H(e)\|^2} \\
&= \frac{\|a_H(e)\|^2}{\|e\|^2}\triangle^2_{\tilde{e}} u_H(y,s) \leq C\|a_H\|^2 \leq C. \quad \square
\end{aligned}$$

Lemma 4.8. *Let $g \in C^2(\overline{B_1})$ be a positive function, and let $u \in C^{4,2}(E_1) \cap C(\overline{E_1})$ be a parabolically convex function satisfying*

$$\begin{aligned}
&-u_t \det D^2 u = g(x), \quad in\ E_1, \\
&-m_1 \leq u_t \leq -m_2,
\end{aligned}$$

and $u(0,0) = 0$, where $E_1 = \{(x,t) \in \mathbb{R}^{n+1}_- : |x|^2 - t < 1\}$. Assume that

$$0 < \mu \leq u \leq \frac{1}{\mu} \quad on\ \partial_p E_1.$$

Then for some $r_0 \in (0,1)$ and $C > 0$, depending only on n, μ, $\min_{\overline{B_1}} g$ and $\|g\|_{C^2(\overline{B_1})}$, we have that

$$|D^2 u| \leq C, \quad in\ E_{r_0}.$$

Proof. We only to show that there exists some $r > 0$, depending only on μ, such that

$$B_{2r} \subset \{x \in B_1 :\ v(x) = u(x,0) < \frac{\mu}{2}\}. \tag{74}$$

Since $Q_{\frac{\mu}{2}}(0) = S_{v(x)}(0, \frac{\mu}{2})$, from Lemma 2.1 in [12] we have for $0 < \lambda < 1$ that

$$\lambda B_{2r} \subset \lambda Q_{\frac{\mu}{2}}(0) \subset Q_{(1-(1-\lambda)\frac{\alpha_n}{2})\frac{\mu}{2}}(0).$$

If $(x,t) \in \lambda B_{2r} \times [-r_1\frac{\mu}{2}, 0]$, then

$$\begin{aligned}
u(x,t) &= u(x,0) - \int_t^0 u_t(x,\tau)d\tau \\
&\leq (1-(1-\lambda)\frac{\alpha_n}{2})\frac{\mu}{2} - m_1 t \\
&\leq (1-(1-\lambda)\frac{\alpha_n}{2} + m_1 r_1)\frac{\mu}{2} \\
&< \frac{\mu}{2}
\end{aligned}$$

for λ and r_1 sufficiently small. Taking $r_0 = \min\{r\lambda, \frac{r_1}{2}\}$, we could get the estimate.

Next, we prove (74). Let $v(\overline{x}) = \frac{\mu}{2}$, by the convexity of v,

$$v(x) \geq v(\overline{x}) + Dv(\overline{x}) \cdot (x - \overline{x}), \quad \forall x \in \overline{B_1}.$$

In particular,

$$0 = v(0) \geq v(\overline{x}) - Dv(\overline{x}) \cdot \overline{x},$$

i.e.,

$$\frac{\mu}{2} = v(\overline{x}) \le |Dv(\overline{x})||\overline{x}|.$$

Taking $x \in \partial B_1$ such that $Dv(\overline{x})$ and $x - \overline{x}$ point the same direction, we have

$$\frac{1}{\mu} \ge v(x) \ge v(\overline{x}) + |Dv(\overline{x})||x - \overline{x}| = v(\overline{x}) + |Dv(\overline{x})|(1 - |\overline{x}|),$$

i.e.,

$$\frac{\frac{1}{\mu} - \frac{\mu}{2}}{1 - |\overline{x}|} \ge |Dv(\overline{x})|.$$

Then we obtain

$$\frac{\mu}{2} \le |Dv(\overline{x})||\overline{x}| \le \frac{(\frac{1}{\mu} - \frac{\mu}{2})|\overline{x}|}{1 - |\overline{x}|},$$

that is,

$$\frac{\mu^2}{2} \le |\overline{x}|.$$

Let $r = \frac{\mu^2}{6}$. (74) is established. □

Remark 4.9. In fact, from the regularity theorem obtained by the first author [30], we are able to get the above conclusion in weaker condition $g \in VMO^{\psi}(\mathbb{R}^n)$.

Proof of Proposition 4.5. For fixed $(x,t) \in \mathbb{R}^{n+1}_-$, let

$$\tilde{u}(z,\tau) = u(z + x, \tau + t) - (u(x,t) + Du(x,t) \cdot z), \quad \text{in } \mathbb{R}^{n+1}_-.$$

Then

$$\tilde{u}(0,0) = 0, \quad \tilde{u} \ge 0 \quad \text{in } \mathbb{R}^{n+1}_-.$$

Since

$$\sup_{e \in E} \sup_{(z,\tau) \in \mathbb{R}^{n+1}_-} \triangle_e^2 \tilde{u}(z,\tau) = \sup_{e \in E} \sup_{(x,t) \in \mathbb{R}^{n+1}_-} \triangle_e^2 u(x,t) \le \gamma,$$

using $\sup_{e \in E} \triangle_e^2 \tilde{u}(0,0) \le \gamma$ and the convexity of $\tilde{u}(\cdot, 0)$, we have

$$\sup_{B_r} \tilde{u}(x,0) \le C(n, m_1, m_2)\gamma r^2, \quad 1 \le \gamma < \infty.$$

On the other hand, for $\overline{z} \in \partial B_r$, from $\sup_{e \in E} \triangle_e^2 \tilde{u}(\frac{\overline{z}}{2}, 0) \le \gamma$, we have

$$\tilde{u}(\frac{\overline{z}}{2} + e, 0) + \tilde{u}(\frac{\overline{z}}{2} - e, 0) - 2\tilde{u}(\frac{\overline{z}}{2}, 0) \le \gamma |e|^2, \quad \forall e \in E.$$

It follows, by the convexity of $\tilde{u}(\cdot, 0)$ and the fact that $\tilde{u}(0,0) = 0$, that

$$\tilde{u}(z,0) \le 2\tilde{u}(\frac{\overline{z}}{2}, 0) + C(n)\gamma \le \tilde{u}(\overline{z}, 0) + C(n)\gamma, \quad \forall z \in \frac{\overline{z}}{2} + (-2,2)^n.$$

Applying Lemma 2.1 to $\tilde{u}(\frac{\overline{z}}{2}, 0)/(\tilde{u}(\overline{z}, 0) + C(n)\gamma)$, taking $\frac{\overline{z}}{|\overline{z}|}$ as e_n, we have

$$\tilde{u}(\overline{z}, 0)^n = \max_{|s| \le |\overline{z}|/2} \tilde{u}(\frac{\overline{z}}{2} + s\frac{\overline{z}}{|\overline{z}|}, 0) \ge (\frac{r \min_{\mathbb{R}^n} f}{m_1 C(n)[\tilde{u}(\tilde{z}, 0) + \gamma)]^n} - 1)(\tilde{u}(\tilde{z}, 0) + C(n)\gamma)^n.$$

If $\tilde{u}(\overline{z}, 0) \le \gamma$, then

$$\tilde{u}(\overline{z}, 0)^n \ge \gamma^n (\frac{r \min_{\mathbb{R}^n} f}{m_1 C(n) \gamma^n} - 1).$$

Fix some suitably large r, depending only on n, γ, $\min_{\mathbb{R}^n} f$ and m_1, such that

$$\gamma^n (\frac{r \min_{\mathbb{R}^n} f}{m_1 C(n) \gamma^n} - 1) \ge 1,$$

we have $\tilde{u}(\overline{z}, 0) \ge 1$. Hence, for such r, we have

$$\min_{\partial B_r} \tilde{u}(z, 0) \ge \min\{\gamma, 1\}.$$

Recall that $E_r = \{(z, \tau) : |z|^2 - \tau \le r^2\}$. From

$$u(x, -r^2) \le u(x, 0) + m_1 r^2 \le C(n, m_1, m_2)\gamma r^2 + m_1 r^2 = C(n, m_1, m_2, \gamma) r^2, \quad x \in \overline{B_r},$$

we then obtain

$$\max_{\partial_p E_r} \tilde{u} \le C(n, m_1, m_2, \gamma) r^2.$$

Similarly, we have

$$\min_{\partial_p E_r} \tilde{u} \ge C(n, m_1, m_2, \gamma).$$

Since

$$-\tilde{u}_\tau \det D^2 \tilde{u}(z, \tau) = f(z + x - [x]),$$

where $[x]$ denotes the integer part of x. We get, by Lemma 4.8, that

$$|D^2 u(x, t)| = |D^2 \tilde{u}(0, 0)| \le C(r).$$

Combining

$$0 < \frac{\min_{\mathbb{R}^n} f}{m_1} \le \det D^2 u \le \frac{\max_{\mathbb{R}^n} f}{m_2},$$

we arrive at the conclusion. $\square$

Proof of Theorem 1.2. For $(x_0, t_0) \in \mathbb{R}^{n+1}_-$, we will show that $u_t(x_0, t_0) = u_t(0, 0)$. Since (x_0, t_0) is arbitrary, u must be have the form $u(x, t) = \tau t + p(x)$, where $\tau = u_t(0, 0) < 0$. Consequently, by (5),

$$\det D^2 p(x) = \det D^2 u(x, t) = \frac{f(x)}{-u_t(x, t)} = \frac{f(x)}{-\tau} := \tilde{f}(x).$$

From Theorem 0.1 in [6], we obtain $p(x)$ is the sum of a quadratic polynomial and a periodic function, i.e.,

$$p(x) = \frac{1}{2} x^T A x + b \cdot x + v(x),$$

with $\det A = ⨍_{\Pi_{i=1}^n [0, a_i]} \tilde{f}$ and $v(x + a_i e_i) = v(x)$. Theorem 1.2 is established.

We may assume $u \in C^{4,2}$. Otherwise, u_t is substituted with $\frac{u(x,t+h) - u(x,t)}{h}$ for $h < 0$. Differentiating (5) with respect to t we get

$$-\frac{(u_t)_t}{u_t} - \operatorname{trace}((D^2 u)^{-1} D^2 u_t) = 0.$$

Condition (7) and Proposition 4.5 yield a uniformly parabolic equation. And by Harnack inequality [21], we see

$$\frac{|u_t(x_0, t_0) - u_t(0, 0)|}{(|x_0|^2 + |t_0|)^\alpha} \le C \frac{\|u_t\|_{L^\infty(\mathbb{R}^{n+1}_-)}}{R^\alpha},$$

for $R > 1$, $R > 2|x_0|$, $R^2 > -2t_0$ and some $0 < \alpha < 1$. Sending $R \to \infty$, we obtain

$$u_t(x_0, t_0) = u_t(0, 0). \quad \square$$

5. Proof of Theorem 1.11

In this section, we give the proof of Theorem 1.11, that is, the local maximum principle for sub-solutions to the following equation:

$$L_\phi u = \frac{u_t}{\phi_t} + \mathrm{trace}((D^2\phi(x,t))^{-1} D^2 u) = 0. \tag{75}$$

We now recall the notion of normalization of the section $S_\phi(x_0|t_0, h)$ given by (24). Let T be the affine transformation that normalizes $S_\phi(x_0|t_0, h)$, that is,

$$B_{\alpha_n}(0) \subset T(S_\phi(x_0|t_0, h)) \subset B_1(0), \quad \alpha_n = n^{-3/2}.$$

And we define the transformation

$$T_p(x,t) = (Tx, \frac{t-t_0}{h}),$$

and its corresponding inverse

$$T_p^{-1}(y,s) = (T^{-1}y, t_0 + sh).$$

In the following, we introduce the notions of normalization of the functions. Set

$$\psi_h(y,s) = \frac{\phi(T_p^{-1}(y,s))}{h} = \frac{\phi(T^{-1}y, t_0+sh)}{h}, \tag{76}$$

and

$$u^*(y,s) = u(T_p^{-1}(y,s)) = u(T^{-1}y, t_0+sh). \tag{77}$$

It is easy to check that

$$S^* = T(S_\phi(x_0|t_0,h)) = S_{\psi_h}(Tx_0|0,1), \tag{78}$$

$$Q^* = T_p(Q_\phi(X_0,h)) = Q_{\psi_h}(T_p(X_0),1). \tag{79}$$

In fact, $\ell_{X_0}(x) = \phi(X_0) + D\phi(X_0)\cdot(x-x_0)$ is a supporting hyperplane of $\phi(\cdot,t_0)$ at $x=x_0$ if and only if $\ell(y) = \psi_h(T_p(X_0)) + \frac{(T^{-1})^t}{h} D\phi(X_0)\cdot(y-Tx_0)$ is a supporting hyperplane of $\psi_h(\cdot, t_0+sh)$ at $y=Tx_0$. Since T normalizes $S_\phi(x_0|t_0,h)$, we see that $|TS_\phi(x_0|t_0,h)| \approx C(n)$. Then we have

$$|\det T|\cdot|S_\phi(x_0|t_0,h)| \approx C(n).$$

Under the normalization, we get

$$u_t = \frac{u_s^*}{h},$$

$$D^2u = T^t D^2u^* T,$$

$$D^2\phi = hT^t D^2\psi_h T \Leftrightarrow (D^2\phi)^{-1} = \frac{T^{-1}(D^2\psi_h)^{-1}(T^{-1})^t}{h},$$

and

$$\phi_t = (\psi_h)_s.$$

It follows from (75) that

$$\frac{1}{h}\frac{u_s^*}{(\psi_h)_s} + \mathrm{trace}(\frac{1}{h}T^{-1}(D^2\psi_h)^{-1}(T^{-1})^t\cdot T^t D^2u^* T) = 0.$$

After simplification, we see that u^* satisfies the following equation:

$$\frac{u_s^*}{(\psi_h)_s} + \mathrm{trace}((D^2\psi_h)^{-1} D^2 u^*) = 0. \tag{80}$$

The parabolic Monge–Ampère measure μ generated by ϕ satisfies the following doubling condition: there exist constants C and $0 < \alpha < 1$ such that

$$\mu(Q_\phi(X,h)) \le C\mu(\alpha Q_\phi(X,h)) \tag{81}$$

for every section $Q_\phi(X,h)$. Let μ^* denote the parabolic Monge–Ampère measure generated by ψ_h. It follows that

$$\begin{aligned}\mu^*(Q^*) &= \int\limits_{T_p(Q_\phi(X_0,h))} -(\psi_h)_s \det D^2\psi_h dyds \\ &= \int\limits_{Q_\phi(X_0,h)} -\phi_t \det D^2\phi \frac{(\det T)^{-2}}{h^n}\frac{\det T}{h} dxdt \\ &= \frac{1}{h^{n+1}\det T}\mu(Q_\phi(X_0,h)).\end{aligned}$$

On the other hand, since μ satisfies doubling condition, μ^* also satisfies the same one. We then define the normalization of ϕ, ϕ^*, by

$$\phi^*(y,s) = \psi_h(y,s) - \overline{\ell}_{(Tx_0,0)}(y) - 1, \tag{82}$$

where $\overline{\ell}_{(Tx_0,0)}(y)$ is the supporting hyperplane of $\psi_h(\cdot,0)$ at $y = Tx_0$. Obviously, the parabolic Monge–Ampère measure generated by ϕ^* is exactly μ^*. Meanwhile $\phi^* = 0$ on $\partial_p Q^*$, and $-1 = \phi^*(Tx_0,0) \le \phi^* \le 0$ on $\overline{Q^*}$. Then we have $\mu^*(Q^*) \approx C(n,\lambda,\Lambda,m_1,m_2)$, i.e.,

$$h^{n+1}\det T \approx C(n,\lambda,\Lambda,m_1,m_2)\mu(Q_\phi(X_0,h)). \tag{83}$$

Lemma 5.1. *([13], Lemma 4.6) Let $Q_\phi(X_0,1)$ be a normalized section. There exist positive constants C and p such that, if $0 < r_1 < r_2 < 1$ and $X' \in Q_\phi(X_0,r_1)$, then*

$$Q_\phi(X',r') \subset Q_\phi(X_0,r_2) \tag{84}$$

for $r' \le \tilde{C}(r_2 - r_1)^p$.

Lemma 5.2. *([12], Lemma 2.1 and Theorem 2.1) There exist $0 < \tau, \lambda < 1$ such that for all x_0, t_0 and $h > 0$,*

$$\beta S_\phi(x_0|t_0,h) \subset S_\phi(x_0|t_0,(1-(1-\beta)\frac{\alpha_n}{2})h), \quad 0<\beta<1,$$

and

$$S_\phi(x_0|t_0,\tau h) \subset \lambda S_\phi(x_0|t_0,h).$$

Lemma 5.3. *Given $\beta > 1$ there exists C depending only on n, λ, Λ, m_1 and m_2 such that*

$$\mu(Q_\phi(X,\beta h)) \le C\beta^{\frac{n+2}{2}}\mu(Q_\phi(X,h)) \tag{85}$$

for any section $Q_\phi(X,h)$.

Proof. By Lemma 3.1 in [11], we have

$$\epsilon_0 S_\phi(x|t,h) \times [-\epsilon_1 h + t, t] \subset Q_\phi(X,h) \subset S_\phi(x|t,h) \times [-\epsilon_2 h + t, t], \tag{86}$$

where ϵ_0, ϵ_1 and ϵ_2 depend on n, m_1 and m_2. Meanwhile from Corollary 3.2.4 in [10], we obtain

$$C_1 h^{\frac{n}{2}} \le |S_\phi(x|t,h)| \le C_2 h^{\frac{n}{2}}, \tag{87}$$

where C_1 and C_2 depend on n, λ, Λ, m_1 and m_2.

(85) is a simple consequence of (86) and (87). □

Lemma 5.4. *Let ϕ satisfy $0<\lambda\le -\phi_t \det D^2\phi \le \Lambda <\infty$ and $-m_1\le \phi_t(x,t)\le -m_2$. Suppose that $X_1=(x_1,t_1)\in Q_\phi(X_0,h)$. Then there exist θ_1 and θ_2 depending only on the n, λ_1, Λ_2, m_1 and m_2 such that*

$$S_\phi(x_0|t_0,h)\subset S_\phi(x_1|t_1,\theta_1 h), \tag{88}$$

and

$$\mu(Q_\phi(X_0,h))\le \frac{\Lambda}{\lambda}\mu(Q_\phi(X_1,\theta_2 h)). \tag{89}$$

Proof. Consider $S_\phi(x_0|t_0,2h)$ and let T be the affine transformation normalizing $S_\phi(x_0|t_0,2h)$ and the function

$$\varphi(y,s)=\frac{1}{h}(\phi-\ell_{X_0})(T^{-1}y,t_0+sh).$$

Then $T_p(Q_\phi(X_0,2h))=Q_\varphi((Tx_0,0),2)$ is normalized. We have $\min_{\overline{Q_\varphi((Tx_0,0),2)}}\varphi=\varphi(Tx_0,0)=0$, $\varphi=2$ on $\partial_p Q_\varphi((Tx_0,0),2)$ and $-m_1\le\varphi_s\le -m_2$.

Let $(y_1,s_1)\in Q_\varphi((Tx_0,0),1)$ then

$$|D\varphi(y_1,s_1)|\le \frac{2}{dist((y_1,s_1),\partial Q_\varphi((Tx_0,0),2)(s_1))}\le C$$

by Theorem 2.1 in [11]. If $y\in S_\varphi(y_0|s_0,1)$ then $\varphi(y,s_0)<1$. And since $m_2\le|\varphi_s|\le m_1$, we get $\varphi(y,s_1)<C$. Now

$$|\ell_{(y_1,s_1)}(y)|=|\varphi(y_1,s_1)+D\varphi(y_1,s_1)(y-y_1)|\le C_1.$$

Hence

$$(\phi-\ell_{(y_1,s_1)})(y,s_1)\le C+C_1:=\theta_1.$$

We conclude that $y\in S_\varphi(y_1|s_1,\theta_1)$. Going back to ϕ we obtain (88) by affine invariance.

By the Lemma 3.1 in [11],

$$Q_\phi(X_0,h)\subset S_\phi(x_0|t_0,h)\times(-\frac{h}{m_2}+t_0,t_0].$$

Since $(x_1,t_1)\in Q_\phi(X_0,h)$, we have, by (88),

$$S_\phi(x_0|t_0,h)\subset S_\phi(x_1|t_1,\theta_1 h).$$

From Lemma 5.2,

$$S_\phi(x_1|t_1,\tau\frac{\theta_1 h}{\tau})\subset\lambda^k S_\phi(x_1|t_1,\frac{\theta_1 h}{\tau^k})\subset S_\phi(x_1|t_1,(1-(1-\lambda^k)\frac{\alpha_n}{2})\frac{\theta_1 h}{\tau^k}),$$

where k will be chosen later. For any $x\in S_\phi(x_0|t_0,h)$, $t_1-t\le\frac{h}{m_2}$

$$\begin{aligned}\phi(x,t)&=\phi(x,t_1)+\int_{t_1}^{t}\phi_t(x,t')dt'\\&\le(1-(1-\lambda^k)\frac{\alpha_n}{2})\frac{\theta_1 h}{\tau^k}-m_1(t-t_1)\\&\le(1-(1-\lambda^k)\frac{\alpha_n}{2}+\frac{m_1\tau^k}{m_2\theta_1})\frac{\theta_1 h}{\tau^k}.\end{aligned}$$

Then we choose k sufficient large such that $1-(1-\lambda^k)\frac{\alpha_n}{2}+\frac{m_1\tau^k}{m_2\theta_1}<1$. Denoting $\theta_2=\frac{\theta_1}{\tau^k}$, we obtain

$$\mu(Q_\phi(X_0,h))\le\frac{\Lambda}{\lambda}\mu(Q_\phi(X_1,\theta_2 h)).\qquad\square$$

The following proposition establishes a crucial property of the super-solutions of $L_\phi u=0$, namely, the uniform critical density of their level sets.

Proposition 5.5. *There are two constants $M_0 > 1$ and $0 < \varepsilon_0 < 1$, depending only on n, λ, Λ, m_1 and m_2, such that for any section $Q_\phi(X_0, h)$ and any nonnegative super-solution u to $L_\phi u = 0$ satisfying*

$$\inf\{u(X) : X \in Q_\phi(X_0, \frac{h}{2})\} \leq 1,$$

we have that

$$\mu(\{X \in Q_\phi(X_0, h) : u(X) < M_0\}) \geq \varepsilon_0 \mu(Q_\phi(X_0, h)). \tag{90}$$

Proof. By the previous argument, $u^*(y, s)$ satisfies

$$\frac{u_s^*}{(\psi_h)_s} + \mathrm{trace}((D^2\psi_h)^{-1} D^2 u^*) \leq 0, \quad \text{in} \quad Q^*, \tag{91}$$

$$\phi^*(y, s) = 0 \quad \text{on} \quad \partial_p Q^*; \quad -1 \leq \phi^*(y, s) \leq 0 \quad \text{in} \quad Q^*; \tag{92}$$

$$-1 \leq \phi^*(y, s) \leq -\frac{1}{2} \quad \text{in} \quad Q^*_{1/2} = T_p(Q_\phi(X_0), \frac{h}{2})) = Q_{\psi_h}(Tx_0, 0), \frac{1}{2}). \tag{93}$$

Consider the auxiliary function

$$w(y, s) = u^*(y, s) + 4\phi^*(y, s).$$

Let $\Gamma(w^-)$ denote the parabolic concave envelope in Q^* of the negative part w and A_w be the contact set, i.e.,

$$A_w = \{(y, s) \in Q^* : \ w < 0, \ w = -\Gamma(w^-)\}.$$

By the geometric–arithmetic mean inequality, we obtain the following estimate on A_w

$$\begin{aligned}
-w_s \det D^2 w &= \frac{-w_s \det D^2 w}{-\phi_s^* \det D^2\phi^*}(-\phi_s^* \det D^2\phi^*) \\
&\leq (\frac{\frac{-w_s}{-\phi_s^*} + tr((D^2\phi^*)^{-1} D^2 w)}{n+1})^{n+1}(-\phi_s^* \det D^2\phi^*) \\
&= (\frac{L_{\phi_h} w}{n+1})^{n+1}(-(\psi_h)_s \det D^2\psi_h) \\
&\leq 4^{n+1}(-(\psi_h)_s \det D^2\psi_h).
\end{aligned}$$

We may assume that

$$u^*(y', s') = \inf\{u^*(y, s) : (y, s) \in Q^*_{1/2}\} \leq 1 \tag{94}$$

where $(y', s') \in \overline{Q^*_{1/2}}$. It was proved in [27] that $\Gamma(w^-)$ is $C^{1,1}$ and $(\sup_{Q^*} w^-)^{n+1}$ is controlled by the volume of the image of A_w under the transformation

$$(y, s) \to (D\Gamma(w^-)(y, s), \Gamma(w^-)(y, s) - yD\Gamma(w^-)(y, s)).$$

By parabolic Alexandrov–Bakelman estimate [25], we have

$$(w^-(y', s'))^{n+1} \leq C(diam(S^*))^n \int_{A_w} |(\Gamma(w^-))_s \det D^2(\Gamma(w^-))| dyds. \tag{95}$$

Obviously, $w \geq -\Gamma(w^-)$ in Q^*. It is easy to check that on A_w

$$D^2 w \geq D^2(-\Gamma(w^-)) \geq 0, \quad w_s \leq (-\Gamma(w^-))_s \leq 0,$$

and

$$\inf_{Q^*_{1/2}} w \leq -1$$

by (93) and (94). It follows that

$$1 \le C \int\limits_{A_w} (-w)_s \det D^2 w dy ds.$$

Noting that $A_w \subset \{(y,s) \in Q^* : u^*(y,s) < 4\}$, we obtain

$$\begin{aligned} 1 &\le C \int\limits_{\{(y,s)\in Q^*:u^*(y,s)<4\}} (-(\psi_h)_s \det D^2 \psi_h) dy ds \\ &= C \int\limits_{\{(x,t)\in Q_\phi(X_0,h):u(x,t)<4\}} (-\phi_t \frac{(\det T)^{-2}}{h^n} \det D^2 \phi \frac{\det T}{h}) dx dt \\ &= \frac{C\mu(\{(x,t) \in Q_\phi(X_0,h) : u(x,t) < 4\})}{h^{n+1} \det T}. \end{aligned}$$

Since $h^{n+1} \det T \approx C(n, \lambda, \Lambda, m_1, m_2)\mu(Q_\phi(X_0, h))$, we have

$$\frac{C(n,\lambda,\Lambda,m_1,m_2)}{C} \mu(Q_\phi(X_0,h)) \le \mu(\{(x,t) \in Q_\phi(X_0,h) : u(x,t) < 4\}),$$

i.e.,

$$\varepsilon_0 \mu(Q_\phi(X_0,h)) \le \mu(\{(x,t) \in Q_\phi(X_0,h) : u(x,t) < M_0\}),$$

where $\varepsilon_0 \in (0, 1)$ and $M_0 = 4$. □

Proposition 5.6. *Let ε_0 and $M_0 > 1$ be the numbers in Proposition 5.5 and $\delta \in (0, 1)$ be a constant. Let u be a nonnegative sub-solution to $L_\phi u = 0$ in the section $Q_\phi(X, h)$ and assume that*

$$\mu(\{Y \in Q_\phi(X,h) : u(Y) > h'\}) \le C_1 (h')^{-1} \mu(Q_\phi(X,h)), \quad \forall h' > 0. \tag{96}$$

Let $\nu = \frac{M_0}{M_0 - \frac{1}{2}} > 1$. Suppose that at a point $X_0 \in Q_\phi(X, \delta h/2)$ and for a positive integer j we have: (a) $u(X_0) \ge \nu^{j-1} M_0$; (b) $(\frac{\rho}{h})^{\frac{n+2}{2}} \ge \frac{C_1 C_2}{\varepsilon_0} (\frac{\nu^j M_0}{2})^{-1}$, for some $\rho < \tilde{C}(\delta/2)^p h$, where $\tilde{C}$ and p are the exponent in Lemma 5.1. Then

$$\sup_{Q_\phi(X_0,\rho)} u > \nu^j M_0. \tag{97}$$

Proof. By renormalizing the section $Q_\phi(X, h)$ as at the beginning of the proof of Proposition 5.5, we may assume that this section is normalized and $h = 1$. Let us assume by contradiction that (97) is false and let

$$\upsilon(x,t) = \frac{\nu^j M_0 - u(x,t)}{\nu^{j-1}(\nu - 1) M_0}.$$

By condition (a) we have $\upsilon(x_0, t_0) \le 1$. Then by Proposition 5.5

$$\mu(\{X \in Q_\phi(X_0,\rho) : \upsilon(X) \ge M_0\}) \le (1 - \varepsilon_0)\mu(Q_\phi(X_0,\rho)), \quad \rho > 0. \tag{98}$$

Let

$$A = \{Y \in Q_\phi(X,\delta) : u(Y) > \frac{\upsilon^j M_0}{2}\}$$

and

$$B = \{Y \in Q_\phi(X_0,\rho) : \upsilon(Y) \ge M_0\}.$$

We claim that

$$Q_\phi(X_0,\rho) \subset A \cup B.$$

In fact, since $X_0 \in Q_\phi(X, \delta/2)$, by Lemma 5.1 $Q_\phi(X_0, \rho) \subset Q_\phi(X, \delta)$ for $\rho < \tilde{C}(\delta/2)^p$, and note that $u(Y) < \nu^j \frac{M_0}{2} \Leftrightarrow \upsilon(Y) > M_0$ by the definition of υ, the claim is easily obtained. Then by (96) and (98) we have

$$\mu(Q_\phi(X_0,\rho)) \le \mu(A)+\mu(B) \le C_1(\frac{\nu^j M_0}{2})^{-1}\mu(Q_\phi(X,\delta)) + (1-\varepsilon_0)\mu(Q_\phi(X_0,\rho))$$
$$< C_1(\frac{\nu^j M_0}{2})^{-1}\mu(Q_\phi(X,1)) + (1-\varepsilon_0)\mu(Q_\phi(X_0,\rho))$$

which implies

$$\mu(Q_\phi(X_0,\rho)) < \frac{C_1}{\varepsilon_0}(\frac{\nu^j M_0}{2})^{-1}\mu(Q_\phi(X,1)). \tag{99}$$

On the other hand, since $X_0 \in Q_\phi(X,1)$, by Lemma 5.3 and Lemma 5.4, we have

$$\mu(Q_\phi(X,1)) \le \frac{\Lambda}{\lambda}\mu(Q_\phi(X_0,\theta_2)) \le C\mu(Q_\phi(X_0,1)) = C\mu(Q_\phi(X_0,\frac{1}{\rho}\rho))$$
$$\le C_2(\frac{1}{\rho})^{\frac{n+2}{2}}\mu(Q_\phi(X_0,\rho)), \quad \text{for } \rho<1,$$

that is,

$$\mu(Q_\phi(X_0,\rho)) \ge \frac{\rho^{\frac{n+2}{2}}}{C_2}\mu(Q_\phi(X,1)).$$

From condition (b),

$$\mu(Q_\phi(X_0,\rho)) \ge \frac{C_1}{\varepsilon_0}(\frac{\nu^j M_0}{2})^{-1}\mu(Q_\phi(X,1)).$$

This is a contradiction to (99). □

Proposition 5.7. *There exists a constant $C>1$ depending only on n, λ, Λ, m_1 and m_2, such that if u is a classical nonnegative sub-solution to $L_\phi u = 0$ in the section $Q_\phi(X,h)$ and satisfies (96) then*

$$\sup_{Q_\phi(X,\frac{\delta h}{3})} u \le C. \tag{100}$$

Proof. By renormalizing the section, we may assume that $Q_\phi(X,h)$ is normalized and $h=1$. Let us take

$$\rho_j = (\frac{C_1C_2}{\varepsilon_0})^{\frac{2}{n+2}}(\frac{\nu^j M_0}{2})^{-\frac{2}{n+2}}, \quad j=1,2,\cdots$$

Since $\nu>1$, we pick m sufficiently large so that

$$\sum_{j\ge m}\rho_j^{1/p} \le \frac{\delta}{100}. \tag{101}$$

We claim that

$$\sup_{Q_\phi(X,\frac{\delta}{4})} u \le \nu^{m-1}M_0.$$

Suppose that the claim is not true. Then there would exist $X_m \in Q_\phi(X,\frac{\delta}{4})$ such that $u(X_m) > \nu^{m-1}M_0$. By the choice of ρ_j we have

$$\mu(Q_\phi(X_m,\rho_m)) \ge \frac{C_1}{\varepsilon_0}(\frac{\nu^m M_0}{2})^{-1}\mu(Q_\phi(X,1)),$$

then by Proposition 5.6,

$$\sup_{Q_\phi(X_m,\rho_m)} u > \nu^m M_0.$$

Consequently, there exists $X_{m+1} \in Q_\phi(X_m, \rho_m)$ such that $u(X_{m+1}) > \nu^m M_0$. Now, $X_m \in Q_\phi(X, \frac{\delta}{4})$ then by Lemma 5.1 $X_{m+1} \in Q_\phi(X, \frac{\delta}{4} + (\frac{\rho_m}{\tilde{C}})^{1/p})$. Again, by the choice of ρ_j and Proposition 5.6, we would have a point $X_{m+2} \in Q_\phi(X_{m+1}, \rho_{m+1})$ such that $u(X_{m+2}) > \nu^{m+1} M_0$, and by Lemma 5.1 we would get $X_{m+2} \in Q_\phi(X, \frac{\delta}{4} + (\frac{\rho_m}{\tilde{C}})^{1/p} + (\frac{\rho_{m+1}}{\tilde{C}})^{1/p})$.

We can then repeat this process, getting a sequence of points $\{X_j\}_{j=m}^\infty$ such that

$$u(X_j) \geq \nu^{j-1} M_0, \quad X_j \in Q_\phi(X_{j-1}, \rho_{j-1}) \subset Q_\phi(X, \frac{\delta}{4} + (\frac{\rho_m}{\tilde{C}})^{1/p} + \cdots + (\frac{\rho_{j-1}}{\tilde{C}})^{1/p}). \tag{102}$$

From (101), we obtain $X_j \in Q_\phi(X, \frac{104}{400}\delta) \subset Q_\phi(X, \frac{\delta}{3})$. Since $\nu > 1$, it follows that $\{u(X_j)\}$ would be an unbounded sequence in $Q_\phi(X, \frac{\delta}{3})$. This is impossible because u is continuous in $\overline{Q_\phi(X, \frac{\delta}{2})}$. $\square$

Proof. (Proof of Theorem 1.11) By normalizing the section $Q_\phi(X, h)$, we consider

$$u_\varepsilon^* = \frac{u^*}{\|u^*\|_{L^1(Q^*, d\mu^*)} + \varepsilon}.$$

We have $\|u_\varepsilon^*\|_{L^1(Q^*, d\mu^*)} \leq 1$ and

$$\begin{aligned} \mu^*(\{Y \in Q^* : u_\varepsilon^*(Y) > h'\}) &\leq \frac{1}{h'} \|u_\varepsilon^*\|_{L^1(Q^*, d\mu^*)} \\ &\leq C_1 (h')^{-1} \mu^*(Q^*), \quad \forall h' > 0. \end{aligned}$$

Applying Proposition 5.6 and Proposition 5.7, we get

$$\sup_{Q^*_{\delta/3}} u_\varepsilon^* \leq C, \tag{103}$$

that is,

$$\sup_{Q^*_{\delta/3}} u^* \leq C(\|u^*\|_{L^1(Q^*,\, d\mu^*)} + \varepsilon),$$

after letting $\varepsilon \to 0$,

$$\sup_{Q^*_{\delta/3}} u^* \leq C \|u^*\|_{L^1(Q^*,\, d\mu^*)}. \tag{104}$$

Rescaling u^*, we obtain

$$\sup_{Q_\phi(X, \frac{\delta h}{3})} u \leq \frac{C \|u\|_{L^1(Q_\phi(X,h),\, d\mu)}}{\mu(Q_\phi(X, h))}. \tag{105}$$

This theorem is proved. $\square$

Conflict of interest statement

None declared.

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