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$\mathcal{D}$-split sequences and derived equivalences

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Abstract

In this paper, we introduce the so-called $\mathcal{D}$-split sequences and show that each $\mathcal{D}$-split sequence gives rise to a derived equivalence via a tilting module. In particular, we obtain derived equivalences from Auslander–Reiten sequences via BB-tilting modules (or from n-almost split sequences via n-BB-tilting modules), and Auslander–Reiten triangles. Further, we recover n-almost split sequences from n-BB-tilting modules over n-Auslander algebras.

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Keywords: $\mathcal{D}$-split sequence; Auslander–Reiten triangle; BB-tilting module; Derived equivalence; Stable equivalence

1. Introduction

Derived equivalences and Auslander–Reiten sequences are fundamental notions in the modern representation theory of algebras and groups. On the one hand, it is well known that derived categories and equivalences are widely used in many aspects of mathematics, in particular, derived equivalences preserve many significant invariants of groups and algebras; for example, the number of irreducible representations, Cartan determinants, Hochschild cohomology groups, algebraic K-theory and G-theory (see [6,9,10], and others). One of the fundamental results on derived categories may be the Morita theory for derived categories established by Rickard in his several papers [24–26], see also [19], which says that two rings A and B are derived-equivalent

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if and only if there is a tilting complex T of A-modules such that B is isomorphic to the endomorphism ring of T. On the other hand, Auslander–Reiten sequences introduced by Auslander and Reiten in 1970's are of significant importance in the modern representation theory, they contain rich combinatorial and homological information on module categories (see [3]). There is a lot of literature on derived categories and Auslander–Reiten sequences individually. Of course, APR-tilting produces very special tilting modules, thus also a relationship between very special Auslander–Reiten sequences and derived equivalences. However, we do not know if there is any relationship between arbitrary Auslander–Reiten sequences and derived equivalences since their introduction. In other words, is it possible to construct derived equivalences from arbitrary Auslander–Reiten sequences or n-almost split sequences or Auslander–Reiten triangles?

In the present paper, we shall provide an affirmative answer to this question and construct derived equivalences from $\mathcal{D}$-split sequences (see Definition 3.1 below). Such sequences include Auslander–Reiten sequences, tilting complement sequences and sequences arising from Cohen–Macaulay modules, and occur very frequently in the representation theory of algebras and groups (see also the examples in Section 3 below). Our result in this direction can be stated in the following general form:

Theorem 1.1. *Let $\mathcal{C}$ be an additive category and M be an object in $\mathcal{C}$. Suppose*

$$X \longrightarrow M' \longrightarrow Y$$

is an $\operatorname{add}(M)$-split sequence in $\mathcal{C}$. Then the endomorphism ring $\operatorname{End}_{\mathcal{C}}(X \oplus M)$ of $X \oplus M$ is derived-equivalent to the endomorphism ring $\operatorname{End}_{\mathcal{C}}(M \oplus Y)$ of $M \oplus Y$ via a tilting module.

As a consequence, we see that, in Theorem 1.1, the finitistic dimension of $\operatorname{End}_{\mathcal{C}}(X \oplus M)$ is finite if and only if so is the finitistic dimension of $\operatorname{End}_{\mathcal{C}}(M \oplus Y)$.

Theorem 1.1 reveals actually a mysterious connection between arbitrary Auslander–Reiten sequences and derived equivalences, namely, we have the following corollary, which also shows that BB-tilting modules, introduced about three decades ago by Brenner and Butler in [5], are closely related to Auslander–Reiten sequences in a very natural way.

Corollary 1.2. *Let A be an Artin algebra.*

(1) *Suppose that $0 \longrightarrow X_i \longrightarrow M_i \longrightarrow X_{i-1} \longrightarrow 0$ is an Auslander–Reiten sequence of finitely generated A-modules for $i = 1, 2, \ldots, n$. Let $M = \bigoplus_{i=1}^{n} M_i$. If $X_0 \notin \operatorname{add}(M)$ and if $X_1, \ldots, X_n$ are pairwise non-isomorphic, then $\operatorname{End}_A(X_n \oplus M)$ is derived-equivalent to $\operatorname{End}_A(M \oplus X_0)$ via an n-BB-tilting module. In particular, if $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an Auslander–Reiten sequence, then $\operatorname{End}_A(X \oplus M)$ is derived-equivalent to $\operatorname{End}_A(M \oplus Y)$ via a BB-tilting module, and they have the same Cartan determinant.*
(2) *If A is self-injective and X is an A-module, then the endomorphism algebra $\operatorname{End}(A \oplus X)$ of $A \oplus X$ and the endomorphism algebra $\operatorname{End}_A(A \oplus \Omega(X))$ of $A \oplus \Omega(X)$ are derived-equivalent, where Ω is the syzygy operator.*

Thus, by Corollary 1.2, or more generally, by Proposition 3.15 in Section 3 below, one can produce a lot of derived equivalences from Auslander–Reiten sequences or n-almost split sequences. We stress that the algebra $\operatorname{End}_A(X \oplus M)$ and the algebra $\operatorname{End}_A(M \oplus Y)$ in Corollary 1.2 may be very different from each other (see the examples in Section 6), though the mesh diagram of the Auslander–Reiten sequence is somehow symmetric. Another result related to Corollary 1.2 is Proposition 5.1 in Section 5 below, which produces derived equivalences from Auslander–Reiten triangles in a triangulated category. In particular, we have

Corollary 1.3. *Let A be a self-injective Artin algebra. Suppose $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an Auslander–Reiten sequence such that $\Omega^{-1}(X) \notin \operatorname{add}(M \oplus Y)$. Then $\underline{\operatorname{End}}_A(X \oplus M)$ and $\underline{\operatorname{End}}_A(M \oplus Y)$ are derived-equivalent, where $\underline{\operatorname{End}}_A(N)$ denotes the stable endomorphism algebra of an A-module N.*

The converse problem of getting Auslander–Reiten sequences from BB-tilting modules is also considered in this paper. However, only partial result is obtained, namely, our Proposition 4.4 in Section 4 below shows that one can recover n-Auslander–Reiten sequences from n-BB-tilting modules over n-Auslander algebras.

The paper is organized as follows: In Section 2, we recall briefly some basic notions and a fundamental result of Rickard on derived categories. Our main result, Theorem 1.1, is proved in Section 3, where we also provide several generalizations of Corollary 1.2; among others is a formulation of Corollary 1.2(1) for n-almost split sequences. In Section 4, we point out that if a $\mathcal{D}$-split sequence is given by an Auslander–Reiten sequence then Theorem 1.1 can be viewed as an alternative version of a BB-tilting module. Thus n-almost split sequences or concatenations of n Auslander–Reiten sequences provide us a natural way to get n-BB-tilting modules (for definition, see Section 4). At the end of this section, we also prove that one can obtain n-almost split sequences from n-BB-titling modules over n-Auslander algebras. In particular, if $n = 1$, we can get Auslander–Reiten sequences from BB-tilting modules over Auslander algebras. In Section 5, we discuss how to get derived equivalences from Auslander–Reiten triangles in triangulated categories. In particular, Corollary 1.3 is proved in this section. In the last section, we present examples to illustrate our main result.

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2. Preliminaries

In this section, we recall some definitions and basic results required in our proofs.

Let $\mathcal{C}$ be an additive category. For two morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in $\mathcal{C}$, the composition of f with g is written as fg, which is a morphism from X to Z. Thus, for an object V in $\mathcal{C}$, we write f^* for the map $\operatorname{Hom}_{\mathcal{C}}(V, f) : \operatorname{Hom}_{\mathcal{C}}(V, X) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(V, Y)$ of abelian groups induced by f. Similarly, we write f_* for the map $\operatorname{Hom}_{\mathcal{C}}(Y, V) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(X, V)$. But for two functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$ of categories, their composition is denoted by GF. For an object X in $\mathcal{C}$, we denote by $\operatorname{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite sums of copies of X.

A complex $X^\bullet$ over $\mathcal{C}$ is a sequence of morphisms d_X^i between objects X^i in $\mathcal{C}$: $\cdots \longrightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \longrightarrow \cdots$, such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$. We write $X^\bullet = (X^i, d_X^i)$. The category of all complexes over $\mathcal{C}$ with the usual complex maps of degree zero is denoted by $\mathscr{C}(\mathcal{C})$. The homotopy and derived categories of complexes over $\mathcal{C}$ are denoted by $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$, respectively. The full subcategory of $\mathscr{C}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$ is denoted by $\mathscr{C}^b(\mathcal{C})$. Similarly, $\mathscr{K}^b(\mathcal{C})$ and $\mathscr{D}^b(\mathcal{C})$ denote the full subcategories consisting of bounded complexes in $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$, respectively.

An object X in a triangulated category $\mathcal{C}$ with a shift functor [1] is called *self-orthogonal* if $\mathrm{Hom}_{\mathcal{C}}(X, X[n]) = 0$ for all integers $n \neq 0$.

Let A be a ring with identity. By an A-module we shall mean a left A-module. We denote by A-Mod the category of all A-modules, by A-mod the category of all finitely presented A-modules, and by A-proj (respectively, A-inj) the category of finitely generated projective (respectively, injective) A-modules. Let X be an A-module. If $f : P \longrightarrow X$ is a projective cover of X with P projective, then the kernel of f is called a *syzygy* of X, denoted by $\Omega(X)$. Dually, if $g : X \longrightarrow I$ is an injective envelope with I injective, then the cokernel of g is called a *co-syzygy* of X, denoted by $\Omega^{-1}(X)$. Note that a syzygy or a co-syzygy of an A-module X is determined, up to isomorphism, uniquely by X. Hence we may speak of the syzygy and the co-syzygy of a module.

It is well known that $\mathscr{K}(A\text{-Mod})$, $\mathscr{K}^b(A\text{-Mod})$, $\mathscr{D}(A\text{-Mod})$ and $\mathscr{D}^b(A\text{-Mod})$ all are triangulated categories. Moreover, it is known that if $X \in \mathscr{K}^b(A\text{-proj})$ or $Y \in \mathscr{K}^b(A\text{-inj})$, then $\mathrm{Hom}_{\mathscr{K}^b(A\text{-Mod})}(X, Z) \simeq \mathrm{Hom}_{\mathscr{D}^b(A\text{-Mod})}(X, Z)$ and $\mathrm{Hom}_{\mathscr{K}^b(A\text{-Mod})}(Z, Y) \simeq \mathrm{Hom}_{\mathscr{D}^b(A\text{-Mod})}(Z, Y)$ for all $Z \in \mathscr{D}^b(A\text{-Mod})$.

For further information on triangulated categories, we refer to [10]. In [24], Rickard proved the following theorem.

Theorem 2.1. *For two rings A and B with identity, the following are equivalent:*

(a) $\mathscr{D}^b(A\text{-Mod})$ *and* $\mathscr{D}^b(B\text{-Mod})$ *are equivalent as triangulated categories;*
(b) $\mathscr{K}^b(A\text{-proj})$ *and* $\mathscr{K}^b(B\text{-proj})$ *are equivalent as triangulated categories;*
(c) $B \simeq \mathrm{End}_{\mathscr{K}^b(A\text{-proj})}(T^\bullet)$, *where $T^\bullet$ is a complex in* $\mathscr{K}^b(A\text{-proj})$ *satisfying*
    (1) *$T^\bullet$ is self-orthogonal in* $\mathscr{K}^b(A\text{-proj})$, *and*
    (2) $\mathrm{add}(T^\bullet)$ *generates* $\mathscr{K}^b(A\text{-proj})$ *as a triangulated category.*

If two rings A and B satisfy the equivalent conditions of Theorem 2.1, then A and B are said to be *derived-equivalent*. A complex $T^\bullet$ in $\mathscr{K}^b(A\text{-proj})$ satisfying the conditions (1) and (2) in Theorem 2.1 is called a *tilting complex* over A. Given a derived equivalence F between A and B, there is a unique (up to isomorphism) tilting complex $T^\bullet$ over A such that $FT^\bullet = B$. This complex $T^\bullet$ is called a tilting complex *associated* to F.

To get derived equivalences or tilting complexes, one may use tilting modules. Recall that a module T over a ring A is called a *tilting module* if

(1) T has a finite projective resolution $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow T \longrightarrow 0$, where each P_i is a finitely generated projective A-module;
(2) $\mathrm{Ext}^i_A(T, T) = 0$ for all $i > 0$, and
(3) there is an exact sequence $0 \longrightarrow A \longrightarrow T^0 \longrightarrow \cdots \longrightarrow T^m \longrightarrow 0$ of A-modules with each T^i in $\mathrm{add}(T)$.

It is well known that each tilting module supplies a derived equivalence. The following result in [8] is a generalization of a result in [10, Theorem 2.10].

Lemma 2.2. *Let A be a ring, ${}_AT$ a tilting A-module and $B = \mathrm{End}_A(T)$. Then A and B are derived-equivalent. In this case, we say that A is derived-equivalent to B via the tilting module ${}_AT$.*

Suppose that A is derived-equivalent to B via a tilting A-module ${}_AT$. Then B^{op} is derived-equivalent to A^{op} via the titling module ${}_{B^{\mathrm{op}}}T$. However, we do not know, in general, whether B is derived-equivalent to A via a tilting B-module.

In Theorem 2.1, if both A and B are left coherent rings, that is, rings for which the kernels of any homomorphisms between finitely generated projective modules are finitely generated, then A-mod and B-mod are abelian categories, and the equivalent conditions in Theorem 2.1 are further equivalent to the condition

(d) $\mathscr{D}^b(A\text{-mod})$ *and* $\mathscr{D}^b(B\text{-mod})$ *are equivalent as triangulated categories.*

A special class of coherent rings is the class of Artin algebras. Recall that an *Artin R-algebra* over a commutative Artin ring R is an R-algebra A such that A is a finitely generated R-module. For the module category over an Artin algebra, there is the notion of Auslander–Reiten sequences, or equivalently, almost split sequences. They play an important role in the modern representation theory of algebras and groups. Recall that a short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in A-mod is called an *Auslander–Reiten sequence* if

(1) the sequence does not split,
(2) X and Z are indecomposable,
(3) for any morphism $h : V \longrightarrow Z$ in A-mod, which is not a split epimorphism, there is a homomorphism $f' : V \longrightarrow Y$ in A-mod such that $h = f'f$, and
(4) for any morphism $h : X \longrightarrow V$ in A-mod, which is not a split monomorphism, there is a homomorphism $f' : Y \longrightarrow V$ in A-mod such that $h = ff'$.

For an introduction to Auslander–Reiten sequences and representations of Artin algebras, we refer the reader to the excellent book [3].

3. $\mathcal{D}$-split sequences and derived equivalences

In this section, we shall construct derived equivalences from Auslander–Reiten sequences. This builds a linkage between Auslander–Reiten sequences (or n-almost split sequences) and derived equivalences. We start first with a general setting by introducing the notion of $\mathcal{D}$-split sequences, which is a generalization of Auslander–Reiten sequences, and then use these sequences to construct derived equivalences between the endomorphism rings of modules involved in $\mathcal{D}$-split sequences. In Section 5, we shall consider the question of getting derived equivalences from Auslander–Reiten triangles.

Now we recall some definitions from [4].

Let $\mathcal{C}$ be a category, and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$, and X an object in $\mathcal{C}$. A morphism $f : D \longrightarrow X$ in $\mathcal{C}$ is called a *right $\mathcal{D}$-approximation* of X if $D \in \mathcal{D}$ and the induced map $\mathrm{Hom}_{\mathcal{C}}(-, f)$: $\mathrm{Hom}_{\mathcal{C}}(D', D) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(D', X)$ is surjective for every object $D' \in \mathcal{D}$. A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is called *right minimal* if any morphism $g : X \longrightarrow X$ with $gf = f$ is an automorphism. A minimal right $\mathcal{D}$-approximation of X is a right $\mathcal{D}$-approximation of X, which is right minimal. Dually, there is the notion of *left $\mathcal{D}$-approximations* and *minimal left*

$\mathcal{D}$*-approximations*. The subcategory $\mathcal{D}$ is called *contravariantly* (respectively, *covariantly*) *finite* in $\mathcal{C}$ if every object in $\mathcal{C}$ has a right (respectively, left) $\mathcal{D}$-approximation. The subcategory $\mathcal{D}$ is called *functorially finite* in $\mathcal{C}$ if $\mathcal{D}$ is both contravariantly and covariantly finite in $\mathcal{C}$.

Let $\mathcal{C}$ be an additive category and $e : X \longrightarrow X$ an idempotent morphism in $\mathcal{C}$. We say that e *splits* if there are objects X' and X'' in $\mathcal{C}$ and an isomorphism $\varphi : X' \oplus X'' \longrightarrow X$ such that $\varphi e = \pi\lambda\varphi$, where $\pi : X' \oplus X'' \longrightarrow X'$ and $\lambda : X' \longrightarrow X' \oplus X''$ are the canonical morphisms. In an arbitrary additive category, all idempotents need not split, but of course, in the case where $\mathcal{C}$ is an abelian category, every idempotent splits. If all idempotents in $\mathcal{C}$ split, then so does every idempotent in a full subcategory $\mathcal{D}$ of $\mathcal{C}$ which is closed under direct summands. Moreover, for an additive category $\mathcal{C}$ such that every idempotent splits, we know that, for each object M in $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(M, -)$ induces an equivalence between $\mathrm{add}(M)$ and $\mathrm{End}_{\mathcal{C}}(M)$-proj.

Definition 3.1. Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. A sequence

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

in $\mathcal{C}$ is called a $\mathcal{D}$-split sequence if

(1) $M \in \mathcal{D}$;
(2) f is a left $\mathcal{D}$-approximation of X, and g is a right $\mathcal{D}$-approximation of Y;
(3) f is a kernel of g, and g is a cokernel of f.

Recall that a morphism $f : Y \longrightarrow X$ in an additive category $\mathcal{C}$ is a *kernel* of a morphism $g : X \longrightarrow Z$ in $\mathcal{C}$ if $fg = 0$, and for any morphism $h : V \longrightarrow X$ in $\mathcal{C}$ with $hg = 0$, there is a unique morphism $h' : V \longrightarrow Y$ such that $h = h'f$. Note that if a morphism has a kernel in $\mathcal{C}$ then it is unique up to isomorphism. A *cokernel* of a given morphism in $\mathcal{C}$ is defined dually. If $f : Y \longrightarrow X$ in $\mathcal{C}$ is the kernel of a morphism $g : X \longrightarrow Z$ in $\mathcal{C}$, then f is a monomorphism, that is, if $h_i : U \longrightarrow Y$ is a morphism in $\mathcal{C}$ for $i = 1, 2$, such that $h_1 f = h_2 f$, then $h_1 = h_2$. Similarly, if $g : X \longrightarrow Z$ in $\mathcal{C}$ is the cokernel of a morphism $f : Y \longrightarrow X$ in $\mathcal{C}$, then g is an epimorphism, that is, if $h_i : Z \longrightarrow V$ is a morphism in $\mathcal{C}$ for $i = 1, 2$, such that $gh_1 = gh_2$, then $h_1 = h_2$.

Notice that $\mathcal{D}$-split sequences may split, whereas Auslander–Reiten sequences never split. Now we give some examples of $\mathcal{D}$-split sequences.

Examples. (a) Let A be an Artin algebra and $\mathcal{C} = A$-mod. Suppose $\mathcal{D}$ is the full subcategory of A-mod consisting of all projective–injective A-modules in $\mathcal{C}$. If $g : M \longrightarrow X$ is a surjective homomorphism in A-mod with $M \in \mathcal{D}$, then the sequence $0 \longrightarrow \ker(g) \longrightarrow M \longrightarrow X \longrightarrow 0$ is a $\mathcal{D}$-split sequence in $\mathcal{C}$, where $\ker(g)$ stands for the kernel of the homomorphism g.

(b) Let A be an Artin algebra and $\mathcal{C} = A$-mod. Suppose $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an Auslander–Reiten sequence. Let N be any module such that $M \in \mathrm{add}(N)$, but neither X nor Y belongs to $\mathrm{add}(N)$. If we take $\mathcal{D} = \mathrm{add}(N)$, then the Auslander–Reiten sequence is a $\mathcal{D}$-split sequence in $\mathcal{C}$.

(c) Let A be an Artin algebra and $M \in A$-mod. Recall that M is an *almost complete tilting module* if M is a partial tilting module (that is, M has finite projective dimension and $\mathrm{Ext}^i_A(M, M) = 0$ for all $i > 0$), and if the number of all non-isomorphic indecomposable direct summands of M equals the number of non-isomorphic simple A-modules minus 1. An indecomposable A-module $X \in A$-mod is called a *tilting complement* to M if $M \oplus X$ is a tilting

A-module. If an almost complete tilting module M is faithful, then there is an exact (not necessarily infinite) sequence

$$0 \longrightarrow X_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots$$

of A-modules such that $M_i \in \mathrm{add}(M)$. Moreover, if we define $X_i = \mathrm{coker}(f_i)$, the cokernel of f_i for $i \geqslant 1$, then $X_i \not\simeq X_j$ for $i \neq j$, $\mathrm{proj.dim}_A(X_i) \geqslant i$ for any i, and $\{X_i \mid i \geqslant 0\}$ is a complete set of non-isomorphic indecomposable tilting complements to M. In addition, each $X_i \longrightarrow M_{i+1}$ is a minimal left $\mathrm{add}(M)$-approximation of X_i and each $M_j \longrightarrow X_j$ is a minimal right $\mathrm{add}(M)$-approximation of X_j. Thus the sequence $0 \longrightarrow X_i \longrightarrow M_{i+1} \longrightarrow X_{i+1} \longrightarrow 0$ is an $\mathrm{add}(M)$-split sequence in A-mod for all $i \geqslant 0$. For further information on almost complete tilting modules and relationship with the generalized Nakayama conjecture, we refer the reader to [7] and [13].

(d) Let X be a Cohen–Macaulay R-module over an arbitrary ring R, that is, $\mathrm{Ext}^i_R(X, R) = 0$ for all $i \geqslant 1$. Then, for any $i \geqslant 0$, the sequence $0 \longrightarrow \Omega^{i+1}(X) \longrightarrow P_i(X) \longrightarrow \Omega^i(X) \longrightarrow 0$ is an $\mathrm{add}({}_RR)$-split sequence in the category of all R-modules, where $P_i(X)$ is the projective cover of $\Omega^i(X)$.

Now we consider some properties of $\mathcal{D}$-split sequences.

Proposition 3.2. *Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$.*

(1) *Suppose that $\mathcal{D}'$ is a full subcategory of $\mathcal{D}$. If a sequence $X \longrightarrow M \longrightarrow Y$ in $\mathcal{C}$ is a $\mathcal{D}$-split sequence with $M \in \mathcal{D}'$, then it is a $\mathcal{D}'$-split sequence in $\mathcal{C}$.*
(2) *If $X \longrightarrow M \xrightarrow{g} Y$ and $X' \longrightarrow M' \xrightarrow{g'} Y'$ are $\mathcal{D}$-split sequences in $\mathcal{C}$ such that both g and g' are right minimal, then $Y \simeq Y'$ if and only if the two sequences are isomorphic. Similarly, if $X \xrightarrow{f} M \longrightarrow Y$ and $X' \xrightarrow{f'} M' \longrightarrow Y'$ are $\mathcal{D}$-split sequences in $\mathcal{C}$ such that both f and f' are left minimal, then $X \simeq X'$ if and only if the two sequences are isomorphic.*

Proof. (1) is clear. We prove the first statement of (2). If the two sequences are isomorphic, then $X \simeq X'$ and $Y \simeq Y'$. Now assume that $\phi : Y \longrightarrow Y'$ is an isomorphism. Then $g\phi$ factorizes through g' since g' is a right $\mathcal{D}$-approximation of Y', and we may write $g\phi = hg'$ for some $h : M \longrightarrow M'$. Similarly, there is a homomorphism $h' : M' \longrightarrow M$ such that $g'\phi^{-1} = h'g$. Thus $hh'g = hg'\phi^{-1} = g\phi\phi^{-1} = g$ and $h'hg' = h'g\phi = g'\phi^{-1}\phi = g'$. Since both g and g' are right minimal, the morphisms hh' and $h'h$ are isomorphisms. It follows easily that h itself is an isomorphism. Since f' is the kernel of g' and since f is the kernel of g, there is a morphism $k : X \longrightarrow X'$ and a morphism $k' : X' \longrightarrow X$ such that $kf' = fh$ and $k'f = f'h^{-1}$. Thus $kk'f = kf'h^{-1} = fhh^{-1} = f$. It follows that $kk' = 1_X$ since f is a monomorphism. Similarly, we have $k'k = 1_{X'}$. Hence k is an isomorphism and the two sequences are isomorphic. Similarly, the other statements in (2) can be proved. □

To get $\mathcal{D}$-split sequences, we may use the following proposition. First, we introduce some notations. Let $\mathcal{D}$ be a full subcategory of a category $\mathcal{C}$. An object C in $\mathcal{C}$ is said to be *generated* (respectively, *co-generated*) by $\mathcal{D}$ if there is an epimorphism $D \longrightarrow C$ (respectively, monomorphism $C \longrightarrow D$) with $D \in \mathcal{D}$. We denote by $\mathscr{F}(\mathcal{D})$ the full subcategory of $\mathcal{C}$ consisting of all objects $C \in \mathcal{C}$ generated by $\mathcal{D}$, and by $\mathscr{S}(\mathcal{D})$ the full subcategory of $\mathcal{C}$ consisting of all objects $C \in \mathcal{C}$ co-generated by $\mathcal{D}$.

Proposition 3.3. *Suppose that A is a ring with identity and $\mathcal{C} = A$-Mod. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. We define $\mathscr{X}(\mathcal{D}) = \{X \in \mathcal{C} \mid \mathrm{Ext}^1_A(X, \mathcal{D}) = 0\}$ and $\mathscr{Y}(\mathcal{D}) = \{Y \in \mathcal{C} \mid \mathrm{Ext}^1_A(\mathcal{D}, Y) = 0\}$.*

(1) *If $\mathcal{D}$ is contravariantly finite in $\mathcal{C}$, then, for any A-module $Y \in \mathscr{F}(\mathcal{D}) \cap \mathscr{X}(\mathcal{D})$, there is a $\mathcal{D}$-split sequence $0 \longrightarrow X \longrightarrow D \longrightarrow Y \longrightarrow 0$ in $\mathcal{C}$.*
(2) *If $\mathcal{D}$ is covariantly finite in $\mathcal{C}$, then, for any A-module $X \in \mathscr{S}(\mathcal{D}) \cap \mathscr{Y}(\mathcal{D})$, there is a $\mathcal{D}$-split sequence $0 \longrightarrow X \longrightarrow D \longrightarrow Y \longrightarrow 0$ in $\mathcal{C}$.*

Proof. (1) Since Y is generated by $\mathcal{D}$, there is a surjective right $\mathcal{D}$-approximation of Y, say $g : M \longrightarrow Y$ with $M \in \mathcal{D}$. Let X be the kernel of g. Then it follows from the exact sequence $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ that the sequence $\mathrm{Hom}_A(M, D') \longrightarrow \mathrm{Hom}_A(X, D') \longrightarrow 0$ is exact since $Y \in \mathscr{X}(\mathcal{D})$. This implies that the homomorphism $X \longrightarrow M$ is a left $\mathcal{D}$-approximation of X. Thus we get a $\mathcal{D}$-split sequence in $\mathcal{C}$. (2) can be proved analogously. $\square$

Our main purpose of introducing $\mathcal{D}$-split sequences is to construct derived equivalences between the endomorphism algebras of objects appearing in $\mathcal{D}$-split sequences. The following lemma is useful in our discussions.

Lemma 3.4. *Let $\mathcal{C}$ be an additive category and M an object in $\mathcal{C}$. Suppose*

$$X \xrightarrow{f} M_n \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{t} M_1 \xrightarrow{g} Y$$

is a (not necessarily exact) sequence of morphisms in $\mathcal{C}$ with $M_i \in \mathrm{add}(M)$ satisfying the following conditions:

(1) *The morphism $f : X \longrightarrow M_n$ is a left $\mathrm{add}(M)$-approximation of X, and the morphism $g : M_1 \longrightarrow Y$ is a right $\mathrm{add}(M)$-approximation of Y.*
(2) *Put $V := X \oplus M$ and $W := M \oplus Y$. There are two induced exact sequences*

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, X) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(V, M_n) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M_1) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{C}}(V, Y),$$

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, W) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(M_1, W) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M_n, W) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(X, W).$$

Then $\mathrm{End}_{\mathcal{C}}(X \oplus M)$ is derived-equivalent to $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ via a tilting module of projective dimension at most n.

Proof. Let Λ be the endomorphism ring of V, and let T be the cokernel of the map $[t\ 0]^* : \mathrm{Hom}_{\mathcal{C}}(V, M_2) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M_1 \oplus M)$, that is, T is the direct sum of $\mathrm{Hom}_A(V, M)$ and the cokernel of $\mathrm{Hom}_A(V, t)$. Then, by (2), we have an exact sequence of Λ-modules:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M_n) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M_2) \\ &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M_1 \oplus M) \longrightarrow T \longrightarrow 0. \end{aligned} \qquad (*)$$

Note that all the Λ-modules appearing in the above exact sequence are finitely generated. Applying $\mathrm{Hom}_\Lambda(-, \mathrm{Hom}_{\mathcal{C}}(V, M))$ to this sequence, we get a sequence which is isomorphic to the following sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda\big(T, \mathrm{Hom}_{\mathcal{C}}(V, M)\big) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M_1 \oplus M, M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M_2, M) \longrightarrow \cdots$$
$$\longrightarrow \mathrm{Hom}_{\mathcal{C}}(M_n, M) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(X, M) \longrightarrow 0.$$

By the second exact sequence in (2) and the fact that f is a left $\mathrm{add}(M)$-approximation of X, we see that this sequence is exact. It follows that $\mathrm{Ext}^i_\Lambda(T, \mathrm{Hom}_{\mathcal{C}}(V, M)) = 0$ for all $i > 0$. Hence $\mathrm{Ext}^i_\Lambda(T, \mathrm{Hom}_{\mathcal{C}}(V, M')) = 0$ for all $i > 0$ and $M' \in \mathrm{add}(M)$. Thus, by applying $\mathrm{Hom}_\Lambda(T, -)$ to the exact sequence $(*)$, we get $\mathrm{Ext}^i_\Lambda(T, T) \simeq \mathrm{Ext}^{i+n}_\Lambda(T, \mathrm{Hom}_{\mathcal{C}}(V, X))$ for all $i > 0$. But $\mathrm{Ext}^{i+n}_\Lambda(T, \mathrm{Hom}_{\mathcal{C}}(V, X)) = 0$ for all $i > 0$ since the projective dimension of T is at most n. Thus $\mathrm{Ext}^i_\Lambda(T, T) = 0$ for all $i > 0$. Also, it follows from the exact sequence $(*)$ that the following sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, X \oplus M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M_n \oplus M) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M_2)$$
$$\longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M_1 \oplus M) \longrightarrow T \longrightarrow 0$$

is exact, where $\mathrm{Hom}_{\mathcal{C}}(V, X \oplus M)$ is just Λ and the other terms are in $\mathrm{add}(T)$. Thus T is a tilting Λ-module of projective dimension at most n.

Next, we show that $\mathrm{End}_\Lambda(T)$ and $\mathrm{End}_{\mathcal{C}}(W)$ are isomorphic. If $n = 1$, we set $V' = X$ and $a = [f, 0] : V' \longrightarrow M_1 \oplus M$. For $n \geqslant 2$, we set $V' = M_2$ and $a = [t, 0] : V' \longrightarrow M_1 \oplus M$. Let $u : V' \longrightarrow V'$ and $v : M_1 \oplus M \longrightarrow M_1 \oplus M$ be two morphisms in $\mathcal{C}$. The morphism pair (u, v) is an endomorphism of the sequence $V' \longrightarrow M_1 \oplus M$ if $ua = av$. Let E be the endomorphism ring of the sequence $V' \longrightarrow M_1 \oplus M$. Let I be the subset of E consisting of those endomorphisms (u, v) such that there exists $h : M_1 \oplus M \longrightarrow V'$ with $ha = v$. It is easy to check that I is an ideal of E. We shall show that $\mathrm{End}_{\mathcal{C}}(W)$ is isomorphic to the quotient ring E/I. Let b be the morphism $\left[\begin{smallmatrix} 0 & g \\ 1_M & 0 \end{smallmatrix}\right] : M_1 \oplus M \longrightarrow W := M \oplus Y$, that is, b is the interchange of the columns of the direct sum of the morphisms g and 1_M. Then, by the second exact sequence of the condition (2), we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(W, W) \xrightarrow{b_*} \mathrm{Hom}_{\mathcal{C}}(M_1 \oplus M, W) \xrightarrow{a_*} \mathrm{Hom}_{\mathcal{C}}\big(V', W\big). \qquad (**)$$

By considering the image of the identity 1_W under the composition $b_* a_*$, we have $ab = 0$. Thus, for each $(u, v) \in E$, we have $avb = uab = 0$, which means that vb is in the kernel of a_*. Therefore, there is a unique map $q : W \longrightarrow W$ such that $bq = vb$. Now, we define $\eta : E \longrightarrow \mathrm{End}_{\mathcal{C}}(W)$ by sending (u, v) to q. Then η is clearly a ring homomorphism. We claim that η is surjective. Indeed, since g is a right $\mathrm{add}(M)$-approximation of Y, it is easy to check that the map b is a right $\mathrm{add}(M)$-approximation of W. Let q be an endomorphism of W. Then there is a morphism $v : M_1 \oplus M \longrightarrow M_1 \oplus M$ such that $vb = bq$. By the first exact sequence in (2), we have the following exact sequence:

$$\mathrm{Hom}_{\mathcal{C}}\big(V', V'\big) \xrightarrow{a^*} \mathrm{Hom}_{\mathcal{C}}\big(V', M_1 \oplus M\big) \xrightarrow{b^*} \mathrm{Hom}_{\mathcal{C}}\big(V', W\big).$$

It follows from $avb = abq = 0$ that av is in the kernel of b^* and there is a map $u : V' \longrightarrow V'$ such that $ua = av$. This implies that (u, v) is in E and $\eta(u, v) = q$. Hence η is surjective.

Now, we determine the kernel of η. Note that, by the first exact sequence in (2), we have an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}\big(M_1 \oplus M, V'\big) \xrightarrow{a^*} \mathrm{Hom}_{\mathcal{C}}(M_1 \oplus M, M_1 \oplus M) \xrightarrow{b^*} \mathrm{Hom}_{\mathcal{C}}(M_1 \oplus M, W).$$

Now, suppose (u, v) is in the kernel of η. Then $vb = 0$, which means that v is in the kernel of b^*. Hence there is a map $h : M_1 \oplus M \longrightarrow V'$ such that $ha = v$. This implies $(u, v) \in I$. On the other hand, if $(u, v) \in I$ and if η sends (u, v) to q, then $bq = vb = hab = 0$, and q is in the kernel of b_*. By the exact sequence $(**)$, we have $q = 0$. Hence I is the kernel of η, and therefore $E/I \simeq \mathrm{End}_{\mathcal{C}}(W)$.

Let $\overline{E}$ be the endomorphism ring of the following complex of Λ-modules:

$$\mathrm{Hom}_{\mathcal{C}}\big(V, V'\big) \xrightarrow{a^*} \mathrm{Hom}_{\mathcal{C}}(V, M_1 \oplus M),$$

and $\overline{I}$ the ideal of $\overline{E}$ consisting of those $(\bar{u}, \bar{v})$ such that $\bar{h}a^* = \bar{v}$ for some $\bar{h} : \mathrm{Hom}_{\mathcal{C}}(V, M_1 \oplus M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, V')$. Similarly, we can show that $\mathrm{End}_{\Lambda}(T)$ is isomorphic to $\overline{E}/\overline{I}$. Finally, the natural map $e : E \longrightarrow \overline{E}$, which sends (u, v) to (u^*, v^*), is clearly an isomorphism of rings, and induces an isomorphism from the ring E/I to the ring $\overline{E}/\overline{I}$. Thus $\mathrm{End}_{\Lambda}(T)$ and $\mathrm{End}_{\mathcal{C}}(W)$ are isomorphic. The proof is completed. □

Remarks. (1) For an Auslander–Reiten sequence $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ in A-mod with A an Artin algebra, the proof that $\mathrm{End}({}_A T)$ of the tilting module T defined in Lemma 3.4 is isomorphic to $\mathrm{End}_A(M \oplus Y)$ can be carried out very easily.

(2) From the proof of Lemma 3.4 we see that if we replace the second exact sequence in (2) by the following two exact sequences

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, M) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(M_1, M) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M_n, M) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(X, M),$$

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(M_1, Y) \xrightarrow{t_*} \mathrm{Hom}_{\mathcal{C}}(M_2, Y),$$

then Lemma 3.4 still holds true. (Here $M_2 = X$ if $n = 1$.) However, in most of cases that we are interested in, the second exact sequence in (2) does exist.

(3) A special case of Lemma 3.4 is the n-almost split sequences in a maximal $(n-1)$-orthogonal subcategory studied in [18]. Let A be a finite-dimensional algebra over a field. Suppose $\mathcal{C}$ is a functorially finite and full subcategory of A-mod. Recall that $\mathcal{C}$ is called a *maximal $(n-1)$-orthogonal* subcategory if $\mathrm{Ext}^i_A(X, Y) = 0$ for all $X, Y \in \mathcal{C}$ and all $0 < i \leqslant n-1$, and $\mathcal{C} = \{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(C, X) = 0$ for $C \in \mathcal{C}$ and $0 < i \leqslant n-1\} = \{Y \in A\text{-mod} \mid \mathrm{Ext}^i_A(Y, C) = 0$ for $C \in \mathcal{C}$ and $0 < i \leqslant n-1\}$. In [18], it is shown that, for any non-projective indecomposable X in $\mathcal{C}$ (respectively, non-injective indecomposable Y in $\mathcal{C}$), there is an exact sequence

$$(*) \quad 0 \longrightarrow Y \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \longrightarrow 0$$

with $C_j \in \mathcal{C}$ and f_j being radical maps such that the following induced sequences of functors are exact on $\mathcal{C}$:

$$0 \longrightarrow \mathcal{C}(-, Y) \longrightarrow \mathcal{C}(-, C_{n-1}) \longrightarrow \cdots \longrightarrow \mathcal{C}(-, C_0) \longrightarrow \mathrm{rad}_{\mathcal{C}}(-, X) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{C}(X, -) \longrightarrow \mathcal{C}(C_0, -) \longrightarrow \cdots \longrightarrow \mathcal{C}(C_{n-1}, -) \longrightarrow \mathrm{rad}_{\mathcal{C}}(Y, -) \longrightarrow 0,$$

where $\mathrm{rad}_{\mathcal{C}}$ stands for the Jacobson radical of the category $\mathcal{C}$. Note also that f_0 is a minimal right almost split morphism and that f_n is a minimal left almost split morphism. The sequence $(*)$ is called an n-almost split sequence in [18]. So, if $Y \notin \mathrm{add}(C_0 \oplus \cdots \oplus C_{n-1})$, then $(*)$ satisfies the conditions of Lemma 3.4.

With Lemma 3.4 in mind, now we can show the significance of $\mathcal{D}$-split sequences for constructing derived equivalences by the following result.

Theorem 3.5. *Let $\mathcal{C}$ be an additive category and M an object in $\mathcal{C}$. Suppose*

$$X \xrightarrow{f} M' \xrightarrow{g} Y$$

is an $\mathrm{add}(M)$-split sequence in $\mathcal{C}$. Then the endomorphism ring $\mathrm{End}_{\mathcal{C}}(X \oplus M)$ of $X \oplus M$ is derived-equivalent to the endomorphism ring $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ of $M \oplus Y$ via a tilting module of projective dimension at most one.

Proof. Let $V = X \oplus M$ and $W = M \oplus Y$. We shall verify the conditions of Lemma 3.4 for $n = 1$. By the definition of a $\mathcal{D}$-split sequence, we see immediately that the condition (1) in Lemma 3.4 is satisfied, while the condition (2) in Lemma 3.4 is implied by the condition (3) in Definition 3.1: In fact, by applying $\mathrm{Hom}_{\mathcal{C}}(V, -)$ to the above sequence, we get a complex of abelian groups

$$(*) \quad 0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, X) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}\big(V, M'\big) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{C}}(V, Y).$$

Since f is a monomorphism, the map f^* is injective. Clearly, the image of the map f^* is contained in the kernel of the map g^*. Since f is a kernel of g, it is easy to see that the kernel of g^* is equal to the image of f^*. Thus $(*)$ is exact. Similarly, we see that the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, W) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}\big(M', W\big) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(X, W)$$

is exact. Thus Theorem 3.5 follows from Lemma 3.4 if we take $n = 1$. $\square$

In Theorem 3.5, the two rings $\mathrm{End}_{\mathcal{C}}(X \oplus M)$ and $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ are linked by a tilting module of projective dimension at most 1. This is precisely the case of classical tilting modules. Thus there is a nice linkage between the torsion theory defined by the tilting module in $\mathrm{End}_{\mathcal{C}}(X \oplus M)$-mod and the one in $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$-mod. For more details, we refer the reader to [5] and [12].

In the following, we deduce some consequences of Theorem 3.5. Since Auslander–Reiten sequences can be viewed as $\mathcal{D}$-split sequences, as explained in Example (b), we have the following corollary.

Corollary 3.6.

(1) *Let A be an Artin algebra, and let* $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ *be an Auslander–Reiten sequence in* A-mod. *Suppose N is an A-module in* A-mod *such that neither X nor Y belongs to* $\mathrm{add}(N)$. *Then* $\mathrm{End}_A(X \oplus N \oplus M)$ *and* $\mathrm{End}_A(N \oplus M \oplus Y)$ *are derived-equivalent. In particular,* $\mathrm{End}_A(X \oplus M)$ *and* $\mathrm{End}_A(M \oplus Y)$ *are derived-equivalent.*
(2) *Let A be a representation-finite Artin algebra, and let Γ be a set of representatives of isomorphism classes of all indecomposable A-modules. If X and $D\,\mathrm{Tr}(X)$ are in Γ, then* $\mathrm{End}_A(\bigoplus_{Y\in\Gamma\setminus\{X\}} Y)$ *and* $\mathrm{End}_A(\bigoplus_{Y\in\Gamma\setminus\{D\,\mathrm{Tr}(X)\}} Y)$ *are derived-equivalent.*

As another consequence of Theorem 3.5, we have the following corollary.

Corollary 3.7. *Let A be an Artin algebra and X a torsion-less A-module, that is, X is a submodule of a projective module in* A-mod. *If $f : X \longrightarrow P$ is a left* $\mathrm{add}({}_AA)$*-approximation of X, then* $\mathrm{End}_A(A \oplus X)$ *is derived-equivalent to* $\mathrm{End}({}_AA \oplus \mathrm{coker}(f))$ *via a tilting module. In particular, if A is a self-injective Artin algebra, then, for any X in* A-mod, *the algebra* $\mathrm{End}_A(A \oplus \Omega(X))$ *is derived-equivalent to the algebra* $\mathrm{End}_A(A \oplus X)$ *via a tilting module.*

Proof. Noting that f is injective, the short exact sequence $0 \longrightarrow X \xrightarrow{f} P \longrightarrow \mathrm{coker}(f) \longrightarrow 0$ is an $\mathrm{add}({}_AA)$-split sequence in A-mod. By Theorem 3.5, the corollary follows. □

As a consequence of Corollary 3.7, we get the following corollary.

Corollary 3.8. *Let A be a self-injective Artin algebra and X an A-module. Then the algebras* $\mathrm{End}_A(A \oplus X)$ *and* $\mathrm{End}_A(A \oplus \tau X)$ *are derived-equivalent, where τ stands for the Auslander–Reiten translation. Thus, for all $n \in \mathbb{Z}$, the algebras* $\mathrm{End}_A(A \oplus \tau^n X)$ *are derived-equivalent.*

Proof. Let ν be the Nakayama functor $D\,\mathrm{Hom}_A(-, A)$. It is known that if A is self-injective then $\tau \simeq \nu\Omega^2$, $\nu(A) = A$ and the Nakayama functor is an equivalence from A-mod to itself. Since the algebra $\mathrm{End}_A(A \oplus \tau X)$ is isomorphic to the algebra $\mathrm{End}_A(A \oplus \Omega^2(X))$, the corollary follows immediately from Corollary 3.7. □

Remark. If A is a finite-dimensional self-injective algebra, then, for any A-module X, it was shown in [22, Corollary 1.2] that all the algebras $\mathrm{End}_A(A \oplus X)$, $\mathrm{End}_A(A \oplus \Omega(X))$ and $\mathrm{End}_A(A \oplus \tau X)$ are stably equivalent of Morita type. Thus they are both derived-equivalent and stably equivalent of Morita type. For further information on stably equivalences of Morita type for general finite-dimensional algebras, we refer the reader to [17,20–22,27,28] and the references therein.

Now, we point out the following consequence of Theorem 3.5, which follows from the fact that derived equivalences preserve the number of non-isomorphic simple modules.

Corollary 3.9. *Let A be an Artin algebra and M an A-module. If* $0 \longrightarrow X \longrightarrow M' \longrightarrow Y \longrightarrow 0$ *is a $\mathcal{D}$-split sequence in* A-mod *with* $\mathcal{D} = \mathrm{add}(M)$, *then X and Y have the same number of non-isomorphic indecomposable direct summands which are not in* $\mathrm{add}(M)$.

Many other invariants of derived equivalences can be used to study the algebras $\mathrm{End}_A(X \oplus M)$ and $\mathrm{End}_A(M \oplus Y)$; for example, $\mathrm{End}_A(X \oplus M)$ has finite global dimension if and only if $\mathrm{End}_A(M \oplus Y)$ has finite global dimension. This follows from the fact that derived equivalences preserve the finiteness of global dimension. In fact, we have the following explicit formula by tilting theory (see [12] and [10, Proposition 3.4, p. 116], for example):

If $0 \longrightarrow X \longrightarrow M' \longrightarrow Y \longrightarrow 0$ is a $\mathcal{D}$-split sequence in A-mod with $\mathcal{D} = \mathrm{add}(M)$ for an A-module M in A-mod, then

$$\mathrm{gl.dim}\big(\mathrm{End}_A(X \oplus M)\big) - 1 \leqslant \mathrm{gl.dim}\big(\mathrm{End}_A(M \oplus Y)\big) \leqslant \mathrm{gl.dim}\big(\mathrm{End}_A(X \oplus M)\big) + 1,$$

where $\mathrm{gl.dim}(A)$ stands for the global dimension of A. Note that the global dimension of $\mathrm{End}_A(X \oplus M)$ may be infinite (see Example 2 in Section 6). Concerning global dimensions and Auslander–Reiten sequences, there is a related result which can be found in [15].

Note that if a derived equivalence between two rings A and B is obtained from a tilting module ${}_AT$, that is, there exists a tilting A-module ${}_AT$ such that $B \simeq \mathrm{End}_A(T)$, then the finitistic dimension of A is finite if and only if the finitistic dimension of B is finite (see [11]). Currently, it is shown in [23] that the finiteness of finitistic dimension is invariant under arbitrary derived equivalences. Recall that the *finitistic dimension* of an Artin algebra A, denoted by $\mathrm{fin.dim}(A)$, is defined to be the supremum of the projective dimensions of finitely generated A-modules of finite projective dimension. The finitistic dimension conjecture states that $\mathrm{fin.dim}(A)$ should be finite for any Artin algebra A. This conjecture has closely been related to many other homological conjectures in the representation theory of algebras. For some advances and further information on the finitistic dimension conjecture, we may refer the reader to the recent paper [29] and the references therein.

Thus we have the following corollary.

Corollary 3.10. *Let $\mathcal{C}$ be an additive category and M an object in $\mathcal{C}$. Suppose $X \xrightarrow{f} M' \xrightarrow{g} Y$ is an $\mathrm{add}(M)$-split sequence in $\mathcal{C}$. Then the finitistic dimension of $\mathrm{End}_{\mathcal{C}}(X \oplus M)$ is finite if and only if the finitistic dimension of $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ is finite.*

If A is an Artin R-algebra over a commutative Artin ring R and M is an A-bimodule, then $A \ltimes M$, the trivial extension of A by M is the R-algebra whose underlying R-module is $A \oplus M$, with multiplication given by

$$(\lambda, m)\big(\lambda', m'\big) = \big(\lambda\lambda', \lambda m' + m\lambda'\big)$$

for $\lambda, \lambda' \in A$, and $m, m' \in M$. It is shown in [25] that if A and B are finite-dimensional algebras over a field k that are derived-equivalent, then $A \ltimes D(A)$ is derived-equivalent to $B \ltimes D(B)$, where $D = \mathrm{Hom}_k(-, k)$. Note that $A \ltimes D(A)$ is a self-injective algebra and that a derived equivalence between two self-injective algebras implies a stable equivalence between them by [25]. For further information on the relationship between derived equivalences and stable equivalences, we refer the reader to the recent papers [16,17]. Since stable equivalences preserve representation dimension (see [1] for definition), we have the following corollary.

Corollary 3.11. *Let Λ be a finite-dimensional algebra over a field k and M a Λ-module in* Λ-mod. *Suppose*

$$X \xrightarrow{f} M' \xrightarrow{g} Y$$

is an add(M)*-split sequence in* Λ-mod*, and let* $A = \mathrm{End}_\Lambda(X \oplus M)$ *and* $B = \mathrm{End}_\Lambda(M \oplus Y)$*. Then* $A \ltimes D(A)$ *and* $B \ltimes D(B)$ *are derived-equivalent. In particular, the representation dimensions of* $A \ltimes D(A)$ *and* $B \ltimes D(B)$ *are equal.*

The following corollary is related to the Auslander–Reiten's conjecture: If X is a module over an Artin algebra A such that $\mathrm{Ext}^i_A(X \oplus A, X \oplus A) = 0$ for all $i \geqslant 1$, then X should be projective. Under the mentioned condition we see that X is a Cohen–Macaulay A-module. Since derived equivalences respect the number of simple modules, we have

Corollary 3.12. *Let* A *be an Artin algebra and* X *an indecomposable Cohen–Macaulay* A*-module. Then* $\mathrm{End}_A(A \oplus X)$ *and* $\mathrm{End}_A(A \oplus \Omega^i(X))$ *are derived-equivalent for all* $i \geqslant 0$*. In particular, if* X *is non-projective, then the* i*-th syzygy of* X *is isomorphic to a direct sum of copies of an indecomposable module* Y_i *for every* $i \geqslant 0$*.*

In the following, we consider several generalizations of Corollary 3.6, namely, we shall deal with the case of a finite family of Auslander–Reiten sequences.

Corollary 3.13. *Let* A *be an Artin algebra, and let* $0 \longrightarrow X_i \longrightarrow M_i \longrightarrow X_{i-1} \longrightarrow 0$ *be an Auslander–Reiten sequence in* A-mod *for* $i = 1, 2, \ldots, n$*. Let* $M = \bigoplus_{i=1}^n M_i$*. Then* $\mathrm{End}_A(X_n \oplus M)$ *is derived-equivalent to* $\mathrm{End}_A(M \oplus X_0)$ *via a tilting module of projective dimension at most* n*.*

Proof. First, we suppose $X_n \in \mathrm{add}(M)$. Then there is an M_i such that X_n is a direct summand of M_i, and therefore there is an irreducible map from X_i to X_n. It follows that there is an irreducible map from $X_0 = \tau^{-i} X_i$ to $X_{n-i} = \tau^{-i} X_n$, where τ stands for the Auslander–Reiten translation. Thus X_0 is a direct summand of M_{n-i+1}, which implies $X_0 \in \mathrm{add}(M)$. Hence $\mathrm{add}(X_n \oplus M) = \mathrm{add}(M) = \mathrm{add}(M \oplus X_0)$. Consequently, the algebras $\mathrm{End}_A(X_n \oplus M)$ and $\mathrm{End}_A(M \oplus X_0)$ are Morita equivalent. Thus $\mathrm{End}_A(X_n \oplus M)$ and $\mathrm{End}_A(M \oplus X_0)$ are, of course, derived-equivalent via a (projective) tilting module.

Next, we assume $X_n \notin \mathrm{add}(M)$. In this case, we claim that there is no integer $i \in \{0, 1, \ldots, n\}$ such that $X_i \in \mathrm{add}(M)$. If $X_0 \in \mathrm{add}(M)$, then there is an M_i, $1 \leqslant i \leqslant n$, such that X_0 is a direct summand of M_i. Thus there is an irreducible map from X_i to X_0. By applying the Auslander–Reiten translation, we see that there is an irreducible map from $X_n = \tau^{n-i} X_i$ to $X_{n-i} = \tau^{n-i} X_0$. Hence X_n is a direct summand of M_{n-i+1}, that is, X_n is in $\mathrm{add}(M)$. This is a contradiction and shows that X_0 does not belong to $\mathrm{add}(M)$. Suppose $X_i \in \mathrm{add}(M)$ for some $0 < i < n$. Then there is an integer $j \in \{1, 2, \ldots, n\}$ such that X_i is a direct summand of M_j. Clearly, $i \neq j$, and there is an irreducible map from X_i to X_{j-1}. On the one hand, if $i > j$, then there is an irreducible map from $X_n = \tau^{n-i} X_i$ to $X_{n-i+j-1} = \tau^{n-i} X_{j-1}$. This implies that X_n is a direct summand of M_{n-i+j}, which is a contradiction. On the other hand, if $i < j$, then there is an irreducible map from $X_0 = \tau^{-i} X_i$ to $X_{j-1-i} = \tau^{-i} X_{j-1}$. It follows that X_0 is a direct summand of M_{j-i}. This is again a contradiction. Hence there is no X_i belonging to $\mathrm{add}(M)$.

Now let m be the minimal integer in $\{0, 1, \ldots, n\}$ such that $X_n \simeq X_m$. If $m = 0$, then $\mathrm{add}(M \oplus X_n) = \mathrm{add}(M \oplus X_0)$. This means that the endomorphism algebras $\mathrm{End}_A(X_n \oplus M)$

and $\mathrm{End}_A(M \oplus X_0)$ are Morita equivalent. Now we assume $m > 0$. Then the A-modules $X_0, X_1, \ldots, X_m$ are pairwise non-isomorphic. We consider the sequence

$$X_m \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow X_0.$$

Since $X_m \notin \mathrm{add}(M)$, every homomorphism from X_m to M factorizes through the map $X_m \longrightarrow M_m$ in the Auslander–Reiten sequence starting at X_m. This means that the map $X_m \longrightarrow M_m$ is a left $\mathrm{add}(M)$-approximation of X_m. Similarly, the map $M_1 \longrightarrow X_0$ is a right $\mathrm{add}(M)$-approximation of X_0. Let $V = X_m \oplus M$. Then $X_i \notin \mathrm{add}(V)$ for all $i = 0, 1, \ldots, m-1$. It follows that we have exact sequences

$$0 \longrightarrow \mathrm{Hom}_A(V, X_i) \longrightarrow \mathrm{Hom}_A(V, M_i) \longrightarrow \mathrm{Hom}_A(V, X_{i-1}) \longrightarrow 0$$

for $i = 1, \ldots, m$. Connecting the above exact sequences, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(V, X_m) \longrightarrow \mathrm{Hom}_A(V, M_m) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(V, M_1) \longrightarrow \mathrm{Hom}_A(V, X_0).$$

This gives the first exact sequence in Lemma 3.4(2). The second exact sequence in Lemma 3.4(2) can be obtained similarly. Thus Corollary 3.13 follows immediately from Lemma 3.4. $\square$

Remark. In Corollary 3.13, if $X_n \notin \mathrm{add}(M)$ and $X_0, X_1, \ldots, X_n$ are pairwise non-isomorphic, then the tilting $\mathrm{End}(X \oplus M)$-module T defined in Lemma 3.4 has projective dimension n. Note that we always have $\mathrm{gl.dim}(\mathrm{End}_A(X \oplus M)) - n \leqslant \mathrm{gl.dim}(\mathrm{End}_A(M \oplus Y)) \leqslant \mathrm{gl.dim}(\mathrm{End}_A(X \oplus M)) + n$.

The following is another type of generalizations of Corollary 3.6.

Proposition 3.14. *Let A be an Artin algebra.*

(1) *Suppose $0 \longrightarrow X_i \longrightarrow M_i \longrightarrow Y_i \longrightarrow 0$ is an Auslander–Reiten sequence for $i = 1, 2, \ldots, n$. Let $X = \bigoplus_i X_i$, $M = \bigoplus_i M_i$ and $Y = \bigoplus_i Y_i$. If $\mathrm{add}(X) \cap \mathrm{add}(M) = 0 = \mathrm{add}(M) \cap \mathrm{add}(Y)$, then $\mathrm{End}_A(X \oplus M)$ and $\mathrm{End}_A(M \oplus Y)$ are derived-equivalent.*
(2) *Suppose $0 \longrightarrow X_1 \longrightarrow X_2 \oplus M_1 \longrightarrow Y_1 \longrightarrow 0$ and $0 \longrightarrow X_2 \longrightarrow Y_1 \oplus M_2 \longrightarrow Y_2 \longrightarrow 0$ are two Auslander–Reiten sequences in A-mod such that neither X_2 is in $\mathrm{add}(M_1)$ nor Y_1 is in $\mathrm{add}(M_2)$. Then $\mathrm{End}_A(X_1 \oplus M_1 \oplus M_2)$ and $\mathrm{End}_A(M_1 \oplus M_2 \oplus Y_2)$ are derived-equivalent.*

Proof. (1) Under our assumptions, the exact sequence $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an $\mathrm{add}(M)$-split sequence in A-mod. Therefore (1) follows from Theorem 3.5.

(2) There is an exact sequence

$$(*) \quad 0 \longrightarrow X_1 \longrightarrow M_1 \oplus M_2 \longrightarrow Y_2 \longrightarrow 0,$$

which can be constructed from the given two Auslander–Reiten sequences. Since Auslander–Reiten quivers of Artin algebras have no loops, the modules X_1 and Y_1 do not belong to $\mathrm{add}(X_2 \oplus M_1)$, and the modules X_2 and Y_2 do not belong to $\mathrm{add}(Y_1 \oplus M_2)$. Note that $X_1 \notin \mathrm{add}(Y_1 \oplus M_2)$ if and only if $Y_2 \notin \mathrm{add}(X_2 \oplus M_1)$. By [14], we see that $X_1 \simeq Y_1$ if and only if $X_2 \simeq Y_2$. Thus it is readily to see that $X_1 \in \mathrm{add}(M_2)$ if and only $Y_2 \in \mathrm{add}(M_1)$. Hence, if $X_1 \in$

add(M_2), or $Y_2 \in$ add(M_1), then the algebras $\mathrm{End}_A(X_1 \oplus M_1 \oplus M_2)$ and $\mathrm{End}_A(M_1 \oplus M_2 \oplus Y_2)$ are Morita equivalent. So we may assume that $X_1 \notin$ add(M_2) and $Y_2 \notin$ add(M_1). Thus, $X_1, Y_2 \notin$ add($M_1 \oplus M_2$), and we can verify that the homomorphism $X_1 \longrightarrow M_1 \oplus M_2$ in $(*)$ is a left add($M_1 \oplus M_2$)-approximation of X_1, and that the homomorphism $M_1 \oplus M_2 \longrightarrow Y_2$ in $(*)$ is a right add($M_1 \oplus M_2$)-approximation of Y_2. Thus $(*)$ is an add($M_1 \oplus M_2$)-split sequence in A-mod, and therefore the conclusion (2) follows from Theorem 3.5. $\square$

Remark. Usually, given two Auslander–Reiten sequences $0 \longrightarrow X_i \longrightarrow M_i \longrightarrow Y_i \longrightarrow 0$ $(1 \leqslant i \leqslant 2)$, we cannot get a derived equivalence between $\mathrm{End}_A(X_1 \oplus X_2 \oplus M_1 \oplus M_2)$ and $\mathrm{End}_A(M_1 \oplus M_2 \oplus Y_1 \oplus Y_2)$. For a counterexample, we refer the reader to Example 2 in the last section.

Now, we mention that, for n-almost split sequences studied in [18], we have a statement similar to Corollary 3.13.

Proposition 3.15. *Let $\mathcal{C}$ be a maximal $(n-1)$-orthogonal subcategory of A-mod with A a finite-dimensional algebra over a field $(n \geqslant 1)$. Suppose that X and Y are two indecomposable A-modules in $\mathcal{C}$ such that the sequence*

$$0 \longrightarrow X \xrightarrow{f} M_n \xrightarrow{t_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{t_2} M_1 \xrightarrow{g} Y \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$. Then $\mathrm{End}_A(X \oplus \bigoplus_{i=1}^n M_i)$ and $\mathrm{End}_A(\bigoplus_{i=1}^n M_i \oplus Y)$ are derived-equivalent.

Proof. Let $M := \bigoplus_{i=1}^n M_i$. Suppose that Y is a direct summand of some M_i. Then there is a canonical projection $\pi : M_i \longrightarrow Y$. Let $t_1 = g$ and $t_{n+1} = f$. We observe that all homomorphisms $t_1, \ldots, t_{n+1}$ are radical maps by the definition of n-almost split sequences. Hence the composition $t_{i+1}\pi$ cannot be a split epimorphism and consequently factorizes through $t_1 = g$, that is, $t_{i+1}\pi = u_1 g$ for a homomorphism $u_1 : M_{i+1} \longrightarrow M_1$. First, we assume $i \neq n$. Then $t_{i+2}u_1 g = t_{i+2}t_{i+1}\pi = 0$. By [18, Theorem 2.5.3], we have $t_{i+2}u_1 = u_2 t_2$ for a homomorphism $u_2 : M_{i+2} \longrightarrow M_2$. Similarly, we get a homomorphism $u_k : M_{i+k} \longrightarrow M_k$ such that $t_{i+k}u_{k-1} = u_k t_k$ for $k = 2, 3, \ldots, n-i$. This allows us to form the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u_{n-i+1}"] & M_n \arrow[r, "t_n"] \arrow[d, "u_{n-i}"] & M_{n-1} \arrow[r] \arrow[d, "u_{n-i-1}"] & \cdots \arrow[r] & M_{i+1} \arrow[r, "t_{i+1}"] \arrow[d, "u_1"] & M_i \arrow[r, "t_i"] \arrow[d, "\pi"] & M_{i-1} \\
M_{n-i+1} \arrow[r, "t_{n-i+1}"] & M_{n-i} \arrow[r, "t_{n-i}"] & M_{n-i-1} \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r, "g"] & Y. &
\end{tikzcd}$$

Note that if $i = n$ then the above diagram still makes sense. We claim that u_{n-i+1} is a split monomorphism. If this is not the case, then the map u_{n-i+1} factorizes through f. This means that there is some map $h_n : M_n \longrightarrow M_{n-i+1}$ such that $fh_n = u_{n-i+1}$. Then we have $f(u_{n-i} - h_n t_{n-i+1}) = fu_{n-i} - u_{n-i+1}t_{n-i+1} = 0$. By [18, Theorem 2.5.3], there is some homomorphism $h_{n-1} : M_{n-1} \longrightarrow M_{n-i}$ such that $t_n h_{n-1} = u_{n-i} - h_n t_{n-i+1}$, that is, $u_{n-i} = t_n h_{n-1} + h_n t_{n-i+1}$.

Similarly, we get $h_k : M_k \longrightarrow M_{k-i+1}$ such that $u_{k-i+1} = h_{k+1}t_{k-i+2} + t_{k-i+1}h_k$ for $k = n-2, n-3, \ldots, i$. Thus $t_{i+1}(\pi - h_i g) = t_{i+1}\pi - (u_1 - h_{i+1}t_2)g = t_{i+1}\pi - u_1 g = 0$. Hence $\pi - h_i g$ factorizes through t_i, say $\pi - h_i g = t_i h_{i-1}$. Then $\pi = h_i g + t_i h_{i-1}$, which is a radical map since both g and t_i are radical maps. This is a contradiction. Hence X is a direct summand of M_{n-i+1} and $\mathrm{add}(X \oplus M) = \mathrm{add}(M) = \mathrm{add}(M \oplus Y)$. Thus, $\mathrm{End}_A(X \oplus M)$ and $\mathrm{End}_A(M \oplus Y)$ are Morita equivalent.

Similarly, if X is a direct summand of some M_i, then Y is a direct summand of M_{n-i+1}. It follows that $\mathrm{End}_A(X \oplus M)$ and $\mathrm{End}_A(M \oplus Y)$ are Morita equivalent.

Now we assume that neither X nor Y is a direct summand of M. We use Lemma 3.4 to show Proposition 3.15. By a property of n-almost split sequences (see [18, Theorem 2.5.3]) and the fact that both X and Y do not lie in $\mathrm{add}(M)$, we see that f is a left $\mathrm{add}(M)$-approximation of X and g is a right $\mathrm{add}(M)$-approximation of Y. It remains to check the condition (2) in Lemma 3.4. However, it follows from [18, Theorem 2.5.3] (see Remark (3) at the end of the proof of Lemma 3.4) that we have two exact sequences

$$0 \longrightarrow \mathrm{Hom}_A(V, X) \xrightarrow{f^*} \mathrm{Hom}_A(V, M_n) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(V, M_1) \xrightarrow{g^*} \mathrm{Hom}_A(V, Y),$$

$$0 \longrightarrow \mathrm{Hom}_A(Y, W) \xrightarrow{g_*} \mathrm{Hom}_A(M_1, W) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(M_n, W) \xrightarrow{f_*} \mathrm{Hom}_A(X, W)$$

for $V := X \oplus M$ and $W := M \oplus Y$. Thus the condition (2) in Lemma 3.4 is satisfied. Consequently, Proposition 3.15 follows from Lemma 3.4. $\square$

4. Auslander–Reiten sequences and BB-tilting modules

In this section, we point out that, when we cofine our consideration to Auslander–Reiten sequences, the tilting modules defining the derived equivalences in Theorem 3.5 are of special form, namely, they are BB-tilting-modules in the sense of Brenner and Butler [5]. This shows that the tilting theory and the Auslander–Reiten theory are so beautifully integrated with each other. We first recall the BB-tilting-module procedure in [5], and then give a generalization of BB-tilting modules, namely, the notion of n-BB-tilting modules.

Let A be an Artin algebra and S a non-injective simple A-module with the following two properties:

(a) $\mathrm{proj.dim}_A(\tau^{-1}S) \leqslant 1$, and
(b) $\mathrm{Ext}^1_A(S, S) = 0$.

Here τ^{-1} stands for the inverse $\mathrm{Tr}\,D$ of the Auslander–Reiten translation, and $\mathrm{proj.dim}_A(S)$ means the projective dimension of S. We denote the projective cover of S by $P(S)$, and assume that ${}_AA = P(S) \oplus P$ such that there is not any direct summand of P isomorphic to $P(S)$. Let $T = \tau^{-1}S \oplus P$. It is well known that T is a tilting module. Such a tilting module is called a BB-*tilting module*. Unfortunately, to date, not much is known about BB-tilting modules. However, if S is a projective non-injective simple module, then $\mathrm{Hom}_A(D(A), S) = 0$, and therefore $\mathrm{proj.dim}_A(\tau^{-1}S) \leqslant 1$. Thus T is a BB-tilting module. This special case was first studied in [2], and the tilting module of this form is called an APR-*tilting module* in literature. It is widely used in the representation theory of algebras. Note that if S is a non-injective, projective simple A-module, then there is an Auslander–Reiten sequence

$$0 \longrightarrow S \longrightarrow P' \longrightarrow \tau^{-1}S \longrightarrow 0$$

in A-mod with P' projective.

Proposition 4.1. *Let A be an Artin algebra, and let $0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$ be an Auslander–Reiten sequence in A-*mod*. Further, let $N \in A$-*mod *such that $X \notin \mathrm{add}(N)$. We define $V := X \oplus N \oplus M$, $\Lambda = \mathrm{End}_A(V)$. Then the Λ-module* $\mathrm{coker}(f^*) \oplus \mathrm{Hom}_A(V, N \oplus M)$ *is a BB-tilting module. In particular, the derived equivalence between* $\mathrm{End}_A(X \oplus M)$ *and* $\mathrm{End}_A(M \oplus Y)$ *in Theorem* 3.5 *is given by a BB-tilting module.*

Proof. Set $\Lambda := \mathrm{End}_A(V)$. From the Auslander–Reiten sequence we have an exact sequence

$$(*) \quad 0 \longrightarrow \mathrm{Hom}_A(V, X) \xrightarrow{f^*} \mathrm{Hom}_A(V, M) \longrightarrow L \longrightarrow 0,$$

where L is the cokernel of $f^* = \mathrm{Hom}_A(V, f)$. (This is a minimal projective presentation of the Λ-module L.) Let $T := L \oplus \mathrm{Hom}_A(V, N \oplus M)$. Then T is a tilting module by the proof of Lemma 3.4. We shall show that T is a BB-tilting Λ-module. To prove this, it is sufficient to show that L is of the form $\tau^{-1}S$ for a simple Λ-module S.

If we apply $\mathrm{Hom}_\Lambda(-, \Lambda)$ to $(*)$, then we get an exact sequence of right Λ-modules:

$$\mathrm{Hom}_\Lambda\big(\mathrm{Hom}_A(V, M), \Lambda\big) \longrightarrow \mathrm{Hom}_\Lambda\big(\mathrm{Hom}_A(V, X), \Lambda\big) \longrightarrow \mathrm{Tr}_\Lambda(L) \longrightarrow 0,$$

which is isomorphic to the following exact sequence

$$\mathrm{Hom}_A(M, V) \xrightarrow{f_*} \mathrm{Hom}_A(X, V) \longrightarrow \mathrm{Tr}_\Lambda(L) \longrightarrow 0,$$

where Tr_Λ stands for the transpose over Λ. Note that the image of the map f_* is the radical of the indecomposable projective right Λ-module $\mathrm{Hom}_A(X, V)$. Thus $\mathrm{Tr}_\Lambda(L)$ is a simple right Λ-module, and consequently, $\tau_\Lambda L$ is isomorphic to the socle S of the indecomposable injective Λ-module $D\,\mathrm{Hom}_A(X, V)$. Hence $L \simeq \tau_\Lambda^{-1}S$. Since X is not a direct summand of M, we see that $\mathrm{Ext}^1_\Lambda(S, S) = 0$. Thus T is a BB-tilting Λ-module. If we take $N = 0$, then the BB-tilting module T induces the derived equivalence between $\mathrm{End}_A(M \oplus X)$ and $\mathrm{End}_A(M \oplus Y)$ in Theorem 3.5. Thus Proposition 4.1 follows. □

Remark. In case of APR-tilting modules, we can see that the Auslander–Reiten sequence

$$0 \longrightarrow S \longrightarrow P' \longrightarrow \tau^{-1}S \longrightarrow 0,$$

just given before Proposition 4.1, defines an APR-tilting module $T := P \oplus \tau^{-1}S$, that this sequence is an $\mathrm{add}(P)$-split sequence in A-mod, and that the derived equivalence between A and $\mathrm{End}_A(T)$ in Theorem 3.5 is given precisely by the APR-tilting module T.

Now, we introduce the notion of n-BB-tilting modules: Let A be an Artin R-algebra. Recall that we denote by Ω^n the n-th syzygy operator, and by Ω^{-n} the n-th co-syzygy operator. As usual, D is the duality of Artin R-algebras. Suppose that S is a simple A-module and n is a positive integer. If S satisfies

(a) $\mathrm{Ext}^j_A(D(A), S) = 0$ for all $0 \leqslant j \leqslant n-1$, and
(b) $\mathrm{Ext}^i_A(S, S) = 0$ for all $1 \leqslant i \leqslant n$,

then we say that S defines an n-BB-tilting module, and that the module $T := \tau^{-1}\Omega^{-n+1}(S) \oplus P$ is an n-BB-*tilting module*, where P is the direct sum of all non-isomorphic indecomposable projective A-modules which are not isomorphic to $P(S)$, the projective cover of S. Note that (a) implies that the injective dimension of S is at least n, and that the case $n = 1$ is just the usual BB-tilting module. The terminology is adjudged by the following lemma.

Lemma 4.2. *If S defines an n-BB-tilting A-module, then $T := \tau^{-1}\Omega^{-n+1}S \oplus P$ is a tilting module of projective dimension at most n.*

Proof. Let ν be the Nakayama functor $D\,\mathrm{Hom}_A(-, {}_AA)$. Suppose that the sequence

$$0 \longrightarrow S \longrightarrow \nu P_0 \longrightarrow \nu P_1 \longrightarrow \cdots \longrightarrow \nu P_n \longrightarrow \cdots$$

is a minimal injective resolution of S with all P_i projective. Since $\mathrm{Ext}^i_A(D(A), S) = 0$ for $0 \leqslant i \leqslant n-1$, we have the following exact sequence by applying $\mathrm{Hom}_A(D(A), -)$ to the injective resolution:

$$0 \longrightarrow \mathrm{Hom}_A\big(D(A), S\big) \longrightarrow \mathrm{Hom}_A\big(D(A), \nu P_0\big) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A\big(D(A), \nu P_n\big) \longrightarrow L \longrightarrow 0,$$

which is isomorphic to the following exact sequence

$$0 \longrightarrow 0 \longrightarrow P_0 \longrightarrow \cdots \longrightarrow P_n \longrightarrow L \longrightarrow 0.$$

This shows that $L \simeq \mathrm{Tr}\, D\Omega_A^{-n+1}(S)$ and the projective dimension of L is at most n. Moreover, we have the following sequence:

$$(*) \quad 0 \longrightarrow \mathrm{Hom}_A(L, P) \longrightarrow \mathrm{Hom}_A(P_n, P) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(P_0, P) \longrightarrow 0.$$

Since $\mathrm{Hom}_A(\nu P_j, \nu P) \simeq \mathrm{Hom}_A(P_j, P)$, we see that $(*)$ is isomorphic to the sequence

$$0 \longrightarrow \mathrm{Hom}_A(L, P) \longrightarrow \mathrm{Hom}_A(\nu P_n, \nu P) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(\nu P_0, \nu P) \longrightarrow 0,$$

which is exact because $\mathrm{Hom}_A(-, \nu P)$ is an exact functor. Note that $\mathrm{Hom}_A(S, \nu P) = 0$ by the definition of P. This shows that $\mathrm{Ext}^i_A(L, P) = 0$ for all $i > 0$. Since $\mathrm{Ext}^i_A(S, S) = 0$ for all $1 \leqslant i \leqslant n$, this means that νP_0 is not a direct summand of νP_i for $1 \leqslant i \leqslant n$. Thus $P(S)$ is not a direct summand of P_i for $1 \leqslant i \leqslant n$, that is, $P_i \in \mathrm{add}(P)$ for all $1 \leqslant i \leqslant n$. Now, if we apply $\mathrm{Hom}_A(L, -)$ to the projective resolution of L, we get $\mathrm{Ext}^{n+i}_A(L, P_0) \simeq \mathrm{Ext}^i_A(L, L)$ for all $i \geqslant 1$. Hence $\mathrm{Ext}^i_A(L, L) = 0$ for all $i \geqslant 1$.

We note that $P_0 = P(S)$ and there is an exact sequence

$$0 \longrightarrow A \longrightarrow P \oplus P_1 \longrightarrow \cdots \longrightarrow L \longrightarrow 0.$$

Altogether, we have shown that T is a tilting module of projective dimension at most n. $\square$

Proposition 4.3.

(1) *Suppose* $0 \longrightarrow X_i \longrightarrow M_i \longrightarrow X_{i-1} \longrightarrow 0$ *is an Auslander–Reiten sequence in* A-mod *for* $i = 1, 2, \ldots, n$. *Let* $M = \bigoplus_{i=1}^{n} M_i$ *and* $V = X_n \oplus M$. *If* $X_n \notin \operatorname{add}(M)$ *and if* $X_0, X_1, \ldots, X_n$ *are pairwise non-isomorphic, then the* $\operatorname{End}_A(V)$*-module* $T := \operatorname{Hom}_A(V, M \oplus X_0)$ *is an* n*-BB-tilting module.*

(2) *Let* $\mathcal{C}$ *be a maximal* $(n-1)$*-orthogonal subcategory of* A-mod *with* A *a finite-dimensional algebra over a field* $(n \geqslant 1)$. *Suppose that* X *and* Y *are two indecomposable* A*-modules in* $\mathcal{C}$ *such that the sequence*

$$0 \longrightarrow X \xrightarrow{f} M_n \xrightarrow{t_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{t_2} M_1 \xrightarrow{g} Y \longrightarrow 0$$

is an n*-almost split sequence in* $\mathcal{C}$. *We define* $M := \bigoplus_{i=1}^{n} M_i$, $V = X \oplus M$, *and* L *to be the image of the map* $\operatorname{Hom}_A(V, g)$. *If* $X \notin \operatorname{add}(M)$, *then* $\operatorname{Hom}_A(V, M) \oplus L$ *is an* n*-BB-tilting* $\operatorname{End}_A(V)$*-module.*

Proof. The proof of (1) is similar to the one of Proposition 4.1. We leave it to the reader.

(2) We shall show that L is isomorphic to $\tau^{-1}\Omega_{\Lambda}^{-n+1}(S)$ with $S = \tau\Omega_{\Lambda}^{n-1}(L)$ being a simple Λ-module. It is easy to see that $D(S) = \operatorname{Tr}\Omega_{\Lambda}^{n-1}(L)$ is a simple right Λ-module. In fact, it is isomorphic to the top of the indecomposable right Λ-module $\operatorname{Hom}_A(X, V)$, and is not injective since $X \notin \operatorname{add}(\bigoplus_j M_j)$. Further, it follows from $X \notin \operatorname{add}(\bigoplus_i M_i)$ that we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_A(Y, V) \longrightarrow \operatorname{Hom}_A(M_1, V) \longrightarrow \operatorname{Hom}_A(M_2, V) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_A(M_n, V) \\ \longrightarrow \operatorname{Hom}_A(X, V) \longrightarrow \operatorname{Tr}\Omega_{\Lambda}^{n-1}(L) = D(S) \longrightarrow 0. \end{aligned}$$

If we apply $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(-, \Lambda)$ to this sequence, we can see that $\operatorname{Ext}^i_{\Lambda^{\mathrm{op}}}(D(S), \Lambda) = 0$ for all $0 \leqslant i \leqslant n-1$. This is just the condition (a) in the definition of n-BB-tilting modules. To see that $\operatorname{Ext}^i_{\Lambda}(S, S) = 0$ for all $1 \leqslant i \leqslant n$, we show that $\operatorname{Ext}^i_{\Lambda^{\mathrm{op}}}(D(S), D(S)) = 0$ for all $1 \leqslant i \leqslant n$. This means that the projective cover $\operatorname{Hom}_A(X, V)$ of the right Λ-module $D(S)$ is not a direct summand of $\operatorname{Hom}_A(M_i, V)$ for all $1 \leqslant i \leqslant n$. However, this follows from the assumption that $X \notin \operatorname{add}(\bigoplus_{j=1}^{n} M_j)$. Thus the condition (b) of n-BB-tilting modules is fulfilled. □

Remarks. (1) One can see that a non-injective simple A-module S defines an n-BB-tilting module if and only if (a′) $\operatorname{proj.dim}_A(\tau^{-1}\Omega^{-n+1}(S)) \leqslant n$, (b′) $\operatorname{Ext}^i_A(S, S) = 0$ for all $1 \leqslant i \leqslant n$ and (c′) $\operatorname{Ext}^i_A(D(A), S) = 0$ for all $1 \leqslant i \leqslant n-1$. Note that if a simple module S defines an n-BB-tilting module then the injective dimension of S is n if and only if $\operatorname{Hom}_A(\tau^{-1}\Omega^{-n+1}(S), A) = 0$.

(2) With the same method as in Proposition 4.3, we can prove the following fact:

Let $\mathcal{C}$ be a maximal $(n-1)$-orthogonal subcategory of A-mod with A a finite-dimensional algebra over a field $(n \geqslant 1)$. Suppose X and Y are two indecomposable A-modules in $\mathcal{C}$ such that the sequence

$$0 \longrightarrow X \xrightarrow{f} M_n \xrightarrow{t_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{t_2} M_1 \xrightarrow{g} Y \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$. We define $M = \bigoplus_{i=1}^{n} M_i$, $V = X \oplus M$, and $U = X \oplus M \oplus Y$. Let Σ be the endomorphism algebra of U. If $X \notin \operatorname{add}(M \oplus Y)$, then $T := \operatorname{Hom}_A(V, U) \oplus S^X$ is an $(n+1)$-BB-tilting right Σ-module, where S^X is the top of the right Σ-module $\mathrm{Hom}_A(X,U)$. If we define $\Delta = \mathrm{End}(T_\Sigma)$, then $\mathrm{Hom}_\Sigma(\mathrm{Hom}_A(V,U)_\Sigma, T_\Sigma) \oplus \mathrm{Hom}_\Sigma(\mathrm{Hom}_A(Y,U)_\Sigma, T_\Sigma)$ is an $(n+1)$-APR-tiling right Δ-module, that is, it is an $(n+1)$-BB-tilting right Δ-module defined by the projective simple right Δ-module $\mathrm{Hom}_\Sigma(S^X, T)$. Note that Δ is a one-point extension of $\mathrm{End}_A(V)$ because $\mathrm{Hom}_\Sigma(S^X, \Sigma) = 0$.

We have seen that each Auslander–Reiten sequence gives rise to a BB-tilting module. The converse question is:

Given a BB-tilting module T over an Artin algebra Λ, can we find an Artin algebra A, an A-module V and an Auslander–Reiten sequence $0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$ in A-mod such that $V = X \oplus M \oplus N$ with $X \notin \mathrm{add}(N)$, $\Lambda \simeq \mathrm{End}_A(V)$ and $T \simeq T'$? Here T' is the BB-tilting module $L \oplus \mathrm{Hom}_A(V, M \oplus N)$ with L the cokernel of $\mathrm{Hom}_A(V,f)$.

At moment, we are not able to answer this question in general, but we have the following partial result.

Let us recall the definition of n-Auslander algebras from [18]. By definition, an n-Auslander algebra Λ is the endomorphism algebra of some maximal $(n-1)$-orthogonal module M over a finite-dimensional algebra A. Again, by definition, the category $\mathrm{add}(M)$ contains all projective modules and all injective modules over A, and has n-almost split sequences.

Proposition 4.4. *Let Λ be an n-Auslander algebra, and let $T := \tau^{-1}\Omega^{-n+1}(S) \oplus P$ be an n-BB-tilting Λ-module. Then there is a finite-dimensional algebra A, a maximal $(n-1)$-orthogonal A-module M, and an n-almost split sequence*

$$0 \longrightarrow X \longrightarrow M_n \longrightarrow \cdots \xrightarrow{f} M_1 \longrightarrow Y \longrightarrow 0$$

in $\mathrm{add}(M)$ such that $\tau^{-1}\Omega^{-n+1}(S)$ is isomorphic to the cokernel of $f^* := \mathrm{Hom}_A(M, f)$.

Proof. Since Λ is an n-Auslander algebra, there is, by definition, a finite-dimensional algebra A and a maximal $(n-1)$-orthogonal A-module M such that $\Lambda = \mathrm{End}_A(M)$. For simplicity, in this proof, we shall denote $\mathrm{Hom}_A(-,-)$ by $(-,-)$.

Since the injective dimension of S is at least n, we have a minimal injective resolution of S

$$0 \longrightarrow S \longrightarrow D(X,M) \xrightarrow{D(g_*)} D(M_n, M) \longrightarrow \cdots \xrightarrow{D(f_*)} D(M_1, M) \longrightarrow \cdots,$$

where the A-modules $X, M_1, \ldots, M_n$ are in $\mathrm{add}(M)$. By the definition of n-BB-tilting modules, we have $\mathrm{Ext}^i_\Lambda(D\Lambda, S) = 0$ for all $0 \leqslant i < n$. By applying $\mathrm{Hom}_\Lambda(D\Lambda, -)$, we get an exact sequence which is isomorphic to the sequence

$$0 \longrightarrow (M,X) \xrightarrow{g^*} (M, M_n) \longrightarrow \cdots \longrightarrow (M, M_2) \xrightarrow{f^*} (M, M_1).$$

By definition, the cokernel of f^* is $\tau^{-1}\Omega^{-n+1}(S)$. Since ${}_AA \in \mathrm{add}(M)$, we get an exact sequence

$$0 \longrightarrow X \xrightarrow{g} M_1 \longrightarrow \cdots \xrightarrow{f} M_1 \longrightarrow Y \longrightarrow 0, \qquad (*)$$

where $Y = \operatorname{coker} f$. Clearly, g is a radical map, and therefore the indecomposable A-module X is not injective. By [18, Theorem 2.5.3], there is an n-almost split sequence in $\operatorname{add}(M)$:

$$0 \longrightarrow X \longrightarrow M'_n \longrightarrow \cdots \longrightarrow M'_1 \longrightarrow Y' \longrightarrow 0. \qquad (**)$$

Now, applying $D(-, M)$ to this sequence, we get the following exact sequence

$$0 \longrightarrow \operatorname{soc} D(X, M)(\simeq S) \longrightarrow D(X, M) \longrightarrow D\big(M'_n, M\big) \longrightarrow \cdots \longrightarrow D\big(Y', M\big) \longrightarrow 0.$$

This gives another minimal injective resolution of S. It follows that the sequences $(*)$ and $(**)$ are isomorphic. Hence the sequence $(*)$ is an n-almost split sequence such that $\tau^{-1}\Omega^{-n+1}(S)$ is isomorphic to $\operatorname{coker}(f^*)$. This finishes the proof. $\square$

5. Auslander–Reiten triangles and derived equivalences

By Corollary 3.6, one can get derived equivalences from Auslander–Reiten sequences. An analogue of Auslander–Reiten sequences in a triangulated category is the notion of Auslander–Reiten triangles. Thus, a natural question rises: is it possible to get derived equivalences from Auslander–Reiten triangles in a triangulated category? In this section, we shall discuss this question. First, let us briefly recall some basic definitions concerning Auslander–Reiten triangles. For more details, we refer the reader to [10].

Let R be a commutative ring. Let $\mathcal{C}$ be a triangulated R-category such that $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ has finite length as an R-module for every X and Y in $\mathcal{C}$. In this case, we say that $\mathcal{C}$ is a Hom-finite triangulated R-category. Suppose further that the category $\mathcal{C}$ is a Krull–Schmidt category. A triangle $X \xrightarrow{f} M \xrightarrow{g} Y \xrightarrow{w} X[1]$ in $\mathcal{C}$ is called an *Auslander–Reiten triangle* if

(AR1) X and Y are indecomposable;
(AR2) $w \neq 0$; and
(AR3) if $t : U \longrightarrow Y$ is not a split epimorphism, then $tw = 0$.

Note that neither f is a monomorphism nor g is an epimorphism in an Auslander–Reiten triangle. This is a difference of Auslander–Reiten triangles from $\mathcal{D}$-split sequences. Thus, Auslander–Reiten triangles in a triangulated category are not $\mathcal{D}$-split sequences. Also, Auslander–Reiten sequences in the module category of an Artin algebra in general may not give us Auslander–Reiten triangles in its derived module category. For Artin algebras, we even don't know whether their stable module categories have triangulated structures except in case that they are self-injective. In this case, Auslander–Reiten sequences can be extended to Auslander–Reiten triangles in their stable module categories. Note that $M = 0$ is possible in an Auslander–Reiten triangle. For example, in the stable module category of $k[x]/(x^2)$ with k a field, we have an Auslander–Reiten triangle $k \longrightarrow 0 \longrightarrow k \xrightarrow{1} \Omega^{-1}(k)$.

Recall that a morphism $f : U \longrightarrow V$ in a category $\mathcal{C}$ is called a *split monomorphism* if there is a morphism $g : V \longrightarrow U$ in $\mathcal{C}$ such that $fg = 1_U$; a *split epimorphism* if $gf = 1_V$; and an *irreducible* morphism if f is neither a split monomorphism nor a split epimorphism, and, for any factorization $f = f_1 f_2$ in $\mathcal{C}$, either f_1 is a split monomorphism or f_2 is a split epimorphism.

Suppose $X \xrightarrow{f} M \xrightarrow{g} Y \xrightarrow{w} X[1]$ is an Auslander–Reiten triangle in a triangulated category $\mathcal{C}$. Then we have the following basic properties:

(1) $fg = 0$ and $gw = 0$. Moreover, if $M \neq 0$, then both f and g are irreducible morphisms.
(2) If $s : X \longrightarrow U$ is not a split monomorphism, then s factorizes through f. Similarly, if $t : V \longrightarrow Y$ is not a split epimorphism, then t factorizes through g.
(3) Let V be an indecomposable object in $\mathcal{C}$. Then V is a direct summand of M if and only if there is an irreducible map from V to Y if and only if there is an irreducible map from X to V.

We mention that in any triangulated category $\mathcal{C}$ the functors $\mathrm{Hom}_{\mathcal{C}}(V, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, V)$ are cohomological functors for each object $V \in \mathcal{C}$ (see [10, Proposition 1.2, p. 4]).

The following is an expected result for Auslander–Reiten triangles.

Proposition 5.1. *Let $\mathcal{C}$ be a* Hom*-finite, Krull–Schmidt, and triangulated R-category. Suppose that $X \xrightarrow{f} M \xrightarrow{g} Y \xrightarrow{w} X[1]$ is an Auslander–Reiten triangle in $\mathcal{C}$ such that $X[1] \notin \mathrm{add}(M \oplus Y)$. If N is an object in $\mathcal{C}$ such that none of X, Y, $X[1]$ and $Y[-1]$ belongs to $\mathrm{add}(N)$, then $\mathrm{End}_{\mathcal{C}}(X \oplus N \oplus M)$ is derived-equivalent to $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus Y)$ via a tilting module. In particular, $\mathrm{End}_{\mathcal{C}}(X \oplus M)$ is derived-equivalent to $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ via a tilting module.*

Proof. First, if X is a direct summand of M, then there is an irreducible map from X to Y. It follows from the property (3) of Auslander–Reiten triangles that Y is a direct summand of M. Similarly, if Y is a direct summand of M, then so is X. Thus, if X or Y is in $\mathrm{add}(M)$, then $\mathrm{add}(N \oplus M \oplus X) = \mathrm{add}(N \oplus M \oplus Y) = \mathrm{add}(N \oplus M)$. In this case, both $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus Y)$ are Morita equivalent to $\mathrm{End}_{\mathcal{C}}(N \oplus M)$, and therefore $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus Y)$ are derived-equivalent. Now, we assume that neither X nor Y lies in $\mathrm{add}(M)$. For simplicity, we set $U := N \oplus M$, $V := X \oplus U$ and $W := U \oplus Y$. Denote by Λ the endomorphism ring of V. Since X and Y are not in $\mathrm{add}(U)$, we see that f is a left $\mathrm{add}(U)$-approximation of X and g is a right $\mathrm{add}(U)$-approximation of Y. To see that the condition (2) in Lemma 3.4 is satisfied, we consider the exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}\big(V, M[-1]\big) \xrightarrow{\delta} \mathrm{Hom}_{\mathcal{C}}\big(V, Y[-1]\big) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, X)$$
$$\longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, Y).$$

We have to show that the map δ is surjective. By assumption, we have $Y[-1] \notin \mathrm{add}(N)$ and $Y[-1] \not\simeq X$ since $Y \not\simeq X[1]$. If $Y[-1] \in \mathrm{add}(M)$, then there is an irreducible map from X to $Y[-1]$ by the property (3), and therefore there is an irreducible map from $X[1]$ to Y. It follows that $X[1]$ is a direct summand of M, which contradicts to our assumption that $X[1] \notin \mathrm{add}(M)$. This shows $Y[-1] \notin \mathrm{add}(M)$. Thus any morphism from V to $Y[-1]$ cannot be a split epimorphism. This implies that the map δ is surjective by the property (2) of Auslander–Reiten triangles since the triangle $X[-1] \longrightarrow M[-1] \longrightarrow Y[-1] \longrightarrow X$ is also an Auslander–Reiten triangle. Hence we have the following desired exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, Y).$$

Similarly, we can get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, W) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M, W) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, W).$$

Thus Proposition 5.1 follows from Lemma 3.4 by taking $n = 1$. $\square$

From Proposition 5.1 we get the following corollary.

Corollary 5.2. *Let A be a self-injective Artin algebra. Suppose $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an Auslander–Reiten sequence such that $\Omega^{-1}(X) \notin \mathrm{add}(M \oplus Y)$. Then $\underline{\mathrm{End}}_A(X \oplus M)$ and $\underline{\mathrm{End}}_A(M \oplus Y)$ are derived-equivalent, where $\underline{\mathrm{End}}_A(N)$ stands for the quotient of $\mathrm{End}_A(N)$ of an A-module N by the ideal of those endomorphisms of N, that factorize through a projective A-module.*

Proof. If A is a self-injective Artin algebra, then every Auslander–Reiten sequence $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ in A-mod can be extended to an Auslander–Reiten triangle

$$X \longrightarrow M \longrightarrow Y \longrightarrow \Omega_A^{-1} X$$

in the triangulated category A-$\underline{\mathrm{mod}}$ (for details, see [10]). Thus Corollary 5.2 follows. $\square$

Note that under the assumptions in Proposition 5.1 the corresponding statement of Proposition 4.1 holds true for Auslander–Reiten triangles.

Let us note that Corollary 5.2 may fail if A is not self-injective; for example, if we take A to be the path algebra (over a field k) of the quiver $2 \longrightarrow 1 \longleftarrow 3$, then there is an Auslander–Reiten sequence

$$0 \longrightarrow P(1) \longrightarrow P(2) \oplus P(3) \longrightarrow I(1) \longrightarrow 0,$$

where $P(i)$ and $I(i)$ stand for the projective and injective modules corresponding to the vertex i, respectively. Clearly, this is a counterexample. Also, this example shows that Corollary 3.8 may fail for non-self-injective algebras.

Finally, we remark that an analogous notion of $\mathcal{D}$-split sequences can be defined for triangulated categories. In this case the exactness condition (3) of Definition 3.1 will be replaced by triangles, we then speak of $\mathcal{D}$-split triangles instead of $\mathcal{D}$-split sequences. For example, mutations in a Calabi–Yau category provide $\mathcal{D}$-split triangles. In fact, let $\mathcal{C}$ be an n-Calabi–Yau category in the sense of Keller, and let T be an $(n-1)$-cluster tilting object with a decomposition $T = T' \oplus T''$ such that $\mathrm{add}(T_1) \cap \mathrm{add}(T_2) = 0$. Then there is a triangle

$$(*) \quad T_3 \xrightarrow{\alpha} T' \xrightarrow{\beta} T_1 \longrightarrow T_3[1]$$

such that β is a minimal right $\mathrm{add}(T_2)$-approximation of T_1. It is easy to check that α is then a minimal left $\mathrm{add}(T_2)$-approximation of T_3. So, the triangle $(*)$ is an $\mathrm{add}(T_2)$-split triangle. The object $T_3 \oplus T_2$ is called the mutation of T_1. We shall discuss this kind of relationship between $\mathcal{D}$-split triangles and derived equivalences in a forthcoming paper.

6. Examples

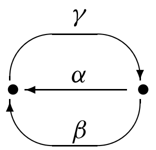
In this section, we illustrate our results with examples.

Example 1. Let k be a field, and let $A = k[x, y]/(x^2, y^2)$. If Y denotes the simple A-module, then there is an Auslander–Reiten sequence

$$0 \longrightarrow X \longrightarrow N \oplus N \longrightarrow Y \longrightarrow 0$$

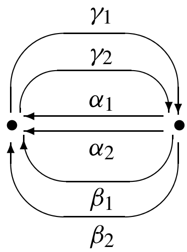
in A-mod. Note that $X = \Omega_A^2(Y)$ and N is the radical of A. By Theorem 1.1 or Corollary 1.2, the two algebras $\mathrm{End}_A(N \oplus Y)$ and $\mathrm{End}_A(N \oplus X)$ are derived-equivalent. Though the local diagram of the Auslander–Reiten sequence is reflectively symmetric, the two algebras $\mathrm{End}_A(N \oplus Y)$ and $\mathrm{End}_A(N \oplus X)$ are very different. This can be seen by the following presentations of the two algebras given by quivers with relations:

$\mathrm{End}_A(N \oplus Y)$



$\alpha\gamma = 0 = \beta\gamma.$

$\mathrm{End}_A(N \oplus X)$

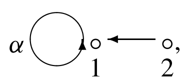


$$\gamma_i\alpha_j = 0 = \gamma_i\beta_j, \quad i \neq j,$$
$$\gamma_1\beta_1 = \gamma_2\beta_2, \qquad \gamma_1\alpha_1 = \gamma_2\alpha_2,$$
$$\alpha_1\gamma_2 = \beta_1\gamma_1, \qquad \alpha_2\gamma_2 = \beta_2\gamma_1.$$

Note that the algebra $\mathrm{End}_A(N \oplus Y)$ is a 7-dimensional algebra of global dimension 2, while the algebra $\mathrm{End}_A(N \oplus X)$ is a 19-dimensional algebra of global dimension 3. Hence the two algebras are not stably equivalent of Morita type since global dimension is invariant under stable equivalences of Morita type (see [27]). A calculation shows that the Cartan determinants of the two algebras equal 1.

Recall that the Cartan matrix of an Artin algebra A is defined as follows: Let $S_1, \dots, S_n$ be a complete list of non-isomorphic simple A-modules, and let P_i be a projective cover of S_i. We denote the multiplicity of S_j in P_i as a composition factor by $[P_i : S_j]$. The Cartan matrix of A is the $n \times n$ matrix $([P_i : S_j])_{1 \leqslant i,j \leqslant n}$, and its determinant is called the *Cartan determinant* of A. It is well known that the Cartan determinant is invariant under derived equivalences.

Example 2. Let A be the following algebra over a field given by quiver with relations:



$\alpha^2 = 0.$

We denote by $P(i)$, $I(i)$ and $S(i)$ the indecomposable projective, injective and simple modules corresponding to the vertex i, respectively. Let V be the 3-dimensional indecomposable module with $S(1)$ as its socle and $S(1) \oplus S(2)$ as its top. The Auslander–Reiten quiver of this algebra can be seen as follows:

$$\begin{tikzcd}
P(2) \arrow[dr] \arrow[rr, dotted, no head] & & I(2) & \\
& I(1) \arrow[ur] \arrow[dr] & & \\
V \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \arrow[dr] & \\
& S(1) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & V \\
& & P(1) \arrow[ur] \arrow[dr] & \\
& & & P(2).
\end{tikzcd}$$

As usual, the horizontal dotted lines denote the Auslander–Reiten translation, and the modules on the vertical dotted lines are identified correspondingly.

For this algebra, there are two Auslander–Reiten sequences:

$$0 \longrightarrow P(1) \longrightarrow V \oplus P(2) \longrightarrow I(1) \longrightarrow 0,$$
$$0 \longrightarrow V \longrightarrow I(1) \oplus S(1) \longrightarrow P(2)/\operatorname{soc}\bigl(P(2)\bigr) \longrightarrow 0.$$

These sequences satisfy the conditions in Proposition 3.14(2). So, we get a derived equivalence between $\operatorname{End}_A(P(1) \oplus P(2) \oplus S(1))$ and $\operatorname{End}_A(P(2) \oplus S(1) \oplus P(2)/\operatorname{soc}(P(2)))$. However, we cannot get a derived equivalence between

$$\operatorname{End}_A\bigl(P(1) \oplus V \oplus P(2) \oplus I(1) \oplus S(1)\bigr) \quad \text{and}$$
$$\operatorname{End}_A\bigl(V \oplus P(2) \oplus S(1) \oplus I(1) \oplus P(2)/\operatorname{soc}\bigl(P(2)\bigr)\bigr)$$

because the Cartan determinant of the former algebra is 1, and the one of the latter is -1. These are two algebras of the form in Proposition 3.14(1). Note that the two Auslander–Reiten sequences do not satisfy the conditions in Proposition 3.14(1).

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