On iterated almost ν-stable derived equivalences

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Abstract

In this paper, by characterizing iterated almost ν-stable derived equivalences, we give several sufficient conditions for a derived equivalence between general finite-dimensional algebras to induce a stable equivalence of Morita type. In particular, we prove the following: Let A and B be two finite-dimensional algebras over a field. Suppose that there is a derived equivalence between A and B induced by a tilting complex $T^{\bullet}$ over A. If each indecomposable projective A-module P without the property "$\nu_A^i P$ is projective for all $i \geqslant 0$" occurs only in the 0-degree term T^0 of $T^{\bullet}$ with multiplicity 1, then A and B are stably equivalent of Morita type.

1 Introduction

This is a continuation of the study on the relationship between derived equivalences and stable equivalences for general finite-dimensional algebras. In [7], we introduced a class of derived equivalences called almost ν-stable derived equivalences. The crucial property [7, Theorem 5.3] is that an almost ν-stable derived equivalence always induces a stable equivalence of Morita type, which generalizes a classical result of Rickard ([13, Corollary 5.5]): Derived equivalent self-injective algebras are stably equivalent of Morita type. The result [7, Theorem 5.3] also gives a sufficient condition for a derived equivalence between general finite-dimensional algebras to induce a stable equivalence of Morita type. Note that many homological dimensions, such as global dimension, finitistic dimension, and representation dimension, are not invariant under derived equivalences in general. But they are all preserved by stable equivalences of Morita type. So, this also helps us to compare the homological dimensions of derived equivalent algebras. For more information about stable equivalences of Morita type, we refer to the papers [3, 9, 10, 7].

Let us first recall the definition of almost ν-stable derived equivalences. Let $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between two finite-dimensional algebras A and B over a field k, where $\mathscr{D}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(B)$ stand for the derived categories of bounded complexes over A and B, respectively. We use F^{-1} to denote a quasi-inverse of F. The functor F is called an *almost ν-stable derived equivalence* if the following hold:

(1) *The tilting complex $T^{\bullet}$ associated to F has the following form:*

$$0 \longrightarrow T^{-n} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow 0$$

In this case, the tilting $\bar{T}^{\bullet}$ associated to F^{-1} has the following form (see [7, Lemma 2.1]):

$$0 \longrightarrow \bar{T}^0 \longrightarrow \bar{T}^1 \longrightarrow \cdots \longrightarrow \bar{T}^n \longrightarrow 0$$

(2) $\mathrm{add}(\bigoplus_{i=1}^n T^{-i}) = \mathrm{add}(\bigoplus_{i=1}^n \nu_A T^{-i})$ *and* $\mathrm{add}(\bigoplus_{i=1}^n \bar{T}^{i}) = \mathrm{add}(\bigoplus_{i=1}^n \nu_B \bar{T}^{i})$, *where ν is the Nakayama functor.*

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In general, the quasi-inverse of an almost ν-stable derived equivalence is not almost ν-stable. This motivates us to look for a more "balanced" notion. If a derived equivalence F is a composition $F_1F_2\cdots F_m$ with F_i or F_i^{-1} being an almost ν-stable derived equivalence for all i, then F is called an *iterated almost* ν*-stable derived equivalence*. By definition, the class of iterated almost ν-stable derived equivalences properly contains the class of almost ν-stable derived equivalences, and is closed under taking compositions and quasi-inverses. Clearly, an iterated almost ν-stable derived equivalence always induces a stable equivalence of Morita type, and therefore the involved algebras have many common homological dimensions. But the problem is:

Question: *Given a derived equivalence F, how to determine whether F is iterated almost ν-stable or not?*

A satisfactory answer to the above question will give us some new sufficient conditions for a derived equivalence between general finite-dimensional algebras to induce a stable equivalence of Morita type. In this paper, we give a complete answer to the above question. Let A be an algebra. We use ν_A-Stp to denote the full subcategory of A-mod consisting of all the projective A-modules P with the property that ν_A^iP is projective for all $i \geqslant 0$. For an A-module X, we write $\mathrm{top}(X)$ for the maximal semi-simple quotient module of X. For a complex $X^\bullet$ over A, we denote by $X^\pm$ the module $\bigoplus_{i\neq 0}X^i$. Our main result is the following theorem.

Theorem 1.1. *Let $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between two finite-dimensional algebras A and B over a field k, and let $T^\bullet$ and $\bar{T}^\bullet$ be tilting complexes associated to F and F^{-1}, respectively. Then the following are equivalent:*

(1) *The functor F is an iterated almost ν-stable derived equivalence.*

(2) $\mathrm{add}(\nu_A T^\pm) = \mathrm{add}({}_AT^\pm)$ *and* $\mathrm{add}(\nu_B\bar{T}^\pm) = \mathrm{add}({}_B\bar{T}^\pm)$.

(3) $T^\pm \in \nu_A$-Stp *and* $\bar{T}^\pm \in \nu_B$-Stp.

(4) *For each indecomposable projective A-module $P \notin \nu_A$-Stp, the image $F(\mathrm{top}(P))$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to a simple B-module.*

(5) *For each indecomposable projective A-module $P \notin \nu_A$-Stp, the following conditions are satisfied:*

(a) *$P \notin \mathrm{add}({}_AT^\pm)$;*

(b) *The multiplicity of P as a direct summand of $\mathrm{b}(T^\bullet)^0$ is 1, where the complex $\mathrm{b}(T^\bullet)$ is a direct sum of all non-isomorphic indecomposable direct summands of $T^\bullet$.*

Moreover, if one of the above equivalent conditions is satisfied, then the algebras A and B are stably equivalent of Morita type.

Theorem 1.1 tells us that, by checking the terms of tilting complexes, we can determine whether a derived equivalence is iterated almost ν-stable or not. If a derived equivalence F between two algebras A and B satisfies one of the equivalent conditions in Theorem 1.1, then F induces a stable equivalence of Morita type between A and B. Thus, one can use Theorem 1.1 to get stable equivalences of Morita type from derived equivalences. Comparing with [7, Theorem 5.3], here we only need to check the terms of the tilting complex $T^\bullet$ associated to the given derived equivalence F, while in [7, Theorem 5.3] one needs to consider both the tilting complex associated to F and that associated to F^{-1}.

Let us remark that the condition (4) arises naturally from a property of stable equivalences of Morita type. Suppose that the algebras A and B are indecomposable finite-dimensional algebras over a perfect field k. Let $\Phi : A\text{-}\underline{\mathrm{mod}} \longrightarrow B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence of Morita type. Then it can be deduced from [4, Proposition 3.4] that $\Phi(\mathrm{top}(P))$ is isomorphic in B-$\underline{\mathrm{mod}}$ to a simple B-module for all indecomposable projective A-modules P not in ν_A-Stp.

This paper is organized as follows. In Section 2, we shall fix some notations and recall some basic facts needed in our proofs. Theorem 1.1 will be proved in Section 3 after several lemmas. In Section 3, we also give a method to construct tilting complexes which induce iterated almost ν-stable derived equivalences.

2 Preliminaries

In this section, we shall recall some basic definitions and facts needed in our later proofs.

Throughout this paper, all algebras are finite-dimensional algebras over a fixed field k. All modules are finitely generated unitary left modules. For an algebra A, the category of A-modules is denoted by A-mod; the full subcategory of A-mod consisting of projective (respectively, injective) modules is denoted by A-proj (respectively, A-inj). The stable module category, denoted by A-$\underline{\text{mod}}$, is the quotient category of A-mod modulo the ideal generated by morphisms factorizing through projective modules. We denote by ν_A the usual Nakayama functor $D\mathrm{Hom}_A(-,A)$, where $D = \mathrm{Hom}_k(-,k)$ is the usual duality. Note that $\nu_A : A\text{-proj} \longrightarrow A\text{-inj}$ is an equivalence.

Let $\mathcal{C}$ be an additive category. The composition of two morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in $\mathcal{C}$ will be denoted by fg. For two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ of categories, their composition is denoted by GF. For an object X in $\mathcal{C}$, $\mathrm{add}(X)$ is the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite direct sums of copies of X.

A complex $X^\bullet$ over $\mathcal{C}$ is a sequence $\cdots \longrightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \cdots$ in $\mathcal{C}$ such that $d_X^i d_X^{i+1} = 0$ for all integers i. The category of complexes over $\mathcal{C}$ is denoted by $\mathscr{C}(\mathcal{C})$. The homotopy category of complexes over $\mathcal{C}$ is denoted by $\mathscr{K}(\mathcal{C})$. When $\mathcal{C}$ is an abelian category, the derived category of complexes over $\mathcal{C}$ is denoted by $\mathscr{D}(\mathcal{C})$. The full subcategory of $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$ is denoted by $\mathscr{K}^{\mathrm{b}}(\mathcal{C})$ and $\mathscr{D}^{\mathrm{b}}(\mathcal{C})$, respectively. As usual, for a given algebra A, we simply write $\mathscr{K}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(A)$ for $\mathscr{K}^{\mathrm{b}}(A\text{-mod})$ and $\mathscr{D}^{\mathrm{b}}(A\text{-mod})$, respectively.

It is well-known that, for an algebra A, $\mathscr{K}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(A)$ are triangulated categories. Moreover, it is known that if $X^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-proj})$ or $Y^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-inj})$, then $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(X^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(X^\bullet, Y^\bullet)$. For basic results on triangulated categories, we refer to Happel's book [5]. Throughout this paper, we use $X^\bullet[n]$ to denote the complex obtained by shifting $X^\bullet$ to the left by n degrees.

Let A be an algebra. A homomorphism $f : X \longrightarrow Y$ of A-modules is called a *radical map* if, for any module Z and homomorphisms $h : Z \longrightarrow X$ and $g : Y \longrightarrow Z$, the composition hfg is not an isomorphism. A complex over A-mod is called a *radical* complex if all its differential maps are radical maps. Every complex over A-mod is isomorphic in the homotopy category $\mathscr{K}(A)$ to a radical complex. It is easy to see that if two radical complexes $X^\bullet$ and $Y^\bullet$ are isomorphic in $\mathscr{K}(A)$, then $X^\bullet$ and $Y^\bullet$ are isomorphic in $\mathscr{C}(A)$.

Two algebras A and B are said to be *derived equivalent* if their derived categories $\mathscr{D}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(B)$ are equivalent as triangulated categories. In [12], Rickard proved that two algebras are derived equivalent if and only if there is a complex $T^\bullet$ in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ satisfying

(1) $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(T^\bullet, T^\bullet[n]) = 0$ for all $n \neq 0$, and

(2) $\mathrm{add}(T^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ as a triangulated category

such that $B \simeq \mathrm{End}(T^\bullet)$. A complex in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ satisfying the above two conditions is called a *tilting complex* over A. It is known that, given a derived equivalence F between A and B, there is a unique (up to isomorphism) tilting complex $T^\bullet$ over A such that $F(T^\bullet) \simeq B$. If $T^\bullet$ is a radical complex, it is called a tilting complex *associated* to F. Note that, for an object $X^\bullet$ in $\mathscr{D}^{\mathrm{b}}(A)$, the image $F(X^\bullet)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to a B-module if and only if $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet, X^\bullet[i]) = 0$ for all $i \neq 0$. By definition, a tilting complex associated to F is unique up to isomorphism in $\mathscr{C}^b(A)$.

The following lemma is useful in our later proof. For the convenience of the reader, we provide a proof.

Lemma 2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two additive categories, and let $F : \mathscr{K}^{\mathrm{b}}(\mathcal{C}) \longrightarrow \mathscr{K}^{\mathrm{b}}(\mathcal{D})$ be a triangle functor. Let $X^\bullet$ be a complex in $\mathscr{K}^{\mathrm{b}}(\mathcal{C})$. For each term X^i, let $Y_i^\bullet$ be a complex isomorphic to $F(X^i)$. Then $F(X^\bullet)$ is isomorphic to a complex $Z^\bullet$ with $Z^m = \bigoplus_{i+j=m} Y_i^j$ for all $m \in \mathbb{Z}$.*

Proof. We use induction on the number of non-zero terms of $X^\bullet$. If $X^\bullet$ has only one non-zero term, then it is obvious. Assume that $X^\bullet$ has more than one non-zero terms. Without loss of generality, we suppose that $X^\bullet$ is the following complex

$$0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow 0$$

with $X^i \neq 0$ for all $i = 0, 1, \cdots, n$. Let $\sigma_{\geqslant 1}X^\bullet$ be the complex $0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow 0$. Then there is a distinguished triangle in $\mathscr{K}^{\mathrm{b}}(\mathcal{C})$:

$$X^0[-1] \longrightarrow \sigma_{\geqslant 1}X^\bullet \longrightarrow X^\bullet \longrightarrow X^0.$$

Applying F, we get a distinguished triangle in $\mathscr{K}^{\mathrm{b}}(\mathcal{D})$:

$$F(X^0[-1]) \longrightarrow F(\sigma_{\geqslant 1}X^\bullet) \longrightarrow F(X^\bullet) \longrightarrow F(X^0).$$

By induction, $F(\sigma_{\geqslant 1}X^\bullet)$ is isomorphic to a complex $U^\bullet$ with $U^m = \bigoplus_{1 \leqslant i \leqslant n, i+j=m} Y_i^j$. Thus, $F(X^\bullet)$ is isomorphic to the mapping cone $Z^\bullet$ of the map from $Y_0^\bullet[-1]$ to $U^\bullet$. Thus, by definition, we have

$$Z^m = \bigoplus_{0 \leqslant i \leqslant n, i+j=m} Y_i^j = \bigoplus_{i+j=m} Y_i^j.$$

This finishes the proof. $\square$

Remark: Let $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between two algebras A and B. F induces an equivalence $F : \mathscr{K}^{\mathrm{b}}(A\text{-proj}) \longrightarrow \mathscr{K}^{\mathrm{b}}(B\text{-proj})$. So, for a bounded complex of projective A-modules, we can use the above lemma to calculate its image under F.

3 Characterizations of iterated almost ν-stable derived equivalences

In this section, we shall give a proof of our main result Theorem 1.1, which characterizes iterated almost ν-stable derived equivalences. For this purpose, we need some lemmas.

Let A be an algebra, and let ν_A-Stp be the full subcategory of A-mod consisting of all projective A-modules P with the property "$\nu_A^i P$ is projective for all $i \geqslant 0$". Note that the property "$\nu_A^i P$ is projective for all $i \geqslant 0$" is equivalent to "$\nu_A^i P$ is projective-injective for all $i \geqslant 0$". So, all the modules in ν_A-Stp are projective-injective. If ${}_AQ$ is a projective A-module such that $\mathrm{add}({}_AQ) = \mathrm{add}(\nu_A Q)$, then clearly ${}_AQ \in \nu_A$-Stp. Recall that for a bounded complex $X^\bullet$ over A, we use $X^\pm$ to denote the A-module $\bigoplus_{i \neq 0} X^i$.

Lemma 3.1. *Let $T^\bullet$ be a tilting complex associated to a derived equivalence $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ between two algebras. Then the following two conditions are equivalent.*

(1) $\mathrm{add}(\nu_A T^\pm) = \mathrm{add}({}_AT^\pm)$;

(2) ${}_AT^\pm \in \nu_A$-Stp.

Proof. $(1) \Rightarrow (2)$. This is clear.

$(2) \Rightarrow (1)$. Let $Q_1 = \bigoplus_{i<0} T^i$. Using the same method in the proof of [7, Lemma 3.1], one can show that $F^{-1}(B)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(A)$ to a complex $X^\bullet$ with $X^i \in \mathrm{add}(\nu_A Q_1)$ for all $i < 0$. Thus, $T^\bullet \simeq X^\bullet$, and there is a quasi-isomorphism $f^\bullet : T^\bullet \longrightarrow X^\bullet$, which induces a quasi-isomorphism

$$\begin{tikzcd}
U^\bullet: & \cdots \arrow[r] & T^{-2} \arrow[r, "d_T^{-2}"] \arrow[d, "f^{-2}"] & T^{-1} \arrow[r, "\pi_T"] \arrow[d, "f^{-1}"] & \mathrm{Im}\, d_T^{-1} \arrow[r] \arrow[d, "f^0|_{\mathrm{Im}\, d_T^{-1}}"] & 0 \\
V^\bullet: & \cdots \arrow[r] & X^{-2} \arrow[r, "d_X^{-2}"] & X^{-1} \arrow[r, "\pi_X"] & \mathrm{Im}\, d_X^{-1} \arrow[r] & 0.
\end{tikzcd}$$

We claim that the canonical epimorphism $\pi_T : T^{-1} \longrightarrow \mathrm{Im}\, d_T^{-1}$ is still a radical map. Otherwise, let $h : Y \longrightarrow T^{-1}$ and $g : \mathrm{Im}\, d_T^{-1} \longrightarrow Y$ be such that $h\pi_T g = 1_Y$. Then Y is isomorphic to a direct summand of T^{-1}, and therefore

Y is an injective module. Thus, g factors through the inclusion $\lambda : \operatorname{Im} d_T^{-1} \longrightarrow T^0$, say $g = \lambda u$. Consequently $1_Y = h\pi_T \lambda u = h d_T^{-1} u$. This means that $d_T^{-1} : T^{-1} \longrightarrow T^0$ is not radical, which is a contradiction. Since T^i and X^i are injective for all $i < 0$, by [7, Lemma 2.2], $U^\bullet$ and $V^\bullet$ are isomorphic in $\mathscr{K}^{\mathrm{b}}(A)$. Thus, T^i is a direct summand of X^i for all $i < 0$, and consequently $Q_1 = \bigoplus_{i<0} T^i \in \operatorname{add}(\nu_A Q_1)$. Since Q_1 and $\nu_A Q_1$ have the same number of non-isomorphic indecomposable direct summands, we have $\operatorname{add}({}_A Q_1) = \operatorname{add}(\nu_A Q_1)$. Let $Q_2 := \bigoplus_{i>0} T^i$. Similarly, we have $\operatorname{add}({}_A Q_2) = \operatorname{add}(\nu_A Q_2)$. Consequently, $\operatorname{add}({}_A T^\pm) = \operatorname{add}({}_A Q_1 \oplus {}_A Q_2) = \operatorname{add}(\nu_A Q_1 \oplus \nu_A Q_2) = \operatorname{add}(\nu_A T^\pm)$. □

In the following, we shall use Lemma 3.1 freely. For instance, in the definition of an almost ν-stable equivalence, the condition $\operatorname{add}(\bigoplus_{i=1}^n T^{-i}) = \operatorname{add}(\bigoplus_{i=1}^n \nu_A T^{-i})$ is equivalent to the condition $T^{-i} \in \nu_A$-Stp for all $i = 1, \cdots, n$.

Lemma 3.2. *Let $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between two algebras A and B, and let $T^\bullet$ and $\bar{T}^\bullet$ be the tilting complexes associated to F and F^{-1}, respectively. If $\operatorname{add}({}_A T^\pm) = \operatorname{add}(\nu_A T^\pm)$ and $\operatorname{add}({}_B \bar{T}^\pm) = \operatorname{add}(\nu_B \bar{T}^\pm)$, then F induces an equivalence between $\mathscr{K}^{\mathrm{b}}(\nu_A\text{-Stp})$ and $\mathscr{K}^{\mathrm{b}}(\nu_B\text{-Stp})$.*

Proof. Let ${}_A E$ (respectively, ${}_B \bar{E}$) be a basic additive generator of ν_A-Stp (respectively, ν_B-Stp). That is, $\operatorname{add}({}_A E) = \nu_A$-Stp. Then $\nu_A E$ is also a basic additive generator of ν_A-Stp. Hence $\nu_A E \simeq {}_A E$. The complex $F({}_A E)$ is isomorphic to a complex $\bar{T}_1^\bullet$ in $\operatorname{add}(\bar{T}^\bullet)$. Since $\nu_A E \simeq {}_A E$, we have $\nu_B \bar{T}_1^\bullet \simeq \bar{T}_1^\bullet$ in $\mathscr{D}^{\mathrm{b}}(B)$. Hence there is a chain map η from $\bar{T}_1^\bullet$ to $\nu_B \bar{T}_1^\bullet$ such that the mapping cone $\operatorname{con}(\eta)$ is acyclic. By our assumption, all $\bar{T}_1^i$ and $\nu_B \bar{T}_1^i$ with $i \neq 0$ are projective-injective since they are all in ν_B-Stp. Hence $\operatorname{con}(\eta)$ splits, and therefore $\nu_B \bar{T}_1^0 \oplus \bar{Q}_1 \simeq \bar{T}_1^0 \oplus \bar{Q}_2$ for some $\bar{Q}_1, \bar{Q}_2 \in \nu_B$-Stp. Hence, $\nu_B \bar{T}_1^0 \in \operatorname{add}(\bar{T}_1^0 \oplus {}_B \bar{E})$. It follows that $\nu_B^i \bar{T}_1^0 \in \operatorname{add}(\bar{T}_1^0 \oplus {}_B \bar{E})$ for all $i \geqslant 0$. Hence $\bar{T}_1^0 \in \nu_B$-Stp, and consequently $\bar{T}_1^\bullet$ is in $\mathscr{K}^{\mathrm{b}}(\nu_B\text{-Stp})$. Similarly, one can show that $F^{-1}({}_B \bar{E})$ is isomorphic to a complex in $\mathscr{K}^{\mathrm{b}}(\nu_A\text{-Stp})$ and the lemma is proved. □

The following lemma is useful in the proof of Theorem 1.1.

Lemma 3.3. *Let $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ and $G : \mathscr{D}^{\mathrm{b}}(B) \longrightarrow \mathscr{D}^{\mathrm{b}}(C)$ be derived equivalences, and let $P^\bullet, \bar{P}^\bullet, Q^\bullet, \bar{Q}^\bullet$, $T^\bullet$, and $\bar{T}^\bullet$ be the tilting complexes associated to F, F^{-1}, G, G^{-1}, GF, and $F^{-1}G^{-1}$ respectively. If the following hold:*

(1) ${}_A P^\pm \in \nu_A$-Stp *and* ${}_B \bar{P}^\pm \in \nu_B$-Stp*;*

(2) ${}_B Q^\pm \in \nu_B$-Stp *and* ${}_C \bar{Q}^\pm \in \nu_C$-Stp*,*

then ${}_A T^\pm \in \nu_A$-Stp *and* ${}_C \bar{T}^\pm \in \nu_C$-Stp.

Proof. We only need to show that $\bar{T}^\pm \in \nu_C$-Stp, the other statement follows by symmetry. By definition, $\bar{T}^\bullet$ is isomorphic to $GF(A) \simeq G(\bar{P}^\bullet)$. Since $\bar{P}^i \in \nu_B$-Stp for all $i \neq 0$, by Lemma 3.2, $G(\bar{P}^i)$ is isomorphic to a complex $Y_i^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\nu_C\text{-Stp})$ for all $i \neq 0$. For $i = 0$, the complex $G(\bar{P}^0)$ is isomorphic to a complex $Y_0^\bullet$ in $\operatorname{add}(\bar{Q}^\bullet)$. By Lemma 2.1, the complex $G(\bar{P}^\bullet)$ is isomorphic to a complex $Z^\bullet$ with $Z^m = \bigoplus_{i+j=m} Y_i^j$. Since all Y_i^j, except Y_0^0, are in ν_C-Stp, we have $Z^\pm \in \nu_C$-Stp. Note that $\bar{T}^\bullet$ and $Z^\bullet$ are complexes in $\mathscr{K}^{\mathrm{b}}(C\text{-proj})$, which are both isomorphic in $\mathscr{D}^{\mathrm{b}}(C)$ to $G(\bar{P}^\bullet)$. Hence $\bar{T}^\bullet$ and $Z^\bullet$ are isomorphic in $\mathscr{K}^{\mathrm{b}}(C\text{-proj})$. Furthermore, since the complex $\bar{T}^\bullet$ is a radical complex, it follows that $\bar{T}^i$ is a direct summand of Z^i for integers i, and consequently $\bar{T}^\pm \in \nu_C$-Stp. □

Finally, we have the following lemma.

Lemma 3.4. *Let $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between two algebras A and B, and let $T^\bullet$ be tilting complex associated to F. If ${}_A T^\pm \in \nu_A$-Stp, then there is an almost ν-stable equivalence $G : \mathscr{D}^{\mathrm{b}}(C) \longrightarrow \mathscr{D}^{\mathrm{b}}(A)$ such that the tilting complex $P^\bullet$ associated to FG satisfies that $P^i \in \nu_C$-Stp for all $i < 0$ and $P^i = 0$ for all $i > 0$.*

Proof. Let ${}_AE$ be an additive generator of ν_A-Stp. That is, $\nu_A\text{-Stp} = \mathrm{add}({}_AE)$. Suppose m is the maximal integer such that $T^m \neq 0$. By a dual statement of [6, Proposition 3.2], there is a tilting complex $Q^\bullet := R^\bullet \oplus {}_AE[-m]$ over A, where $R^\bullet$ is of the form: $R^\bullet : 0 \longrightarrow A \longrightarrow R^1 \longrightarrow \cdots \longrightarrow R^m \longrightarrow 0$ with $R^i \in \nu_A$-Stp for all $i > 0$. Let C be the endomorphism algebra of $Q^\bullet$, and let $H : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(C)$ be a derived equivalence induced by the tilting complex $Q^\bullet$. It is easy to see that $H({}_AE) \simeq {}_CP[m]$ for some ${}_CP \in \nu_C$-Stp, and $H(A)$ is isomorphic to a complex $S^\bullet$: $0 \longrightarrow S^{-m} \longrightarrow \cdots \longrightarrow S^{-1} \longrightarrow S^0 \longrightarrow 0$ with $S^i \in \nu_C$-Stp for all $i < 0$. Let G is a quasi-inverse of H. Then $S^\bullet$ is a tilting complex associated to G. By Lemma 3.1, we see that G is almost ν-stable.

Now let $Y_i^\bullet := H(T^i)$ for each integer i. Since $T^\pm \in \nu_A$-Stp, for each integer $i \neq 0$, we have $Y_i^\bullet \simeq P_i[m]$ for some $P_i \in \nu_C$-Stp. Moreover, $Y_i^\bullet = 0$ for all $i > m$ since $T^i = 0$ for all $i > m$. The complex $Y_0^\bullet$ has the property that $Y_0^i = 0$ for all $i > 0$ and $Y_0^i \in \nu_C$-Stp for all $i < 0$. By Lemma 2.1, the complex $H(T^\bullet)$ is isomorphic to a complex $Z^\bullet$ with $Z^t = \bigoplus_{i+j=t} Y_i^j$. It follows that $Z^t = 0$ for all $t > 0$ and $Z^t \in \nu_C$-Stp for all $t < 0$. Since $FG(H(T^\bullet)) \simeq F(T^\bullet) \simeq B \simeq FG(P^\bullet)$ in $\mathscr{D}^{\mathrm{b}}(B)$, the complex $Z^\bullet$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(C)$ to the tilting complex $P^\bullet$ associated to FG. It follows that $Z^\bullet$ and $P^\bullet$ are isomorphic in $\mathscr{K}^{\mathrm{b}}(C\text{-proj})$. Since $P^\bullet$ is a radical complex, the term P^i is a direct summand of Z^i for all i, and consequently $P^\bullet$ has the desired property. □

We are now in the position to give a proof of our main result.

Proof of Theorem 1.1. $(1) \Rightarrow (2)$. Note that the condition (2) clearly holds for almost ν-stable derived equivalences and their quasi-inverses. Thus, $(1) \Rightarrow (2)$ follows immediately from Lemma 3.3.

$(2) \Leftrightarrow (3)$. This follows from Lemma 3.1.

$(3) \Rightarrow (1)$. By Lemma 3.4, there is an almost ν-stable derived equivalence $G : \mathscr{D}^{\mathrm{b}}(C) \longrightarrow \mathscr{D}^{\mathrm{b}}(A)$ such that the tilting complex $P^\bullet$ associated to FG has the property that $P^i = 0$ for all $i > 0$ and $P^i \in \nu_C$-Stp for all $i < 0$. Let $\bar{P}^\bullet$ be the tilting complex associated to $G^{-1}F^{-1}$. It follows from Lemma 3.3 that ${}_B\bar{P}^\pm \in \nu_B$-Stp. Since $P^i = 0$ for all $i > 0$, by [7, Lemma 2.1], one get $\bar{P}^i = 0$ for all $i < 0$. Using Lemma 3.1, we see that $\mathrm{add}(\bigoplus_{i<0} P^i) = \mathrm{add}(\bigoplus_{i<0} \nu_C P^i)$ and $\mathrm{add}(\bigoplus_{i>0} \bar{P}^i) = \mathrm{add}(\bigoplus_{i>0} \nu_B \bar{P}^i)$. This implies that FG is an almost ν-stable derived equivalence. Thus, $F \simeq (FG)G^{-1}$ is an iterated almost ν-stable derived equivalence.

$(4) \Rightarrow (5)$. For each indecomposable projective A-module P not in ν_A-Stp, since $F(\mathrm{top}(P))$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to a simple B-module, we have $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, \mathrm{top}(P)[i]) = 0$ for all $i \neq 0$. This implies that P is not a direct summand of $T^\pm$. This proves (a) of condition (5). It follows from the definition of $\mathrm{b}(T^\bullet)$ that $F(\mathrm{b}(T^\bullet))$ is isomorphic to a basic projective generator M of B. Since $F(\mathrm{top}(P))$ is a simple B-module, we deduce that

$$\begin{aligned}
\mathrm{Hom}_A(\mathrm{b}(T^\bullet)^0, \mathrm{top}(P)) &\simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(\mathrm{b}(T^\bullet), \mathrm{top}(P)) \\
&\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(\mathrm{b}(T^\bullet), \mathrm{top}(P)) \\
&\simeq \mathrm{Hom}_B(M, F(\mathrm{top}(P))) \\
&\simeq \mathrm{End}_B(F(\mathrm{top}(P))) \simeq \mathrm{End}_A(\mathrm{top}(P))
\end{aligned}$$

is one-dimensional over the division ring $\mathrm{End}_A(\mathrm{top}(P))$. It follows that the multiplicity of P as a direct summand of $\mathrm{b}(T^\bullet)^0$ is 1.

$(5) \Rightarrow (4)$. By condition (a), we see that $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet, \mathrm{top}(P)[i]) = 0$ for all $i \neq 0$. Hence $F(\mathrm{top}(P))$ is isomorphic to a B-module X. By condition (b), up to isomorphism, there is only one indecomposable direct summand $T_P^\bullet$ of $T^\bullet$ such that $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T_P^\bullet, \mathrm{top}(P)) \neq 0$. Equivalently, up to isomorphism, there is only one indecomposable projective B-module $\bar{P}$ such that $\mathrm{Hom}_B(\bar{P}, X) \neq 0$. This means that X only contains $\mathrm{top}(\bar{P})$ as composition factors. If X is not a simple B-module, then there is a nonzero map $X \longrightarrow \mathrm{soc}(X) \longrightarrow X$ in $\mathrm{End}_B(X)$ which is not an isomorphism. This contradicts to the fact that $\mathrm{End}_B(X) \simeq \mathrm{End}_A(\mathrm{top}(P))$ is a division ring. Hence $X \simeq F(\mathrm{top}(P))$ is a simple B-module.

$(3) \Rightarrow (4)$. Let ${}_AE$ and ${}_B\bar{E}$ be additive generators of ν_A-Stp and ν_B-Stp, respectively. That is, $\mathrm{add}({}_AE) = \nu_A$-Stp and $\mathrm{add}({}_B\bar{E}) = \nu_B$-Stp. Let P be an indecomposable projective A-module not in ν_A-Stp. Then it is clear that $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet, \mathrm{top}(P)[i]) = 0$ for all $i \neq 0$ since $T^\pm \in \nu_A$-Stp, and consequently $F(\mathrm{top}(P))$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to a B-module X. By Lemma 3.2, the complex $F^{-1}({}_B\bar{E})$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(A)$ to a complex $E^{\bullet}$ in $\mathscr{K}^{\mathrm{b}}(\nu_A\text{-Stp})$. Hence

$$\mathrm{Hom}_B({}_B\bar{E},X) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(F^{-1}({}_B\bar{E}),\mathrm{top}(P)) \simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(E^{\bullet},\mathrm{top}(P)) = 0.$$

If ${}_BX$ is not simple, then there is a short exact sequence $0 \longrightarrow \bar{U} \longrightarrow X \longrightarrow \bar{V} \longrightarrow 0$ in B-mod with $\bar{U},\bar{V}$ non-zero. Applying $\mathrm{Hom}_B({}_B\bar{E},-)$, we get that $\mathrm{Hom}_B({}_B\bar{E},\bar{U}) = 0 = \mathrm{Hom}_B({}_B\bar{E},\bar{V})$, and consequently $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{T}^{\bullet},\bar{U}[i]) = 0 = \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{T}^{\bullet},\bar{V}[i])$ for all $i \neq 0$ since $\bar{T}^{\pm} \in \nu_B$-Stp. Hence $F^{-1}(\bar{U})$ and $F^{-1}(\bar{V})$ are isomorphic to some A-modules U and V, respectively. Thus, we get a distinguished triangle

$$U \longrightarrow \mathrm{top}(P) \longrightarrow V \longrightarrow U[1]$$

in $\mathscr{D}^{\mathrm{b}}(A)$ by applying F^{-1} to the distinguished triangle $\bar{U} \longrightarrow X \longrightarrow \bar{V} \longrightarrow \bar{U}[1]$. Applying $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(A,-)$ to the above triangle, we get an exact sequence $0 \longrightarrow U \longrightarrow \mathrm{top}(P) \longrightarrow V \longrightarrow 0$ with non-zero A-modules U and V. This contradicts to the fact that $\mathrm{top}(P)$ is a simple A-module. Hence $F(\mathrm{top}(P)) \simeq X$ is a simple B-module.

(4) $\Rightarrow$ (3). For each indecomposable projective A-module P not in ν_A-Stp, since $F(\mathrm{top}(P))$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to a simple B-module, we have

$$\mathrm{Hom}_A(T^i,\mathrm{top}(P)) \simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(T^{\bullet},\mathrm{top}(P)[-i]) \simeq 0$$

for all $i \neq 0$. This implies that $T^i \in \nu_A$-Stp for all $i \neq 0$, that is, $T^{\pm} \in \nu_A$-Stp. It remains to show that $\bar{T}^{\pm} \in \nu_B$-Stp. Let ${}_AQ$ be a direct sum of all non-isomorphic indecomposable projective A-modules not in ν_A-Stp. Then $F(\mathrm{top}(Q))$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to a semi-simple B-module. Let ${}_B\bar{Q}$ be a projective cover of $F(\mathrm{top}(Q))$. That is, $F(\mathrm{top}(Q)) \simeq \mathrm{top}(\bar{Q})$. Now we set ${}_BW$ to be a direct sum of all non-isomorphic indecomposable projective B-modules not in $\mathrm{add}({}_B\bar{Q})$. Then $\bar{Q} \oplus W$ is a basic projective generator of B. There are isomorphisms

$$\begin{aligned}\mathrm{Hom}_B(\bar{T}^i,\mathrm{top}(\bar{Q})) &\simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{T}^{\bullet},\mathrm{top}(\bar{Q})[-i]) \\ &\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{T}^{\bullet},\mathrm{top}(\bar{Q})[-i]) \\ &\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}({}_AA,\mathrm{top}(Q)[-i]) = 0\end{aligned}$$

for all $i \neq 0$. This means that none of the indecomposable direct summands of $\bar{T}^{\pm}$ are in $\mathrm{add}({}_B\bar{Q})$, or equivalently, $\bar{T}^{\pm} \in \mathrm{add}({}_BW)$. It remains to show ${}_BW \in \nu_B$-Stp. Note that

$$\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(F^{-1}({}_BW),\mathrm{top}(Q)[i]) \simeq \mathrm{Hom}_B({}_BW,\mathrm{top}(\bar{Q})[i]) = 0$$

for all integers i. It follows that $F^{-1}({}_BW)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(A)$ to a radical complex in $\mathscr{K}^{\mathrm{b}}(\nu_A\text{-Stp})$. Using the same method in the proof [1, Theorem 2.1], one can show that $\nu_B^i W$ is a projective B-module for all $i \geqslant 0$. Hence ${}_BW \in \nu_B$-Stp. This finishes the proof. $\square$

Remark: (1) The condition (5) in Theorem 1.1 provides a convenient way to check whether a given derived equivalence is iterated almost ν-stable or not.

(2) Let P be a projective A-module. The condition $\mathrm{add}({}_AP) = \mathrm{add}(\nu_AP)$ is equivalent to saying that P is projective-injective and $\mathrm{add}(\mathrm{top}(P)) = \mathrm{add}(\mathrm{soc}(P))$.

(3) It is interesting to know whether Theorem 1.1 holds for general Artin algebras. Note that the only problem is the step "(4) $\Rightarrow$ (3)", where the method in the proof of [1, Theorem 2.1] does not work for general Artin algebras. In particular, for general Artin algebras, the conditions (1), (2) and (3) in Theorem 1.1 are still equivalent.

As a consequence of Theorem 1.1, together with [7, Theorem 5.3, and Corollary 1.2], we have a corollary.

Corollary 3.5. *Let $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between two finite-dimensional algebras over a field. If one of the equivalent conditions in* Theorem 1.1 *is satisfied, then the algebras A and B are stably equivalent of Morita type. Moreover, the algebras A and B have the same finitistic dimension, global dimension, representation dimension and dominant dimension.*

In the rest of this section, we give a method to construct tilting complexes which induce iterated almost ν-stable derived equivalences.

Let us recall from [2] the definition of approximations. Let $\mathcal{C}$ be a category, and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$, and X an object in $\mathcal{C}$. A morphism $f : D \longrightarrow X$ in $\mathcal{C}$ is called a *right $\mathcal{D}$-approximation* of X if $D \in \mathcal{D}$ and the induced map $\mathrm{Hom}_{\mathcal{C}}(D', f)$: $\mathrm{Hom}_{\mathcal{C}}(D', D) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(D', X)$ is surjective for every object $D' \in \mathcal{D}$. Dually, one can define *left $\mathcal{D}$-approximations*.

Let A be an algebra, and let P, Q be two projective A-modules satisfying the following two conditions:

(1) $\mathrm{add}({}_AP) = \mathrm{add}(\nu_A P)$, $\mathrm{add}({}_AQ) = \mathrm{add}(\nu_A Q)$;

(2) $\mathrm{Hom}_A(P, Q) = 0$.

For each positive integer r, we can form the following complex:

$$0 \longrightarrow P^{-r} \xrightarrow{f_r} P^{-r+1} \longrightarrow \cdots \longrightarrow P^{-1} \xrightarrow{f_1} A \longrightarrow 0,$$

where $f_1 : P^{-1} \longrightarrow A$ is a right $\mathrm{add}({}_AP)$-approximation of A, and $f_{i+1} : P^{-i-1} \longrightarrow P^{-i}$ is a right $\mathrm{add}({}_AP)$-approximation of $\mathrm{Ker}(f_i)$ for $i = 1, \cdots, r-1$. Similarly, we can form a complex

$$0 \longrightarrow A \xrightarrow{g_1} Q^1 \longrightarrow \cdots \longrightarrow Q^{s-1} \xrightarrow{g_s} Q^s \longrightarrow 0,$$

where g_1 is a left $\mathrm{add}({}_AQ)$-approximation of A, and g_{i+1} is a left $\mathrm{add}({}_AQ)$-approximation of $\mathrm{Coker}(g_i)$ for $i = 1, 2, \cdots, s-1$. Since $\mathrm{Hom}_A(P, Q) = 0$, gluing the two complexes together, we get a complex

$$0 \longrightarrow P^{-r} \longrightarrow \cdots \longrightarrow P^{-1} \xrightarrow{f_1} A \xrightarrow{g_1} Q^1 \longrightarrow \cdots \longrightarrow Q^s \longrightarrow 0,$$

where A is in degree zero. We denote this complex by $T^{\bullet}_{P,Q}$, and let $T^{\bullet} := T^{\bullet}_{P,Q} \oplus P[r] \oplus Q[-s]$.

Proposition 3.6. *Keeping the notations above, the complex $T^{\bullet}$ is a tilting complex that induces an iterated almost ν-stable derived equivalence between the algebras A and* $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(T^{\bullet})$.

Proof. By the construction of $T^{\bullet}$, we have

$$T^i = \begin{cases} P^{-r} \oplus P, & i = -r; \\ P^i, & -r < i < 0; \\ A, & i = 0; \\ Q^i, & 0 < i < s; \\ Q^s \oplus Q, & i = s; \\ 0 & \text{otherwise.} \end{cases} \quad \text{, and } d_T^i = \begin{cases} \begin{bmatrix} f_r \\ 0 \end{bmatrix}, & i = -r; \\ f_{-i}, & -r < i < 0; \\ g_{i+1}, & 0 \leqslant i < s-1; \\ \begin{bmatrix} g_s \\ 0 \end{bmatrix}, & i = s-1; \\ 0 & \text{otherwise.} \end{cases}$$

We first show that $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(T^{\bullet}, T^{\bullet}[i]) = 0$ for all $i \neq 0$. Assume that i is a positive integer. Let $u^{\bullet}$ be a morphism in $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(T^{\bullet}, T^{\bullet}[i])$. Then we have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & T^{-i-1} \arrow[r, "d_T^{-i-1}"] \arrow[d, "u^{-i-1}"] & T^{-i} \arrow[r, "d_T^{-i}"] \arrow[d, "u^{-i}"] & T^{-i+1} \arrow[r, "d_T^{-i+1}"] \arrow[d, "u^{-i+1}"] & \cdots \arrow[r] & T^{-1} \arrow[r, "d^{-1}"] \arrow[d, "u^{-1}"] & T^0 \arrow[r, "d_T^0"] \arrow[d, "u^0"] & T^1 \arrow[r, "d_T^1"] \arrow[d, "u^1"] & \cdots \\
\cdots \arrow[r] & T^{-1} \arrow[r, "d_T^{-1}"] & T^0 \arrow[r, "d_T^0"] & T^1 \arrow[r, "d_T^1"] & \cdots \arrow[r] & T^{i-1} \arrow[r, "d_T^{i-1}"] & T^i \arrow[r, "d_T^i"] & T^{i+1} \arrow[r, "d_T^{i+1}"] & \cdots
\end{tikzcd}$$

Since $\mathrm{Hom}_A(P,Q)=0$, we have $u^k=0$ for all $-i<k<0$. By definition, $T^{-i}\in \mathrm{add}({}_AP)$. Since $d_T^{-1}=f_1$ is a right $\mathrm{add}({}_AP)$-approximation, there is a map $h^{-i}: T^{-i}\longrightarrow T^{-1}$ such that $u^{-i}=h^{-i}d_T^{-1}$. Thus,

$$(u^{-i-1}-d_T^{-i-1}h^{-i})d_T^{-1}=d_T^{-i-1}u^{-i}-d_T^{-i-1}h^{-i}d_T^{-1}=d_T^{-i-1}u^{-i}-d_T^{-i-1}u^{-i}=0.$$

Since d_T^{-2} is a right $\mathrm{add}({}_AP)$-approximation of $\mathrm{Ker}(d_T^{-1})$, there is a map $h^{-i-1}: T^{-i-1}\longrightarrow T^{-2}$ such that $u^{-i-1}-d_T^{-i-1}h^{-i}=h^{-i-1}d_T^{-2}$, that is $u^{-i-1}=d_T^{-i-1}h^{-i}+h^{-i-1}d_T^{-2}$. Similarly, for each integer $k<-i-1$, there are maps $h^{k+1}: T^{k+1}\longrightarrow T^{k+i}$ and $h^k: T^k\longrightarrow T^{k+i-1}$ such that $u^k=d_T^kh^{k+1}+h^kd_T^{k+i-1}$. Defining $h^k=0$ for all $-i<k\leqslant 0$, we have $u^k=d_T^kh^{k+1}+h^kd_T^{k+i-1}$ for all $k<0$. Similarly, we can prove that $u^k=d_T^kh^{k+1}+h^kd_T^{k+i-1}$ for $k\geqslant 0$. It follows that $u^\bullet=0$ in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$. Hence $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(T^\bullet,T^\bullet[i])=0$ for all $i>0$. By an analogous proof, one get $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(T^\bullet,T^\bullet[i])=0$ for all $i<0$. Finally, since $P[r]$ and $Q[-s]$ are in $\mathrm{add}(T^\bullet)$, we deduce that ${}_AA$ is in the triangulated subcategory of $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ generated by $\mathrm{add}(T^\bullet)$. Hence $\mathrm{add}(T^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ as a triangulated category, and consequently $T^\bullet$ is a tilting complex over A. It follows from the condition (5) in Theorem 1.1 that the tilting complex $T^\bullet$ induces an iterated almost ν-stable derived equivalence between A and $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(T^\bullet)$. $\square$

To illustrate Proposition 3.6, we give an example. Let k be a field, and let A be the finite-dimensional k-algebra given by the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, shift left, "\alpha"] & \underset{2}{\bullet} \arrow[l, shift left, "\alpha'"] \arrow[r, shift left, "\beta"] & \underset{3}{\bullet} \arrow[l, shift left, "\beta'"] \arrow[r, shift left, "\gamma"] & \underset{4}{\bullet} \arrow[l, shift left, "\gamma'"] \end{tikzcd}$$

with relations $\alpha'\alpha=\beta\beta'=\alpha\beta=\beta\gamma=\beta'\alpha'=\gamma'\beta'=\beta'\beta-\gamma\gamma'=0$. We use P_i to denote the indecomposable projective A-module corresponding to the vertex i for $i=1,2,3,4$. The Loewy structure of the projective A-modules can be listed as follows.

$$P_1:\ \begin{matrix}1\\2\\1\end{matrix}\qquad P_2:\ \begin{matrix}&2&\\1&&3\end{matrix}\qquad P_3:\ \begin{matrix}&3&\\2&&4\\&3&\end{matrix}\qquad P_4:\ \begin{matrix}4\\3\\4\end{matrix}$$

Let $P:=P_1$ and $Q:=P_3\oplus P_4$. Then we have $\mathrm{add}({}_AP)=\mathrm{add}(\nu_AP)$, $\mathrm{add}({}_AQ)=\mathrm{add}(\nu_AQ)$, and $\mathrm{Hom}_A(P,Q)=0$. Using Proposition 3.6, we have a tilting complex $T^\bullet$ over A. The indecomposable direct summands of $T^\bullet$ are:

$$\begin{array}{ll} T_1^\bullet: & 0\longrightarrow P_1\longrightarrow 0\\ T_2^\bullet: & 0\longrightarrow P_1\longrightarrow P_2\longrightarrow P_3\longrightarrow 0\\ T_3^\bullet: & \qquad\qquad\qquad\ \ 0\longrightarrow P_3\longrightarrow 0\\ T_4^\bullet: & \qquad\qquad\qquad\ \ 0\longrightarrow P_4\longrightarrow 0 \end{array}$$

A calculation shows that the algebra $B:=\mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet)$ is given by the quiver

$$\begin{tikzcd} & & \overset{3}{\bullet} \arrow[dl, "\delta"'] & \\ \underset{1}{\bullet} \arrow[r, shift left, "\alpha"] & \underset{2}{\bullet} \arrow[l, shift left, "\alpha'"] \arrow[rr, "\beta"'] & & \underset{4}{\bullet} \arrow[ul, "\gamma"'] \end{tikzcd}$$

with relations $\alpha'\alpha=\alpha\beta=\delta\alpha'=\beta\gamma\delta=\gamma\delta\beta\gamma=0$. By Proposition 3.6, $T^\bullet$ induces an iterated almost ν-stable derived equivalence between A and B. Therefore, A and B are also stably equivalent of Morita type.

Finally, let us remark that one can inductively construct iterated almost ν-stable derived equivalences from given ones, as we have done for almost ν-stable derived equivalences in [7] and [8].

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References

[1] S. AL-NOFAYEE and J. RICKARD, Rigidity of tilting complexes and derived equivalence for self-injective algebras. preprint, 2004.

[2] M. AUSLANDER and S. O. SMALØ, Preprojective modules over Artin algebras. J. Algebra 66(1980), 61-122.

[3] M. BROUÉ, Equivalences of blocks of group algebras. In: "*Finite dimensional algebras and related topics*", edited by V. Dlab and L. L. Scott, Kluwer, 1994, 1-26.

[4] A. DUGAS and R. MARTÍNEZ-VILLA, A note on stable equivalences of Morita type. J. Pure Appl. Algebra 208(2)(2007), 421-433.

[5] D. HAPPEL, *Triangulated categories in the representation theory of finite dimensional algebras.* Cambridge Univ. Press, Cambridge, 1988.

[6] M. HOSHINO and Y. KATO, Tilting complexes associated with a sequence of idempotents. J. Algebra 183(2003), 105-124.

[7] W. HU and C. C. XI, Derived equivalences and stable equivalences of Morita type, I. Nagoya Math. J. 200 (2010), 107-152.

[8] W. HU and C. C. XI, Derived equivalences for Φ-Auslander-Yoneda algebras. preprint 1-27, 2009. Available at: http://arxiv.org/abs/0912.0647

[9] Y. M. LIU and C. C. XI, Constructions of stable equivalences of Morita type for finite dimensional algebras, I. Trans. Amer. Math. Soc. 358(2006), no.6, 2537-2560.

[10] Y. M. LIU and C. C. XI, Constructions of stable equivalences of Morita type for finite dimensional algebras, II. Math. Z. 251(2005), 21-39.

[11] Y. M. LIU and C. C. XI, Constructions of stable equivalences of Morita type for finite dimensional algebras, III. J. London Math. Soc. 76 (2007), 567-585.

[12] J. RICKARD, Morita theory for derived categories. J. London Math. Soc. 39(1989), 436-456.

[13] J. RICKARD, Derived equivalence as derived functors. J. London Math. Soc. 43(1991), 37-48.