Derived equivalences, restriction to self-injective subalgebras and invariance of homological dimensions

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Abstract

Unlike Hochschild (co)homology and K-theory, global and dominant dimensions of algebras are far from being invariant under derived equivalences in general. We show that, however, global dimension and dominant dimension are derived invariant when restricting to a class of algebras with anti-automorphisms preserving simples. Such anti-automorphisms exist for all cellular algebras and in particular for many finite dimensional algebras arising in algebraic Lie theory. Both dimensions then can be characterised intrinsically inside certain derived categories. As an application, the global and dominant dimensions of blocks of classical and quantised Schur algebras are completely determined. On the way, a restriction theorem is proved, and used, which says that derived equivalences between algebras with positive ν-dominant dimension always restrict to derived equivalences between their associated self-injective algebras, which under this assumption do exist.

Contents

1 Introduction

This article is about the problem when homological dimensions are preserved by derived equivalences. Positive results are obtained for global and dominant dimension, and applied to Schur algebras.

Homological dimensions are used to describe and to compare algebras and their representation theories. In particular, the global dimension of an algebra, defined in terms of lengths of projective resolutions of modules, is considered to be a fundamental invariant, often used to define and to characterise classes of algebras. Dominant dimension, defined in terms of projective modules occuring in injective coresolutions, is used to describe, and to measure the quality of, double centraliser properties as in the Morita-Tachikawa correspondence and in Schur-Weyl duality, and to formulate Nakayama's conjecture. The interplay of global and dominant dimensions is crucial in Auslander's correspondence and Iyama's higher Auslander correspondence and in Auslander's definition of representation dimension.

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Equivalences of derived categories are widely used, both to connect different areas of mathematics and to compare different situations in particular areas. In representation theory, derived equivalences provide a very general form of tilting from one algebra to another. While various homological and K-theoretic invariants are preserved, homological dimensions appear to be rarely invariant under derived equivalences. For instance, an algebra derived equivalent to a path algebra of a quiver need not have the same global dimension, and two derived equivalent path algebras need not have the same dominant dimension. The difference of global dimensions is bounded above by the length of the tilting complex inducing the derived equivalence. For dominant dimensions, this is not true and it even seems to be unknown whether finiteness of dominant dimension is preserved. It is an open problem to describe minimal or maximal values these dimensions take in the derived equivalence class of a given finite dimensional algebra. The only class of derived equivalences known to preserve global and dominant dimension are the (iterated) almost ν-stable derived equivalences introduced in [19, 23], which induce stable equivalences of Morita type, which in turn are known to preserve these dimensions.

In this article, a class of algebras is identified having the property that both global and dominant dimension are preserved by derived equivalences connecting members of this class. An algebra A in this class is required to admit an anti-automorphism with itself such that each simple A-module is isomorphic to the simple module obtained by twisting the A-action by the anti-automorphism and then dualising. For instance, an anti-automorphism fixing each element of a complete set of pairwise orthogonal primitive idempotents satisfies this assumption.

Theorem A (Theorem 5.10 and Corollary 5.14)**.** *Let A and B be finite dimensional algebras over a splitting field k, and suppose A admits an anti-automorphism preserving simple modules. Suppose A and B are derived equivalent. Then*

$$\operatorname{gldim}(A) \leqslant \operatorname{gldim}(B).$$

In particular, $\operatorname{gldim}(A)$ *is minimal in the derived equivalence class of A.*

When B also admits an anti-automorphism preserving simple modules, then $\operatorname{gldim}(A) = \operatorname{gldim}(B)$.

When both A and B admit anti-automorphisms preserving simple modules and in addition the dominant dimensions of A and B are both positive, then

$$\operatorname{domdim}(A) = \operatorname{domdim}(B).$$

In the course of the proof, global and dominant dimension will be identified as properties of certain complexes in the derived categories of the enveloping algebras of A and B, respectively, see Corollary 5.12. Dropping the respective assumptions, the two claims become wrong.

The class of algebras with anti-automorphisms preserving simples is quite large; in particular, it contains all cellular algebras, thus making the results applicable to many algebras arising, for instance, in algebraic Lie theory. This is illustrated by an application of Theorem A to classical and quantised Schur algebras $S_q(n,r)$ with $n \geqslant r$. While global and dominant dimensions of these algebras are known [39, 10, 14, 17], nothing was known for their blocks, i.e. the algebra direct summands.

These blocks depend on two combinatorial parameters, weight and core (where the core depends on the classical or quantum characteristic). Apart from giving precise formulae, Theorem 6.4 also shows that the global dimension does not depend on the core, and the dominant dimension is independent of both parameters, except that the semisimple case (weight zero) is obviously different. For simplicity, we only state the classical case here.

Theorem B (Theorem 6.4 specialised to classical Schur algebras)**.** *Let $n \geqslant r$. Let $\mathbf{B}_{\tau,w}$ be a block of the Schur algebra $S(n,r)$, with p-core τ and weight w. Then its global dimension equals*

$$\operatorname{gldim}(\mathbf{B}_{\tau,w}) = 2(pw - d_p(w)),$$

and its dominant dimension equals

$$\operatorname{domdim}(\mathbf{B}_{\tau,w}) = \begin{cases} \infty, & w = 0; \\ 2(p-1), & w > 0. \end{cases}$$

where p is the characteristic of the ground field, and $d_p(w)$ is the sum of the digits in the p-adic expansion of w.

The proof uses Theorem A and derived equivalences constructed by Chuang and Rouquier [7].

In the proof of Theorem A a new feature of the derived module category is coming up, which also explains the assumption needed for invariance of dominant dimension. There is a characteristic subcategory of $\mathscr{D}^{\mathrm{b}}(A)$ closely related to a maximal self-injective centraliser subalgebra eAe of A. Here, a subcategory is called *characteristic* when every derived equivalence from A to some algebra B sends it to the similarly defined subcategory of $\mathscr{D}^{\mathrm{b}}(B)$. Known examples of characteristic subcategories are the bounded homotopy category of projectives in the bounded derived module category as well as the bounded derived category as a subcategory of the bounded above or below derived categories.

The definition of the new characteristic subcategory involves a particular kind of projective modules: A projective A-module P is called strongly projective-injective if $\nu_A^i P$ is projective for all $i \geqslant 0$. Here, ν_A is the Nakayama functor. The additive full subcategory of all strongly projective-injective A-modules is denoted by A-stp. Using strongly

projective-injective modules instead of projective-injective ones in injective coresolutions, the ν-dominant dimension of A is defined as a version of dominant dimension. When $\nu\text{-domdim}(A) \geqslant 1$, there is a minimal faithful strongly projective-injective module Ae whose endomorphism ring eAe is called the *associated self-injective (sub)algebra* of A.

Theorem C (Theorem 4.3 and Corollary 4.4)**.** *Let A and B be algebras of ν-dominant dimension at least one, and let eAe and fBf be their associated self-injective subalgebras. Suppose A and B are derived equivalent. Then any derived equivalence $F : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ restricts to an equivalence*

$$\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp}) \to \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{stp})$$

of triangulated subcategories. Moreover, there are equivalences of triangulated categories

$$\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp}) \simeq \mathscr{K}^{\mathrm{b}}(eAe\text{-}\mathsf{proj}) \ \text{ and } \ \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{stp}) \simeq \mathscr{K}^{\mathrm{b}}(fBf\text{-}\mathsf{proj}).$$

In particular, the associated self-injective algebras of A and B are derived equivalent:

$$\mathscr{D}^{\mathrm{b}}(eAe) \simeq \mathscr{D}^{\mathrm{b}}(fBf).$$

The assumption on ν-dominant dimension is satisfied in particular by the Morita algebras defined by Kerner and Yamagata [25], generalising gendo-symmetric algebras [15, 16] and including many Lie theoretic examples such as Schur algebras. As an application of Theorem C, the difference of the dominant dimension of two derived equivalent k-algebras both of ν-dominant dimension at least one is shown to be bounded above by the minimal length of a tilting complex associated with a derived equivalence (Theorem 5.2). Another consequence is that two self-injective algebras of finite representation type are seen to be derived equivalent if and only if their Auslander algebras are derived equivalent.

Theorem C may be compared with a result by Martinez-Villa [29]. He proved that the Auslander-Reiten conjecture on stable equivalences preserving the number of non-projective simple modules of finite dimensional algebras holds true provided it does so for self-injective algebras. To pass from a general algebra to a self-injective one he used the associated self-injective algebra.

The structure of this article is as follows: Section 2 fixes general conventions and definitions and proves auxiliary results on homological dimensions and on classes of algebras, including a characterisation of Morita algebras by having ν-dominant dimension at least two (Proposition 2.9). Section 3 collects and proves technical results on derived equivalences. Section 4 is devoted to describing $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$ and then proving Theorem C. Section 5 starts by discussing how global and dominant dimension change under general derived equivalences. In the second subsection, almost self-injective algebras are introduced, which differ from their associated self-injective algebras by not more than one simple module. It is shown that derived equivalences between such algebras always are iterated almost ν-stable and thus p-reserve global and dominant dimension (Corollary 5.8). Allowing the algebra to have two or more simples that are not representations of the associated self-injective algebra, leads to the appearance of derived equivalences that are not almost ν-stable any more, and such examples do exist for Schur algebras. Therefore, the main part of Section 5 focusses on the large and applicable class of algebras with an anti-automorphism preserving simples, proving Theorem A. The final Section 6 then concentrates on applications to Schur algebras, proving Theorem B and its general form, Theorem 6.4.

2 Three homological invariants and two classes of algebras

After recalling two major homological invariants of algebras, global dimension and dominant dimension, a new variation of dominant dimension, ν-dominant dimension, is introduced that will turn out to provide a crucial assumption in the main results. In the second subsection, the two main classes of algebras considered here will be discussed and related to ν-dominant dimension; these are the Morita algebras, which will get characterised in terms of ν-dominant dimension, and the new class of almost self-injective algebras.

2.1 General conventions

Throughout, k is an arbitrary field of any characteristic. Algebras are *finite dimensional* k-vector spaces and, unless stated otherwise, modules are finitely generated left modules. When A is an algebra, A^{op} denotes the opposite algebra of A, and A^{e} is the enveloping algebra $A \otimes_k A^{\mathrm{op}}$. Let A-Mod (respectively A-mod) be the category of all (respectively all finitely generated) left A-modules, and A-proj (respectively A-inj) the full subcategory of A-mod whose objects are the projective (respectively injective) left A-modules. Let D be the usual k-duality functor $\mathrm{Hom}_k(-,k) : A\text{-}\mathsf{mod} \to A^{\mathrm{op}}\text{-}\mathsf{mod}$ and $\nu_A = \mathrm{D}\,\mathrm{Hom}_A(-,A) : A\text{-}\mathsf{proj} \to A\text{-}\mathsf{inj}$ the Nakayama functor.

We follow the conventions from [3]. Let $\mathcal{C}$ be an additive category. An object X in $\mathcal{C}$ is called *strongly indecomposable* if $\mathrm{End}_{\mathcal{C}}(X)$ is a local ring. An object Y in $\mathcal{C}$ is called *basic* if Y is a direct sum of strongly indecomposable

objects of multiplicity one each. For an object M in $\mathcal{C}$, we write $\mathrm{add}(M)$ for the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite direct sum of copies of M. By $f \cdot g$ or fg we denote the composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$. A morphism $h : X \to Y$ is said to be *radical* in $\mathcal{C}$, if for morphisms $\alpha : Z \to X$ and $\beta : Y \to Z$, the composition $\alpha \cdot h \cdot \beta$ never is an isomorphism. In contrast to the composition rule for morphisms, we write $\mathcal{G} \circ \mathcal{F}$ for the composition of two functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{E}$ between additive categories.

2.2 Global dimension and two dominant dimensions

Given a finite dimensional k-algebra A, there are many homological invariants around to measure the complexity of A from different points of view, and global dimension is the most widely used one. By definition, the global dimension of A, denoted by $\mathrm{gldim} A$, is the smallest number g or ∞ such that $\mathrm{Ext}^i_A(M,N) = 0$ for any $i > g$ and all $M, N \in A\text{-mod}$. The following (well-known) characterisation can be found for instance in [14, Corollary 3.8].

Lemma 2.1. *Let A be a k-algebra. If $\mathrm{gldim} A < \infty$, then $\mathrm{gldim} A$ is the largest number g such that $\mathrm{Ext}^g_A({}_A\mathrm{D}(A), {}_AA) \neq 0$.*

Dominant dimension was introduced by Nakayama, and developed later mainly by Morita and Tachikawa, see [41] for more information, and [17, 14, 15, 16, 13] for a recent development partly motivating our aims and results.

Definition 2.2. *Let A be a k-algebra. The* dominant dimension *of A, denoted by $\mathrm{domdim}(A)$, is defined to be the largest number $d \geqslant 0$ (or ∞) such that in a minimal injective resolution $0 \to {}_AA \to I^0 \to I^1 \to I^2 \to \cdots$ of the left regular A-module, I^i is projective for all $i < d$ (or ∞).*

Thus, I^0 not being projective, that is $\mathrm{domdim}(A) = 0$, is equivalent to A not having a faithful projective-injective module. The module $I \in A\text{-mod}$ is projective and injective if and only if so is $\mathrm{D}(I)$ in $A^{\mathrm{op}}\text{-mod}$. It follows that $\mathrm{domdim}(A) = \mathrm{domdim}(A^{\mathrm{op}})$ and thus $\mathrm{domdim}(A)$ can be defined alternatively via right A-modules. If $\mathrm{domdim}(A) \geqslant 1$, then there exists a unique (up to isomorphism) minimal faithful right A-module (and also a unique up to isomorphism minimal faithful left A-module). It must be projective and injective, hence of the form eA for some idempotent e in A. If $\mathrm{domdim}(A) \geqslant 2$, then eA is a faithful balanced bimodule, i.e., there is a double centralizer property, namely $A \cong \mathrm{End}_{eAe}(eA)$ canonically. Algebras of infinite dominant dimension are conjectured to be self-injective (this is the celebrated *Nakayama conjecture*), see [41]. The following characterisation of dominant dimension is due to Müller.

Proposition 2.3 (Müller [32])**.** *Let A be a k-algebra of dominant dimension at least 2. Let eA be a minimal faithful right A-module and $n \geqslant 2$ be an integer. Then $\mathrm{domdim}(A) \geqslant n$ if and only if $\mathrm{Ext}^i_{eAe}(eA, eA) = 0$ for $1 \leqslant i \leqslant n-2$.*

Of particular interest later on will be certain derived equivalences, (iterated) almost ν-stable derived equivalences, defined in [19, 20]. Here, a certain subclass of both projective and injective (projective-injective for short) modules is crucial. This motivates the following variation of dominant dimension, which is crucial for our main results.

Definition 2.4. *Let A be a k-algebra and $\nu_A = \mathrm{D}\mathrm{Hom}_A(-, A) : A\text{-mod} \to A\text{-mod}$ be the Nakayama functor. A projective A-module P is said to be* strongly projective-injective *if $\nu_A^i(P)$ is projective for all $i > 0$. By $A\text{-}\mathsf{stp}$ we denote the full subcategory of $A\text{-}\mathsf{proj}$ consisting of strongly projective-injective A-modules.*

In [23], strongly projective-injective modules are called ν-stably projective; since this may be misunderstood as implying ν-stable, we use a different terminology here. Strongly projective-injective modules are projective and injective, which justifies their name. An easy proof goes as follows, see also [23, Lemma 2.3]. First note that P is strongly projective-injective if and only if so is each of its direct summands. Thus we may assume that P is indecomposable. Since the Nakayama functor ν_A sends indecomposable projective modules to indecomposable injective modules, it follows that $\nu_A^i(P)$ are indecomposable projective-injective for all $i > 0$. But there are only finitely many indecomposable objects in $A\text{-proj}$, so there must exist $0 < a < b$ such that $\nu_A^a(P) \cong \nu_A^b(P)$. Using again that $\nu_A : A\text{-proj} \to A\text{-inj}$ is an equivalence, we deduce that $P \cong \nu_A^{b-a}(P)$. In particular, P is both projective and injective.

Definition 2.5. *Let A be a k-algebra. The* ν-dominant dimension *of A, denoted by $\nu\text{-}\mathrm{domdim}(A)$, is defined to be the largest number $d \geqslant 0$ (or ∞) such that in a minimal injective resolution $0 \to {}_AA \to I^0 \to I^1 \to I^2 \to \cdots$ of the left regular A-module, I^i is strongly projective-injective for all $i < d$ (or ∞).*

By definition, $\nu\text{-}\mathrm{domdim}(A) \leqslant \mathrm{domdim}(A)$, but in general there is no equality. Here is an example illustrating the difference between these two dimensions. Let A be the path algebra kQ of the quiver $Q : 1 \longrightarrow 2$. Then P_1 is projective-injective, but not strongly projective-injective, since $\nu_A(P_1) \cong \mathrm{D}\mathrm{Hom}_A(P_1, A) \cong \mathrm{D}(e_1A) \cong I_1$ and I_1 is not projective. As a result, $\mathrm{domdim}(A) = 1$, while $\nu\text{-}\mathrm{domdim}(A) = 0$.

In our context, ν-dominant dimension is important, since it allows to identify particular self-injective centraliser subalgebras:

Lemma 2.6. *Let A be a k-algebra. If $\nu\text{-}\mathrm{domdim}(A) \geqslant 1$, then all projective-injective A-modules are strongly projective-injective, and thus $\nu\text{-}\mathrm{domdim}(A) = \mathrm{domdim}(A)$. In this case, endomorphism rings of minimal faithful left A-modules are self-injective.*

Suppose $\nu\text{-domdim}(A) \geqslant 1$. Then, a minimal faithful left A-module is of the form Ae, and strongly projective-injective. We will use its endomorphism ring eAe as 'the largest self-injective centraliser subalgebra'.

Proof. Since $\nu\text{-domdim}(A) \geqslant 1$, the injective envelope I of ${}_AA$ is strongly projective-injective. Let P be an indecomposable projective-injective A-module. The composition $P \hookrightarrow A \hookrightarrow I$ is a split monomorphism. Thus P is a direct summand of I, and in particular strongly projective-injective. Consequently, all projective-injective A-modules are strongly projective-injective. Hence the two dominant dimensions coincide.

Let Ae be a minimal faithful left A-module. By assumption, it is strongly projective-injective. Hence $\mathrm{D}(eA) \cong \nu_A(Ae)$ belongs to $\mathrm{add}(Ae)$, and in particular ${}_{eAe}\mathrm{D}(eAe) = e\,\mathrm{D}(eA) \in e\,\mathrm{add}({}_AAe) = \mathrm{add}\,{}_{eAe}(eAe)$, that is, eAe is self-injective. □

The endomorphism ring of a strongly projective-injective A-module in general may not be self-injective, even when assuming $\nu\text{-domdim}(A) \geqslant 1$. For instance, let A be the self-injective Nakayama algebra with cyclic quiver, three simple modules and $\mathrm{rad}(A)^2 = 0$. Then each indecomposable projective module is injective as well, and even strongly projective-injective, but the endomorphism ring of a sum of two non-isomorphic indecomposable projective modules never is self-injective.

The proof of Lemma 2.6 works not only for Ae, but also for any direct sum of copies of Ae.

Definition 2.7. *Let A be a k-algebra with $\nu\text{-domdim}(A) \geqslant 1$ and let Ae be a minimal faithful left A-module. Then the centraliser algebra eAe is called the* associated self-injective (sub)algebra of A.

These associated self-injective subalgebras have been introduced and strongly used before in work of Martinez-Villa [28, 29] reducing validity of the Auslander-Reiten conjecture on stable equivalences preserving the number of non-projective simple modules to the case of self-injective algebras. There, the setup is more general and the associated self-injective algebras have been allowed to be zero, which does not make sense in our situation as we need a strong connection between the given algebra and its associated self-injective subalgebra. The term 'associated self-injective algebra' first occurred in [12].

2.3 Morita algebras and almost self-injective algebras

The term 'Morita algebras' (not related to Morita rings occurring in Morita contexts) was coined by Kerner and Yamagata in [25], when they investigated algebras first studied by Morita [33] as endomorphism rings of generators over self-injective algebras. The subclass of Morita algebras consisting of endomorphism rings of generators over symmetric algebras, called gendo-symmetric algebras, was introduced and studied independently in [15, 16].

Definition 2.8 ([25, 15, 16])**.** *A k-algebra A is called a* Morita algebra *if A is isomorphic to $\mathrm{End}_H(H \oplus M)$ for some self-injective algebra H and some module $M \in H\text{-}\mathsf{mod}$. A is called* gendo-symmetric *if in addition H is symmetric.*

Gendo-symmetric algebras form a large class of algebras, cutting across traditional boundaries such as finite or infinite global dimension. Examples of finite global dimension include classical and quantised Schur algebras $S(n,r)$ (with $n \geqslant r$), blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ and many other algebras occurring in algebraic Lie theory and elsewhere. Examples of infinite global dimension include symmetric algebras, for instance group algebras and Hecke algebras. Morita algebras include in addition self-injective, and in particular Frobenius algebras, and their Auslander algebras. Morita algebras have been characterised in several ways, see [25, 16, 13]. For our purposes, a new characterisation is needed in terms of ν-dominant dimension (Section 2.2):

Proposition 2.9. *Let A be a k-algebra. Then $\nu\text{-domdim}(A) \geqslant 2$ if and only if A is a Morita algebra.*

Proof. Suppose $\nu\text{-domdim}(A) \geqslant 2$. Then by Lemma 2.6, the minimal faithful A-module Ae is (strongly) projective-injective and its endomorphism ring eAe is self-injective. Moreover, A has dominant dimension at least two, which implies a double centraliser property on Ae, between A and eAe. Therefore, A is a Morita algebra.

Conversely, if A is a Morita algebra, then A is isomorphic to $\mathrm{End}_H(M)$ for some self-injective algebra H and a generator M in $H\text{-}\mathsf{mod}$. Let E be the direct sum of all pairwise non-isomorphic indecomposable projective H-modules.

Claim. $\mathrm{Hom}_H(M,E)$ *is a strongly projective-injective (left) A-module.*

Proof. By definition, E is a direct summand of M and $\mathrm{add}({}_HE) = H\text{-}\mathsf{proj}$. Moreover, $\nu_H E \cong E$ as left H-modules and there are isomorphisms of A-modules

$$\begin{aligned}\nu_A\mathrm{Hom}_H(M,E) &= \mathrm{D}\,\mathrm{Hom}_A\big(\mathrm{Hom}_H(M,E),\mathrm{Hom}_H(M,M)\big)\\ &\overset{(1)}{\cong} \mathrm{D}\,\mathrm{Hom}_H(E,M) \overset{(2)}{\cong} \mathrm{D}(\mathrm{Hom}_H(E,H)\otimes_H M)\\ &\overset{(3)}{\cong} \mathrm{Hom}_H(M,\nu_H E) \overset{(4)}{\cong} \mathrm{Hom}_H(M,E).\end{aligned}$$

Here, the isomorphism (1) follows from $E \in \operatorname{add} M$. The isomorphism (2) uses $\operatorname{add}({}_H E) = H\text{-proj}$ and the isomorphism (3) follows from tensor-hom adjointness. (4) uses $\nu_H E \cong E$. This proves the claim.

Now we construct an injective presentation of the left regular A-module ${}_A A$ (or $\operatorname{End}_H(M)$) as follows: take an injective presentation $0 \to {}_H M \to P_1 \to P_2$ of M and apply $\operatorname{Hom}_H(M,-)$ to obtain the exact sequence $0 \to \operatorname{Hom}_H(M,M) \to \operatorname{Hom}_H(M,P_1) \to \operatorname{Hom}_H(M,P_2)$ of left A-modules. Note that both P_1 and P_2 are projective H-modules and thus belong to $\operatorname{add}({}_H E)$. Therefore, $\operatorname{Hom}_H(M,P_i) \in A\text{-stp}$ for $i = 1,2$, and so $\nu\text{-domdim}(A) \geqslant 2$. □

Corollary 2.10. *Let A be a Morita algebra. Then* $\operatorname{domdim}(A) = \nu\text{-domdim}(A)$.

Proof. This follows immediately from Lemma 2.6 and Proposition 2.9. □

Gendo-symmetric algebras appeared first in [15], see [16] for further information. In our context, there is the following characterisation:

Proposition 2.11 ([15, 16])**.** *Let A be a k-algebra. Then A is gendo-symmetric if and only if $\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$ as A-bimodules. If A is gendo-symmetric, then* $\operatorname{domdim}(A) = \sup\{s \mid \operatorname{Ext}^i_A(\mathrm{D}(A),A) = 0,\ 1 \leqslant i \leqslant s-2\}$.

Proof. The first claim follows from [15, Theorem 3.2] and the second one from [15, Proposition 3.3]. Alternatively, the characterisation of $\operatorname{domdim}(A)$ also follows from Proposition 2.3 combined with the following claim:

Claim. Let e be an idempotent in A such that eA is a minimal faithful right A-module. Then there are isomorphisms $\operatorname{Ext}^i_A(\mathrm{D}(A),A) \cong \operatorname{Ext}^i_{eAe}(eA,eA)$ in A-mod for $0 \leqslant i \leqslant \operatorname{domdim}(A) - 1$.

Proof. Since A is a gendo-symmetric algebra, dualising both sides of the isomorphism $\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$ yields an isomorphism $\operatorname{Hom}_A(\mathrm{D}(A),A) \cong A$ as A-bimodules. Thus $\operatorname{Hom}_A(\mathrm{D}(A),Ae)$ is isomorphic to Ae, and in particular it is projective-injective. Now, for a minimal injective resolution $\mathscr{E}:\ 0 \to {}_A A \to I^0 \to I^1 \to I^2 \to \cdots$ of the left regular A-module, the first $\operatorname{domdim}(A)$-terms are projective-injective, hence belong to $\operatorname{add}({}_A Ae)$. Applying $\operatorname{Hom}_A(\mathrm{D}(A),-)$ to the sequence $\mathscr{E}$, and comparing the cohomologies of $\operatorname{Hom}_A(\mathrm{D}(A),\mathscr{E})$ and $\operatorname{Hom}_{eAe}(eA,e\mathscr{E})$, proves the claim. □

The second class of algebras we are going to study generalises self-injective algebras.

Definition 2.12. *An algebra A is called an* almost self-injective algebra*, if $\nu\text{-domdim}(A) \geqslant 1$ and there is at most one indecomposable projective A-module that is not injective.*

Among the examples are Schur algebras of finite representation type, which have finite global dimension and which are Morita algebras. Schur algebras, and thus Morita algebras, in general are not almost self-injective. Schur algebras of finite representation type are examples of gendo-Brauer tree algebras described and classified in [5]. These algebras are representation-finite gendo-symmetric and in addition biserial; the corresponding symmetric algebras are Brauer tree algebras. Conversely, almost self-injective algebras need not be Morita algebras as the following example illustrates. Let A be the k-algebra given by the quiver

$$2 \underset{\alpha}{\overset{\delta}{\rightleftarrows}} 1 \underset{\theta}{\overset{\beta}{\rightleftarrows}} 3$$

and relations $\{\delta\alpha, \alpha\delta, \theta\beta, \beta\theta\}$. The Loewy series of the indecomposable projective left A-modules are

$$P_1 = \begin{array}{ccc} & 1 & \\ 2 & & 3 \end{array} \qquad P_2 = \begin{array}{c} 2 \\ 1 \\ 3 \end{array} \qquad P_3 = \begin{array}{c} 3 \\ 1 \\ 2 \end{array}$$

Then both P_2 and P_3 are strongly projective-injective, and $\nu\text{-domdim}(A) = 1$. Thus A is an almost self-injective algebra, but not a Morita algebra.

3 Derived equivalences

After recalling fundamental facts of derived Morita theory, basic properties of standard equivalences will be shown and then almost ν-stable derived equivalences will be explained, thus providing crucial tools for proofs later on.

Let $\mathcal{C}$ be an additive category. A complex $X^\bullet$ over $\mathcal{C}$ is a sequence of morphisms d^i_X in $\mathcal{C}$ of the form

$$\cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}_X} X^i \xrightarrow{d^i_X} X^{i+1} \xrightarrow{d^{i+1}_X} \cdots$$

with $d^i_X d^{i+1}_X = 0$ for all $i \in \mathbb{Z}$. We call $X^\bullet$ a *radical* complex if all d^i_X are radical morphisms. We denote by $\mathscr{C}(\mathcal{C})$ (respectively $\mathscr{C}^{\mathrm{b}}(\mathcal{C})$) the category of complexes (respectively bounded complexes) over $\mathcal{C}$, and by $\mathscr{K}(\mathcal{C})$ (respectively

$\mathscr{K}^{\mathrm{b}}(\mathcal{C})$) the corresponding homotopy category. $\mathscr{D}(\mathcal{C})$ (respectively $\mathscr{D}^{\mathrm{b}}(\mathcal{C})$) is the derived category of complexes (respectively bounded complexes) over $\mathcal{C}$ when $\mathcal{C}$ is abelian. Homotopy categories and derived categories are prominent examples of *triangulated categories*.

For an algebra A, we write $\mathscr{C}^*(A)$, $\mathscr{K}^*(A)$ and $\mathscr{D}^*(A)$ for $\mathscr{C}^*(A\text{-mod})$, $\mathscr{K}^*(A\text{-mod})$ and $\mathscr{D}^*(A\text{-mod})$ respectively, where $*$ stands for blank or b.

Lemma 3.1. *Let A be an algebra. Then:*

(1) *Every complex of A-modules is isomorphic to a radical complex in the homotopy category $\mathscr{K}(A)$.*

(2) *Two radical complexes are isomorphic in the homotopy category $\mathscr{K}(A)$ if and only if so they are in $\mathscr{C}(A)$.*

(3) *For two complexes $X^\bullet$ and $Y^\bullet$, if there exists an integer n such that $X^\bullet$ has no cohomology in degrees larger than n (i.e., $\mathrm{H}^i(X^\bullet)=0$ for $i>n$), and $Y^\bullet$ has no cohomology in degrees smaller than n (i.e., $\mathrm{H}^i(Y^\bullet)=0$ for $i<n$), then*

$$\mathrm{Hom}_{\mathscr{D}(A)}(X^\bullet, Y^\bullet) \cong \mathrm{Hom}_A(\mathrm{H}^n(X^\bullet), \mathrm{H}^n(Y^\bullet)).$$

In particular, for any complex $Z^\bullet$ of A-modules, $\mathrm{Hom}_{\mathscr{D}(A)}(A, Z^\bullet[i])$ is isomorphic to its i-th cohomology $\mathrm{H}^i(Z^\bullet)$ for all i.

Proof. The first two statements are taken from [20, p. 112-113]; the remaining statements can be shown by using truncations and Cartan-Eilenberg resolutions of complexes. □

The tensor product of two complexes $X^\bullet$ and $Y^\bullet$ in $\mathscr{C}(A^{\mathrm{e}}\text{-Mod})$ is defined to be the total complex of the double complex with its (i,j)-term $X^i \otimes_A Y^j$, and their tensor product in $\mathscr{D}(A^{\mathrm{e}}\text{-Mod})$ is the tensor product of their projective resolutions in $\mathscr{C}(A^{\mathrm{e}}\text{-Mod})$ [38].

Let $X^\bullet$ be a complex over $\mathcal{C}$ of the form $\cdots \longrightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \cdots$. Then the *brutal truncations* of $X^\bullet$ are defined by cutting off the left or right hand part of the complex: $X^\bullet_{\geqslant i} = \sigma_{\geqslant i}(X^\bullet) : \cdots \longrightarrow 0 \longrightarrow X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \cdots$ and $X^\bullet_{<i} = \sigma_{<i}(X^\bullet) : \cdots \longrightarrow X^{i-2} \xrightarrow{d_X^{i-2}} X^{i-1} \longrightarrow 0 \longrightarrow \cdots$. There is an exact sequence of complexes $0 \to \sigma_{\geqslant i}(X^\bullet) \to X^\bullet \to \sigma_{<i}(X^\bullet) \to 0$, which also defines a triangle.

3.1 Derived equivalences and tilting complexes

Derived equivalences are by definition equivalences of derived categories that preserve the triangulated structures, that is shift and triangles. Two algebras *A and B are derived equivalent* if there is a derived equivalence between their derived categories. Despite their importance, derived equivalences are still rather unknown and even basic questions are still open. The equivalence relation between algebras defined by derived equivalence does, however, admit a very satisfactory theory, known as *Morita theory for derived categories*, due to Rickard and (more generally for dg algebras) to Keller.

Theorem 3.2 (Rickard [34], Keller [24])**.** *Let A and B be two k-algebras. The following statements are equivalent.*

(1) *$\mathscr{D}(A\text{-Mod})$ and $\mathscr{D}(B\text{-Mod})$ are equivalent as triangulated categories.*

(2) *$\mathscr{D}^{\mathrm{b}}(A\text{-Mod})$ and $\mathscr{D}^{\mathrm{b}}(B\text{-Mod})$ are equivalent as triangulated categories.*

(3) *$\mathscr{D}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(B)$ are equivalent as triangulated categories.*

(4) *$\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ and $\mathscr{K}^{\mathrm{b}}(B\text{-proj})$ are equivalent as triangulated categories.*

(5) *There exists a complex $T^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-proj})$ such that $\mathrm{End}_{\mathscr{D}(A)}(T^\bullet) \cong B$ and*

   (a) *$\mathrm{Hom}_{\mathscr{D}(A)}(T^\bullet, T^\bullet[n]) = 0$ unless $n=0$;*

   (b) *$\mathrm{add}(T^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ as triangulated category.*

The complex $T^\bullet$ in (5) is called a *tilting complex*. For any derived equivalence from $\mathscr{D}(A\text{-Mod})$ to $\mathscr{D}(B\text{-Mod})$, the image of A in $\mathscr{D}(B\text{-Mod})$ is a tilting complex, and so is the preimage of B in $\mathscr{D}(A\text{-Mod})$. It is not known whether the equivalences in (1)-(4) determine each other uniquely (see [34, Section 7]). To fix the ambiguity, Rickard [36] associated to each derived equivalence a *standard derived equivalence*. A complex $\Delta^\bullet \in \mathscr{D}^{\mathrm{b}}((B \otimes_k A^{\mathrm{op}})\text{-Mod})$ is called a *two-sided tilting complex* if and only if

$$\Delta^\bullet \otimes^{\mathbf{L}}_A \Theta^\bullet \cong {}_BB_B \quad \text{and} \quad \Theta^\bullet \otimes^{\mathbf{L}}_B \Delta^\bullet \cong {}_AA_A$$

for some complex $\Theta^\bullet \in \mathscr{D}^{\mathrm{b}}((A \otimes_k B^{\mathrm{op}})\text{-Mod})$. The complex $\Theta^\bullet$ is called an *inverse* of $\Delta^\bullet$. The functor $\Delta^\bullet \otimes^{\mathbf{L}}_A - : \mathscr{D}(A\text{-Mod}) \to \mathscr{D}(B\text{-Mod})$ (respectively, $-\otimes^{\mathbf{L}}_B \Delta^\bullet : \mathscr{D}(B^{\mathrm{op}}\text{-Mod}) \to \mathscr{D}(A^{\mathrm{op}}\text{-Mod})$) is a triangle equivalence with $\Theta^\bullet \otimes^{\mathbf{L}}_B -$ (respectively, $-\otimes^{\mathbf{L}}_A \Theta^\bullet$) as a quasi-inverse. Such a derived equivalence is called a *standard derived equivalence*. It has been proved in [36] that each derived equivalence $\mathcal{F} : \mathscr{D}^{\mathrm{b}}(A\text{-Mod}) \cong \mathscr{D}^{\mathrm{b}}(B\text{-Mod})$ induces a derived equivalence $\hat{\mathcal{F}}$ from $\mathscr{D}^{\mathrm{b}}((B \otimes_k B^{\mathrm{op}})\text{-Mod})$ to $\mathscr{D}^{\mathrm{b}}((B \otimes_k A^{\mathrm{op}})\text{-Mod})$, and the image $\Delta^\bullet$ of ${}_BB_B$ under $\hat{\mathcal{F}}$ is a two-sided tilting complex such that $\Delta^\bullet \otimes^{\mathbf{L}}_A X^\bullet \cong \mathcal{F}(X^\bullet)$ for all $X^\bullet \in \mathscr{D}^{\mathrm{b}}(A\text{-Mod})$. It is not known whether $\mathcal{F}$ and $\Delta^\bullet \otimes^{\mathbf{L}}_A -$ agree on morphisms.

Two-sided tilting complexes can be used to provide further derived equivalences. Let $\Delta^\bullet$ be a two-sided tilting complex in $\mathscr{D}^{\mathrm{b}}((B\otimes A^{\mathrm{op}})\text{-Mod})$ with an inverse $\Theta^\bullet$. Then, for each algebra C, the functor

$$\Delta^\bullet\otimes^{\mathbf{L}}_A - : \mathscr{D}((A\otimes_k C^{\mathrm{op}})\text{-Mod}) \to \mathscr{D}((B\otimes_k C^{\mathrm{op}})\text{-Mod})$$

is a triangle equivalence with $\Theta^\bullet\otimes^{\mathbf{L}}_B -$ as a quasi-inverse, and the functor

$$\check{\mathcal{F}} := \Delta^\bullet\otimes^{\mathbf{L}}_A - \otimes^{\mathbf{L}}_A \Theta^\bullet : \mathscr{D}(A^{\mathrm{e}}\text{-Mod}) \longrightarrow \mathscr{D}(B^{\mathrm{e}}\text{-Mod}) \tag{1}$$

defines a derived equivalence with $\Theta^\bullet\otimes^{\mathbf{L}}_B - \otimes^{\mathbf{L}}_B \Delta^\bullet$ as a quasi-inverse, see [36] for more details.

3.2 Standard derived equivalences

Here are some properties of the standard derived equivalence $\check{\mathcal{F}}$, to be used later on.

Lemma 3.3. *Let A and B be derived equivalent k-algebras. Let $\check{\mathcal{F}}$ be a standard derived equivalence from $\mathscr{D}(A^{\mathrm{e}}\text{-Mod})$ to $\mathscr{D}(B^{\mathrm{e}}\text{-Mod})$ defined by a two-sided tilting complex $\Delta^\bullet$ in $\mathscr{D}(B\otimes A^{\mathrm{op}})$ as defined above in (1). Then for any complexes $X^\bullet$ and $Y^\bullet$ in $\mathscr{D}(A^{\mathrm{e}}\text{-Mod})$, there are isomorphisms in $\mathscr{D}(B^{\mathrm{e}}\text{-Mod})$*

(1) $\check{\mathcal{F}}(A)\cong B$, $\check{\mathcal{F}}(\mathrm{D}(A))\cong \mathrm{D}(B)$;
(2) $\check{\mathcal{F}}(X^\bullet\otimes^{\mathbf{L}}_A Y^\bullet)\cong \check{\mathcal{F}}(X^\bullet)\otimes^{\mathbf{L}}_B \check{\mathcal{F}}(Y^\bullet)$;
(3) $\check{\mathcal{F}}\big(\mathbf{R}\mathrm{Hom}_A({}_AX^\bullet, {}_AY^\bullet)\big)\cong \mathbf{R}\mathrm{Hom}_B({}_B\check{\mathcal{F}}(X^\bullet), {}_B\check{\mathcal{F}}(Y^\bullet))$;
(4) $\check{\mathcal{F}}(\mathrm{D}(X^\bullet))\cong \mathrm{D}(\check{\mathcal{F}}(X^\bullet))$.

Proof. (1) follows from the isomorphism $\Delta^\bullet\otimes^{\mathbf{L}}_A A\otimes^{\mathbf{L}}_A\Theta^\bullet\cong B$ and (4), to be proved below, and (2) follows from [36, Proposition 5.2]. To show (3) and (4), note that the derived functor $\mathbf{R}\mathrm{Hom}_B(\Delta^\bullet,-)$ from $\mathscr{D}((B\otimes A^{\mathrm{op}})\text{-Mod})$ to $\mathscr{D}(A^{\mathrm{e}}\text{-Mod})$ is right adjoint to the derived functor $\Delta^\bullet\otimes^{\mathbf{L}}_A -$ (see [31, 38]). Thus it is naturally isomorphic to the derived functor $\Theta^\bullet\otimes^{\mathbf{L}}_B -$. Similarly, the two derived functors $\mathbf{R}\mathrm{Hom}_A(\Theta^\bullet,-)$ and $\Delta^\bullet\otimes^{\mathbf{L}}_A -$ are naturally isomorphic. (3) then follows by a series of isomorphisms in $\mathscr{D}(B^{\mathrm{e}}\text{-Mod})$:

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_B({}_B\check{\mathcal{F}}(X^\bullet), {}_B\check{\mathcal{F}}(Y^\bullet)) &= \mathbf{R}\mathrm{Hom}_B(\Delta^\bullet\otimes^{\mathbf{L}}_A X^\bullet\otimes^{\mathbf{L}}_A\Theta^\bullet, \Delta^\bullet\otimes^{\mathbf{L}}_A Y^\bullet\otimes^{\mathbf{L}}_A\Theta^\bullet)\\
&\overset{(*)}{\cong} \mathbf{R}\mathrm{Hom}_A(X^\bullet\otimes^{\mathbf{L}}_A\Theta^\bullet, \mathbf{R}\mathrm{Hom}_B(\Delta^\bullet, \Delta^\bullet\otimes^{\mathbf{L}}_A Y^\bullet\otimes^{\mathbf{L}}_A\Theta^\bullet))\\
&\cong \mathbf{R}\mathrm{Hom}_A(X^\bullet\otimes^{\mathbf{L}}_A\Theta^\bullet, \Theta^\bullet\otimes^{\mathbf{L}}_B\Delta^\bullet\otimes^{\mathbf{L}}_A Y^\bullet\otimes^{\mathbf{L}}_A\Theta^\bullet)\\
&\cong \mathbf{R}\mathrm{Hom}_A(X^\bullet\otimes^{\mathbf{L}}_A\Theta^\bullet, Y^\bullet\otimes^{\mathbf{L}}_A\Theta^\bullet)\\
&\overset{(*)}{\cong} \mathbf{R}\mathrm{Hom}_A(\Theta^\bullet, \mathbf{R}\mathrm{Hom}_A(X^\bullet, Y^\bullet\otimes^{\mathbf{L}}_A\Theta^\bullet))\\
&\cong \Delta^\bullet\otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(X^\bullet, Y^\bullet\otimes^{\mathbf{L}}_A\Theta^\bullet)\\
&\overset{(\dagger)}{\cong} \Delta^\bullet\otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A({}_AX^\bullet, {}_AY^\bullet)\otimes^{\mathbf{L}}_A\Theta^\bullet\\
&= \check{\mathcal{F}}(\mathbf{R}\mathrm{Hom}_A({}_AX^\bullet, {}_AY^\bullet)).
\end{aligned}$$

Here the isomorphisms marked by $(*)$ follow by tensor-hom adjointness, and the isomorphism marked by $(\dagger)$ follows from ${}_A\Theta^\bullet\in\mathscr{K}^{\mathrm{b}}(A\text{-proj})$.

To prove (4), observe that $\mathrm{D}(X^\bullet)\cong\mathbf{R}\mathrm{Hom}_A(X^\bullet,\mathrm{D}(A))$ in $\mathscr{D}(A^{\mathrm{e}}\text{-Mod})$. Thus by (1) and (3)

$$\begin{aligned}
\check{\mathcal{F}}(\mathrm{D}(X^\bullet)) \cong \check{\mathcal{F}}(\mathbf{R}\mathrm{Hom}_A({}_AX^\bullet, {}_A\mathrm{D}(A))) &\cong \mathbf{R}\mathrm{Hom}_B({}_B\check{\mathcal{F}}(X), {}_B\check{\mathcal{F}}(\mathrm{D}(A)))\\
&\cong \mathbf{R}\mathrm{Hom}_B({}_B\check{\mathcal{F}}(X), {}_B\mathrm{D}(B)) \cong \mathrm{D}(\check{\mathcal{F}}(X^\bullet)),
\end{aligned}$$

in $\mathscr{D}(B^{\mathrm{e}}\text{-Mod})$. □

Lemma 3.4. *Let $\mathcal{T}$ be a triangulated category, and let $\xi_i : X_i\to Y\xrightarrow{f_i} Z\to X_i[1], i=1,2$ be triangles in $\mathcal{T}$. If one of the following conditions is satisfied*

(1) $\mathrm{Hom}_{\mathcal{T}}(Y,X_i)=0=\mathrm{Hom}_{\mathcal{T}}(Y,X_i[1])$ *for* $i=1,2$;
(2) $\mathrm{Hom}_{\mathcal{T}}(X_i,Z)=0=\mathrm{Hom}_{\mathcal{T}}(X_i[1],Z)$ *for* $i=1,2$,

then ξ_1 and ξ_2 are isomorphic.

Proof. Assume that condition (1) is satisfied. Applying $\mathrm{Hom}_{\mathcal{T}}(Y,-)$ to ξ_i $(i=1,2)$ yields isomorphisms $\mathrm{Hom}_{\mathcal{T}}(Y,f_i)$: $\mathrm{Hom}_{\mathcal{T}}(Y,Y)\to\mathrm{Hom}_{\mathcal{T}}(Y,Z)$. This means that each morphism $g:Y\to Z$ factorises uniquely through f_i. In particular,

$f_1 = hf_2$ and $f_2 = h'f_1$ for some $h, h' \in \mathrm{End}_{\mathcal{T}}(Y)$. It follows that $f_1 = hh'f_1$ and $f_2 = h'hf_2$, and thus by uniqueness, $hh' = 1_Y = h'h$. Hence $h : Y \to Y$ is an isomorphism, and the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f_1} & Z \\ {\scriptstyle h}\downarrow & & \| \\ Y & \xrightarrow{f_2} & Z \end{array}$$

extends to an isomorphism between ξ_1 and ξ_2. The proof is similar when assuming (2). □

Derived equivalences, by definition, preserve all triangles, especially the following type. Let $e = e^2$ be an idempotent in the k-algebra A. There are canonical triangles associated to e in $\mathscr{D}(A^{\mathrm{e}}\text{-Mod})$:

$$\begin{aligned} &(\mathbf{CT1})\ \xi_e^A : \quad U_A(e) \longrightarrow Ae \otimes^{\mathbf{L}}_{eAe} eA \xrightarrow{\pi_e} A \longrightarrow U_A(e)[1] \\ &(\mathbf{CT2})\ \eta_e^A : \quad A \xrightarrow{\rho_e} \mathbf{R}\mathrm{Hom}_{eAe}(eA, eA) \longrightarrow V_A(e) \longrightarrow A[1] \end{aligned}$$

where $\pi_e : Ae \otimes^{\mathbf{L}}_{eAe} eA \to A$ is induced by the multiplication map $Ae \otimes_{eAe} eA \to A$, and ρ_e is induced by the canonical morphism $A \to \mathrm{End}_{eAe}(eA)$. The triangle ξ_e^A plays a crucial role in the analysis of recollements of derived categories [31]. The following result implies that the property of being a canonical triangle is preserved under certain derived equivalences.

Lemma 3.5. *Let A and B be derived equivalent k-algebras, and ${}_B\Delta_A^\bullet$ a two-sided tilting complex with inverse ${}_A\Theta_B^\bullet$. Let e and f be idempotents in A and B respectively. Assume that the standard derived equivalence $\Delta^\bullet \otimes^{\mathbf{L}}_A - : \mathscr{D}(A\text{-Mod}) \to \mathscr{D}(B\text{-Mod})$ restricts to a triangle equivalence $\mathscr{K}^{\mathrm{b}}(\mathrm{add}({}_AAe)) \cong \mathscr{K}^{\mathrm{b}}(\mathrm{add}({}_BBf))$. Then $\Delta^\bullet \otimes^{\mathbf{L}}_A \xi_e^A \otimes^{\mathbf{L}}_A \Theta^\bullet \cong \xi_f^B$ and $\Delta^\bullet \otimes^{\mathbf{L}}_A \eta_e^A \otimes^{\mathbf{L}}_A \Theta^\bullet \cong \eta_f^B$ as triangles in $\mathscr{D}(B^{\mathrm{e}}\text{-Mod})$.*

Proof. Since $eA \otimes_A - : \mathrm{add}({}_AAe) \to eAe\text{-proj}$ is an equivalence of additive categories with a quasi-inverse $Ae \otimes_{eAe} - : eAe\text{-proj} \to \mathrm{add}({}_AAe)$, it follows that $eA \otimes^{\mathbf{L}}_A - : \mathscr{K}^{\mathrm{b}}(\mathrm{add}({}_AAe)) \to \mathscr{K}^{\mathrm{b}}(eAe\text{-proj})$ is an equivalence of triangulated categories with a quasi-inverse $Ae \otimes^{\mathbf{L}}_{eAe} -$. Similarly, the derived functor $fB \otimes^{\mathbf{L}}_B -$ defines an equivalence from $\mathscr{K}^{\mathrm{b}}(\mathrm{add}({}_BBf))$ to $\mathscr{K}^{\mathrm{b}}(fBf\text{-proj})$ with a quasi-inverse $Bf \otimes^{\mathbf{L}}_{fBf} -$. Note that $\Theta^\bullet \cong \mathbf{R}\mathrm{Hom}_B(\Delta^\bullet, B)$ and $\Delta^\bullet \cong \mathbf{R}\mathrm{Hom}_A(\Theta^\bullet, A)$ (see [36]).

Claim.

(a)	$\Delta^\bullet e \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}({}_BBf))$,	$\Theta^\bullet f \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}({}_AAe))$,
(b)	$f\Delta^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(eA_A))$,	$e\Theta^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(fB_B))$,
(c)	$f\Delta^\bullet e \otimes^{\mathbf{L}}_{eAe} e\Theta^\bullet f \cong fBf$,	$e\Theta^\bullet f \otimes^{\mathbf{L}}_{fBf} f\Delta^\bullet e \cong eAe$.

Proof of claim. By assumption, $\Delta^\bullet e \cong \Delta^\bullet \otimes^{\mathbf{L}}_A Ae$ belongs to $\mathscr{K}^{\mathrm{b}}(\mathrm{add}({}_BBf))$. Similarly $\Theta^\bullet f \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}({}_AAe))$, hence (a).

There are the following isomorphisms

$$f\Delta^\bullet \cong \mathbf{R}\mathrm{Hom}_B(Bf, \Delta^\bullet) \cong \mathbf{R}\mathrm{Hom}_B(Bf, \mathbf{R}\mathrm{Hom}_A(\Theta^\bullet, A)) \cong \mathbf{R}\mathrm{Hom}_A(\Theta^\bullet f, A).$$

It follows that $f\Delta^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(eA_A))$. By symmetry, also $e\Theta^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(fB_B))$, hence (b).

To prove the first statement in (c), consider the isomorphisms

$$f\Delta^\bullet e \otimes^{\mathbf{L}}_{eAe} e\Theta^\bullet f \cong f\Delta^\bullet \otimes^{\mathbf{L}}_A Ae \otimes^{\mathbf{L}}_{eAe} eA \otimes^{\mathbf{L}}_A \Theta^\bullet f \cong fB \otimes^{\mathbf{L}}_B \Delta^\bullet \otimes^{\mathbf{L}}_A \Theta^\bullet \otimes^{\mathbf{L}}_B Bf \cong fBf$$

which use $eA \otimes^{\mathbf{L}}_A - : \mathscr{K}^{\mathrm{b}}(\mathrm{add}({}_AAe)) \to \mathscr{K}^{\mathrm{b}}(eAe\text{-proj})$ being an equivalence with quasi-inverse $Ae \otimes^{\mathbf{L}}_{eAe} -$ and $\Delta^\bullet \otimes^{\mathbf{L}}_A \Theta^\bullet \cong {}_BB_B$. The second statement in (c) follows similarly.

Now, the isomorphisms (the second one using (a) and (c))

$$\begin{aligned} \Delta^\bullet \otimes^{\mathbf{L}}_A Ae \otimes^{\mathbf{L}}_{eAe} eA \otimes^{\mathbf{L}}_A \Theta^\bullet &\cong \Delta^\bullet e \otimes^{\mathbf{L}}_{eAe} e\Theta^\bullet \\ &\cong Bf \otimes^{\mathbf{L}}_{fBf} f\Delta^\bullet e \otimes^{\mathbf{L}}_{eAe} e\Theta^\bullet f \otimes^{\mathbf{L}}_{fBf} fB \\ &\cong Bf \otimes^{\mathbf{L}}_{fBf} fB \end{aligned}$$

combined with Lemma 3.3 (1) yield that $\Delta^\bullet \otimes^{\mathbf{L}}_A \xi_e^A \otimes^{\mathbf{L}}_A \Theta^\bullet$ is a triangle of the following form

$$\delta_f^B : \qquad U_B(f)' \longrightarrow Bf \otimes^{\mathbf{L}}_{fBf} fB \xrightarrow{\varepsilon} B \longrightarrow U_B(f)'[1]$$

in $\mathscr{D}(B^{\mathrm{e}}\text{-Mod})$, where $U_B(f)' = \Delta^\bullet \otimes^{\mathbf{L}}_A U_A(e) \otimes^{\mathbf{L}}_A \Theta^\bullet$. So, the triangles ξ_f^B and δ_f^B have at least two terms in common. To identify them as triangles, note that $eU_A(e) = 0$ and $fU_B(f)' = f\Delta^\bullet \otimes^{\mathbf{L}}_A U_A(e) \otimes^{\mathbf{L}}_A \Theta^\bullet = 0$ since $f\Delta^\bullet$ belongs to $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(eA_A))$ by (b). Then

$$\mathrm{Hom}_{\mathscr{D}(B^{\mathrm{e}}\text{-Mod})}(Bf \otimes^{\mathbf{L}}_{fBf} fB, U_B(f)[i]) = 0 = \mathrm{Hom}_{\mathscr{D}(B^{\mathrm{e}}\text{-Mod})}(Bf \otimes^{\mathbf{L}}_{fBf} fB, U_B(f)'[i]), \forall i \in \mathbb{Z}.$$

Here, the vanishing follows by adjointness. Thus ξ_f^B and δ_f^B are isomorphic triangles in $\mathscr{D}(B^{\mathrm{e}}\text{-}\mathsf{Mod})$ by Lemma 3.4. So, ξ_f^B is as claimed.

It remains to check the claim about η_f^B.

Claim. The canonical triangle η_e^A is isomorphic to $\mathbf{R}\mathrm{Hom}_A(\xi_e^A, A)$.

Proof of claim. Applying $\mathbf{R}\mathrm{Hom}_A(-,A)$ to ξ_e^A results in a triangle

$$\eta' : A \longrightarrow \mathbf{R}\mathrm{Hom}_A(Ae \otimes^{\mathbf{L}}_{eAe} eA, A) \longrightarrow \mathbf{R}\mathrm{Hom}_A(U_A(e), A) \longrightarrow A[1],$$

and $\mathbf{R}\mathrm{Hom}_A(Ae \otimes^{\mathbf{L}}_{eAe} eA, A) \cong \mathbf{R}\mathrm{Hom}_{eAe}(eA, eA)$ by adjointness. Let $V' := \mathbf{R}\mathrm{Hom}_A(U(e), A)$. By definition, $U_A(e)e = 0$. Using adjointness again, this implies $\mathbf{R}\mathrm{Hom}_A(Ae, V') = 0$, that is, $eV' = 0$. Similarly, $eV_A(e) = 0$. It follows that

$$\mathrm{Hom}_{\mathscr{D}(A^{\mathrm{e}}\text{-}\mathsf{Mod})}(V'[i], \mathbf{R}\mathrm{Hom}_{eAe}(eA, eA)) = 0 = \mathrm{Hom}_{\mathscr{D}(A^{\mathrm{e}}\text{-}\mathsf{Mod})}(V_A(e)[i], \mathbf{R}\mathrm{Hom}_{eAe}(eA, eA))$$

for all $i \in \mathbb{Z}$. By Lemma 3.4, the triangles η' and η_e^A are isomorphic, which proves the claim.

By the first part of the proof, $\Delta^\bullet \otimes^{\mathbf{L}}_A \xi_e^A \otimes^{\mathbf{L}}_A \Theta^\bullet \cong \xi_f^B$. Applying Lemma 3.3 (3) shows that $\Delta^\bullet \otimes^{\mathbf{L}}_A \eta_e^A \otimes^{\mathbf{L}}_A \Theta^\bullet \cong \eta_f^B$. □

3.3 Almost ν-stable derived equivalences

Derived equivalences in general fail to preserve homological invariants such as global or dominant dimension. In this respect, stable equivalences of Morita type behave much better. Unfortunately, derived equivalences between algebras that are not self-injective, in general do not induce stable equivalences. The problem of finding derived equivalences, which do induce stable equivalences of Morita type, has been addressed in [20] by introducing a new class of derived equivalences, called *almost* ν*-stable derived equivalences*, and relating them with stable equivalences. As a crucial feature, these derived equivalences preserve many homological invariants.

Definition 3.6 ([20])**.** *Let $\mathcal{F} : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between two k-algebras A and B. We call $\mathcal{F}$ an* almost ν-stable derived equivalence *if the following conditions are satisfied.*

(1) *The radical tilting complex $\mathcal{F}(A) = (\overline{T}^i, \overline{d}^i)_{i\in\mathbb{Z}}$ in $\mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$ has nonzero terms only in positive degrees, that is, $\overline{T}^i = 0$ for all $i < 0$; the radical tilting complex $\mathcal{F}^{-1}(B) = (T^i, d^i)_{i\in\mathbb{Z}}$ in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ has nonzero terms only in negative degrees, that is, $T^i = 0$ for all $i > 0$.*

(2) $\mathrm{add}(\bigoplus_{i<0} T^i) = \mathrm{add}(\bigoplus_{i<0} \nu_A T^i)$ *and* $\mathrm{add}(\bigoplus_{i>0} \overline{T}^i) = \mathrm{add}(\bigoplus_{i>0} \nu_B \overline{T}^i)$.

Note that we may assume without loss of generality that the tilting complex $\mathcal{F}(A)$ is radical by Lemma 3.1. The two conditions for $T^\bullet$ are equivalent to those for $\overline{T}^\bullet$ [20]. To generalise this type of derived equivalence, but to keep many interesting properties, the following *iterated almost* ν*-stable derived equivalences* have been introduced.

Definition 3.7 ([19])**.** *Let $\mathcal{F} : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between two k-algebras A and B. We call $\mathcal{F}$ an* iterated almost ν-stable derived equivalence*, if there exists a sequence of derived equivalences $\mathcal{F}_i : \mathscr{D}(A_i) \cong \mathscr{D}(A_{i+1})$ of k-algebras A_i $(0 \leqslant i \leqslant N)$ for some $N \in \mathbb{N}$ with $A_0 = A$ and $A_{N+1} = B$ such that each $\mathcal{F}_i$ or $\mathcal{F}_i^{-1}$ is an almost ν-stable derived equivalence and $\mathcal{F} \cong \mathcal{F}_N \circ \cdots \circ \mathcal{F}_0$.*

In Section 4 we will see that all derived equivalences between certain classes of algebras are iterated almost ν-stable derived equivalences. This will make full use of the characterisations developed in [19]. The crucial property in our context is:

Proposition 3.8 ([19])**.** *Let $\mathcal{F} : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ be an iterated almost* ν*-stable derived equivalences between k-algebras. Then* $\mathrm{gldim}(A) = \mathrm{gldim}(B)$ *and* $\mathrm{domdim}(A) = \mathrm{domdim}(B)$.

4 Restriction from Morita algebras to self-injective algebras

Theorem C, to be proved in this section, states that derived equivalences between two algebras restrict to derived equivalences between their associated self-injective centraliser subalgebras, provided the two given algebras have ν-dominant dimension at least one. A subcategory of the bounded homotopy category of projective modules will be defined and shown to be invariant under derived equivalence, under the assumption on ν-dominant dimension. This will be the key ingredient in the proof of Theorem C.

Recall that for an algebra A with $\nu\text{-}\mathrm{domdim}(A) \geqslant 1$ and minimal faithful left module Ae, the algebra $H = H_A = eAe$ is called the *associated self-injective algebra of* A. Then the category $A\text{-}\mathsf{stp}$ is additively generated by Ae. That H is self-injective has been shown in the proof of Lemma 2.6.

Suppose now that A and B are derived equivalent k-algebras. In general, it may happen that $A\text{-}\mathsf{stp} \neq \{0\}$ and $B\text{-}\mathsf{stp} = \{0\}$, and it is not known how to define a non-zero associated self-injective algebra for B in this case. An

example of such a situation has been given, for different reasons, in [35, Section 5], where both algebras have ν-dominant dimension zero. Therefore, it seems difficult to deduce any connection between the associated self-injective algebras of A and B in general without assuming both algebras have ν-dominant dimension at least one.

If we, however, assume that both A and B have ν-dominant dimension at least 1, then Theorem 4.3 below shows that any derived equivalence between A and B restricts to a derived equivalence between their associated self-injective algebras. The main tool for showing is the following subcategory of the homotopy category.

Definition 4.1. *Let A be a k-algebra and ν_A be the Nakayama functor. Define*

$$\mathscr{X}_A := \big\{P^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj}) \mid P^\bullet \cong \nu_A(P^\bullet) \text{ in } \mathscr{D}^{\mathrm{b}}(A)\big\}.$$

Note that for $P^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$, the complex $\nu_A(P^\bullet)$ is defined componentwise. When A has arbitrary ν-dominant dimension, a complex in $\mathscr{X}_A$ need not be isomorphic in $\mathscr{K}(A\text{-}\mathsf{mod})$ to a complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$; this is illustrated by an example below. However:

Proposition 4.2. *Let A be a k-algebra with $\nu\text{-}\operatorname{domdim}(A) \geqslant 1$. Then $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$ is the smallest triangulated full subcategory of $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ that contains $\mathscr{X}_A$ and is closed under taking direct summands. In particular, every complex in $\mathscr{X}_A$ is isomorphic in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ to a complex in the category $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$.*

Proof. Let $\mathsf{thick}(\mathscr{X}_A)$ be the smallest triangulated full subcategory of $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ which contains $\mathscr{X}_A$ and is closed under taking direct summands. Let E be a basic additive generator of $A\text{-}\mathsf{stp}$. Then $\nu_A(E)$ is isomorphic to E in $A\text{-}\mathsf{mod}$, because $\nu_A(E)$ is again basic and strongly projective-injective and has the same number of indecomposable direct summands as E. Therefore E belongs to $\mathscr{X}_A$. Since $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$ is the smallest triangulated full subcategory of $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ which contains E and is closed under taking direct summands, it follows that $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp}) \subseteq \mathsf{thick}(\mathscr{X}_A)$.

To finish the proof, we need to show $\mathscr{X}_A \subseteq \mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$, that is, every radical complex $P^\bullet = (P^i, d^i)_{i\in\mathbb{Z}}$ in $\mathscr{X}_A$ is isomorphic in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ to a complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$. Without loss of generality, we assume that $\inf\{l \mid P^l \neq 0\} = 0$. Let $n = \sup\{r \mid P^r \neq 0\}$. So the complex $P^\bullet$ is of the form

$$\cdots \longrightarrow 0 \longrightarrow P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots \to P^{n-1} \xrightarrow{d^{n-1}} P^n \longrightarrow 0 \longrightarrow \cdots.$$

We will prove by induction on n that $P^\bullet$ is isomorphic in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ to a complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$. If $n = 0$, then $\nu_A(P^\bullet) \cong P^\bullet$ in $\mathscr{D}^{\mathrm{b}}(A)$ implies that $\nu_A(P^0) \cong P^0$ in $A\text{-}\mathsf{mod}$, and so $P^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$. In general, we first prove:

Claim. P^0 is strongly projective-injective.

Proof. Since $P^\bullet$ is a radical complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ and $\nu_A : A\text{-}\mathsf{proj} \to A\text{-}\mathsf{inj}$ is an equivalence, it follows that $\nu_A(P^\bullet)$ is a radical complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{inj})$. Let $f^\bullet = \{f^i\} : P^\bullet \to \nu_A(P^\bullet)$ be a quasi-isomorphism. Then $\nu_A(P^\bullet)$ is an injective resolution of $P^\bullet$. By the construction of Cartan-Eilenberg injective resolutions $P^\bullet$ admits an injective resolution $I^\bullet$ with the properties: $I^\bullet$ is quasi-isomorphic to $P^\bullet$, and $I^i = 0$ for $i < 0$ and I^0 is the injective envelope of P^0. By the uniqueness of injective resolutions up to homotopy, the radical complex $\nu_A(P^\bullet)$ and the complex $I^\bullet$ are isomorphic in $\mathscr{K}(A\text{-}\mathsf{inj})$, and therefore $\nu_A(P^0)$ is a direct summand of I^0 by Lemma 3.1. Since $\nu\text{-}\operatorname{domdim}(A) \geqslant 1$, the injective envelope I^0 of P^0 is strongly projective-injective. It follows that $\nu_A(P^0)$ and hence P^0 are strongly projective-injective as well.

Claim. $f^0 : P^0 \to \nu_A(P^0)$ *is an isomorphism of A-modules.*

Proof. Let $\operatorname{Cone}(f^\bullet)$ be the mapping cone of $f^\bullet$, a complex of the form

$$\operatorname{Cone}(f^\bullet): \qquad 0 \longrightarrow P^0 \xrightarrow{[-d^0, f^0]} P^1 \oplus \nu_A(P^0) \longrightarrow \cdots \longrightarrow \nu_A(P^n) \longrightarrow 0$$

where P^0 is placed in degree -1. Since $f^\bullet$ is a quasi-isomorphism, it follows that $\operatorname{Cone}(f^\bullet)$ is an acyclic complex, and thus the morphism $[-d^0, f^0] : P^0 \to P^1 \oplus \nu_A(P^0)$ splits in $A\text{-}\mathsf{mod}$ because P^0 is injective. Let $u : P^1 \longrightarrow P^0$ and $v : \nu_A(P^0) \longrightarrow P^0$ be morphisms in $A\text{-}\mathsf{mod}$ such that $-d^0u + f^0v = 1_{P^0}$. Then $f^0v = 1 + d^0u$ is invertible in $\operatorname{End}_A(P^0)$ by the assumption that d^0 is a radical morphism. As a result, f^0 is a split monomorphism, and even an isomorphism as P^0 and $\nu_A(P^0)$ have the same number of indecomposable direct summands.

Now, let $P^\bullet_{\geqslant 1} := \sigma_{\geqslant 1}(P^\bullet)$ be the brutal truncation of $P^\bullet$ and let $f^\bullet_{\geqslant 1} : P^\bullet_{\geqslant 1} \longrightarrow \nu_A(P^\bullet_{\geqslant 1})$ be the corresponding truncation of the chain morphism $f^\bullet$. Starting from the triangle $P^0[-1] \longrightarrow P^\bullet_{\geqslant 1} \longrightarrow P^\bullet \longrightarrow P^0$, the following commutative diagram in $\mathscr{D}(A)$ gives a morphism of triangles in $\mathscr{D}(A)$

$$\begin{array}{ccccccc}
P^0[-1] & \longrightarrow & P^\bullet_{\geqslant 1} & \longrightarrow & P^\bullet & \longrightarrow & P^0 \\
\downarrow{\scriptstyle f^0[-1]} & & \downarrow{\scriptstyle f^\bullet_{\geqslant 1}} & & \downarrow{\scriptstyle f^\bullet} & & \downarrow{\scriptstyle f^0} \\
\nu_A(P^0)[-1] & \longrightarrow & \nu_A(P^\bullet_{\geqslant 1}) & \longrightarrow & \nu_A(P^\bullet) & \longrightarrow & \nu_A(P^0).
\end{array}$$

Since $f^\bullet$ is a quasi-isomorphism (by assumption) and so is f^0 (by the arguments above), it follows that $f^\bullet_{\geqslant 1}$ is a quasi-isomorphism. Therefore $P^\bullet_{\geqslant 1} \in \mathscr{X}_A$, and by induction $P^\bullet_{\geqslant 1}$ is isomorphic in $\mathscr{K}(A)$ to a complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$. Using $P^0 \in A\text{-}\mathsf{stp}$ implies that $P^\bullet$ is isomorphic in $\mathscr{K}(A)$ to a complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$. □

Example. Without the assumption on ν-dominant dimension, Proposition 4.2 may fail in general: Let A be the k-algebra given by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightleftarrows}} 3$$

and relations $\{\beta\delta, \alpha\beta\alpha, \delta\gamma\delta, \alpha\beta - \delta\gamma\}$. The indecomposable projective left A-modules are

$$P_1 = \begin{array}{ccc} & 1 & \\ & 2 & \\ 1 & & 3 \end{array} \qquad P_2 = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \\ & & 3 \end{array} \qquad P_3 = \begin{array}{c} 3 \\ 2 \\ 3 \end{array}$$

The indecomposable injective left A-modules are

$$I_1 = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \qquad I_2 = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array} \qquad I_3 = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \\ & & 3 \end{array}$$

Let $P^\bullet$ be the complex $0 \to P_1 \xrightarrow{d} P_2 \to 0$, where d is the unique (up to scalar) non-zero map, and P_1 is placed in degree zero. Then $\nu_A(P^\bullet)$ is a complex of the form $0 \to I_1 \to I_2 \to 0$. The obvious surjective maps $P_1 \to I_1$ and $P_2 \to I_2$ define a chain map from $P^\bullet$ to $\nu_A(P^\bullet)$ and is a quasi-isomorphism. As a result, $P^\bullet \in \mathscr{X}_A$, but $P^\bullet$ does not belong to $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$, since $A\text{-}\mathsf{stp} = \{0\}$.

Theorem 4.3. *Let A and B be derived equivalent k-algebras, both of ν-dominant dimension at least 1. Then any derived equivalence $\mathcal{F} : \mathscr{D}^{\mathrm{b}}(A) \xrightarrow{\sim} \mathscr{D}^{\mathrm{b}}(B)$ restricts to an equivalence $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp}) \xrightarrow{\sim} \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{stp})$ of triangulated subcategories.*

Proof. The equivalence $\mathcal{F}$ induces $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj}) \xrightarrow{\sim} \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$ as triangulated subcategories, and

$$\mathcal{F}(\nu_A(P^\bullet)) \cong \nu_B(\mathcal{F}(P^\bullet))$$

in $\mathscr{D}^{\mathrm{b}}(B)$, for any $P^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$. Thus $\mathcal{F}(\mathscr{X}_A) \subseteq \mathscr{X}_B$ and so $\mathcal{F}(\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})) \subseteq \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{stp})$ by Proposition 4.2. Let $\mathcal{G}$ be a quasi-inverse of $\mathcal{F}$. The same arguments applied to $\mathcal{G}$ imply $\mathcal{G}(\mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{stp})) \subseteq \mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$. Therefore, $\mathcal{F}$ induces an equivalence $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp}) \xrightarrow{\sim} \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{stp})$. □

Remark. Theorem 4.3 has two predecessors: A special case, stated only for gendo-symmetric algebras, was proved in [17], where it was used to relate the Hochschild cohomology of A and that of its associated self-injective algebra. A result in the same spirit as Theorem 4.3 was obtained in [20] without any restriction on algebras, but assuming $\mathcal{F}$ to be an (iterated) almost ν-stable derived equivalence.

Corollary 4.4 (Theorem C)**.** *Let A and B be derived equivalent k-algebras, both of ν-dominant dimension at least 1. Then the associated self-injective algebras of A and B are derived equivalent. In particular, every derived equivalence of Morita algebras induces a derived equivalence of their associated self-injective algebras.*

Proof. Let H be an associated self-injective algebra of A. Then by definition $A\text{-}\mathsf{stp} \cong H\text{-}\mathsf{proj}$ as additive categories, and thus $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp}) \cong \mathscr{K}^{\mathrm{b}}(H\text{-}\mathsf{proj})$ as triangulated categories. The statement then follows directly from Theorem 4.3. □

An application of Corollary 4.4 is to Auslander algebras: A (finite dimensional) k-algebra A is said to be of *finite representation type*, if there are only finitely many indecomposable A-modules (up to isomorphism). The *Auslander algebra* Γ_A of A is defined to be the endomorphism ring of the direct sum of all pairwise non-isomorphic indecomposable left A-modules.

Corollary 4.5. *Let A and B be self-injective k-algebras, both of finite representation type. Let Γ_A and Γ_B be the Auslander algebras of A and B respectively. Then*

(1) *Γ_A and Γ_B are derived equivalent if and only if A and B are derived equivalent.*

(2) *If Γ_A and Γ_B are derived equivalent, then they are stably equivalent of Morita type.*

Proof. (1) If A and B are derived equivalent, then by [22, Corollary 3.13], their Auslander algebras Γ_A and Γ_B are derived equivalent. Conversely, if Γ_A and Γ_B are derived equivalent, then by Corollary 4.4 A and B are derived equivalent, since A and B are the associated self-injective algebras of Γ_A and Γ_B respectively.

(2) If Γ_A and Γ_B are derived equivalent, then A and B are derived equivalent by (1), and thus stably equivalent of Morita type by [36, Corollary 5.5]. Now [27, Theorem 1.1] implies that Γ_A and Γ_B are stably equivalent of Morita type. □

5 Invariance of homological dimensions

In this section, Theorem A and further invariance results will be proven. First, for almost self-injective algebras the approach is to show that derived equivalences have a special form; they are iterated almost ν-stable and therefore preserve both global and dominant dimension. For the algebras we are mainly interested in, algebras with an anti-automorphism preserving simples, a new and rather different approach is needed, identifying the two homological dimensions inside the derived category. Before addressing invariance, in the first subsection the general question is addressed how much homological dimensions can vary under derived equivalences.

5.1 Variance of homological dimensions under derived equivalences

As it is well-known, the difference of global dimensions of two derived equivalent algebras is bounded by the length of a tilting complex inducing a derived equivalence, see for example [18, Section 12.5(b)]. More precisely, let A be a k-algebra, and define the *length* of a radical complex $X^\bullet$ in $\mathscr{K}^{\mathrm{b}}(A)$ to be

$$\ell(X^\bullet) = \sup\{t \mid X^t \neq 0\} - \inf\{b \mid X^b \neq 0\} + 1.$$

The length of an arbitrary complex $Y^\bullet$ in $\mathscr{K}^{\mathrm{b}}(A)$ is defined to be the length of the unique radical complex that is isomorphic to $Y^\bullet$ in $\mathscr{K}^{\mathrm{b}}(A)$ (Lemma 3.1).

Proposition 5.1 ([18, Section 12.5(b)])**.** *Let $\mathcal{F} : \mathscr{D}^{\mathrm{b}}(A) \xrightarrow{\sim} \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between k-algebras. Then* $|\operatorname{gldim}(A) - \operatorname{gldim}(B)| \leqslant \ell(\mathcal{F}(A)) - 1$.

Naively, one may expect that dominant dimension behaves similarly under derived equivalences. Here is a counterexample: Let $n \geqslant 2$ be an integer, and let A be the k-algebra given by the quiver

$$1 \xleftarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xleftarrow{\alpha_3} \cdots \xleftarrow{\alpha_{2n-1}} 2n \xleftarrow{\alpha_{2n}} 2n+1$$

and relations $\alpha_i\alpha_{i+1} = 0$ $(1 \leqslant i \leqslant 2n-1, i \neq n)$. Let S_i denote the simple left A-module corresponding to the vertex i and P_i be the projective cover of S_i. The modules P_i are projective-injective for $i \neq 1, n+1$. The projective dimensions of the simple modules are $\operatorname{pd} S_1 = 0$, $\operatorname{pd} S_i = \operatorname{pd} S_{i+n} = i-1$ for $2 \leqslant i \leqslant n+1$, and the minimal injective resolution of the left regular A-module is of the form

$$0 \to A \to (\bigoplus_{i \neq 1, n+1} P_i) \oplus P_2 \oplus P_{n+2} \to P_3 \oplus P_{n+3} \to \cdots \to P_n \oplus P_{2n} \to P_{n+2} \oplus P_{2n+1}$$
$$\to S_{2n+1} \oplus P_{n+2}/S_n \to 0$$

Consequently, $\operatorname{domdim}(A) = \operatorname{gldim}(A) = n$. Let $T := \tau^{-1}S_1 \oplus P_2 \oplus \cdots P_{2n+1}$ be the APR-tilting module (see, for instance, [4, VI.2.8]) associated with the projective simple A-module S_1, and let $B = \operatorname{End}_A(T)$. Then $\operatorname{pd} T = 1$ and therefore the derived equivalence between A and B induced by T is given by a two-term tilting complex. By direct computation, B is seen to be isomorphic to the k-algebra given by the same quiver as A but with different relations $\alpha_i\alpha_{i+1} = 0$ for $2 \leqslant i \leqslant 2n$, $i \neq n$. As a result, $\operatorname{domdim}(B) = 1$ and the difference between dominant dimensions of A and B is $(n-1)$, although the derived equivalence is induced by a tilting module.

Although this example smashes any hope to bound the difference of dominant dimensions of derived equivalent algebras in terms of lengths of tilting complexes, there are still some cases where both global dimension and dominant dimension behave nicely, see [20, 19] and [17]. Note that both algebras in the example above are of ν-dominant dimension 0. This suggests to restrict attention to algebras of ν-dominant dimension at least 1 - a restriction that has been needed for Theorem 4.3 and that will be further justified by the invariance results later on.

Theorem 5.2. *Let A and B be k-algebras, both of ν-dominant dimension at least 1 and derived equivalent by $\mathcal{F} : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$. Then* $|\operatorname{domdim}(A) - \operatorname{domdim}(B)| \leqslant \ell(\mathcal{F}(A)) - 1$.

Proof. Let $m = \operatorname{domdim}(B)$ and $n = \ell(\mathcal{F}(A)) - 1$. Since $\ell(\mathcal{F}^{-1}(B)) = \ell(\mathcal{F}(A)) = n+1$ by [20, Lemma 2.1], it is enough to show that $\operatorname{domdim}(A) \geqslant m - n$. If $m \leqslant n+1$, there is nothing left to prove. Assume that $\mathcal{F}$ is a standard derived equivalence, $m > n+1$ and $P^\bullet = \mathcal{F}(A)$ is a radical tilting complex in $\mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$ of the following form (up to degree shift)

$$0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \cdots \longrightarrow P^{n-2} \longrightarrow P^{n-1} \longrightarrow P^n \longrightarrow 0$$

where P^0 is nonzero and placed in degree 0. Take a Cartan-Eilenberg injective resolution $I^\bullet$ of $P^\bullet$ that is the total complex of the double complex obtained by taking minimal injective resolutions of P^i for all i. Since $\nu\text{-}\operatorname{domdim}(B) \geqslant 1$ and $m \geqslant n+1$, the modules I^i are strongly projective-injective for $0 \leqslant i \leqslant m-1$ by Lemma 2.6. Let $I^\bullet_{<m}$ and $I^\bullet_{\geqslant m}$ be the brutal truncations of $I^\bullet$. By definition of brutal truncation, there is a triangle in $\mathscr{D}(B)$

$$(*) \qquad I^\bullet_{<m}[-1] \longrightarrow I^\bullet_{\geqslant m} \longrightarrow I^\bullet \longrightarrow I^\bullet_{<m}.$$

The standard derived equivalence $\mathcal{F}$ lifts to a derived equivalence (denoted by $\mathcal{F}$ again) from $\mathscr{D}(A)$ to $\mathscr{D}(B)$. Applying $\mathcal{F}^{-1}$ to the triangle $(*)$, we obtain the triangle in $\mathscr{D}(A)$

$$\mathcal{F}^{-1}(I^{\bullet}_{<m})[-1] \longrightarrow \mathcal{F}^{-1}(I^{\bullet}_{\geqslant m}) \longrightarrow \mathcal{F}^{-1}(I^{\bullet}) \longrightarrow \mathcal{F}^{-1}(I^{\bullet}_{<m}) \tag{2}$$

and therefore the long exact sequence

$$\cdots \to \mathrm{H}^{i-1}(\mathcal{F}^{-1}(I^{\bullet}_{<m})) \to \mathrm{H}^{i}(\mathcal{F}^{-1}(I^{\bullet}_{\geqslant m})) \to \mathrm{H}^{i}(\mathcal{F}^{-1}(I^{\bullet})) \to \mathrm{H}^{i}(\mathcal{F}^{-1}(I^{\bullet}_{<m})) \to \cdots \tag{3}$$

Note that $\mathcal{F}^{-1}(I^{\bullet}) \cong \mathcal{F}^{-1}(P^{\bullet}) \cong \mathcal{F}^{-1} \circ \mathcal{F}(A) \cong A$ in $\mathscr{D}(A)$, and $\mathcal{F}^{-1}(I^{\bullet}_{<m})$ belongs to $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$ by the construction of $I^{\bullet}_{<m}$ and Theorem 4.3. Moreover, Lemma 3.1 implies for $i \leqslant m-n-1$,

$$\mathrm{H}^{i}(\mathcal{F}^{-1}(I^{\bullet}_{\geqslant m})) \cong \mathrm{Hom}_{\mathscr{D}(A)}(A, \mathcal{F}^{-1}(I^{\bullet}_{\geqslant m})[i]) \cong \mathrm{Hom}_{\mathscr{D}(B)}(P^{\bullet}, I^{\bullet}_{\geqslant m}[i]) = 0$$

since $(I^{\bullet}_{\geqslant m}[i])^{p} = 0$ for $p \leqslant n$ and $i \leqslant m-n-1$. Therefore, from the long exact sequence (3), $\mathrm{H}^{i}(\mathcal{F}^{-1}(I^{\bullet}_{<m})) = 0$ for $i < 0$, $\mathrm{H}^{0}(\mathcal{F}^{-1}(I^{\bullet}_{<m})) \cong A$, and $\mathrm{H}^{i}(\mathcal{F}^{-1}(I^{\bullet}_{<m})) \cong \mathrm{H}^{i+1}(\mathcal{F}^{-1}(I^{\bullet}_{\geqslant m})) = 0$ for $1 \leqslant i \leqslant m-n-2$. Hence $\mathcal{F}^{-1}(I^{\bullet}_{<m})$ is isomorphic to a radical complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$ of the form

$$0 \longrightarrow E^{0} \longrightarrow E^{1} \longrightarrow \cdots \longrightarrow E^{m-n-2} \longrightarrow E^{m-n-1} \longrightarrow \cdots$$

such that $0 \to A \to E^{0} \to E^{1} \to \cdots \to E^{m-n-2} \to E^{m-n-1}$ is exact. Consequently the dominant dimension of A is at least $m-n$, as desired. □

A special case of Theorem 5.2 is:

Corollary 5.3. *Let A and B be Morita algebras. If there is a derived equivalence $\mathcal{F} : \mathscr{D}^{\mathrm{b}}(A) \xrightarrow{\sim} \mathscr{D}^{\mathrm{b}}(B)$, then $|\,\mathrm{domdim}(A) - \mathrm{domdim}(B)| \leqslant \ell(\mathcal{F}(A)) - 1$.*

A special case of Corollary 5.3, where $\mathcal{F}$ is induced by a tilting module, was proved in [6].

5.2 Almost self-injective algebras

Interactions between derived equivalences and stable equivalences frequently seem to be of particular interest, see [20, 23] and the references therein. [36, Corollary 5.5] and Corollary 4.5 state that, for self-injective algebras and Auslander algebras of finite representation type self-injective algebras, derived equivalences imply stable equivalences of Morita type. The following theorem implies that the same holds for almost self-injective algebras, by characterising all derived equivalences among them as iterated almost ν-stable derived equivalences.

Theorem 5.4. *Let A and B be derived equivalent almost self-injective algebras. Then every derived equivalence $\mathcal{F} : \mathscr{D}^{\mathrm{b}}(A) \xrightarrow{\sim} \mathscr{D}^{\mathrm{b}}(B)$ is an iterated almost ν-stable derived equivalence (up to shifts). In particular, derived equivalent almost self-injective algebras are stably equivalent of Morita type.*

To prove this theorem, we need some preparations. First, we recall some basics on $\mathcal{D}$-split sequences introduced in [21], see also [22]. Let $\mathcal{A}$ be an additive category and let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$. A morphism $f : X \to M$ in $\mathcal{A}$ is a *right $\mathcal{X}$-approximation*, if $X \in \mathcal{X}$ and for any $X' \in \mathcal{X}$, the canonical morphism $\mathrm{Hom}_{\mathcal{A}}(X', X) \to \mathrm{Hom}_{\mathcal{A}}(X', M)$ is an epimorphism. We call f *right minimal* if an equality $\alpha \cdot f = f$ implies that α is an isomorphism, for $\alpha \in \mathrm{End}_{\mathcal{A}}(X)$. Left $\mathcal{X}$-approximations and left minimal morphisms are defined similarly. Let $\mathcal{C}$ be a triangulated category and let $\mathcal{D}$ be a full (not necessarily triangulated) additive subcategory of $\mathcal{C}$. A triangle in $\mathcal{C}$

$$X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} X[1]$$

is called a *$\mathcal{D}$-split triangle* if f is a left $\mathcal{D}$-approximation and g is a right $\mathcal{D}$-approximation. A full subcategory $\mathcal{T}$ of $\mathcal{C}$ is called a *tilting subcategory* if $\mathrm{Hom}_{\mathcal{C}}(\mathcal{T}, \mathcal{T}[i]) = 0$ for all $i \neq 0$ and $\mathcal{C}$ itself is the only triangulated subcategory of $\mathcal{C}$ that contains $\mathcal{T}$ and is closed under taking direct summands. An object T in $\mathcal{C}$ is a *tilting object* if $\mathrm{add}(T)$ is a tilting subcategory of $\mathcal{C}$. For example, all tilting complexes over an algebra A are tilting objects in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$.

Lemma 5.5. *Let $\mathcal{C}$ be a triangulated category, and $\mathcal{D}$ an additive full subcategory of $\mathcal{C}$. Let $X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} X[1]$ be a $\mathcal{D}$-split triangle. Then:*

(1) *Suppose that f is left minimal and g is right minimal, and that $X \cong \bigoplus_{i=1}^{n} X_i$ and $Y \cong \bigoplus_{i=1}^{m} Y_i$ are decompositions of X and Y into strongly indecomposable direct summands. Then $m = n$. In particular, if indecomposable objects in $\mathcal{C}$ are strongly indecomposable, then X is indecomposable if and only if so is Y.*

(2) *If $\mathcal{D} \cup \{X\}$ is a tilting subcategory of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}}(\mathcal{D}, f)$ is injective, then $\mathcal{D} \cup \{Y\}$ is also a tilting subcategory of $\mathcal{C}$.*

(3) *If $\mathcal{D} \cup \{Y\}$ is a tilting subcategory of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}}(g, \mathcal{D})$ is injective, then $\mathcal{D} \cup \{X\}$ is also a tilting subcategory of $\mathcal{C}$.*

Proof. (2) and (3) follow from [1, Theorem 2.32]. It remains to prove (1). If $D \cong 0$, then $Y \cong X[1]$ and we are done. Now we assume that $D \ncong 0$. Then $X \ncong 0$ and $Y \ncong 0$, since otherwise $f = 0$ or $g = 0$, which contradicts the minimality of f and g. To proceed, let $I_X = \{\alpha \in \mathrm{End}_{\mathcal{C}}(X) \mid \alpha \text{ factors through } f\}$ and $I_Y = \{\gamma \in \mathrm{End}_{\mathcal{C}}(Y) \mid \gamma \text{ factors through } g\}$.

Claim. I_X is a two-sided ideal of $\mathrm{End}_{\mathcal{C}}(X)$ and it is contained in the Jacobson radical J_X of $\mathrm{End}_{\mathcal{C}}(X)$.

Proof. For any $\alpha \in I_X$ and $\theta \in \mathrm{End}_{\mathcal{C}}(X)$, there exist $u \in \mathrm{Hom}_{\mathcal{C}}(D,X)$ with $\alpha = f \cdot u$, and $\omega \in \mathrm{End}_{\mathcal{C}}(D)$ with $\theta \cdot f = f \cdot \omega$ since f is a left $\mathcal{D}$-approximation. Therefore, $\theta \cdot \alpha = \theta \cdot f \cdot u = f \cdot (\omega \cdot u)$, which implies $\theta \cdot \alpha \in I_X$. Similarly $\alpha \cdot \theta \in I_X$ and so I_X is a two-sided ideal of $\mathrm{End}_{\mathcal{C}}(X)$. To see $I_X \subseteq J_X$, it suffices to show, by [3, Theorem 15.3, p. 166], that $1-\alpha$ is invertible in $\mathrm{End}_{\mathcal{C}}(X)$ for each $\alpha \in I_X$. Let $\alpha = f \cdot u$ for some $u \in \mathrm{Hom}_{\mathcal{C}}(D,X)$. In the diagram in $\mathcal{C}$

$$\begin{array}{ccccccc} Y[-1] & \xrightarrow{-h[-1]} & X & \xrightarrow{f} & D & \xrightarrow{g} & Y \\ \| & & \downarrow{\scriptstyle 1-\alpha} & & \downarrow{\scriptstyle \beta} & & \| \\ Y[-1] & \xrightarrow{-h[-1]} & X & \xrightarrow{f} & D & \xrightarrow{g} & Y \end{array}$$

the first square commutes because $h[-1] \cdot \alpha = h[-1] \cdot f \cdot u = 0$, and β exists so that the other squares commute by axioms of triangulated categories. Since g is a right minimal morphism, β is an isomorphism, and thus again by axioms of triangulated categories, $1-\alpha$ is an isomorphism.

By similar arguments, I_Y is a two-sided ideal of $\mathrm{End}_{\mathcal{C}}(Y)$ and it is contained in the Jacobson radical J_Y of $\mathrm{End}_{\mathcal{C}}(Y)$.

Claim. There is an algebra isomorphism $\mathrm{End}_{\mathcal{C}}(X)/I_X \cong \mathrm{End}_{\mathcal{C}}(Y)/I_Y$.

Proof. We first construct a ring homomorphism $\phi : \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(Y)/I_Y$ as follows. For each $\alpha \in \mathrm{End}_{\mathcal{C}}(X)$, there exists a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccccc} X & \xrightarrow{f} & D & \xrightarrow{g} & Y & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha[1]} \\ X & \xrightarrow{f} & D & \xrightarrow{g} & Y & \xrightarrow{h} & X[1] \end{array}$$

where β exists since f is a left $\mathcal{D}$-approximation, and γ exists by axioms of triangulated categories. If $\beta' \in \mathrm{End}_{\mathcal{C}}(D)$ and $\gamma' \in \mathrm{End}_{\mathcal{C}}(Y)$ are different choices such that $f \cdot \beta' = \alpha \cdot f$ and $g \cdot \gamma' = \beta' \cdot g$, then $(\gamma - \gamma') \cdot h = 0$ which implies that $\gamma - \gamma' = u \cdot g$ for some $u \in \mathrm{Hom}_{\mathcal{C}}(Y,D)$. In other words, the image of γ in the quotient ring $\mathrm{End}_{\mathcal{C}}(Y)/I_Y$ is well-defined, and thus $\phi(\alpha)$ is well-defined. Then ϕ is a ring homomorphism. It is surjective since g is a right $\mathcal{D}$-approximation.

Claim. The kernel of ϕ equals I_X.

Proof. For any $\alpha \in I_X$, there exists $v \in \mathrm{Hom}_{\mathcal{C}}(D,X)$ with $\alpha = f \cdot v$. So $\phi(\alpha) \cdot h = h \cdot \alpha[1] = h \cdot f[1] \cdot v[1] = 0$ which implies that $\phi(\alpha)$ factors through g and thus $I_X \subseteq \ker(\phi)$. On the other hand, for any $\alpha \in \ker(\phi)$, there exists $u \in \mathrm{Hom}_{\mathcal{C}}(Y,D)$ with $\phi(\alpha) = u \cdot g$, and so $h[-1] \cdot \alpha = \phi(\alpha)[-1] \cdot h[-1] = (u \cdot g \cdot h)[-1] = 0$ which implies that α factors through f and thus $\ker(\phi) \subseteq I_X$.

Altogether, ϕ induces an isomorphism $\mathrm{End}_{\mathcal{C}}(X)/I_X \cong \mathrm{End}_{\mathcal{C}}(Y)/I_Y$, and hence further an isomorphism

$$\mathrm{End}_{\mathcal{C}}(X)/J_X \cong \mathrm{End}_{\mathcal{C}}(Y)/J_Y.$$

Since both X and Y are decomposed into strongly indecomposable direct summands, it follows that both rings $\mathrm{End}_{\mathcal{C}}(X)$ and $\mathrm{End}_{\mathcal{C}}(Y)$ are semi-perfect rings by [3, Theorem 27.6(b), p. 304], and therefore the isomorphism above implies that $\mathrm{End}_{\mathcal{C}}(X)$ and $\mathrm{End}_{\mathcal{C}}(Y)$ have the same number of simple left modules, or equivalently the same number of indecomposable projective left modules. Consequently, X and Y have the same number of indecomposable direct summands, that is $m = n$. □

Lemma 5.6. *Let A be an almost self-injective algebra, but not self-injective. Let E be an additive generator of A-stp and let P be the unique indecomposable projective left A-module such that A-proj $= \mathrm{add}(P \oplus E)$. Let $T^\bullet = (T^i, d^i)$ be an indecomposable radical complex in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$. If $T^\bullet \oplus E$ is a tilting complex over A, then at least one of the following two assertions holds true:*

(1) *There exists $r \leqslant 0$ such that $T^r \cong P$, $T^i = 0$ for all $i > 0$ and for all $i < r$, and $T^i \neq 0$ in A-stp for all $r < i \leqslant 0$;*

(2) *There exists $s \geqslant 0$ such that $T^s \cong P$, $T^i = 0$ for all $i < 0$ and for all $i > s$, and $T^i \neq 0$ in A-stp for all $0 \leqslant i < s$.*

Proof. Special case. The complex $T^\bullet$ has only one nonzero term, that is, $T^\bullet = Q[m]$ for some $Q \in A$-proj and $m \in \mathbb{Z}$. Then Q is indecomposable since $T^\bullet$ is indecomposable. If $Q \in A$-stp, then $T^\bullet \oplus E \in \mathscr{K}^{\mathrm{b}}(A\text{-stp}) \subsetneq \mathscr{K}^{\mathrm{b}}(A\text{-proj})$ which contradicts the assumption that $T^\bullet \oplus E$, as a tilting complex, generates $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$. Since each indecomposable projective left A-module is either isomorphic to P or strongly projective-injective, by the definition of almost self-injective algebras, Q must be isomorphic to P. We still have to show that $m = 0$. Assume $m \neq 0$. Then by the

self-orthogonality of tilting complexes, $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})}(P[m]\oplus E, E[m]) = 0$, which implies $\mathrm{Hom}_A(P,E) = 0$, a contradiction to $\nu\text{-}\mathrm{domdim}(A) \geqslant 1$. So, $T^\bullet = P$, which satisfies both conditions (1) and (2).

In the *general case*, $T^\bullet = (T^i, d^i)$ is an indecomposable radical complex of the following form

$$0 \longrightarrow T^r \xrightarrow{d^r} T^{r+1} \longrightarrow \cdots \xrightarrow{d^{s-1}} T^s \longrightarrow 0$$

where T^i are nonzero projective left A-modules and $r < s$. Using the self-orthogonality of the tilting complex $T^\bullet \oplus E$, we are going to check:

Claim. (a) *T^r and T^s have no nonzero common direct summands;*
(b) *if $r \neq 0$, then $T^r \in \mathrm{add}(P)$; and*
(c) *if $s \neq 0$, then $T^s \in \mathrm{add}(P)$.*

Proof. To see (a), assume on the contrary that K is a nonzero common direct summand of both T^r and T^s. Let u be a split epimorphism from T^r to K and v be a split monomorphism from K to T^s. Then the composition $u \cdot v$ from T^r to T^s defines a nonzero morphism in $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})}(T^\bullet, T^\bullet[s-r])$, because $T^\bullet$ is a radical complex. But this contradicts the assumption that $T^\bullet$ is self-orthogonal and $s-r > 0$, which forces any morphism from $T^\bullet$ to $T^\bullet[s-r]$ in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ to be zero.

Similarly, (b) and (c) follow since for any $m \neq 0$,

$$\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})}(T^\bullet, E[m]) = 0 = \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})}(E, T^\bullet[m]).$$

As a consequence of the claim, $s = 0$ or $r = 0$. Indeed, if $r \neq 0$ and $s \neq 0$, then by (b) and (c), T^r and T^s have a common direct summand P, which contradicts (a).

We will finish the proof by analysing these two cases.

Case $r < s = 0$. By (b), $T^r \in \mathrm{add}(P)$ and then $T^0 \in \mathrm{add}(E)$ by (a). Let $T^\bullet_{<0}$ be the brutal truncation of $T^\bullet$, which by definition provides the following triangle in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$:

$$T^\bullet_{<0}[-1] \xrightarrow{f} T^0 \xrightarrow{g} T^\bullet \longrightarrow T^\bullet_{<0} \tag{4}$$

where f is the chain morphism induced by $d^{-1} : T^{-1} \to T^0$, and g is the chain morphism induced by $\mathrm{id} : T^0 \to T^0$. Applying $\mathrm{Hom}_{\mathscr{K}(A)}(-, E)$ to this triangle gives the short exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, E) \xrightarrow{g^*} \mathrm{Hom}_{\mathscr{K}(A)}(T^0, E) \xrightarrow{f^*} \mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet_{<0}[-1], E) \to 0$$

since $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet_{<0}, E) = 0$ trivially and $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet[-1], E) = 0$ by $T^\bullet \oplus E$ being tilting. In particular, g^* is injective, and f is a left $\mathrm{add}(E)$-approximation.

Claim. The morphism f is left minimal.

Proof. For any $u : T^0 \to T^0$ with $f \cdot u = f$, we have $d^{-1} \cdot u = d^{-1}$. Thus u defines a chain morphism $\boldsymbol{\gamma}^\bullet : T^\bullet \to T^\bullet$ in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ with $\gamma^0 = u$ and $\gamma^i = \mathrm{id}$ for all $i \neq 0$. Now $T^\bullet$ being an indecomposable complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ implies that $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})}(T^\bullet)$ is a local k-algebra, and therefore $\boldsymbol{\gamma}^\bullet$ is either nilpotent or invertible. If $\boldsymbol{\gamma}^\bullet$ is nilpotent in $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})}(T^\bullet)$, say $(\boldsymbol{\gamma}^\bullet)^m = 0$ for some $m \in \mathbb{Z}$, then there exists a homotopy morphism $s^\bullet = \{s^i\} : T^\bullet \to T^\bullet[-1]$ such that $(\gamma^i)^m = s^i \cdot d^{i-1} + d^i \cdot s^{i+1}$. By definition of $T^\bullet$, the map d^{r-1} is zero. Thus, $\mathrm{id} = d^r \cdot s^{r+1}$ and hence d^r is a split monomorphism. But this contradicts $T^\bullet$ being a radical complex. It follows that f is a left minimal morphism.

Applying $\mathrm{Hom}_{\mathscr{K}(A)}(E, -)$ to the triangle (4) implies that g is a right $\mathrm{add}(E)$-approximation. Since g induces an isomorphism $\mathrm{Hom}_{\mathscr{K}(A)}(T^0, T^\bullet) \cong \mathrm{Hom}_A(T^0, T^0)$, g is a right minimal morphism. Consequently, the triangle (4) is a $\mathrm{add}(E)$-split sequence, with g^* being an injective morphism. Thus, by Lemma 5.5 (1) and (3), $T^\bullet_{<0}$ is an indecomposable radical complex, and $T^\bullet_{<0}[-1] \oplus E$ is a tilting complex over A. Replacing $T^\bullet$ by $T^\bullet_{<0}[-1]$ and repeating the same arguments $(r-1)$ times, we finally get that T^r is indecomposable and $T^r \oplus E$ is a tilting complex over A, whereby T^r is isomorphic to P and the assertion (1) holds.

Case $0 = r < s$. By similar arguments as in the previous case, the assertion (2) in the statement can be verified for $T^\bullet$. Since the case $r = s = 0$ has been shown already, all cases have been settled. □

With these preparations, we are now able to prove Theorem 5.4.

Proof of Theorem 5.4. First we assume that one of A or B is self-injective. When the ground field is algebraically closed, being self-injective is invariant under derived equivalence, by [2]. Under our assumption $\nu\text{-}\mathrm{domdim} \geqslant 1$, invariance of being self-injective can be shown as follows, without any restriction on the ground field. By assumption, both A and B have $\nu\text{-}\mathrm{domdim} \geqslant 1$. Then A is self-injective if and only if $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj}) = \mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp})$, and similarly for B. By Theorem 4.3, the given derived equivalence $\mathcal{F}$ induces an equivalence $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{stp}) \xrightarrow{\sim} \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{stp})$ of triangulated

subcategories. Hence, both A and B are self-injective. Now, [21, Proposition 3.8] implies that $\mathcal{F}$ is an almost ν-stable derived equivalence up to degree shift.

Next, we assume that neither A nor B is self-injective. Let E_A and E_B be basic additive generators of A-stp and B-stp respectively. Then the number of indecomposable direct summands of E_B is exactly one less than the number of the simple left B-modules. Since $\nu\text{-domdim}(A) \geqslant 1$ and $\nu\text{-domdim}(B) \geqslant 1$, the derived equivalence $\mathcal{F}$ induces an equivalence $\mathscr{K}^{\mathrm{b}}(A\text{-stp}) \xrightarrow{\sim} \mathscr{K}^{\mathrm{b}}(B\text{-stp})$ of triangulated subcategories, again by Theorem 4.3. Let $\mathcal{E}^\bullet$ be the image of E_A in $\mathscr{K}^{\mathrm{b}}(B\text{-proj})$ under the equivalence. Then $\mathrm{add}(\mathcal{E}^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(B\text{-stp})$, and $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B\text{-stp})}(\mathcal{E}^\bullet, \mathcal{E}^\bullet[i]) = 0$ unless $i = 0$. Without loss of generality, we assume that $\mathcal{E}^\bullet$ is, up to degree shift, a radical complex of the form

$$\mathcal{E}^\bullet: \qquad 0 \longrightarrow S^{-n} \longrightarrow \cdots \longrightarrow S^{-1} \longrightarrow S^0 \longrightarrow 0.$$

where $S^i \in \mathrm{add}(E_B)$ for all i. By [23, Proposition 4.1], there exists a complex $X^\bullet$ in $\mathscr{K}^{\mathrm{b}}(B\text{-proj})$ such that $X^\bullet \oplus \mathcal{E}^\bullet$ is a basic tilting complex over B, inducing an almost ν-stable derived equivalence

$$\mathcal{G}: \mathscr{D}^{\mathrm{b}}(B) \xrightarrow{\sim} \mathscr{D}^{\mathrm{b}}(C)$$

where $C = \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(B\text{-proj})}(X^\bullet \oplus \mathcal{E}^\bullet)$. Let $T^\bullet$ be a radical complex such that $\mathcal{F}(T^\bullet) \cong X^\bullet$. Then $(\mathcal{G} \circ \mathcal{F})(T^\bullet \oplus E_A) \cong C$. In particular, $T^\bullet \oplus E_A$ is a tilting complex over A and $T^\bullet$ is indecomposable. Hence by Lemma 5.6 and [19, Theorem 1.3(5)], $\mathcal{G} \circ \mathcal{F}$ is an iterated almost ν-stable derived equivalence up to degree shift, and thus $\mathcal{F} = \mathcal{G}^{-1} \circ (\mathcal{G} \circ \mathcal{F})$ is an iterated almost ν-stable derived equivalence. □

Corollary 5.7. *Let A and B be self-injective k-algebras and let X (respectively Y) be an indecomposable left A-module (respectively left B-module). If $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}_B(B \oplus Y)$ are derived equivalent, then every derived equivalence between them is an iterated almost ν-stable derived equivalence (up to degree shift), and the two endomorphism algebras are stably equivalent of Morita type.*

Proof. By Definition 2.8, both endomorphism algebras are almost self-injective algebras. The statements then follow directly from Theorem 5.4. □

Example. Theorem 5.4 may fail if the algebras A and B are not assumed to be almost self-injective. Here is an example. Let $\Lambda = k[x,y]/(x^2, y^2)$, and let S be the unique simple Λ-module. Then Λ is a self-injective k-algebra and the Auslander-Reiten sequence

$$0 \to \Omega^2 S \to \mathrm{rad}(\Lambda) \oplus \mathrm{rad}(\Lambda) \to S \to 0$$

is an $\mathrm{add}(\Lambda \oplus \mathrm{rad}(\Lambda))$-split sequence in the sense of [21]. By [21, Theorem 1.1], the endomorphism algebras $A := \mathrm{End}_\Lambda(\Lambda \oplus \mathrm{rad}(\Lambda) \oplus S)$ and $B := \mathrm{End}_\Lambda(\Lambda \oplus \mathrm{rad}(\Lambda) \oplus \Omega^2 S)$ are derived equivalent. By direct checking, both A and B are seen to have ν-dominant dimension at least two, but none of them is an almost self-injective algebra. Since $\mathrm{gldim}(A) = 2$ and $\mathrm{gldim}(B) = 3$, the algebras A and B cannot be stably equivalent of Morita type.

Combining Proposition 3.8 and Theorem 5.4 yields the following result:

Corollary 5.8. *Derived equivalences between almost self-injective algebras preserve both global dimension and dominant dimension.*

5.3 Algebras with anti-automorphisms preserving simples

Many algebras have anti-automorphisms that preserve simple modules. Prominent examples are involutory such anti-automorphisms, often called dualities, which are fundamental ingredients of the definition of cellular algebras. Among the examples are group algebras of symmetric groups in any characteristic, many Hecke algebras, Brauer algebras, Temperley-Lieb algebras and many classes of Schur algebras. In particular, quasi-hereditary algebras with duality occur frequently in algebraic Lie theory. They have been studied in [14, 16, 17] from the point of view of dominant dimension, motivating the concept of gendo-symmetric algebras and our present investigation of invariance properties. The anti-automorphisms defining dualities often occur as shadows of Cartan involutions. A much larger class of algebras (with anti-automorphisms not required to be involutions) will be shown now to satisfy homological invariance properties. After establishing various elementary properties of the class of algebras with anti-automorphism preserving simples, both global and dominant dimension will get identified explicitly in the derived category, in terms of complexes occurring in canonical triangles.

Now we state precisely what we mean by an anti-automorphism preserving simples, and then we collect basic properties of algebras with such anti-automorphisms. See also [9, 14, 30] and the references therein for further discussion of the special case of dualities. Let A be a k-algebra and let $\gamma: A \to A$ be an algebra anti-automorphism. For each (left) A-module M, the γ-twist of M, denoted by ${}^\gamma M$, is defined to be the right A-module that equals M as a k-vector space, and is equipped with the right A-action:

$$m \cdot a = \gamma(a)m \qquad \forall\, a \in A, m \in M$$

The notation $\cdot$ is reserved for this action. We define the twist of a right A-module or an A-bimodule similarly. For an A-bimodule M, there are two different right A-module structures on ${}^{\gamma}M$: the usual one from the right A-action on M and the twisted one from the left A-action on M. To emphasize the difference, we write $({}^{\gamma}M)_A$ for the former and ${}^{\gamma}({}_AM)$ for the latter. If for every simple A-module S, the k-dual of ${}^{\gamma}S$ is isomorphic to S as A-modules, then we say that γ *fixes all simple modules*.

A k-algebra will be called k-split, if the endomorphism rings of simple A-modules are just k. Then simple A-bimodules are tensor products of simple left A-modules with simple right A-modules.

Lemma 5.9. *Let A be a split k-algebra with an anti-automorphism ω fixing all simple A-modules. Then:*

(1) *For each primitive idempotent e in A, the idempotent $\omega(e)$ is conjugate to e in A.*

(2) *For each projective-injective A-module P, the A-module $\nu_A(P)$ is projective-injective.*

(3) $\operatorname{domdim}(A) = \nu\text{-}\operatorname{domdim}(A)$. *In particular, A is a Morita algebra if and only if* $\operatorname{domdim}(A) \geqslant 2$.

(4) ${}^{\omega}A^{\omega} \cong A$, ${}^{\omega}\mathrm{D}(A)^{\omega} \cong \mathrm{D}(A)$, *and*
$\mathrm{D}({}^{\omega}L^{\omega}) \cong L$ *in* A^{e}-mod, *for every simple A-bimodule L.*

(5) *Let e be a basic idempotent such that* $\operatorname{add}(Ae) = A$-stp. *Then in* A^{e}-Mod,

$$\operatorname{Ext}^i_{eAe}(eA, eA) \cong \operatorname{Ext}^i_{eAe}(Ae, Ae), \qquad \text{for all } i \geqslant 0.$$

(6) *If M is a non-zero A-bimodule with ${}^{\omega}M^{\omega} \cong M$ in* A^{e}-mod, *then* $\operatorname{Hom}_{A^{\mathrm{e}}}(M, \mathrm{D}(M)) \neq 0$.

(7) *If M and N are A-bimodules with ${}^{\omega}M^{\omega} \cong M$ and ${}^{\omega}N^{\omega} \cong N$ in* A^{e}-mod, *then in* A^{e}-mod

$$\operatorname{Ext}^i_A({}_AM, {}_AN) \cong {}^{\omega}\operatorname{Ext}^i_A(M_A, N_A)^{\omega}, \qquad \text{for all } i \geqslant 0.$$

(8) *If $M^{\bullet}$ is a complex in $\mathscr{D}(A^{\mathrm{e}})$ and m is an integer such that:* (a) ${}^{\omega}H^m(M^{\bullet})^{\omega} \cong H^m(M^{\bullet}) \neq 0$, *and* (b) $H^i(M^{\bullet}) = 0$ *for all $i > m$, then* $m = \max\{d | \operatorname{Hom}_{\mathscr{D}(A^{\mathrm{e}})}(M^{\bullet}[d], (\mathrm{D}M^{\bullet})[-d]) \neq 0\}$.

Proof. (1) Let $\iota : {}^{\omega^{-1}}(Ae) \to \omega(e)A$ be the k-linear map defined by $\iota(ae) = \omega(e)\omega(a)$, for any $ae \in Ae$. Then $\iota(ae \cdot x) = \iota(\omega^{-1}(x)ae) = \omega(e)\omega(a)x$ for any $x \in A$. (Note that here Ae gets twisted by $\gamma = \omega^{-1}$.) So ι is a right A-module isomorphism. Now for each simple left A-module S,

$$\begin{aligned}\mathrm{D}(\omega(e)S) \cong \operatorname{Hom}_A(\omega(e)A, \mathrm{D}(S)) &\cong \operatorname{Hom}_A({}^{\omega^{-1}}(Ae), \mathrm{D}(S)) \\ &\cong \operatorname{Hom}_A(Ae, \mathrm{D}(S)^{\omega}) \cong \operatorname{Hom}_A(Ae, S) \cong eS\end{aligned}$$

by the assumption on ω. Therefore, e and $\omega(e)$ are conjugate idempotents in A.

(2) We may assume that P is an indecomposable projective-injective A-module, thus of the form Ae for some idempotent e. Then $\nu_A(P) = \mathrm{D}(eA)$ and $eA \cong \omega(e)A \cong {}^{\omega^{-1}}(Ae)$ by (1). Therefore, eA is projective-injective and $\nu_A(P) \cong \mathrm{D}(eA)$ is so, too.

(3) Let Ae be a basic projective-injective A-module such that every projective-injective A-module P belongs to $\operatorname{add}(Ae)$. Then $\nu_A(Ae) \cong Ae$ as A-modules by (2). In other words, every projective-injective A-module is a strongly projective-injective A-module. So by definition, $\operatorname{domdim}(A) = \nu\text{-}\operatorname{domdim}(A)$. The claim about Morita algebras then follows from Proposition 2.9.

(4) Let $\theta : A \to {}^{\omega}A^{\omega}$ be the k-linear morphism defined by: $\theta(a) = \omega(a)$ for $a \in A$. Then for any $x, y \in A$, there are equalities $\theta(xay) = \omega(y)\omega(a)\omega(x) = x \cdot \theta(a) \cdot y$ in ${}^{\omega}A^{\omega}$. Thus θ is an A-bimodule isomorphism and so is the k-dual of θ, that is, ${}^{\omega}\mathrm{D}(A)^{\omega} \cong \mathrm{D}({}^{\omega}A^{\omega}) \cong \mathrm{D}(A)$ as A-bimodules. Since A is a split k-algebra, it follows that every simple A-bimodule L is of the form ${}_AS \otimes \mathrm{D}(S')_A$, where S and S' are simple left A-modules. Thus, there is a series of isomorphisms of A-bimodules

$$\begin{aligned}\mathrm{D}({}^{\omega}L^{\omega}) &\cong \mathrm{D}({}^{\omega}S \otimes \mathrm{D}(S')^{\omega}) \cong \mathrm{D}({}^{\omega}S \otimes \mathrm{D}({}^{\omega}S')) \cong \mathrm{D}({}^{\omega}S \otimes S') \\ &\cong \operatorname{Hom}_k(S', \mathrm{D}({}^{\omega}S)) \cong \operatorname{Hom}_k(S', S) \cong S \otimes \mathrm{D}(S') \cong L\end{aligned}$$

as the anti-automorphism ω fixes simple A-modules by assumption.

(5) Since $\operatorname{domdim}(A) \geqslant 2$, the minimal faithful A-module Ae is basic and projective-injective. By (2), $\nu_A(Ae) \cong \mathrm{D}(eA)$ is also basic and projective-injective. As a result, $\mathrm{D}(eA) \cong Ae$ as A-modules, and therefore $\mathrm{D}(eA)_{\tau} \cong Ae$ as (A, eAe)-bimodules for some automorphism τ of eAe. So for all $i \geqslant 0$

$$\operatorname{Ext}^i_{eAe}(eA, eA) \cong \operatorname{Ext}^i_{eAe}(\mathrm{D}(eA), \mathrm{D}(eA)) \cong \operatorname{Ext}^i_{eAe}(\mathrm{D}(eA)_{\tau}, \mathrm{D}(eA)_{\tau}) \cong \operatorname{Ext}^i_{eAe}(Ae, Ae).$$

(6) Since $M \neq 0$, there exists a simple A-bimodule L such that $\operatorname{Hom}_{A^{\mathrm{e}}}(M, L) \neq 0$. Therefore

$$\operatorname{Hom}_{A^{\mathrm{e}}}(L, \mathrm{D}(M)) \cong \operatorname{Hom}_{A^{\mathrm{e}}}(\mathrm{D}({}^{\omega}L^{\omega}), \mathrm{D}({}^{\omega}M^{\omega})) \cong \operatorname{Hom}_{A^{\mathrm{e}}}({}^{\omega}M^{\omega}, {}^{\omega}L^{\omega}) \cong \operatorname{Hom}_{A^{\mathrm{e}}}(M, L) \neq 0$$

Thus the composition of an epimorphism from M to L followed by a monomorphism from L to $\mathrm{D}(M)$ in A^{e}-mod defines a nonzero morphism from M to $\mathrm{D}(M)$. In particular, $\operatorname{Hom}_{A^{\mathrm{e}}}(M, \mathrm{D}(M)) \neq 0$.

(7) Since ${}^{\omega}M^{\omega} \cong M$ and ${}^{\omega}N^{\omega} \cong N$ in A^e-mod as A-bimodules, it follows that

$$\mathrm{Ext}^i_A({}_AM,{}_AN) \cong \mathrm{Ext}^i_A({}^{\omega}_AM^{\omega},{}^{\omega}_AN^{\omega}) \cong \mathrm{Ext}^i_A({}^{\omega}(M_A),{}^{\omega}(M_A)) \cong {}^{\omega}\mathrm{Ext}^i_A(M_A,N_A)^{\omega}$$

as A-modules for all $i \geqslant 0$.

(8) By definition, the complex $M^{\bullet}[m]$ has vanishing cohomology in degrees larger than 0, and $(\mathrm{D}M^{\bullet})[-m]$ has vanishing cohomology in degrees smaller than 0. This implies that $\mathrm{Hom}_{\mathscr{D}(A^{\mathrm{e}})}(M^{\bullet}[d],(\mathrm{D}M^{\bullet})[-d]) = 0$ for all $d > m$. Moreover, by Lemma 3.1(3), $\mathrm{Hom}_{\mathscr{D}(A^{\mathrm{e}})}(M^{\bullet}[m],(\mathrm{D}M^{\bullet})[-m])$ is isomorphic to $\mathrm{Hom}_{A^{\mathrm{e}}}(H^m(M^{\bullet}),\mathrm{D}H^m(M^{\bullet}))$, which is nonzero by (6). □

Theorem 5.10 (Theorem A)**.** *Let A and B be derived equivalent split k-algebras.*

(a) *If both A and B have anti-automorphisms fixing all simple modules, then* $\mathrm{gldim}(A) = \mathrm{gldim}(B)$.

(b) *If furthermore both A and B have dominant dimension at least* 1*, then* $\mathrm{domdim}(A) = \mathrm{domdim}(B)$.

For the proof of the theorem, we need the following lemma.

Lemma 5.11. *Let A be a k-split algebra with an anti-automorphism ω fixing all simple A-modules, and let e be an idempotent such that Ae is basic and* $\mathrm{add}(Ae) = A\text{-}\mathsf{stp}$*. Then, in the canonical triangle*

$$(\mathbf{CT2})\ \eta^A_e: \quad A \xrightarrow{\rho_e} \mathbf{R}\mathrm{Hom}_{eAe}(eA,eA) \longrightarrow V_A(e) \longrightarrow A[1],$$

for each integer i, we have ${}^{\omega}H^i(V_A(e))^{\omega} \cong H^i(V_A(e))$ as A-bimodules.

Proof. First of all, it follows from Lemma 5.9(1) that $A\omega(e)$ and Ae are isomorphic left A-modules, and consequently $\omega(e)$ and e are conjugate in A, namely, there is an invertible element $u \in A$ such that $\omega(e) = ueu^{-1}$. We fix this notation throughout the proof.

Let $\eta : A \longrightarrow \mathrm{Hom}_{eAe}(eA,eA)$ be the canonical map sending each element $a \in A$ to the map $r_a : ex \mapsto exa$. From the long exact sequence of cohomologies induced by the canonical triangle **CT2**, it follows that

$$H^i(V_A(e)) \cong \begin{cases} 0 & \text{when } i < -1; \\ \mathrm{Ker}\,\eta & \text{when } i = -1; \\ \mathrm{Coker}\,\eta & \text{when } i = 0; \\ \mathrm{Ext}^i_{eAe}(eA,eA) & \text{when } i > 0. \end{cases}$$

For each $i \geqslant 0$, we have the following isomorphisms of A-bimodules

$$\begin{aligned} \mathrm{Ext}^i_{eAe}(eA,eA) &\cong \mathrm{Ext}^i_{eAe}(e{}^{\omega}A^{\omega},e{}^{\omega}A^{\omega}) \cong \mathrm{Ext}^i_{\omega(e)A\omega(e)}({}^{\omega}A\omega(e),{}^{\omega}A\omega(e)) \\ &\cong {}^{\omega}\mathrm{Ext}^i_{ueAeu^{-1}}(Aeu^{-1},Aeu^{-1})^{\omega} \cong {}^{\omega}\mathrm{Ext}^i_{eAe}(Ae,Ae)^{\omega}, \\ &\cong {}^{\omega}\mathrm{Ext}^i_{eAe}(eA,eA)^{\omega}, \end{aligned}$$

where the last isomorphism follows from Lemma 5.9 (5). Thus, for each integer $i > 0$, there is an isomorphism ${}^{\omega}H^i(V_A(e))^{\omega} \cong H^i(V_A(e))$ of A-bimodules.

It remains to consider the cohomologies of $V_A(e)$ in degrees 0 and -1. For this purpose, we need four A-bimodules isomorphisms. We write down these isomorphisms explicitly, and it is straightforward to check that they are really A-bimodule homomorphisms. The first is

$$c_{\omega} : \mathrm{Hom}_{eAe}(eA,eA) \longrightarrow {}^{\omega}\mathrm{Hom}_{\omega(e)A\omega(e)}(A\omega(e),A\omega(e))^{\omega}, f \mapsto \omega^{-1}f\omega.$$

Note that $\omega(e) = ueu^{-1}$ and $A\omega(e) = Aeu^{-1}$. Let $r_u : Aeu^{-1} \longrightarrow Ae$ be the map sending xeu^{-1} to xe. The second isomorphism reads as follows.

$$c_u : \mathrm{Hom}_{\omega(e)A\omega(e)}(A\omega(e),A\omega(e)) \longrightarrow \mathrm{Hom}_{eAe}(Ae,Ae), g \mapsto r_u^{-1}gr_u.$$

By assumption Ae is basic and $\mathrm{add}(Ae) = A\text{-}\mathsf{stp}$. Then $\nu_A Ae$ is again in $\mathrm{add}(Ae)$, basic and has the same number of indecomposable direct summands as Ae. Hence $\nu_A Ae \cong Ae$ as left A-modules. It follows that $\mathrm{D}(eA) = \nu_A Ae \cong Ae$ as left A-modules. Equivalently, $\mathrm{D}(Ae) \cong eA$ as right A-modules, and there is some automorphism σ of eAe such that $\mathrm{D}(Ae) \cong {}_{\sigma}eA$ as eAe-A-bimodules. Let $\tau : \mathrm{D}(Ae) \to eA$ be such an isomorphism. The third isomorphism is the canonical map

$$\mathrm{D} : \mathrm{Hom}_{eAe}(Ae,Ae) \longrightarrow \mathrm{Hom}_{eAe}(\mathrm{D}(Ae),\mathrm{D}(Ae)), h \mapsto \mathrm{D}(h),$$

and the fourth isomorphism is a composition

$$c_{\tau} : \mathrm{Hom}_{eAe}(\mathrm{D}(Ae),\mathrm{D}(Ae)) \longrightarrow \mathrm{Hom}_{eAe}({}_{\sigma}eA,{}_{\sigma}eA) \longrightarrow \mathrm{Hom}_{eAe}(eA,eA), h \mapsto \tau^{-1}h\tau.$$

From the proof of Lemma 5.9 (4), the map $\omega : A \to {}^{\omega}A^{\omega}$ is an A-bimodule isomorphism. Then it is straightforward to check that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \mathrm{Hom}_{eAe}(eA, eA) \\ \Big\downarrow{\scriptstyle \omega} & & \Big\downarrow{\scriptstyle c_\omega} \\ & & {}^{\omega}\mathrm{Hom}_{\omega(e)A\omega(e)}(A\omega(e), A\omega(e))^{\omega} \\ & & \Big\downarrow{\scriptstyle c_u} \\ & & {}^{\omega}\mathrm{Hom}_{eAe}(Ae, Ae)^{\omega} \\ & & \Big\downarrow{\scriptstyle \mathrm{D}} \\ & & {}^{\omega}\mathrm{Hom}_{eAe}(\mathrm{D}(Ae), \mathrm{D}(Ae))^{\omega} \\ & & \Big\downarrow{\scriptstyle c_\tau} \\ {}^{\omega}A^{\omega} & \xrightarrow{\eta} & {}^{\omega}\mathrm{Hom}_{eAe}(eA, eA)^{\omega} \end{array} \qquad (\star)$$

is commutative. Actually, for each $a \in A$, the image of a under $\eta c_\omega c_u \mathrm{D}$ is $\mathrm{D}(r_u^{-1}\omega^{-1} r_a \omega r_u)$, which sends each $\alpha \in \mathrm{D}(Ae)$ to $r_u^{-1}\omega^{-1} r_a \omega r_u \alpha$. The image of a under $\omega\eta$ is $r_{\omega(a)}$. Then one can check that there is a commutative diagram

$$\begin{array}{ccc} eA & \xrightarrow{r_{\omega(a)}} & eA \\ \Big\downarrow{\scriptstyle \tau^{-1}} & & \Big\downarrow{\scriptstyle \tau^{-1}} \\ \mathrm{D}(Ae) & \xrightarrow{\mathrm{D}(r_u^{-1}\omega^{-1} r_a \omega r_u)} & \mathrm{D}(Ae) \end{array}$$

of right A-modules. This means precisely that the image of a under $\eta c_\omega c_u \mathrm{D}\, c_\tau$ is the same as its image under $\omega\eta$. Thus, the diagram $(\star)$ is commutative, and induces another commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}\,\eta & \longrightarrow & A & \xrightarrow{\eta} & \mathrm{Hom}_{eAe}(eA, eA) & \longrightarrow & \mathrm{Coker}\,\eta & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \omega} & & \Big\downarrow{\scriptstyle c_\omega c_u \mathrm{D}\, c_\tau} & & \Big\downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & {}^{\omega}(\mathrm{Ker}\,\eta)^{\omega} & \longrightarrow & {}^{\omega}A^{\omega} & \xrightarrow{\eta} & {}^{\omega}\mathrm{Hom}_{eAe}(eA, eA)^{\omega} & \longrightarrow & {}^{\omega}(\mathrm{Coker}\,\eta)^{\omega} & \longrightarrow & 0 \end{array}$$

where the rows are exact and vertical maps are A-bimodule isomorphisms. This finishes the proof. □

Proof of Theorem 5.10. (a) Proposition 5.1 implies that if A or B has infinite global dimension, then so does the other and we are done. Now assume A and B to have finite global dimension and $\mathscr{D}^{\mathrm{b}}(A) \cong \mathscr{D}^{\mathrm{b}}(B)$ to be a standard derived equivalence. Let g and $\bar{g}$ be the global dimensions of A and B respectively, and let $R_A^\bullet$ denote the bounded complex $\mathbf{R}\mathrm{Hom}_A({}_A\mathrm{D}(A), {}_AA)$ in $\mathscr{D}^{\mathrm{b}}(A^{\mathrm{e}})$, and $R_B^\bullet$ the bounded complex $\mathbf{R}\mathrm{Hom}_B({}_B\mathrm{D}(B), {}_BB)$ in $\mathscr{D}^{\mathrm{b}}(B^{\mathrm{e}})$. Then Lemma 2.1 can be reformulated as

$$g = \max\{i \mid \mathrm{H}^i(R_A^\bullet) \neq 0\}, \qquad \bar{g} = \max\{i \mid \mathrm{H}^i(R_B^\bullet) \neq 0\}.$$

By Lemma 5.9 (4) and (7) and by k-duality,

$$\mathrm{Ext}_A^g({}_A\mathrm{D}(A), {}_AA) \cong {}^{\omega}\mathrm{Ext}_A^g(\mathrm{D}(A)_A, A_A)^{\omega} \cong {}^{\omega}\mathrm{Ext}_A^g({}_A\mathrm{D}(A), {}_AA)^{\omega}$$

as A-bimodules. Note that $H^m(R_A^\bullet) \cong \mathrm{Ext}_A^g({}_A\mathrm{D}(A), {}_AA)$. Hence, by Lemma 5.9(8), we get

$$g = \max\{i \mid \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A^{\mathrm{e}})}(R_A^\bullet[i], (\mathrm{D}R_A^\bullet)[-i]) \neq 0\}.$$

Similarly $\bar{g} = \max\{i \mid \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B^{\mathrm{e}})}(R_B^\bullet[i], (\mathrm{D}R_B^\bullet)[-i]) \neq 0\}$. Let $\mathcal{F} : \mathscr{D}^{\mathrm{b}}(A^{\mathrm{e}}) \cong \mathscr{D}^{\mathrm{b}}(B^{\mathrm{e}})$ be the derived equivalence induced from the given standard derived equivalence between A and B. Then $\mathcal{F}(R_A^\bullet) \cong R_B^\bullet$ and $\mathcal{F}(\mathrm{D}R_A^\bullet) \cong \mathrm{D}R_B^\bullet$ in $\mathscr{D}^{\mathrm{b}}(B^{\mathrm{e}})$ by Lemma 3.3 (1), (3) and (4). Hence, $g = \bar{g}$, that is, A and B have the same global dimension.

(b) Let d and $\bar{d}$ be the ν-dominant dimensions of A and B respectively. Since both d and $\bar{d}$ are at least one by assumption, Lemma 5.9 (3) (or Lemma 2.6) implies that there is no need to distinguish between dominant dimension and ν-dominant dimension. Let e and f be basic idempotents in A and B, respectively, such that $\mathrm{add}(Ae) = A\text{-}\mathsf{stp}$ and $\mathrm{add}(Bf) = B\text{-}\mathsf{stp}$. Let $\eta_e^A : A \to \mathbf{R}\mathrm{Hom}_{eAe}(eA, eA) \to V_A(e) \to A[1]$ and $\eta_f^B : B \to \mathbf{R}\mathrm{Hom}_{fBf}(fB, fB) \to V_B(f) \to B[1]$ be the canonical triangles (denoted by (CT2) in 3.2) associated to the idempotents e and f respectively.

et $m := \min\{i | H^i(V_A(e)) \neq 0\}$. We claim that $d = m+1$. As $d \geqslant 1$, the module eA is a faithful right A-module and η is injective. In this case $H^{-1}(V_A(e)) = 0$, and $H^0(V_A(e)) \cong \mathrm{Coker}\,\eta$ vanishes if and only if η is also surjective, if and only if $d \geqslant 2$ (see [32]). This implies that $m = 0$ when $d = 1$. If $d \geqslant 2$, then it follows from Müller's characterisation (Proposition 2.3) that $d = m+1$.

Without loss of generality, the derived equivalence $\mathscr{D}(A) \cong \mathscr{D}(B)$ can be assumed to be standard and to be given by a two-sided tilting complex $\Delta^\bullet$ in $\mathscr{D}(B \otimes A^{\mathrm{op}})$. Let $\mathcal{F} : \mathscr{D}(A^{\mathrm{e}}) \to \mathscr{D}(B^{\mathrm{e}})$ be the induced standard derived equivalence and $\Theta^\bullet = \mathbf{R}\mathrm{Hom}_{\mathscr{D}(B)}(\Delta^\bullet, B)$. Then by Theorem 4.3 and Lemma 3.5,

$$\Delta^\bullet \otimes^{\mathbf{L}}_A \eta^A_e \otimes^{\mathbf{L}}_A \Theta^\bullet \cong \eta^B_f$$

as triangles in $\mathscr{D}(B^{\mathrm{e}})$, and in particular, $\mathcal{F}(V_A(e)) \cong V_B(f)$ in $\mathscr{D}(B^{\mathrm{e}})$.

Thus, it follows from Lemma 5.11 and Lemma 5.9(8) that

$$1 - d = \max\{i | \mathrm{Hom}_{\mathscr{D}(A^{\mathrm{e}})}((\mathrm{D}V_A(e))[i], V_A(e)[-i]) \neq 0\}, \text{ and}$$

$$1 - \bar{d} = \max\{i | \mathrm{Hom}_{\mathscr{D}(B^{\mathrm{e}})}((\mathrm{D}V_B(f))[i], V_B(f)[-i]) \neq 0\}.$$

Since $\mathcal{F}(V_A(e)) \cong V_B(f)$ and $\mathcal{F}(\mathrm{D}V_A(e)) \cong \mathrm{D}V_B(f)$, one has $1 - d = 1 - \bar{d}$, and $d = \bar{d}$. □

The characterisations of global and dominant dimension, respectively, inside the derived category of the enveloping algebras, provided by the proof may be of independent interest.

Corollary 5.12. *Let A be an algebra with an anti-automorphism preserving simples.*

(a) *Let $R^\bullet_A$ denote the bounded complex $\mathbf{R}\mathrm{Hom}_A({}_A\mathrm{D}(A), {}_AA)$ in $\mathscr{D}^{\mathrm{b}}(A^{\mathrm{e}})$. Then the global dimension of A, if finite, is determined in the category $\mathscr{D}^{\mathrm{b}}(A^{\mathrm{e}})$ as*

$$\mathrm{gldim}(A) = \max\{i \mid \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A^{\mathrm{e}})}(R^\bullet_A[i], (\mathrm{D}R^\bullet_A)[-i]) \neq 0\}.$$

(b) *Suppose e is an idempotent in A such that Ae is basic and* $\mathrm{add}(Ae) = A\text{-}\mathsf{stp}$. *Then the dominant dimension of A is determined in the category $\mathscr{D}^{\mathrm{b}}(A^{\mathrm{e}})$ by the canonical triangle*

$$(\mathbf{CT2})\ \eta^A_e : \quad A \xrightarrow{\rho_e} \mathbf{R}\mathrm{Hom}_{eAe}(eA, eA) \longrightarrow V_A(e) \longrightarrow A[1], \text{as}$$

$$\mathrm{domdim}(A) = 1 - \max\{i \mid \mathrm{Hom}_{\mathscr{D}(A^{\mathrm{e}})}(\mathrm{D}(V_A(e))[i], V_A(e)[-i]) \neq 0\}.$$

Proof. The proof of Theorem 5.10 implies that (a) holds, and that (b) holds in case that $\mathrm{domdim}(A) \geqslant 1$. Suppose that $\mathrm{domdim}(A) = 0$. Then eA cannot be a faithful A-module, and thus the canonical map $\eta : A \to \mathrm{Hom}_{eAe}(eA, eA)$ is not injective. It follows that $H^{-1}(V_A(e))$, which is $\mathrm{Ker}\,\eta$, is nonzero. Then, in this case, (b) follows from Lemma 5.9(8). □

In the current context, the complexes $R^\bullet_A$ used to identify global dimension are preserved at least by standard derived equivalences.

Theorem C (Corollary 4.4 and Theorem 4.3) is a crucial ingredient of the above proof of Theorem 5.10. Together with Lemma 3.5, they guarantee that the canonical triangle **(CT2)** is preserved under standard derived equivalences between algebras with positive ν-dominant dimensions.

Corollary 5.13. *Let A and B be algebras with anti-automorphisms preserving simples. Then a standard derived equivalence from A to B induces a standard equivalence from $\mathscr{D}^{\mathrm{b}}(A^{\mathrm{e}})$ to $\mathscr{D}^{\mathrm{b}}(B^{\mathrm{e}})$ that sends $R^\bullet_A$ to $R^\bullet_B$.*

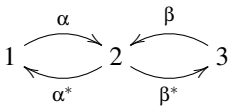
When an algebra A with an anti-automorphism preserving simples is derived equivalent to an algebra B, possibly without such an anti-automorphism, the proof of Theorem 5.10 provides an inequality:

Corollary 5.14. *Let A be an algebra with an anti-automorphism preserving simples and suppose A is derived equivalent to an algebra B. Then there is an inequality* $\mathrm{gldim}(B) \geqslant \mathrm{gldim}(A)$.

Proof. The characterisation of $\mathrm{gldim}(A) = g$ given in the proof of Theorem 5.10 remains valid, by Corollaries 5.12 and 5.13. Then the non-vanishing over A of $0 \neq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A^{\mathrm{e}})}(R^\bullet_A[g], (\mathrm{D}R^\bullet_A)[-g]) \cong \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B^{\mathrm{e}})}(R^\bullet_B[g], (\mathrm{D}R^\bullet_B)[-g]) \neq 0\}$, implies a contradiction to $\mathrm{gldim}(B) < g$. □

The following example shows that the assumption in Theorem 5.10, requiring both algebras to admit an anti-automorphism preserving simples, cannot be relaxed; the inequality in Corollary 5.14 is in general not an equality. In this example, the algebra A has an anti-automorphism, while B does not. The algebras A and B are derived equivalent. The global dimension of B is strictly bigger than that of A.

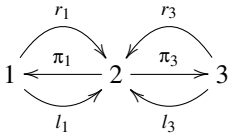
Let the algebra A be given by the quiver



with relations $\{\alpha^*\alpha, \beta^*\beta, \beta^*\alpha, \alpha^*\beta\}$. This is a *dual extension* (defined by C.C.Xi [40]) of the path algebra of $1 \to 2 \leftarrow 3$. It is a quasi-hereditary cellular algebra, and has global dimension 2. Let P_i be the indecomposable projective A-modules corresponding to the vertex i, and let $T^\bullet$ be the direct sum of three indecomposable complexes $P_1^\bullet$, $P_2^\bullet$ and $P_3^\bullet$, where $P_2^\bullet := P_2[1]$ and, for each $i \in \{1,3\}$, the complex $P_i^\bullet$ is the complex

$$0 \to P_2 \to P_i \to 0$$

with P_2 in degree -1. Then $T^\bullet$ is a tilting complex, inducing a derived equivalence between A and the endomorphism algebra B of $T^\bullet$. The algebra B has the following quiver



with relations $\{l_1\pi_1 - l_3\pi_3, r_1\pi_1 - r_3\pi_3, \pi_1 l_i, \pi_3 r_i, i = 1,3\}$. The algebra A has an anti-automorphism fixing simple modules, while B cannot have such an anti-automorphism. (If an algebra Λ has such an anti-automorphism, then $\mathrm{Ext}^1_\Lambda(U,V) \cong \mathrm{Ext}^1_\Lambda(V,U)$ for all simple Λ-modules U,V. This implies that, in the quiver of the algebra Λ, for any two vertices a and b, the number of arrows from a to b coincides with the number of arrows in the opposite direction, from b to a). A direct calculation shows that the algebra B has global dimension 3, which is strictly larger than $\mathrm{gldim}(A)$.

Cellular algebras by definition have anti-automorphisms preserving simples; hence this large and widely studied class of algebras is covered by Theorem 5.10.

Corollary 5.15. *Suppose that A and B are derived equivalent cellular algebras over a field. Then A and B have the same global dimension. If both A and B have positive dominant dimension, then their dominant dimensions are equal.*

In the next Section, it will be shown how to use this result to compute homological dimensions of blocks of quantised Schur algebras.

6 Homological dimensions of q-Schur algebras and their blocks

Global dimensions of classical and quantised Schur algebras $S(n,r)$ (with $n \geqslant r$) have been determined by Totaro [39] and Donkin [10]. Dominant dimensions of these algebras have been obtained more recently in [17, 14]. The aim of this section is to determine the global and dominant dimensions of all blocks of these algebras, that is, of the indecomposable algebra direct summands.

To recall some basics on q-Schur algebras and their blocks (see [10] and the references therein), let k be a field of characteristic 0 or p, and q a non-zero element in k. Let ℓ be the smallest integer such that $1+q+\cdots+q^{\ell-1} = 0$, and set $\ell = 0$ if no such integer exists. For a natural number r, let Σ_r be the symmetric group on r letters and let $\mathcal{H}_q(r)$ be the associated Hecke algebra that is defined by generators $\{T_1, \ldots, T_{r-1}\}$ and relations

$$\begin{aligned} (T_i+1)(T_i-q) &= 0, && (1 \leqslant i \leqslant r-1);\\ T_iT_j &= T_jT_i, && (|i-j|>1);\\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, && (1 \leqslant i \leqslant r-2). \end{aligned}$$

For each composition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of r, that is, a sequence of n non-negative integers summing up to r, let $\mathcal{H}_q(\Sigma_\lambda)$ be the associated parabolic Hecke algebra that is isomorphic to $\mathcal{H}_q(\lambda_1) \otimes_k \cdots \otimes_k \mathcal{H}_q(\lambda_n)$ as k-algebras. Then $\mathcal{H}_q(\Sigma_\lambda)$ naturally can be seen as a k-subalgebra of $\mathcal{H}_q(r)$. The algebra $\mathcal{H}_q(\Sigma_\lambda)$ has a trivial module k with all T_i's acting as scalar q. Let M^λ be the permutation module over $\mathcal{H}_q(r)$ that is defined as the induced module $\mathcal{H}_q(r) \otimes_{\mathcal{H}_q(\Sigma_\lambda)} k$. The q-Schur algebra $S_q(n,r)$ is then defined to be the endomorphism ring $\mathrm{End}_{\mathcal{H}_q(r)}(\bigoplus_\lambda M^\lambda)$ where λ ranges over all compositions $(\lambda_1, \ldots, \lambda_n)$ of r into n parts, where n is any natural number. For each composition λ of r, there is a unique associated composition $\overline{\lambda}$ (called *partition*) obtained by rearranging the entries of λ weakly decreasing; the permutation module $M^{\overline{\lambda}}$ is isomorphic to M^λ and is a direct sum of the (indecomposable) Young modules Y^μ with multiplicities $K_{\overline{\lambda},\mu}$ (p-Kostka number) where μ ranges over all partitions of r. These multiplicities are known to satisfy $K_{\overline{\lambda},\overline{\lambda}} = 1$ and $K_{\overline{\lambda},\mu} = 0$ unless $\mu \trianglerighteq \overline{\lambda}$ in the dominance ordering on partitions.

When q is not a root of unity, then the q-Schur algebra $S_q(n,r)$ is semisimple, and thus the global dimensions of $S_q(n,r)$ and its blocks are 0, and the dominant dimensions of $S_q(n,r)$ and its blocks are ∞. In the following, we always assume that q is a root of unity. If $p = 0$ and $r = r_{-1} + \ell r_0$ is the ℓ-adic expansion of r, then we set $d_{\ell,p}(r) = r_{-1} + r_0$; if $p > 0$, $r = r_{-1} + \ell r'$ and $r' = r_0 + pr_1 + p^2 r_2 + \cdots$ are the ℓ-adic expansion of r and the p-adic expansion of r' respectively, then we set $d_{\ell,p}(r) = r_{-1} + r_0 + r_1 + r_2 + \cdots$. The global dimension of $S_q(n,r)$ for $n \geqslant r$ in this case has been given by Totaro [39] for $q = 1$, that is for the classical Schur algebra, and by Donkin [10] in general.

Theorem 6.1 ([10, 39]). *If q is a root of unity and $n \geqslant r$, then* $\operatorname{gldim} S_q(n,r) = 2(r - d_{\ell,p}(r))$.

A lower bound for the dominant dimension of $S_q(n,r)$ has been obtained (implicitly) by Kleshchev and Nakano [26] for $q = 1$ and by Donkin [11, Proposition 10.5] in general. It was shown later in [17, 14] that this lower bound is an upper bound, too.

Theorem 6.2 ([11, 17, 14, 26]). *If q is a root of unity and $n \geqslant r$, then* $\operatorname{domdim} S_q(n,r) = 2(\ell - 1)$.

To determine the global and dominant dimensions of blocks of the q-Schur algebras, we will use the setup of algebras with a duality. Each q-Schur algebra has an anti-automorphism that fixes all simple modules, and there is a block decomposition that is invariant under the involution

$$S_q(n,r) \cong \bigoplus_{(\tau,w)} \mathbf{B}_{\tau,w}$$

where $0 \leqslant w \leqslant r$ and τ ranges over all ℓ-core partitions of $r - w\ell$. Moreover, for $m, n \geqslant r$, the two blocks $\mathbf{B}_{\tau,w}$ of $S_q(n,r)$ and $\mathbf{B}_{\tau',w'}$ of $S_q(m,r)$ are derived equivalent if $w = w'$ [7].

Theorem 6.3. *If q is a root of unity and $n \geqslant r$, then the global dimension of the block $\mathbf{B}_{\tau,w}$ is equal to $2(\ell w - d_{\ell,p}(\ell w))$.*

Proof. Chuang and Rouquier have shown in [7], that $\mathbf{B}_{\tau,w}$ and $\mathbf{B}_{\emptyset,w}$ are derived equivalent. As all block algebras of q-Schur algebras have anti-automorphisms fixing all simple modules, Theorem 5.10 can be applied and we get $\operatorname{gldim} \mathbf{B}_{\tau,w} = \operatorname{gldim} \mathbf{B}_{\emptyset,w}$.

For each natural number s, set $g(s) = 2(\ell s - d_{\ell,p}(\ell s))$. Then $g(s+1) = 2(\ell s + \ell - d_{\ell,p}(\ell s + \ell)) = g(s) + 2(\ell + d_{\ell,p}(\ell s) - d_{\ell,p}(\ell s + \ell)) \geqslant g(s) + 2(\ell - 1)$. In particular, $g(s) > g(s')$ if $s > s'$.

Now we are going to compute the global dimension of $\mathbf{B}_{\tau,w}$ by induction on w; we have to show that it equals $g(w)$. If $w = 0$, then the block algebra $\mathbf{B}_{\tau,w}$ is semisimple ([10]), and thus $\operatorname{gldim} \mathbf{B}_{\tau,w} = 0 = g(w)$. Now we assume that $w > 0$. Note that the q-Schur algebra $S_q(\ell w, \ell w)$ is of global dimension $g(w)$ by Theorem 6.1, and has a block subalgebra $\mathbf{B}_{\emptyset,w}$. It follows that $\operatorname{gldim} \mathbf{B}_{\emptyset,w} = g(w)$ since all other block subalgebras of $S_q(\ell w, \ell w)$ are $\mathbf{B}_{\tau,w'}$ with $w' < w$, and thus $\operatorname{gldim} \mathbf{B}_{\tau,w'} = g(w') < g(w) = \operatorname{gldim} \mathbf{B}_{\emptyset,w}$ by induction. □

In terms of the cover theory introduced by Rouquier [37], the q-Schur algebra $S_q(n,r)$ is a quasi-hereditary cover of the Hecke algebra $\mathcal{H}_q(r)$ of covering degree $(\ell - 1)$ by Theorem 6.2 and [14], that is, $S_q(n,r)$ is a $(\ell-1)$-cover, but not an ℓ-cover of $\mathcal{H}_q(r)$. The following result implies a particular property of the cover; each block of $S_q(n,r)$ is a quasi-hereditary cover of the corresponding block of $\mathcal{H}_q(r)$, of the same dominant dimension. This property may be formulated as saying that the covering is uniform of covering degree $\ell - 1$.

Theorem 6.4. *If q is a root of unity and $n \geqslant r$, then the dominant dimension of $\mathbf{B}_{\tau,w}$ satisfies*

$$\operatorname{domdim} \mathbf{B}_{\tau,w} = \begin{cases} \infty & \text{when } w = 0; \\ 2(\ell - 1) & \text{when } w \neq 0. \end{cases}$$

Proof. If $w = 0$, then $\mathbf{B}_{\tau,w}$ is semisimple and has dominant dimension infinity. Now we assume that $w > 0$. Since q is a root of unity, it follows that $\ell \geqslant 2$, and thus by Theorem 6.2

$$\operatorname{domdim} \mathbf{B}_{\tau,w} \geqslant \operatorname{domdim} S_q(n,r) = 2(\ell - 1) \geqslant 2. \tag{5}$$

Note that all block algebras have anti-automorphisms fixing all simple modules. By [7], the block subalgebra $\mathbf{B}_{\tau,w}$ of $S_q(n,r)$ and the principal block subalgebra $\mathbf{B}_{\emptyset,w}$ of $S_q(\ell w, \ell w)$ are derived equivalent, and thus have the same dominant dimension by Theorem 5.10. Therefore we only need to show $\operatorname{domdim} \mathbf{B}_{\emptyset,w} = 2(\ell - 1)$.

Let e be an idempotent in $\mathbf{B}_{\emptyset,w}$ such that $\mathbf{B}_{\emptyset,w}e$ is a minimal faithful $\mathbf{B}_{\emptyset,w}$-module. Then $\mathbf{b}_w = e\mathbf{B}_{\emptyset,w}e$ is a block subalgebra of $\mathcal{H}_q(\ell w)$ and the $\mathbf{b}_w$-module $e\mathbf{B}_{\emptyset,w}$ is isomorphic to a direct sum of those Young modules Y^μ that belong to the block $\mathbf{b}_w$ (see [10]). By Proposition 2.3 and the inequality (5), to finish the proof, it suffices to show:

Claim. There exist Young modules Y^λ and Y^μ of $\mathcal{H}_q(\ell w)$ that belong to $\mathbf{b}_w$ such that $\operatorname{Ext}^{2(\ell-1)-1}_{\mathbf{b}_w}(Y^\lambda, Y^\mu) \neq 0$.

Proof. Set $\nu = (\ell, 1, \ldots, 1)$ and $\mu = (\ell w)$, which are two partitions of ℓw. Then the Young module Y^μ belongs to the block $\mathbf{b}_w$ and by definition $Y^\mu = M^\mu = k$. As a result, by Mackey's decomposition theorem

$$\operatorname{Ext}^{2(\ell-1)-1}_{\mathcal{H}_q(r)}(M^\nu, Y^\mu) \cong \operatorname{Ext}^{2(\ell-1)-1}_{\mathcal{H}_q(r)}(\mathcal{H}_q(r) \otimes_{\mathcal{H}_q(\Sigma_\nu)} k, k) \cong \operatorname{Ext}^{2(\ell-1)-1}_{\mathcal{H}_q(\Sigma_\nu)}(k,k) \neq 0.$$

Here, the first isomorphism uses the definition of permutation modules and the second one uses adjointness; non-vanishing of the third extension space follows by identifying Σ_ν with Σ_ℓ and then using the known cohomology of $\mathcal{H}_q(\Sigma_\ell)$. So there is a direct summand Y^λ of M^ν such that $\operatorname{Ext}^{2(\ell-1)-1}_{\mathcal{H}_q(r)}(Y^\lambda, Y^\mu) \neq 0$. In this case, the Young module Y^λ must belong to the block $\mathbf{b}_w$, too. This proves the claim and the Theorem. □

Remark. (1) When k has characteristic zero or bigger than the weight w, the dominant dimension of the blocks $\mathbf{B}_{\tau,w}$ have been determined in [17] by using Chuang and Tan's complete description of the corresponding Rouquier blocks [8].

(2) All blocks $\mathbf{B}_{\tau,2}$ of the q-Schur algebra $S_q(n,r)$ are also stably equivalent of Morita type, and hence have the same global and dominant dimensions, as well as the same representation dimension. Indeed, by carefully examining the tilting complexes constructed by Chuang and Rouquier [7], we see that the induced derived equivalences are almost ν-stable. However, the derived equivalences between general blocks constructed by Chuang and Rouquier [7] are not almost ν-stable. For instance, when $p = 2$, the group algebras of S_6 and of S_7, respectively, each have a unique block of p-weight 3. These two blocks correspond to each other under the reflection s_0 of the Weyl group of the affine Kac-Moody algebra $\widehat{sl_2}$. The derived equivalences constructed in [7], relating these two blocks, are not almost ν-stable.

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