On iterated almost $\nu$-stable derived equivalences

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Abstract

In this paper, by characterizing iterated almost $\nu$-stable derived equivalences, we give several sufficient conditions for a derived equivalence between general finite-dimensional algebras to induce a stable equivalence of Morita type. In particular, we prove the following: Let $A$ and $B$ be two finite-dimensional algebras over a field. Suppose that there is a derived equivalence between $A$ and $B$ induced by a tilting complex $T^\bullet$ over $A$. If each indecomposable projective $A$-module $P$ without the property "$\nu_i A P$ is projective for all $i \geq 0$" occurs only in the 0-degree term $T^0$ of $T^\bullet$ with multiplicity 1, then $A$ and $B$ are stably equivalent of Morita type.

1 Introduction

This is a continuation of the study on the relationship between derived equivalences and stable equivalences for general finite-dimensional algebras. In [7], we introduced a class of derived equivalences called almost $\nu$-stable derived equivalences. The crucial property [7, Theorem 5.3] is that an almost $\nu$-stable derived equivalence always induces a stable equivalence of Morita type, which generalizes a classical result of Rickard ([13, Corollary 5.5]): Derived equivalent self-injective algebras are stably equivalent of Morita type. The result [7, Theorem 5.3] also gives a sufficient condition for a derived equivalence between general finite-dimensional algebras to induce a stable equivalence of Morita type. Note that many homological dimensions, such as global dimension, finitistic dimension, and representation dimension, are not invariant under derived equivalences in general. But they are all preserved by stable equivalences of Morita type. So, this also helps us to compare the homological dimensions of derived equivalent algebras. For more information about stable equivalences of Morita type, we refer to the papers [3, 9, 10, 7].

Let us first recall the definition of almost $\nu$-stable derived equivalences. Let $F : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B)$ be a derived equivalence between two finite-dimensional algebras $A$ and $B$ over a field $k$, where $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ stand for the derived categories of bounded complexes over $A$ and $B$, respectively. We use $F^{-1}$ to denote a quasi-inverse of $F$. The functor $F$ is called an almost $\nu$-stable derived equivalence if the following hold:

1. The tilting complex $T^\bullet$ associated to $F$ has the following form:

$$0 \rightarrow T^{-n} \rightarrow \cdots \rightarrow T^{-1} \rightarrow T^0 \rightarrow 0$$

In this case, the tilting $\tilde{T}^\bullet$ associated to $F^{-1}$ has the following form (see [7, Lemma 2.1]):

$$0 \rightarrow \tilde{T}^0 \rightarrow \tilde{T}^1 \rightarrow \cdots \rightarrow \tilde{T}^n \rightarrow 0$$

2. $\text{add}(\bigoplus_{i=1}^n T^{-i}) = \text{add}(\bigoplus_{i=1}^n \nu_A T^{-i})$ and $\text{add}(\bigoplus_{i=1}^n \tilde{T}^i) = \text{add}(\bigoplus_{i=1}^n \nu_B \tilde{T}^i)$, where $\nu$ is the Nakayama functor.

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In general, the quasi-inverse of an almost \( \nu \)-stable derived equivalence is not almost \( \nu \)-stable. This motivates us to look for a more “balanced” notion. If a derived equivalence \( F \) is a composition \( F_1F_2\cdots F_m \) with \( F_i \) or \( F_i^{-1} \) being an almost \( \nu \)-stable derived equivalence for all \( i \), then \( F \) is called an \emph{iterated almost \( \nu \)-stable derived equivalence}. By definition, the class of iterated almost \( \nu \)-stable derived equivalences properly contains the class of almost \( \nu \)-stable derived equivalences, and is closed under taking compositions and quasi-inverses. Clearly, an iterated almost \( \nu \)-stable derived equivalence always induces a stable equivalence of Morita type, and therefore the involved algebras have many common homological dimensions. But the problem is:

**Question:** Given a derived equivalence \( F \), how to determine whether \( F \) is iterated almost \( \nu \)-stable or not?

A satisfactory answer to the above question will give us some new sufficient conditions for a derived equivalence between general finite-dimensional algebras to induce a stable equivalence of Morita type. In this paper, we give a complete answer to the above question. Let \( A \) be an algebra. We use \( \nu_A\)-Stp to denote the full subcategory of \( A\text{-mod} \) consisting of all the projective \( A \)-modules \( P \) with the property that \( \nu_A^iP \) is projective for all \( i \geq 0 \). For an \( A \)-module \( X \), we write \( \text{top}(X) \) for the maximal semi-simple quotient module of \( X \). For a complex \( X^\bullet \) over \( A \), we denote by \( X^\pm \) the module \( \bigoplus_{i \neq 0} X^i \). Our main result is the following theorem.

**Theorem 1.1.** Let \( F : \mathcal{D}^b(A) \to \mathcal{D}^b(B) \) be a derived equivalence between two finite-dimensional algebras \( A \) and \( B \) over a field \( k \), and let \( T^\bullet \) and \( \bar{T}^\bullet \) be tilting complexes associated to \( F \) and \( F^{-1} \), respectively. Then the following are equivalent:

1. The functor \( F \) is an iterated almost \( \nu \)-stable derived equivalence.
2. \( \text{add}(\nu_AT^\pm) = \text{add}(\nu_BT^\pm) \) and \( \text{add}(\nu_B\bar{T}^\pm) = \text{add}(\nu_A\bar{T}^\pm) \).
3. \( T^\pm \in \nu_A\text{-Stp} \) and \( \bar{T}^\pm \in \nu_B\text{-Stp} \).
4. For each indecomposable projective \( A \)-module \( P \notin \nu_A\text{-Stp} \), the image \( F(\text{top}(P)) \) is isomorphic in \( \mathcal{D}^b(B) \) to a simple \( B \)-module.
5. For each indecomposable projective \( A \)-module \( P \notin \nu_A\text{-Stp} \), the following conditions are satisfied:
   a. \( P \notin \text{add}(\nu_AT^\pm) \);
   b. The multiplicity of \( P \) as a direct summand of \( \text{b}(T^\bullet)^0 \) is 1, where the complex \( \text{b}(T^\bullet) \) is a direct sum of all non-isomorphic indecomposable direct summands of \( T^\bullet \).

Moreover, if one of the above equivalent conditions is satisfied, then the algebras \( A \) and \( B \) are stably equivalent of Morita type.

Theorem 1.1 tells us that, by checking the terms of tilting complexes, we can determine whether a derived equivalence is iterated almost \( \nu \)-stable or not. If a derived equivalence \( F \) between two algebras \( A \) and \( B \) satisfies one of the equivalent conditions in Theorem 1.1, then \( F \) induces a stable equivalence of Morita type between \( A \) and \( B \). Thus, one can use Theorem 1.1 to get stable equivalences of Morita type from derived equivalences. Comparing with [7, Theorem 5.3], here we only need to check the terms of the tilting complex \( T^\bullet \) associated to the given derived equivalence \( F \), while in [7, Theorem 5.3] one needs to consider both the tilting complex associated to \( F \) and that associated to \( F^{-1} \).

Let us remark that the condition (4) arises naturally from a property of stable equivalences of Morita type. Suppose that the algebras \( A \) and \( B \) are indecomposable finite-dimensional algebras over a perfect field \( k \). Let \( \Phi : A\text{-mod} \to B\text{-mod} \) be a stable equivalence of Morita type. Then it can be deduced from [4, Proposition 3.4] that \( \Phi(\text{top}(P)) \) is isomorphic in \( B\text{-mod} \) to a simple \( B \)-module for all indecomposable projective \( A \)-modules \( P \) not in \( \nu_A\text{-Stp} \).

This paper is organized as follows. In Section 2, we shall fix some notations and recall some basic facts needed in our proofs. Theorem 1.1 will be proved in Section 3 after several lemmas. In Section 3, we also give a method to construct tilting complexes which induce iterated almost \( \nu \)-stable derived equivalences.
2 Preliminaries

In this section, we shall recall some basic definitions and facts needed in our later proofs.

Throughout this paper, all algebras are finite-dimensional algebras over a fixed field \(k\). All modules are finitely generated unitary left modules. For an algebra \(A\), the category of \(A\)-modules is denoted by \(A\text{-mod}\); the full subcategory of \(A\)-mod consisting of projective (respectively, injective) modules is denoted by \(A\text{-proj}\) (respectively, \(A\text{-inj}\)). The stable module category, denoted by \(A\text{-mod}_{\text{st}}\), is the quotient category of \(A\text{-mod}\) modulo the ideal generated by morphisms factorizing through projective modules. We denote by \(\nu_A\) the usual Nakayama functor \(\text{DHom}_A(\cdot,A)\), where \(D = \text{Hom}_k(\cdot,k)\) is the usual duality. Note that \(\nu_A : A\text{-proj} \rightarrow A\text{-inj}\) is an equivalence.

Let \(C\) be an additive category. The composition of two morphisms \(f : X \rightarrow Y\) and \(g : Y \rightarrow Z\) in \(C\) will be denoted by \(fg\). For two functors \(F : C \rightarrow D\) and \(G : D \rightarrow E\) of categories, their composition is denoted by \(GF\).

For an object \(X\) in \(C\), \(\text{add}(X)\) is the full subcategory of \(C\) consisting of all direct summands of finite direct sums of copies of \(X\).

A complex \(X^*\) over \(C\) is a sequence \(\cdots \rightarrow X_{i-1} \xrightarrow{d_{i-1}} X_i \xrightarrow{d_i} X_{i+1} \xrightarrow{d_{i+1}} \cdots\) in \(C\) such that \(d_id_{i+1} = 0\) for all integers \(i\). The category of complexes over \(C\) is denoted by \(\mathcal{C}(C)\). The homotopy category of complexes over \(C\) is denoted by \(\mathcal{K}(C)\). When \(C\) is an abelian category, the derived category of complexes over \(C\) is denoted by \(\mathcal{D}(C)\). The full subcategory of \(\mathcal{K}(C)\) and \(\mathcal{D}(C)\) consisting of bounded complexes over \(C\) is denoted by \(\mathcal{K}^b(C)\) and \(\mathcal{D}^b(C)\), respectively. As usual, for a given algebra \(A\), we simply write \(\mathcal{K}^b(A)\) and \(\mathcal{D}^b(A)\) for \(\mathcal{K}^b(A\text{-mod})\) and \(\mathcal{D}^b(A\text{-mod})\), respectively.

It is well-known that, for an algebra \(A\), \(\mathcal{K}^b(A)\) and \(\mathcal{D}^b(A)\) are triangulated categories. Moreover, it is known that if \(X^* \in \mathcal{K}^b(A\text{-proj})\) or \(Y^* \in \mathcal{K}^b(A\text{-inj})\), then \(\text{Hom}_{\mathcal{K}^b(A)}(X^*, Y^*) \simeq \text{Hom}_{\mathcal{D}^b(A)}(X^*, Y^*)\). For basic results on triangulated categories, we refer to Happel’s book [5]. Throughout this paper, we use \(X^*[n]\) to denote the complex obtained by shifting \(X^*\) to the left by \(n\) degrees.

Let \(A\) be an algebra. A homomorphism \(f : X \rightarrow Y\) of \(A\)-modules is called a radical map if, for any module \(Z\) and homomorphisms \(h : Z \rightarrow X\) and \(g : Y \rightarrow Z\), the composition \(hfg\) is not an isomorphism. A complex over \(A\text{-mod}\) is called a radical complex if all its differential maps are radical maps. Every complex over \(A\text{-mod}\) is isomorphic in the homotopy category \(\mathcal{K}(A)\) to a radical complex. It is easy to see that if two radical complexes \(X^*\) and \(Y^*\) are isomorphic in \(\mathcal{K}(A)\), then \(X^*\) and \(Y^*\) are isomorphic in \(\mathcal{D}(A)\).

Two algebras \(A\) and \(B\) are said to be derived equivalent if their derived categories \(\mathcal{D}^b(A)\) and \(\mathcal{D}^b(B)\) are equivalent as triangulated categories. In [12], Rickard proved that two algebras are derived equivalent if and only if there is a complex \(T^*\) in \(\mathcal{K}^b(A\text{-proj})\) satisfying

1. \(\text{Hom}_{\mathcal{K}^b(A\text{-proj})}(T^*, T^*[n]) = 0\) for all \(n \neq 0\), and
2. \(\text{add}(T^*)\) generates \(\mathcal{K}^b(A\text{-proj})\) as a triangulated category

such that \(B \simeq \text{End}(T^*)\). A complex in \(\mathcal{K}^b(A\text{-proj})\) satisfying the above two conditions is called a tilting complex over \(A\). It is known that, given a derived equivalence \(F\) between \(A\) and \(B\), there is a unique (up to isomorphism) tilting complex \(T^*\) over \(A\) such that \(F(T^*) \simeq B\). If \(T^*\) is a radical complex, it is called a tilting complex associated to \(F\).

Note that, for an object \(X^*\) in \(\mathcal{D}^b(A)\), the image \(F(X^*)\) is isomorphic in \(\mathcal{D}^b(B)\) to a \(B\)-module if and only if \(\text{Hom}_{\mathcal{D}^b(A)}(T^*, X^*[i]) = 0\) for all \(i \neq 0\). By definition, a tilting complex associated to \(F\) is unique up to isomorphism in \(\mathcal{D}(A)\).

The following lemma is useful in our later proof. For the convenience of the reader, we provide a proof.

**Lemma 2.1.** Let \(C\) and \(D\) be two additive categories, and let \(F : \mathcal{K}^b(C) \rightarrow \mathcal{K}^b(D)\) be a triangle functor. Let \(X^*\) be a complex in \(\mathcal{K}^b(C)\). For each term \(X^i\), let \(Y^*\) be a complex isomorphic to \(F(X^i)\). Then \(F(X^*)\) is isomorphic to a complex \(Z^*\) with \(Z^m = \bigoplus_{i+j=m} Y^i\) for all \(m \in \mathbb{Z}\).

**Proof.** We use induction on the number of non-zero terms of \(X^*\). If \(X^*\) has only one non-zero term, then it is obvious. Assume that \(X^*\) has more than one non-zero terms. Without loss of generality, we suppose that \(X^*\) is
the following complex

$$0 \rightarrow X^0 \rightarrow X^1 \rightarrow \cdots \rightarrow X^n \rightarrow 0$$

with $X^i \neq 0$ for all $i = 0, 1, \cdots, n$. Let $\sigma_{\geq 1} X^*$ be the complex $0 \rightarrow X^1 \rightarrow \cdots \rightarrow X^n \rightarrow 0$. Then there is a distinguished triangle in $\mathcal{K}^b(C)$:

$$X^0[-1] \rightarrow \sigma_{\geq 1} X^* \rightarrow X^* \rightarrow X^0.$$

Applying $F$, we get a distinguished triangle in $\mathcal{K}^b(D)$:

$$F(X^0[-1]) \rightarrow F(\sigma_{\geq 1} X^*) \rightarrow F(X^*) \rightarrow F(X^0).$$

By induction, $F(\sigma_{\geq 1} X^*)$ is isomorphic to a complex $U^*$ with $U^m = \bigoplus_{1 \leq i \leq n, j = m} Y_i^j$. Thus, $F(X^*)$ is isomorphic to the mapping cone $Z^*$ of the map from $Y_0^*[-1]$ to $U^*$. Thus, by definition, we have

$$Z^m = \bigoplus_{0 \leq i \leq n, j = m} Y_i^j = \bigoplus_{i+j=m} Y_i^j.$$

This finishes the proof. \(\square\)

Remark: Let $F : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B)$ be a derived equivalence between two algebras $A$ and $B$. $F$ induces an equivalence $F : \mathcal{K}^b(A\text{-proj}) \rightarrow \mathcal{K}^b(B\text{-proj})$. So, for a bounded complex of projective $A$-modules, we can use the above lemma to calculate its image under $F$.

3 Characterizations of iterated almost $\nu$-stable derived equivalences

In this section, we shall give a proof of our main result Theorem 1.1, which characterizes iterated almost $\nu$-stable derived equivalences. For this purpose, we need some lemmas.

Let $A$ be an algebra, and let $\nu_A\text{-St}_p$ be the full subcategory of $A\text{-mod}$ consisting of all projective $A$-modules $P$ with the property “$\nu^i_A P$ is projective for all $i \geq 0$”. Note that the property “$\nu^i_A P$ is projective for all $i \geq 0$” is equivalent to “$\nu^i_A P$ is projective-injective for all $i \geq 0$”. So, all the modules in $\nu_A\text{-St}_p$ are projective-injective. If $A Q$ is a projective $A$-module such that $\text{add}(A Q) = \text{add}(\nu_A Q)$, then clearly $A Q \in \nu_A\text{-St}_p$. Recall that for a bounded complex $X^*$ over $A$, we use $X^{\pm}$ to denote the $A$-module $\bigoplus_{i \neq 0} X^i$.

Lemma 3.1. Let $T^*$ be a tilting complex associated to a derived equivalence $F : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B)$ between two algebras. Then the following two conditions are equivalent.

1. $\text{add}(\nu_A T^\pm) = \text{add}(T^\pm)$;
2. $T^{\pm} \in \nu_A\text{-St}_p$.

Proof. (1) $\Rightarrow$ (2). This is clear.

(2) $\Rightarrow$ (1). Let $Q_1 = \bigoplus_{i \neq 0} T^i$. Using the same method in the proof of [7, Lemma 3.1], one can show that $F^{-1}(B)$ is isomorphic in $\mathcal{D}^b(A)$ to a complex $X^*$ with $X^i \in \text{add}(\nu_A Q_1)$ for all $i < 0$. Thus, $T^* \simeq X^*$, and there is a quasi-isomorphism $f^* : T^* \rightarrow X^*$, which induces a quasi-isomorphism

$$U^* : \cdots \rightarrow T^{-2} \xrightarrow{d_{T^{-2}}} T^{-1} \xrightarrow{\pi_T} \text{Im} d_{T^{-1}} \rightarrow 0$$

$$V^* : \cdots \rightarrow X^{-2} \xrightarrow{d_{X^{-2}}} X^{-1} \xrightarrow{\pi_X} \text{Im} d_{X^{-1}} \rightarrow 0.$$

We claim that the canonical epimorphism $\pi_T : T^{-1} \rightarrow \text{Im} d_{T^{-1}}$ is still a radical map. Otherwise, let $h : Y \rightarrow T^{-1}$ and $g : \text{Im} d_{T^{-1}} \rightarrow Y$ be such that $h \pi_T g = 1_Y$. Then $Y$ is isomorphic to a direct summand of $T^{-1}$, and therefore
Y is an injective module. Thus, \( g \) factors through the inclusion \( \lambda : \text{Im} d_T^{-1} \rightarrow T^0 \), say \( g = \lambda u \). Consequently \( 1_Y = \lambda h \pi \lambda^2 = h d_T^{-1}u \). This means that \( d_T^{-1}: T^{-1} \rightarrow T^0 \) is not radical, which is a contradiction. Since \( T^i \) and \( X^i \) are injective for all \( i < 0 \), by [7, Lemma 2.2], \( U^\bullet \) and \( V^\bullet \) are isomorphic in \( \mathcal{X}^b(A) \). Thus, \( T^i \) is a direct summand of \( X^i \) for all \( i < 0 \), and consequently \( Q_i = \bigoplus_{i \geq 0} T_i \in \text{add}(v_AQ_i) \). Since \( Q_i \) and \( v_AQ_i \) have the same number of non-isomorphic indecomposable direct summands, we have \( \text{add}(A Q_i) = \text{add}(v_A Q_i) \). Let \( Q := \bigoplus_{i > 0} T_i \). Similarly, we have \( \text{add}(A Q) = \text{add}(v_A Q) \). Consequently, \( \text{add}(A T^\pm) = \text{add}(A Q_1 \oplus A Q_2) = \text{add}(v_A Q_1 \oplus v_A Q_2) = \text{add}(v_A T^\pm) \).

In the following, we shall use Lemma 3.1 freely. For instance, in the definition of an almost \( \nu \)-stable equivalence, the condition \( \text{add}\left(\bigoplus_{i=1}^n T^{-i}\right) = \text{add}\left(\bigoplus_{i=1}^n v_A T^{-i}\right) \) is equivalent to the condition \( T^{-i} \in v_A-\text{Stp} \) for all \( i = 1, \ldots, n \).

**Lemma 3.2.** Let \( F : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B) \) be a derived equivalence between two algebras \( A \) and \( B \), and let \( T^* \) and \( \bar{T}^* \) be the tilting complexes associated to \( F \) and \( F^{-1} \), respectively. If \( \text{add}(A T^\pm) = \text{add}(v_A T^\pm) \) and \( \text{add}(B \bar{T}^\pm) = \text{add}(v_B \bar{T}^\pm) \), then \( F \) induces an equivalence between \( \mathcal{X}^b(v_A-\text{Stp}) \) and \( \mathcal{X}^b(v_B-\text{Stp}) \).

**Proof.** Let \( A E \) (respectively, \( B E \)) be a basic additive generator of \( v_A-\text{Stp} \) (respectively, \( v_B-\text{Stp} \)). That is, \( \text{add}(A E) = v_A-\text{Stp} \). Then \( v_A E \) is also a basic additive generator of \( v_A-\text{Stp} \). Hence \( \text{con}(\eta) = \lambda E \). The complex \( F(\eta) \) is isomorphic to a complex \( \tau_1 \eta \) in \( \text{add}(\mathcal{T}^?) \). Since \( \text{con}(\eta) \) splits, and therefore \( v_B \bar{\eta} \tau_1 \eta \in \mathcal{X}^b(B) \). Hence \( \text{add}(\text{con}(\eta)) \) is acyclic. By our assumption, all \( \tau_1 \eta \) and \( v_B \eta \eta \) with \( \tau_1 \) are projective-injective since they are all in \( v_B-\text{Stp} \). Hence \( \text{con}(\eta) \) splits, and therefore \( v_B \bar{\eta} \tau_1 \eta \in \text{add}(\mathcal{T}^?) \). It follows that \( v_B \eta \tau_1 \eta \in \text{add}(\mathcal{T}^?) \) for all \( i \geq 0 \). Hence \( \mathcal{T}^? \in v_B-\text{Stp} \), and consequently \( \tau_1 \eta \in \text{add}(\mathcal{T}^?) \). Similarly, one can show that \( F^{-1}(B E) \) is isomorphic to a complex in \( \mathcal{X}^b(v_A-\text{Stp}) \) and the lemma is proved.

The following lemma is useful in the proof of Theorem 1.1.

**Lemma 3.3.** Let \( F : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B) \) and \( G : \mathcal{D}^b(B) \rightarrow \mathcal{D}^b(C) \) be derived equivalences, and let \( P^*, \bar{P}^*, Q^*, \bar{Q}^*, T^*, \bar{T}^* \) be the tilting complexes associated to \( F, F^{-1}, G, G^{-1}, GF, \) and \( F^{-1}G^{-1} \) respectively. If the following hold:

1. \( A P^* \in v_A-\text{Stp} \) and \( B \bar{P}^* \in v_B-\text{Stp} \);
2. \( A Q^* \in v_A-\text{Stp} \) and \( C \bar{Q}^* \in v_C-\text{Stp} \);

then \( A \bar{T}^* \in v_A-\text{Stp} \) and \( C \bar{T}^* \in v_C-\text{Stp} \).

**Proof.** We only need to show that \( \bar{T}^* \in v_C-\text{Stp} \), the other statement follows by symmetry. By definition, \( \bar{T}^* \) is isomorphic to \( GF(A) \simeq G(\bar{P}^*) \). Since \( \bar{P}^* \in v_B-\text{Stp} \) for all \( i \neq 0 \), by Lemma 3.2, \( G(\bar{P}^*) \) is isomorphic to a complex \( Y_i^* \) in \( \mathcal{X}^b(\text{Vc-Stp}) \). For all \( i \neq 0 \). For \( i = 0 \), the complex \( G(\bar{P}^*) \) is isomorphic to a complex \( Y_0^* \) in \( \text{add}(\bar{Q}^*) \). By Lemma 2.1, the complex \( G(\bar{P}^*) \) is isomorphic to a complex \( Z^* \) with \( Z^* = \bigoplus_{i+j=n} Y_i^j \). Since all \( Y_i^0 \) are in \( \text{Vc-Stp} \), we have \( Z^* \in v_C-\text{Stp} \). Note that \( \bar{T}^* \) and \( Z^* \) are complexes in \( \mathcal{X}^b(\text{C-proj}) \), which are both isomorphic in \( \mathcal{X}^b(\text{C-proj}) \). Hence \( \bar{T}^* \) and \( Z^* \) are isomorphic in \( \mathcal{X}^b(\text{C-proj}) \). Furthermore, since the complex \( \bar{T}^* \) is a radical complex, it follows that \( \bar{T}^i \) is a direct summand of \( Z^* \) for integers \( i \), and consequently \( \bar{T}^* \in v_C-\text{Stp} \).

Finally, we have the following lemma.

**Lemma 3.4.** Let \( F : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B) \) be a derived equivalence between two algebras \( A \) and \( B \), and let \( T^* \) be tilting complex associated to \( F \). If \( A T^* \in v_A-\text{Stp} \), then there is an almost \( \nu \)-stable equivalence \( G : \mathcal{D}^b(C) \rightarrow \mathcal{D}^b(A) \) such that the tilting complex \( P^* \) associated to \( FG \) satisfies that \( P^i \in v_C-\text{Stp} \) for all \( i < 0 \) and \( P^i \in v_C-\text{Stp} \) for all \( i > 0 \).
Proof. Let $AE$ be an additive generator of $v_A$-Stp. That is, $v_A$-Stp = $\text{add}(AE)$. Suppose $m$ is the maximal integer such that $T^m \neq 0$. By a dual statement of [6, Proposition 3.2], there is a tilting complex $Q^* := R^* \oplus AE[-m]$ over $A$, where $R^*$ is of the form: $R^* : 0 \to A \to R^1 \to \cdots \to R^m \to 0$ with $R^i \in v_A$-Stp for all $i > 0$. Let $C$ be the endomorphism algebra of $Q^*$, and let $H : \mathcal{D}(A) \to \mathcal{D}(B)$ be a derived equivalence induced by the tilting complex $Q^*$. It is easy to see that $H(A) \simeq CP[m]$ for some $CP \in v_C$-Stp, and $H(A)$ is isomorphic to a complex $S^* : 0 \to S^{-m} \to \cdots \to S^{-1} \to S^0 \to 0$ with $S^i \in v_C$-Stp for all $i < 0$. Let $G$ is a quasi-inverse of $H$. Then $S^*$ is a tilting complex associated to $G$. By Lemma 3.1, we see that $G$ is almost $v$-stable.

Now let $Y^*_i := H(T^i)$ for each integer $i$. Since $T^i = v_A$-Stp, for each integer $i \neq 0$, we have $Y^*_i \simeq P_i[m]$ for some $P_i \in v_C$-Stp. Moreover, $Y^*_i \neq 0$ for all $i > m$ since $T^i = 0$ for all $i > m$. The complex $Y^*_0$ has the property that $Y^*_i = 0$ for all $i > 0$ and $Y^*_0 \in v_C$-Stp for all $i < 0$. By Lemma 2.1, the complex $H(T^*)$ is isomorphic to a complex $Z^*$ with $Z^* \rightleftharpoons \bigoplus_{i,j} Y^*_i$. It follows that $Z^* = 0$ for all $t > 0$ and $Z^* \in v_C$-Stp for all $t < 0$. Since $FG(H(T^*)) \simeq F(T^*) \simeq B \simeq FG(P^*)$ in $\mathcal{D}^b(B)$, the complex $Z^*$ is isomorphic in $\mathcal{D}^b(C)$ to the tilting complex $P^*$ associated to $FG$.

It follows that $Z^*$ and $P^*$ are isomorphic in $\mathcal{X}^b(C)$-proj. Since $P^*$ is a radical complex, the term $P^i$ is a direct summand of $Z^i$ for all $i$, and consequently $P^*$ has the desired property. \qed

We are now in the position to give a proof of our main result.

Proof of Theorem 1.1. (1) $\Rightarrow$ (2). Note that the condition (2) clearly holds for almost $v$-stable derived equivalences and their quasi-inverses. Thus, (1) $\Rightarrow$ (2) follows immediately from Lemma 3.3.

(2) $\Leftrightarrow$ (3). This follows from Lemma 3.1.

(3) $\Rightarrow$ (1). By Lemma 3.4, there is an almost $v$-stable derived equivalence $G : \mathcal{D}^b(C) \to \mathcal{D}^b(A)$ such that the tilting complex $P^*$ associated to $FG$ has the property that $P^i = 0$ for all $i > 0$ and $P^i \in v_C$-Stp for all $i < 0$. Let $P^*$ be the tilting complex associated to $G^{-1}F^{-1}$. It follows from Lemma 3.3 that $bP^{\pm} \in v_B$-Stp. Since $P^{\pm} = 0$ for all $i > 0$, by [7, Lemma 2.1], we get $P^i = 0$ for all $i < 0$. Using Lemma 3.1, we see that add($\bigoplus_{i < 0} P^i$) = add($\bigoplus_{i < 0} v_C P^i$) and add($\bigoplus_{i > 0} P^i$) = add($\bigoplus_{i > 0} v_B P^i$). This implies that $FG$ is an almost $v$-stable derived equivalence. Thus, $F \simeq (FG)G^{-1}$ is an iterated almost $v$-stable derived equivalence.

(4) $\Rightarrow$ (5). For each indecomposable projective $A$-module $P$ not in $v_A$-Stp, since $F(top(P))$ is isomorphic in $\mathcal{D}^b(B)$ to a simple $B$-module, we have Hom$_{\mathcal{X}^b(A)}(T^*, top(P))[i]) = 0$ for all $i \neq 0$. This implies that $P$ is not a direct summand of $T^\pm$. This proves (a) of condition (5). It follows from the definition of $b(T^*)$ that $F(b(T^*))$ is isomorphic to a basic projective generator $M$ of $B$. Since $F(top(P))$ is a simple $B$-module, we deduce that

$\text{Hom}_A(b(T^*)^0, top(P)) \simeq \text{Hom}_{\mathcal{X}^b(A)}(b(T^*), top(P)) \simeq \text{Hom}_{\mathcal{X}^b(A)}(b(T^*), top(P)) \simeq \text{End}_B(F(top(P))) \simeq \text{End}_A(top(P))$ is one-dimensional over the division ring $\text{End}_A(top(P))$. It follows that the multiplicity of $P$ as a direct summand of $b(T^*)^0$ is 1.

(5) $\Rightarrow$ (4). By condition (a), we see that Hom$_{\mathcal{X}^b(A)}(T^*, top(P))[i]) = 0$ for all $i \neq 0$. Hence $F(top(P))$ is isomorphic to a $B$-module $X$. By condition (b), up to isomorphism, there is only one indecomposable direct summand $T^p$ of $T^*$ such that Hom$_{\mathcal{X}^b(A)}(T^p, top(P)) \neq 0$. Equivalently, up to isomorphism, there is only one indecomposable projective $B$-module $P$ such that Hom$_B(P, X) \neq 0$. This means that $X$ only contains top($P$) as composition factors. If $X$ is not a simple $B$-module, then there is a nonzero map $X \to \text{soc}(X) \to X$ in $\text{End}_B(X)$ which is not an isomorphism. This contradicts the fact that $\text{End}_B(X) \simeq \text{End}_A(top(P))$ is a division ring. Hence $X \simeq F(top(P))$ is a simple $B$-module.

(3) $\Rightarrow$ (4). Let $AE$ and $BE$ be additive generators of $v_A$-Stp and $v_B$-Stp, respectively. That is, add($AE) = v_A$-Stp and add($BE) = v_B$-Stp. Let $P$ be an indecomposable projective $A$-module not in $v_A$-Stp. Then it is clear that Hom$_{\mathcal{X}^b(A)}(T^*, top(P))[i]) = 0$ for all $i \neq 0$ since $T^\pm \in v_A$-Stp, and consequently $F(top(P))$ is isomorphic
in \( D(B) \) to a \( B \)-module \( X \). By Lemma 3.2, the complex \( F^{-1}(b\tilde{E}) \) is isomorphic in \( D(A) \) to a complex \( E^* \) in \( K(A) \). Hence
\[
\text{Hom}_{\mathcal{A}}(b\tilde{E}, X) \simeq \text{Hom}_{\mathcal{A}}(F^{-1}(b\tilde{E}), \text{top}(P)) \simeq \text{Hom}_{\mathcal{A}}(E^*, \text{top}(P)) = 0.
\]

If \( bX \) is not simple, then there is a short exact sequence \( 0 \to \tilde{U} \to \bar{X} \to \bar{V} \to 0 \) in \( \text{mod}-B \) with \( \tilde{U}, \bar{V} \) non-zero.

By applying \( \text{Hom}_{\mathcal{A}}(b\tilde{E}, -) \), we get that \( \text{Hom}_{\mathcal{A}}(b\tilde{E}, \bar{U}) = 0 = \text{Hom}_{\mathcal{A}}(b\tilde{E}, \bar{V}) \), and consequently \( \text{Hom}_{\mathcal{A}}(b\tilde{E}, \bar{W}) = 0 = \text{Hom}_{\mathcal{A}}(b\tilde{E}, \bar{W}[i]) \) for all \( i \neq 0 \) since \( T^\pm \in \mathcal{A} \)-Stp. Hence \( F^{-1}(\bar{U}) \) and \( F^{-1}(\bar{V}) \) are isomorphic to some \( A \)-modules \( U \) and \( V \), respectively. Thus, we get a distinguished triangle
\[
U \to \text{top}(P) \to V \to U[1]
\]
in \( D(A) \) by applying \( F^{-1} \) to the distinguished triangle \( \tilde{U} \to \bar{X} \to \bar{V} \to \tilde{U}[1] \). Applying \( \text{Hom}_{\mathcal{A}}(A, -) \) to the above triangle, we get an exact sequence \( 0 \to \tilde{U} \to \text{top}(P) \to V \to 0 \) with non-zero \( A \)-modules \( U \) and \( V \). This contradicts to the fact that \( \text{top}(P) \) is a simple \( A \)-module. Hence \( F(\text{top}(P)) \simeq X \) is a simple \( B \)-module.

(4) \( \Rightarrow \) (3). For each indecomposable projective \( A \)-module \( P \) not in \( \mathcal{A} \)-Stp, since \( F(\text{top}(P)) \) is isomorphic in \( D(B) \) to a simple \( B \)-module, we have
\[
\text{Hom}_{\mathcal{A}}(T^i, \text{top}(P)) \simeq \text{Hom}_{\mathcal{A}}(\text{proj}(T^i), \text{top}(P)[-i]) \simeq 0
\]
for all \( i \neq 0 \). This implies that \( T^i \in \mathcal{A} \)-Stp for all \( i \neq 0 \), that is, \( T^\pm \in \mathcal{A} \)-Stp. It remains to show that \( T^\pm \in \mathcal{A} \)-Stp. Let \( \mathcal{A}Q \) be a direct sum of all non-isomorphic indecomposable projective \( A \)-modules not in \( \mathcal{A} \)-Stp. Then \( F(\text{top}(P)) \) is isomorphic in \( D(B) \) to a semi-simple \( B \)-module. Let \( \mathcal{A}Q \) be a projective cover of \( F(\text{top}(P)) \). That is, \( F(\text{top}(P)) \simeq \text{top}(\mathcal{A}Q) \). Now we set \( b\mathcal{A}W \) to be a direct sum of all non-isomorphic indecomposable projective \( B \)-modules not in \( \mathcal{A} \)-Stp. Then \( \mathcal{A}Q \oplus \mathcal{A}W \) is a basic projective generator of \( B \). There are isomorphisms
\[
\text{Hom}_{\mathcal{A}}(\mathcal{A}Q, \text{top}(P)) \simeq \text{Hom}_{\mathcal{A}}(\mathcal{A}W, \text{top}(P)[i]) \simeq 0
\]
for all \( i \neq 0 \). This means that none of the indecomposable direct summands of \( \mathcal{A}Q \) are in \( \mathcal{A} \)-Stp, or equivalently, \( T^\pm \in \mathcal{A} \)-Stp. It remains to show that \( b\mathcal{A}W \in \mathcal{A} \)-Stp. Note that
\[
\text{Hom}_{\mathcal{A}}(F^{-1}(b\mathcal{A}W), \text{top}(P)[i]) \simeq \text{Hom}_{\mathcal{A}}(b\mathcal{A}W, \text{top}(\mathcal{A}Q)[i]) = 0
\]
for all integers \( i \). It follows that \( F^{-1}(b\mathcal{A}W) \) is isomorphic in \( D(A) \) to a radical complex in \( \mathcal{A} \)-Stp. Using the same method in the proof [1, Theorem 2.1], one can show that \( b\mathcal{A}W \) is a projective \( B \)-module for all \( i \geq 0 \). Hence \( b\mathcal{A}W \in \mathcal{A} \)-Stp. This finishes the proof.

Remark: (1) The condition (5) in Theorem 1.1 provides a convenient way to check whether a given derived equivalence is iterated almost \( v \)-stable or not.

(2) Let \( P \) be a projective \( A \)-module. The condition \( \text{add}(\mathcal{A}P) = \text{add}(\mathcal{A}P) \) is equivalent to saying that \( P \) is projective-injective and \( \text{add}(P) = \text{add}(\text{soc}(P)) \).

(3) It is interesting to know whether Theorem 1.1 holds for general Artin algebras. Note that the only problem is the step “(4) \( \Rightarrow \) (3)”, where the method in the proof of [1, Theorem 2.1] does not work for general Artin algebras. In particular, for general Artin algebras, the conditions (1), (2) and (3) in Theorem 1.1 are still equivalent.

As a consequence of Theorem 1.1, together with [7, Theorem 5.3, and Corollary 1.2], we have a corollary.
**Corollary 3.5.** Let $F : \mathcal{D}^b(A) \to \mathcal{D}^b(B)$ be a derived equivalence between two finite-dimensional algebras over a field. If one of the equivalent conditions in Theorem 1.1 is satisfied, then the algebras $A$ and $B$ are stably equivalent of Morita type. Moreover, the algebras $A$ and $B$ have the same finitistic dimension, global dimension, representation dimension and dominant dimension.

In the rest of this section, we give a method to construct tilting complexes which induce iterated almost $\nu$-stable derived equivalences.

Let us recall from [2] the definition of approximations. Let $C$ be a category, and let $\mathcal{D}$ be a full subcategory of $C$, and $X$ an object in $C$. A morphism $f : D \to X$ in $C$ is called a right $\mathcal{D}$-approximation of $X$ if $D \in \mathcal{D}$ and the induced $\text{Hom}_C(D', f) : \text{Hom}_C(D', D) \to \text{Hom}_C(D', X)$ is surjective for every object $D' \in \mathcal{D}$. Dually, one can define left $\mathcal{D}$-approximations.

Let $A$ be an algebra, and let $P, Q$ be two projective $A$-modules satisfying the following two conditions:

1. $\text{add}(A) = \text{add}(\text{add}(A))$;
2. $\text{Hom}_A(P, Q) = 0$.

For each positive integer $r$, we can form the following complex:

\[
0 \to P^{-r} \xrightarrow{f_1} P^{-r+1} \to \cdots \to P^{-1} \xrightarrow{f_1} A \to 0,
\]

where $f_1 : P^{-1} \to A$ is a right $\text{add}(A)$-approximation of $A$, and $f_{i+1} : P^{-i-1} \to P^{-i}$ is a right $\text{add}(A)$-approximation of $\text{Ker}(f_i)$ for $i = 1, \cdots, r - 1$. Similarly, we can form a complex

\[
0 \to A \xrightarrow{g_1} Q^1 \to \cdots \to Q^{s-1} \xrightarrow{g_s} Q^s \to 0,
\]

where $g_1$ is a left $\text{add}(A)$-approximation of $A$, and $g_{i+1}$ is a left $\text{add}(A)$-approximation of $\text{Coker}(g_i)$ for $i = 1, 2, \cdots, s - 1$. Since $\text{Hom}_A(P, Q) = 0$, gluing the two complexes together, we get a complex

\[
0 \to P^{-r} \to \cdots \to P^{-1} \xrightarrow{f_1} A \xrightarrow{g_1} Q^1 \to \cdots \to Q^s \to 0,
\]

where $A$ is in degree zero. We denote this complex by $T^\bullet_{P, Q}$, and let $T^\bullet := T^\bullet_{P, Q} \oplus P[r] \oplus Q[-s]$.

**Proposition 3.6.** Keeping the notations above, the complex $T^\bullet$ is a tilting complex that induces an iterated almost $\nu$-stable derived equivalence between the algebras $A$ and $\text{End}_{\mathcal{A}^{\nu}(A\text{-proj})}(T^\bullet)$.

**Proof.** By the construction of $T^\bullet$, we have

\[
T^i = \begin{cases}
P^{-i} \oplus P, & i = -r; \\
P^i, & -r < i < 0; \\
A_i, & i = 0; \\
Q^i, & 0 < i < s; \\
Q^i \oplus Q, & i = s; \\
0 & \text{otherwise},
\end{cases}
\]

and $d^T_i = \begin{cases}
[f_1], & i = -r; \\
f_{i-1}, & -r < i < 0; \\
g_{i+1}, & 0 \leq i < s - 1; \\
g_r, & i = s - 1; \\
0 & \text{otherwise}.
\end{cases}$

We first show that $\text{Hom}_{\mathcal{A}^{\nu}(A\text{-proj})}(T^\bullet, T^\bullet[i]) = 0$ for all $i \neq 0$. Assume that $i$ is a positive integer. Let $u^i$ be a morphism in $\text{Hom}_{\mathcal{A}^{\nu}(A\text{-proj})}(T^\bullet, T^\bullet[i])$. Then we have the following commutative diagram

\[
\cdots \to T^{-i-1} \xrightarrow{d^{-i-1}} T^{-i} \xrightarrow{d^{-i}} T^{-i+1} \xrightarrow{d^{-i+1}} \cdots \to T^{-1} \xrightarrow{d^{-1}} T^0 \xrightarrow{d_0} T^1 \xrightarrow{d_1} \cdots \to T^{-i-1} \xrightarrow{d^{-i-1}} T^{-i} \xrightarrow{d^{-i}} T^{-i+1} \xrightarrow{d^{-i+1}} \cdots
\]

\[
\cdots \to T^{-i} \xrightarrow{d^{-i}} T^{-i+1} \xrightarrow{d^{-i+1}} \cdots \to T^{-1} \xrightarrow{d^{-1}} T^0 \xrightarrow{d_0} T^1 \xrightarrow{d_1} \cdots \to T^{-i-1} \xrightarrow{d^{-i-1}} T^{-i} \xrightarrow{d^{-i}} T^{-i+1} \xrightarrow{d^{-i+1}} \cdots
\]
Since $\text{Hom}_A(P,Q) = 0$, we have $u^i = 0$ for all $-i < k < 0$. By definition, $T^{-i} \in \text{add}(A_P)$. Since $d_T^{-1} = f_1$ is a right add$(A_P)$-approximation, there is a map $h^{-1}: T^{-i} \to T^{-1}$ such that $u^{-i} = h^{-1}d_T^{-1}$. Thus,

\[(u^{-i-1} - d_T^{-i-1}h^{-1})d_T^{-1} = d_T^{-i-1}u^{-i} - d_T^{-i-1}h^{-1}d_T^{-1} = d_T^{-i-1}u^{-i} - d_T^{-i-1}u^{-i} = 0.\]

Since $d_T^{-2}$ is a right add$(A_P)$-approximation of Ker$(d_T^{-1})$, there is a map $h^{-1}: T^{-i} \to T^{-2}$ such that $u^{-i-1} - d_T^{-i-1}h^{-1} = h^{-i-1}d_T^{-2}$, that is $u^{-i-1} = d_T^{-i-1}h^{-i-1}d_T^{-2}$. Similarly, for each integer $k < -i - 1$, there are maps $h^{k+1}: T^{k+1} \to T^{k+i}$ and $h^k: T^k \to T^{k+i-1}$ such that $u^k = h^k h^{k+1} + h^{k+1} d_T^{-2}$. Defining $h^k = 0$ for all $-i < k < 0$, we have $u^k = h^k h^{k+1} + h^{k+1} d_T^{-2}$ for all $k < 0$. Similarly, we can prove that $u^0 = 0$ in $\mathcal{X}^b(A\text{-proj})$. Hence $\text{Hom}_{\mathcal{X}^b(A\text{-proj})}(T^i, T^i[i]) = 0$ for all $i > 0$. By an analogous proof, one get $\text{Hom}_{\mathcal{X}^b(A\text{-proj})}(T^i, T^i[i]) = 0$ for all $i < 0$. Finally, since $P[r] \in \text{add}(T^*)$, we deduce that $A\text{-A}$ is in the triangulated subcategory of $\mathcal{X}^b(A\text{-proj})$ generated by $\text{add}(T^*)$. Hence $\text{add}(T^*)$ generates $\mathcal{X}^b(A\text{-proj})$ as a triangulated category, and consequently $T^*$ is a tilting complex over $A$. It follows from the condition (5) in Theorem 1.1 that the tilting complex $T^*$ induces an iterated almost $v$-stable derived equivalence between $A$ and $\text{End}_{\mathcal{X}^b(A\text{-proj})}(T^*)$.

To illustrate Proposition 3.6, we give an example. Let $k$ be a field, and let $A$ be the finite-dimensional $k$-algebra given by the quiver

\[
\begin{array}{ccccccc}
\bullet & \alpha & \bullet & \beta & \bullet & \gamma & \bullet \\
1 & 2 & 3 & 4
\end{array}
\]

with relations $\alpha'\alpha = \beta'\beta = \gamma'\gamma = 0$. We use $P_i$ to denote the indecomposable projective $A$-module corresponding to the vertex $i$ for $i = 1, 2, 3, 4$. The Loewy structure of the projective $A$-modules can be listed as follows.

\[
P_1: 1 \quad P_2: 2 \quad P_3: 2 \quad P_4: 4
\]

Let $P := P_3 \oplus P_4$. Then we have add$(A_P) = \text{add}(v_A P)$, add$(A_Q) = \text{add}(v_A Q)$, and Hom$_A(P, Q) = 0$. Using Proposition 3.6, we have a tilting complex $T^*$ over $A$. The indecomposable direct summands of $T^*$ are:

\[
\begin{align*}
T^*_1 & : 0 \to P_1 \to 0 \\
T^*_2 & : 0 \to P_1 \to P_2 \to P_3 \to 0 \\
T^*_3 & : 0 \to P_3 \to 0 \\
T^*_4 & : 0 \to P_4 \to 0
\end{align*}
\]

A calculation shows that the algebra $B := \text{End}_{\mathcal{X}^b(A)}(T^*)$ is given by the quiver

\[
\begin{array}{ccccccc}
\bullet & \alpha & \bullet & \beta & \bullet & \gamma \\
1 & 2 & 3 & 4
\end{array}
\]

with relations $\alpha'\alpha = \beta'\beta = \gamma'\gamma = 0$. By Proposition 3.6, $T^*$ induces an iterated almost $v$-stable derived equivalence between $A$ and $B$. Therefore, $A$ and $B$ are also stably equivalent of Morita type.

Finally, let us remark that one can inductively construct iterated almost $v$-stable derived equivalences from given ones, as we have done for almost $v$-stable derived equivalences in [7] and [8].

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References


