A CHARACTERIZATION OF ALGEBRAS GENERATED BY IDEMPOTENTS

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Abstract. We study algebras generated by idempotents. For finite dimensional algebras over a field, it turns out that these algebras can be characterized by their irreducible modules homologically. Particularly, we obtain that a finite dimensional algebra over an algebraically closed field is generated by idempotents if and only if Ext$_A^1(S, S) = 0$ for all 1-dimensional $A$-modules $S$.

1. Introduction

Throughout $F$ denotes an arbitrary field and let $A$ be an associative algebra over $F$ with unity $1$. In this paper, we aim to provide, to some extent, an easier criterion to determine whether or not the algebra $A$ is generated by idempotents. This topic is motivated by recent work on algebra representation theory and linear preserving problems.

In algebra representation theory, idempotents are of crucial importance, since one of the main objectives is to give a complete set of (central) primitive idempotents of a finite dimensional algebra $A$, which relates to the graded structure, block structure, etc., of $A$ (see [7]). In many specific problems, it is important to know the subalgebras generated by some given idempotents. Laffey [13], Rowen and Segev [15] classified the associative algebras generated by two idempotents. Recently, more results related to associative or nonassociative algebras generated by idempotents are considered, see [2, 8, 10, 11, 14, 15, 17] and the references therein.

In the field of linear preserving problems, many linear maps or operators are characterized by their local properties through idempotents. Based on the work on commutativity preserving maps [4], zero product preserving maps [5] and other related linear maps, Brešar, Grašič and Sanchez [3] introduced the notion of zero product determined algebras (see Definition 2.1 below). It is proved that zero product determined algebras are helpful to study algebra homomorphisms and derivations, see [1, 2, 6] and the references therein. In [2, Theorem 3.7], Brešar proved that a finite dimensional algebra is zero product determined if and only if it is generated by idempotents. He then raised the problem of finding other classes of algebras which are generated by idempotents.

Motivated by the aforementioned results, we want to find a method to determine whether an algebra is generated by idempotents through representation theory. If
the algebra \( A \) is generated by idempotents, it is natural to study the special properties of irreducible \( A \)-modules and principal indecomposable \( A \)-modules. We will introduce a tool called zero action determined modules (see Definition 2.1 below) and prove that if the finite dimensional algebra \( A \) is generated by idempotents, then \( \dim V > \dim \text{End}_A(V) \) or \( \dim \text{End}_A(V) = 1 \) for each irreducible \( A \)-module \( V \) and \( \text{Ext}^1_A(S,S) = 0 \) for 1-dimensional \( A \)-module \( S \). Much to our surprise, the above conditions of irreducible \( A \)-modules are also sufficient for \( A \) being generated by idempotents.

2. Algebras Generated by Idempotents

Throughout this paper, \( F \) is a fixed field, and all algebras are unital associative algebras over \( F \). The tensor product \( \otimes_F \) over \( F \) will be abbreviated as \( \otimes \), and the dimension of an \( F \)-space \( V \) is denoted by \( \dim V \).

In this section, we shall prove the main theorem of this paper, which shows that whether \( A \) is generated by idempotents depends only on its irreducible modules.

Let us begin by giving a concept.

**Definition 2.1.** Let \( A \) be a unital associative algebra over \( F \). An \( A \)-module \( V \) is said to be zero action determined (zad for short) if for every \( F \)-bilinear map \( f: A \times V \to F \) with the property that

\[
f(a,m) = 0 \quad \text{whenever} \quad am = 0, \tag{2.1}
\]

then there exists a linear map \( \Phi: V \to F \) such that

\[
f(x,v) = \Phi(xv) \quad \text{for all} \quad x \in A, v \in V. \tag{2.2}
\]

If the regular \( A \)-module \( AA \) is zad, we say that \( A \) is zero product determined (zpd for short).

An observation shows that the equation (2.2) is equivalent to

\[
f(xy,v) = f(x,yv) \quad \text{for all} \quad x,y \in A, v \in V, \tag{2.3}
\]

as well as to

\[
f(x,v) = f(1,xv) \quad \text{for all} \quad x \in A, v \in V. \tag{2.4}
\]

In the following we will use (2.3) or (2.4) instead of (2.2) without comments.

We remark that the notion of a zpd algebra coincides with that introduced in [3], which can be deduced from the following lemma.

**Lemma 2.2.** Let \( V \) be an \( A \)-module. Then \( V \) is zad if and only if for every bilinear map \( f: A \times V \to F \) with the property that

\[
f(a,m) = 0 \quad \text{whenever} \quad am = 0, \tag{2.1}
\]

then there exists a linear map \( \Phi: V \to F \) such that

\[
f(x,v) = \Phi(xv) \quad \text{for all} \quad x \in A, v \in V. \tag{2.2}
\]

**Proof.** We only need to prove the necessity. Let \( U \) be an arbitrary \( F \)-space and \( f: A \times V \to U \) be an arbitrary \( F \)-bilinear map satisfying \( f(a,m) = 0 \) whenever \( am = 0 \). Take an arbitrary linear functional \( \alpha \) on \( U \) and define \( f_\alpha := \alpha \circ f: A \times V \to F \). It is clear that \( f_\alpha \) satisfies (2.1) and hence \( f_\alpha(x,v) = f_\alpha(1,xv) \) for all \( x \in A, v \in V \).
Since $\alpha$ is an arbitrary linear functional on $U$, this actually implies that $f(x,v) = f(1, xv)$ for all $x \in A, v \in V$. Indeed, if $f(x,v) \neq f(1, xv)$ for some $x \in A, v \in V$, that is, $w = f(x,v) - f(1, xv) \neq 0$, applying Zorn’s Lemma to the set of subspaces $L$ of $U$ with $w \notin L$ (partially ordered by inclusion) gives rise to a subspace $M$ of $U$ such that $Fw \oplus M = U$. Thus, there is a linear functional $\alpha$ on $U$ such that $\alpha(w) = 1 \neq 0$ and consequently $f_\alpha(x,v) \neq f_\alpha(1, xv)$, which is a contradiction.

Define a linear map $\Phi : V \to U$ by $\Phi(v) = f(1,v)$. It is clear that $f(x,v) = f(1,xv) = \Phi(xv)$ for all $x \in A, v \in V$. \hfill\box

The following proposition shows that zad modules can play a role in linear preserving problems. Let $V$ and $U$ be two $A$-modules. A linear map $\varphi : V \to U$ is called a zero action preserving map, if for every $a \in A, m \in V$, there is $a\varphi(m) = 0$ whenever $am = 0$.

**Proposition 2.3.** Let $V$ and $U$ be two $A$-modules. Let $\varphi : V \to U$ be a zero action preserving linear map. If $V$ is zad, then $\varphi$ is a homomorphism of $A$-modules.

**Proof.** Define a bilinear map $f : A \times V \to U$ by $f(x,v) = x \varphi(v)$. Then Lemma 2.2 implies that there exists a linear map $\Phi : V \to U$ such that $f(x,v) = \Phi(xv)$ for all $x \in A, v \in V$. Hence $x\varphi(v) = f(x,v) = f(1,xv) = \varphi(xv)$.

The following technical lemma is useful to discuss the properties of zad modules.

**Lemma 2.4.** The $A$-module $V$ is zad if and only if

$$x \otimes v - 1 \otimes xv \in F\text{-span}\{a \otimes m \in A \otimes V \mid am = 0\}$$

for every $x \in A, v \in V$.

**Proof.** We first prove the "if" part. Suppose that $f : A \times V \to F$ is a bilinear map satisfying (2.1). The bilinearity of $f$ implies that there exists a unique linear map $\alpha_f : A \otimes V \to F$ such that $\alpha_f(x \otimes v) = f(x,v)$. Since $f$ satisfies (2.1), $\alpha_f$ vanishes on the subspace $F$-span$\{a \otimes m \in A \otimes V \mid am = 0\}$. By the hypothesis, we have $\alpha_f(x \otimes v - 1 \otimes xv) = 0$ for every $x \in A, v \in V$. In other words, $f$ satisfies (2.4) and thus $V$ is a zad $A$-module.

To prove the converse, we assume that there exist $a \in A, m \in V$ such that $a \otimes m - 1 \otimes am \notin F$-span$\{x \otimes v \in A \otimes V \mid xv = 0\}$. Similarly, as we have done in the proof of Lemma 2.2, one can find a linear functional $\alpha$ on $A \otimes V$ such that $\alpha$ vanishes on the subspace $F$-span$\{x \otimes v \in A \otimes V \mid xv = 0\}$ but $\alpha(a \otimes m - 1 \otimes am) \neq 0$. Then the bilinear map $f : A \times V \to F$ defined by $f(x,v) = \alpha(x \otimes v)$ satisfies (2.1) but not (2.4) since $f(a,m) \neq f(1, am)$, a contradiction. \hfill\box

For later use, we collect some properties of zad modules as follows.

**Proposition 2.5.** We have that

(i) the algebra $A$ is zpd if and only if every $A$-module is zad;

(ii) a homomorphic image of a zad $A$-module is also zad;

(iii) the direct sum of a family of $A$-modules is zad if and only if every summand is zad.

**Proof.** (i) Let $A$ be zpd. For every $A$-module $V$, let $f : A \times V \to F$ be an arbitrary bilinear map satisfying (2.1). For each $v \in V$, we can define a bilinear map $f_v : A \times A \to F$ by $f_v(x,y) = f(x,yv)$. It is clear that $f_v(a,b) = 0$ whenever $ab = 0$. Since $A$ is zpd, $f_v(x,1) = f_v(1,x)$ for all $x \in A$. In other words, $f(x,v) = f(1,xv)$
for all $x \in A, v \in V$ and hence $V$ is zad by (2.4). The converse is clear from the Definition 2.1.

(ii) Let $\varphi : V \to U$ be a surjective homomorphism of $A$-modules. If $V$ is zad, then $x \otimes v - 1 \otimes xv \in F$-span$\{a \otimes m \in A \otimes V \mid am = 0\}$ for all $x \in A, v \in V$ by Lemma 2.4. For every $x \in A, u \in U$, the surjection of $\varphi$ shows that $x \otimes u - 1 \otimes xu \in F$-span$\{a \otimes \varphi(m) \in A \otimes U \mid am = 0\}$. Note that $F$-span$\{a \otimes \varphi(m) \in A \otimes U \mid am = 0\}$ is a subset of $F$-span$\{a \otimes u \in A \otimes U \mid au = 0\}$, since $a\varphi(m) = \varphi(am) = 0$. Then again using Lemma 2.4, we obtain that $U$ is a zad $A$-module.

(iii) By (ii), one only need to prove the “if” part. Assume that $V_i, i \in I$ is a family of zad $A$-modules, and $V = \bigoplus_{i \in I} V_i$ is the direct sum. Suppose that $f : A \times V \to F$ is a bilinear map satisfying (2.1). For each $i \in I$, let $\theta_i : V_i \to V$ be the natural injection, and define $f_i : A \times V_i \to F$ by sending $(a, v_i)$ to $f(a, \theta_i(v_i))$.

It follows easily that $f_i$ is a bilinear map and satisfies (2.1). Since $V_i$ is zad, we have $f_i(x, v_i) = f_i(1, xv_i)$ for all $x \in A$ and $v_i \in V_i$. Now for each $x \in A$ and $(v_i)_{i \in I} \in V$, where at most finitely many $v_i$ are nonzero, one has

$$f(x, (v_i)_{i \in I}) = \sum_{i \in I} f(x, \theta_i(v_i)) = \sum_{i \in I} f_i(x, v_i) = \sum_{i \in I} f_i(1, xv_i) = f(1, x(v_i)_{i \in I}).$$

This proves that $V$ is zad. \hfill $\square$

We now want to study when an irreducible module is zad. Let $V$ be an $A$-module. We denote $\text{Ann}_A(V)$ the annihilator ideal of $V$ in $A$, i.e.,

$$\text{Ann}_A(V) = \{ a \in A \mid av = 0, \forall v \in V \}.$$ 

Let $J$ be an ideal of $A$ contained in $\text{Ann}_A(V)$. Then $V$ becomes an $A/J$-module in a natural way. The following useful lemma will be frequently used in later proofs.

**Lemma 2.6.** Let $A$ be an algebra and $V$ be an $A$-module. Suppose that $J$ is an ideal of $A$ contained in $\text{Ann}_A(V)$. Then $V$ is a zad $A$-module if and only if $V$ is a zad $A/J$-module.

**Proof.** For simplicity, we write $\tilde{A}$ for $A/J$.

($\Rightarrow$). For every bilinear map $f : A \times V \to F$ satisfying (2.1), we define a bilinear map $\tilde{f} : \tilde{A} \times V \to F$ by $\tilde{f}(\tilde{x}, v) = f(x, v)$, where $\tilde{x}$ denote the canonical image of $x$ in $\tilde{A}$. Clearly, $\tilde{f}$ satisfies (2.1) and hence $\tilde{f}(\tilde{x}, v) = \tilde{f}(1, xv)$ for all $x \in A, v \in V$. Therefore $\tilde{f}(\tilde{x}, v) = f(1, xv) = f(1, \tilde{v})$, i.e., $f$ satisfies (2.4).

($\Leftarrow$). For every bilinear map $f : A \times V \to F$ satisfying (2.1), we define $\tilde{f} : \tilde{A} \times V \to F$ by $\tilde{f}(\tilde{x}, v) = f(x, v)$. We claim that $\tilde{f}$ is well-defined. In fact, for any $x_1, x_2 \in A$, if $\tilde{x}_1 = \tilde{x}_2$, we have $x_1 - x_2 \in J \subseteq \text{Ann}_A(V)$. Then $(x_1 - x_2)v = 0$ for all $v \in V$, and hence $f(x_1 - x_2, v) = 0$. In other words, $f(x_1, v) = f(x_2, v)$ for all $v \in V$ and $\tilde{f}$ is well-defined. Now it is easy to check that $\tilde{f}$ is a bilinear map satisfying (2.1). Since $V$ is a zad $\tilde{A}$-module, $\tilde{f}(\tilde{x}, v) = \tilde{f}(\tilde{1}, \tilde{v}) = \tilde{f}(\tilde{1}, xv)$ for all $x \in A, v \in V$. Therefore, $f$ satisfies (2.4). \hfill $\square$

**Proposition 2.7.** Let $V$ be a finite dimensional irreducible $A$-module. Then $V$ is zad if and only if $\dim V > \dim \text{End}_A(V)$ or $\dim \text{End}_A(V) = 1$.

**Proof.** Since $V$ is finite dimensional, then $\tilde{A} := A/\text{Ann}_A(V)$ is also finite dimensional, which has a faithful irreducible module. Therefore, $\tilde{A}$ is a simple algebra. The well-known Wedderburn-Artin theorem implies that $\tilde{A}$ is isomorphic to the full matrix algebra $M_n(D)$, with $D$ a division $F$-algebra and $n \geq 1$. 


(⇒). If \( V \) is a zad \( A \)-module, then \( V \) is a zad \( \bar{A} \)-module by Lemma 2.6. From Proposition 2.5 (iii), we deduce that the algebra \( \bar{A} \) is zero product determined. Assume that \( \dim \operatorname{End}_A(V) \neq 1 \). Then \( \dim D > 1 \) since \( \operatorname{End}_A(V) = \operatorname{End}_A(V) \cong D^{op} \), the opposite algebra of \( D \). We have known that \( M_n(D) \) is zero product determined for \( n \geq 2 \) by [2, Proposition 2.18], see also [3], and \( D \) is zero product determined if and only if \( D = F \) by [2, Proposition 2.8]. Therefore, \( n \geq 2 \) and \( \dim V = n \dim D > \dim \operatorname{End}_A(V) \)

(⇐). If \( \dim \operatorname{End}_A(V) = 1 \), then \( D = F \) since \( \operatorname{End}_A(V) = D^{op} \), which in turn implies that \( \bar{A} \cong M_n(F) \) is a zero product determined algebra by [2, Propositions 2.8 and 2.18]. If \( \dim V > \dim \operatorname{End}_A(V) \), then \( n \geq 2 \), and \( \bar{A} \cong M_n(D) \) is a zero product determined algebra by [2, Proposition 2.18]. Hence, in both cases, \( V \) is a zad \( A \)-module by Proposition 2.5 (i) and Lemma 2.6. This completes the proof of the theorem.

We should remark that up to now, the algebra \( A \) may be infinite dimensional. However, for the rest of this section, all algebras are assumed to be finite dimensional \( F \)-algebras, unless specified otherwise. If \( A \) is generated by idempotents, we want to know what properties should the irreducible \( A \)-modules have, which in turn motivates the following result.

**Proposition 2.8.** Let \( A \) be a finite dimensional \( F \)-algebra generated by idempotents. Then \( \dim V > \dim \operatorname{End}_A(V) \) or \( \dim \operatorname{End}_A(V) = 1 \) for each irreducible \( A \)-module \( V \) and \( \operatorname{Ext}^1_A(S,S) = 0 \) for all 1-dimensional \( A \)-modules \( S \).

To prove Proposition 2.8, we need some technical lemmas and the first one is a generalization of [2, Proposition 2.8]. Recall that an \( F \)-algebra is local if it has a unique maximal left ideal, which coincides with the unique maximal ideal.

**Lemma 2.9.** Let \( B \) be a finite dimensional zero product determined \( F \)-algebra. If \( B \) is local, then \( B = F \).

**Proof.** By [2, Theorem 3.7], the algebra \( B \) is generated by idempotents, and therefore \( B = F \), since it has to be generated by the only nonzero idempotent 1.

**Lemma 2.10.** Let \( B \) be a finite dimensional local \( F \)-algebra, and let \( X \) be a nonzero cyclic \( B \)-module. Then \( X \) is zad as a \( B \)-module if and only if \( X \) is 1-dimensional.

**Proof.** We only need to prove the necessity by Proposition 2.7. Let \( m \) be the unique maximal ideal of \( B \). We denote \( D := B/m \). Then \( D \) is a division algebra. We first prove that \( D \cong F \). Actually, by Nakayama’s Lemma, the quotient \( X/mX \) is a nonzero cyclic \( D \)-module. This implies that \( X/mX \cong D \) as \( D \)-modules. By Proposition 2.5 (ii) and Lemma 2.6, the quotient \( X/mX \) is a zad \( D \)-module. This implies that \( D \) is a zero product determined algebra. It follows from Lemma 2.9 (or [2, Proposition 2.8]) that \( D \cong F \). Therefore, \( B = F \oplus m \) as an \( F \)-space.

To show \( X \) is 1-dimensional, it suffices to show that \( mX = 0 \). Suppose contrarily that \( mX \neq 0 \). Then, by Nakayama’s Lemma, \( m^2X \neq mX \), so \( X/m^2X \) is not isomorphic to \( X/mX \) and has dimension > 1. Since \( X \) is a cyclic \( B \)-module, \( X/m^2X \) is also a cyclic \( B \)-module. For simplicity, we denote \( X/m^2X \) by \( X \). Then \( m^2X = 0 \). Let \( \bar{x} \) be a generator of \( X \). Then there is a surjective \( B \)-module homomorphism \( \pi : B \to \bar{X} \) sending \( b \) to \( b\bar{x} \) with kernel \( \operatorname{Ann}_B(\bar{x}) := \{ b \in B \mid b\bar{x} = 0 \} \). We claim that \( \operatorname{Ann}_B(\bar{X}) = \operatorname{Ann}_B(\bar{x}) \). The inclusions \( \operatorname{Ann}_B(\bar{x}) \subseteq \operatorname{Ann}_B(\bar{X}) \subseteq m \) are obvious. Conversely, since \( \operatorname{Ann}_B(\bar{x})m \subseteq m^2 \subseteq \operatorname{Ann}_B(\bar{x}) \) and \( B = F \oplus m \) as an \( F \)-space, one
gets \( \text{Ann}_B(\bar{x})B \subseteq \text{Ann}_B(\bar{x}) \). It follows that \( \text{Ann}_B(\bar{x})B \bar{x} = 0 \), that is, \( \text{Ann}_B(\bar{x})X = 0 \). Hence \( \text{Ann}_B(\bar{x}) \subseteq \text{Ann}_B(\bar{X}) \) and consequently \( \text{Ann}_B(\bar{x}) = \text{Ann}_B(\bar{X}) \). It follows that \( \bar{X} \cong B/\ker \pi = B/\text{Ann}_B(\bar{x}) = B/\text{Ann}_B(\bar{X}) \).

This means that \( \bar{X} \) is isomorphic to the regular module of \( B/\text{Ann}_B(\bar{X}) \). By Proposition 2.5 (i) and (ii), Lemma 2.6, we deduce that \( B/\text{Ann}_B(\bar{X}) \) is a zero product determined algebra, which is local. Then Lemma 2.9 forces that \( B/\text{Ann}_B(\bar{X}) \cong F \) and \( \bar{X} \cong F \), which is a contradiction since \( \dim \bar{X} > 1 \).

**Proof of Proposition 2.8.** Since \( A \) is generated by idempotents, it is zero product determined by [2, Proposition 2.16]. Therefore, \( \dim V > \dim \text{End}_A(V) \) or \( \dim \text{End}_A(V) = 1 \) for each irreducible \( A \)-module \( V \) by Proposition 2.5 (i) and Proposition 2.7.

Let \( e \) be a primitive idempotent such that \( A \neq A(1 - e)A \). We first claim that the algebra \( A/A(1 - e)A \) is local. In fact, the algebra \( eAe \) is local since \( e \) is primitive. Restricting the natural homomorphism \( A \to A/A(1 - e)A \) to the subalgebra \( eAe \), we get an algebra homomorphism \( \varphi : eAe \to A/A(1 - e)A \). For any \( a + A(1 - e)A \), it is clear that \( eae + A(1 - e)A = a + A(1 - e)A \), since \( a = eae + ea(1 - e) + (1 - e)ae + (1 - e)a(1 - e) \). Hence \( \varphi \) is surjective and \( A/A(1 - e)A \) is a local algebra.

Let \( R \) be the Jacobson radical of \( A \). Then each 1-dimensional \( A \)-module \( S \) is of the form \( S \cong Ae/Re \) with \( e \) a primitive idempotent. Assume that \( \text{Ext}^1_A(S, S) \neq 0 \). This means that there is a non-split short exact sequence

\[
0 \to S \to X \to S \to 0
\]

of \( A \)-modules. In other words, \( X \) is not isomorphic to the direct sum \( S \oplus S \). This implies that \( X \) is a 2-dimensional cyclic \( A \)-module. By Proposition 2.5 (i), \( X \) is also a zad \( A \)-module. Since \( S \) is 1-dimensional, the multiplicity of \( Ae \) as a direct summand of the regular \( A \)-module \( A \) is 1. This means there is no direct summands of \( A(1 - e) \) having \( S \) as a quotient. Hence \( (1 - e)S \cong \text{Hom}_A(A(1 - e), S) = 0 \) as \( F \)-spaces, or equivalently, \( A(1 - e)AS = 0 \). Particularly \( A \neq A(1 - e)A \) since \( S \neq 0 \), and thus \( A/A(1 - e)A \) is a local algebra. Applying the exact functor \( \text{Hom}_A(A(1 - e), \cdot) \) to short exact sequence \( 0 \to S \to X \to S \to 0 \), we obtain \( (1 - e)X \cong \text{Hom}_A(A(1 - e), X) = 0 \). Thus \( X \) is a zad module over the local algebra \( A/A(1 - e)A \) by Lemma 2.6. This leads to a contradiction by Lemma 2.10. \hfill \Box

Much to our surprise, the converse of Proposition 2.8 is also true.

**Theorem 2.11.** Let \( A \) be a finite dimensional \( F \)-algebra. Then \( A \) is generated by idempotents if and only if \( \dim V > \dim \text{End}_A(V) \) or \( \dim \text{End}_A(V) = 1 \) for each irreducible \( A \)-module \( V \) and \( \text{Ext}^1_A(S, S) = 0 \) for all 1-dimensional \( A \)-modules \( S \).

We only need to prove the sufficiency by Proposition 2.8 and the proof is organized by a series of lemmas. Let us begin by introducing some notations. Let \( A \) be a \( F \)-algebra, and let \( I \) be the ideal generated by all commutators of the form \( [e, x] \), where \( e \) is an idempotent and \( x \) is an arbitrary element in \( A \). Let \( E \) be the subalgebra generated by all idempotents. The task is to show that \( A = E \).

**Lemma 2.12.** Let \( e_1, e_2 \in A \) be two idempotents such that \( e_1e_2 = 0 = e_2e_1 \). Then every \( A \)-module homomorphism \( \psi : Ae_1 \to Ae_2 \) has \( \text{Im} \psi \subseteq Ie_2 \).
Proof. It is clear that $\psi(e_1) = e_1 \psi(e_1) = e_1 x e_2$ for some $x \in A$. Since $e_1 x e_2 = [e_1, x e_2] e_2 \in Ie_2$, the image $\psi(a e_1) = a \psi(e_1)$ of any element $a e_1$ belongs to $Ie_2$. □

Lemma 2.13. Let $e \in A$ be a primitive idempotent. If $Ae/Te$ is a nonzero zad $A$-module, then $Ae/Te \cong F$.

Proof. Notice that each element $(1-e)ae$ in $(1-e)Ae$ is equal to the commutator $[1-e, ae]$. This implies that $A(1-e)Ae \subseteq Ie$. Hence $A(1-e)Ae \subseteq \text{Ann}_A(Ae/Te)$, and thus, by Lemma 2.6, $Ae/Te$ is a nonzero zad cyclic module over $A/A(1-e)Ae$. Particularly $A/A(1-e)Ae$ is nonzero, and hence it is local by the proof of Proposition 2.8. Therefore $Ae/Te$ has dimension 1 by Lemma 2.10. □

From now on, we assume the algebra $A$ fulfills the assumptions of Theorem 2.11.

Lemma 2.14. Let $R$ be the Jacobson radical of $A$. Then $R \subseteq I$.

Proof. We only need to prove $Re \subseteq Ie$ for any primitive idempotent $e$. There are two cases occurring.

Case (i): Assume that $Ae$ is isomorphic to a direct summand of $A(1-e)$. Then there are $A$-module homomorphisms $g : Ae \to A(1-e)$ and $h : A(1-e) \to Ae$ such that $hg = \text{Id}_{Ae}$. By Lemma 2.12, we see that $Ae \cong \text{Im} \, h \subseteq Ie$, which in turn implies that $Re \subseteq Ie$.

Case (ii): Assume that $Ae$ is not isomorphic to any direct summand of $A(1-e)$. In this case, the multiplicity of principal indecomposable projective module $Ae$ as a direct summand of the regular module $Ae$ is 1. This means that the simple quotient $Ae/Re$ is 1-dimensional over its endomorphism algebra. In particular, $\dim Ae/Re = \dim \text{End}_A(Ae/Re)$ as $F$-spaces. Hence we have $\text{End}_A(Ae/Re) \cong F$ by the hypothesis. Thus $Ae/Re$ is a 1-dimensional zad $A$-module, by Proposition 2.7, and is the unique quotient module of $Ae$. Denote $Ae/Re$ by $S$. In this case, it is well-known that $\dim \text{Ext}^1_A(S, S)$ coincides with the multiplicity of $S$ as a direct summand of $Re/R^2e$. Since $\text{Ext}^1_A(S, S) = 0$, $Re/R^2e$ does not have $S$ as a direct summand. Hence $Ae$ is not a direct summand of the projective cover $P$ of $Re/R^2e$, which coincides with the projective cover of $Re$. Write $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ as the direct sum of indecomposable projective modules. Then each $P_i$ is a direct summand of $A(1-e)$ for $1 \leq i \leq n$. Denote $\psi_i$ the composition of maps $A(1-e) \to P_i \to P \to Re$. Let $\iota : Re \to Ae$ be the inclusion map. Then by Lemma 2.12, we have $\text{Im}(\psi_i) \subseteq Ie$ for all $i = 1, \cdots, n$. It follows that $Re = \sum_{i=1}^n \text{Im}(\psi_i) \subseteq Ie$. □

Lemma 2.15. With notations as above, we have $A = E$.

Proof. It is known that $I \subseteq E$ (see [2, Lemma 3.1]) and consequently $Ie \subseteq Ee \subseteq Ae$ for any primitive idempotent $e$. We only need to prove $Ee = Ae$. If $Ie = Ae$, the result is clear. Suppose that $Ie \neq Ae$. Then $Ae/Te \neq 0$. Since $Ae/Re$ is zad by Proposition 2.7 and $Re \subseteq Ie$ by Lemma 2.14, it follows that $Ae/Te$ is a quotient module of $Ae/Re$, and is again zad as an $A$-module. By Lemma 2.13, $Ae/Te$ is 1-dimensional. Now $e \notin Ie$ since $Ie \neq Ae$. However $e \in Ee$. This means that $Ee/Ie$ is a nonzero subspace of the 1-dimensional space $Ae/Ie$ and has to be the whole $Ae/Ie$. Hence $Ee = Ae$. □

We have completed the proof of Theorem 2.11 by Lemma 2.15. The reader may have found that the condition of $A$ being finite dimensional is not strictly necessary in Theorem 2.11.
Remark 2.16. The proof of Theorem 2.11 also works under the condition that $A$ is a semiperfect $F$-algebra with $A/R$ finite dimensional. Notice that the local algebras and artinian algebras are special examples of semiperfect algebras (see [16, Section 4.6]).

Recall that a finite dimensional $F$-algebra $A$ is splitting if every irreducible $A$-module has endomorphism algebra $F$. This happens, for example, when $F$ is algebraically closed.

Corollary 2.17. Let $A$ be a finite dimensional splitting $F$-algebra. Then $A$ is generated by idempotents if and only if $\text{Ext}^1_A(S,S) = 0$ for all 1-dimensional irreducible $A$-modules $S$.

Proof. In this case $\text{dim End}_A(V) = 1$ for all irreducible $A$-modules $V$. The corollary follows immediately from Theorem 2.11. □

3. Applications

Finally, we list two corollaries of Theorem 2.11, showing that many classes of algebras are generated by idempotents. Recall that the global dimension of an algebra $A$ is the supremum of projective dimensions of $A$-module.

Corollary 3.1. Let $A$ be a finite dimensional splitting $F$-algebra. If $A$ has finite global dimension, then $A$ is generated by idempotents.

Proof. Since $A$ is splitting, then $\text{dim End}_A(V) = 1$ for each irreducible $A$-module $V$. If $A$ has finite global dimension, then it follows from [12, Corollary 5.6] that every 1-dimensional $A$-module $S$ satisfies $\text{Ext}^1_A(S,S) = 0$. By Theorem 2.11, we deduce that $A$ is zero product determined. □

By [9, Appendix, Statement 9], a quasi-hereditary algebra is of finite global dimension. Hence Corollary 3.1 immediately implies the following

Corollary 3.2. If $A$ is a quasi-hereditary algebra over a splitting field $F$, then $A$ is generated by idempotents. Particularly, $q$-Schur algebras and symplectic $q$-Schur algebras are generated by idempotents.

Usually, a finite dimensional algebra $A$ generated by idempotents possesses a 'good' $\mathbb{Z}$-graded structure. Corollary 3.2 shows the possibility to study the graded representations of (symplectic) $q$-Schur algebras and the categorification of quantized universal enveloping algebras.

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