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The representation dimension of $k[x, y]/(x^2, y^n)$ ☆

Thorsten Holm [a,*], Wei Hu [a,b]

[a] *Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, England, UK*
[b] *School of Mathematical Sciences, Beijing Normal University, 100875 Beijing, PR China*

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Abstract

The representation dimension of an Artin algebra was defined by M. Auslander in 1970. The precise value is not known in general, and is very hard to compute even for small examples. For group algebras, it is known in the case of cyclic Sylow subgroups. For some group algebras (in characteristic 2) of rank at least 3 the precise value of the representation dimension follows from recent work of R. Rouquier. There is a gap for group algebras of rank 2. In this paper we show that for all $n \geqslant 0$ and any field k the commutative algebras $k[x, y]/(x^2, y^{2+n})$ have representation dimension 3. For the proof, we give an explicit inductive construction of a suitable generator–cogenerator. As a consequence, we obtain that the group algebras in characteristic 2 of the groups $C_2 \times C_{2^m}$ have representation dimension 3. Note that for $m \geqslant 3$ these group algebras have wild representation type.

1. Introduction

The representation dimension of an Artin algebra was defined by M. Auslander [1] as a way of measuring homologically how far an algebra is from being of finite representation

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* Corresponding author.
E-mail addresses: tholm@maths.leeds.ac.uk (T. Holm), hwxbest@163.com (W. Hu).

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type. He then showed that the representation dimension of an algebra is at most 2 if and only if the algebra has finite representation type.

For three decades it remained unclear whether Auslander's philosophy and hopes were justified, since the precise value of the representation dimension of an algebra is very hard to determine. This situation changed dramatically over the last few years. For instance, O. Iyama proved that the representation dimension is always finite [7]. Moreover, it was recently shown by X. Guo [4] that the representation dimension is invariant under stable equivalences. (For stable equivalences of Morita type this was already shown by C. Xi [11].) In particular, for selfinjective algebras the representation dimension is invariant under derived equivalences.

A breakthrough was obtained by R. Rouquier [9,10]. As a consequence of a general theory of dimensions of triangulated categories he obtained the first examples of algebras with representation dimension > 3. More than that, Rouquier obtained that the representation dimension is unbounded, by showing that for any $n \geqslant 3$ the exterior algebra of an n-dimensional vector space has representation dimension $n + 1$.

So Auslander's philosophy is working; we get a new division of algebras according to the size of their representation dimension. This new division certainly does not run along the lines of the classical division into finite, tame and wild representation type. For instance, already Auslander showed that all finite-dimensional hereditary algebras have representation dimension at most 3, not depending on their representation type.

In this paper, we will provide more examples of algebras of wild representation type having representation dimension 3. Our main motivation comes from group algebras. It is well known that the representation type of a modular group algebra kG of a finite group over a field k of characteristic p is determined by a Sylow p-subgroup P of G [2]: kG has finite representation type if and only if P is cyclic (see also [5]); kG has tame representation type precisely when $p = 2$ and P is dihedral, semidihedral or quaternion. In all other cases kG has wild representation type. Hence, in the cyclic case, the representation dimension of kG is at most 2. It is also known that all group algebras of tame representation type (and related algebras) have representation dimension 3. (Unfortunately, there is no complete proof in the literature, but see [6] and [8, 4.4.2].)

From Rouquier's work one gets that for any $n \geqslant 3$ the mod 2 group algebra of an elementary abelian group $C_2 \times \cdots \times C_2$ of order 2^n has representation dimension $n + 1$. Although one might hope to be able to adapt these methods for p odd, they certainly do not work for groups of rank 2.

The main aim of this paper is to give some partial results on the rank 2 case. Note that the mod p group algebra of $C_{p^m} \times C_{p^n}$ is isomorphic to the commutative ring $k[x, y]/(x^{p^m}, y^{p^n})$. So the natural class of algebras to study are the commutative rings $A_{n,m} := k[x, y]/(x^m, y^n)$, for natural numbers $n, m \geqslant 2$.

Our main result in this paper gives the precise value of the representation dimension for the algebras $A_{2,m}$. Note that this result is independent of the ground field.

Theorem 1.1. *Let k be a field and let $\Lambda_n = k[x, y]/(x^2, y^{2+n})$ for every non-negative integer n. Then the representation dimension of Λ_n is 3.*

As a consequence we get the following partial answer on the 'rank-2-gap' for group algebras.

Corollary 1.2. *Let k be a field of characteristic* 2*. Then for any $n \geqslant 1$ the group algebra $k(C_2 \times C_{2^n})$ has representation dimension* 3*.*

The paper is organized as follows. In Section 2 we recall the necessary definitions and background on representation dimensions. In particular, we outline the method for determining the precise value of the representation dimension. The final Section 3 then contains the proof of the main result.

2. Computing representation dimensions

Throughout this paper, by an algebra we mean a finite-dimensional algebra over a fixed field k. A finitely generated A-module M is called a *generator–cogenerator* for A if all projective indecomposable A-modules and all injective indecomposable A-modules occur as direct summands of M.

Auslander's basic idea was to study homological properties of endomorphism rings of modules (instead of studying modules directly). For any algebra A the *representation dimension* is defined to be

$$\mathrm{rep.dim}(A) := \inf\bigl\{\mathrm{gl.dim}\bigl(\mathrm{End}_A(M)\bigr) \mid M \text{ generator–cogenerator}\bigr\}.$$

It was shown by O. Iyama [7] that $\mathrm{rep.dim}(A)$ is always finite. Auslander's fundamental result states that $\mathrm{rep.dim}(A) \leqslant 2$ if and only if A has finite representation type.

The definition indicates how to determine upper bounds for $\mathrm{rep.dim}(A)$; if one can show for some generator–cogenerator M that $\mathrm{gl.dim}(\mathrm{End}_A(M)) \leqslant m$ then $\mathrm{rep.dim}(A) \leqslant m$. Of course, the notoriously difficult problem is to find a suitable module M with endomorphism ring of small global dimension. However, this is exactly the strategy we will pursue successfully in the proof of our main theorem. We will explain in Section 3 below how to construct inductively generator–cogenerators for the algebras $k[x, y]/(x^2, y^{2+n})$ with endomorphism rings of global dimension 3.

We now describe the general method used later for how to determine the global dimension of the endomorphism ring of a generator–cogenerator M.

The identity of $\mathrm{End}_A(M)$ is the sum of the "identity maps" on the indecomposable direct summands of M. Hence we have primitive idempotents of $\mathrm{End}_A(M)$ corresponding to the summands of M. For any indecomposable summand T of M we denote the corresponding simple $\mathrm{End}_A(M)$-module by E_T. The corresponding indecomposable projective $\mathrm{End}_A(M)$-module Q_T is given by all homomorphisms from M to T, for abbreviation denoted by $Q_T = \mathrm{Hom}_A(M, T) =: (M, T)$. To prove that $\mathrm{gl.dim}(\mathrm{End}_A(M)) \leqslant 3$, we explicitly construct a projective resolution with length $\leqslant 3$ for every simple module E_T. The general method is as follows. Recall that for a finitely generated A-module M, $\mathrm{add}\, M$ denotes the full subcategory of mod A consisting of direct sums of direct summands of M.

For any indecomposable summand T of M, we first construct a suitable exact sequence $0 \to K \to N_1 \to T$ with $N_1 \in \operatorname{add} M$ with the following property:

($*$) Every homomorphism from an indecomposable summand of M to T, except the multiples of the identity on T, factors through N_1.

Applying the functor $(M, -)$, we get another short exact sequence

$$0 \to (M, K) \to (M, N_1) \to (M, T).$$

If the cokernel of $(M, N_1) \to (M, T)$ is 1-dimensional then the cokernel is E_T (because (M, T) is projective), i.e. we get the initial part of a projective resolution of the simple $\operatorname{End}_A(M)$-module E_T.

If $K \in \operatorname{add} M$, then also (M, K) is projective and we have constructed a projective resolution of E_T of length 2. Otherwise, we construct another suitable short exact sequence $0 \to K' \to N_2 \to K$ with $N_2 \in \operatorname{add} M$ with the following property:

($**$) Every map from an indecomposable summand of M to K factors through N_2.

Applying the functor $(M, -)$, we get a short exact sequence

$$0 \to (M, K') \to (M, N_2) \to (M, K) \to 0.$$

If it happens that $K' \in \operatorname{add} M$, we get a projective resolution

$$0 \to Q_{K'} \to Q_{N_2} \to Q_{N_1} \to Q_T \to E_T \to 0$$

and $\operatorname{proj.dim} E_T \leqslant 3$.

The crucial aspect in this strategy is to come up with a generator–cogenerator M for which this process really stops at this stage, i.e. for which $K' \in \operatorname{add} M$ in the second step above.

3. Proof of the main theorem

For any $n \geqslant 1$, let Λ_n denote the (commutative) algebra $k[x, y]/(x^2, y^{n+2})$. The quotient algebra $\Lambda_n/\operatorname{soc}(\Lambda_n)$ will be denoted by A_n.

In order to prove Theorem 1.1 it suffices to show that $\operatorname{rep.dim}(A_n) = 3$, due to the following general result. Recall that an algebra A is called *basic* if all simple A-modules are of dimension 1.

Lemma 3.1. [3, Proposition 1.2] *Let Λ be a basic algebra, and let P be an indecomposable projective-injective Λ-module. Define $A = \Lambda/\operatorname{soc}(P)$. If* $\operatorname{rep.dim}(A) \leqslant 3$*, then* $\operatorname{rep.dim}(\Lambda) \leqslant 3$.

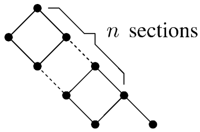


Fig. 1.

Note that our algebras $k[x, y]/(x^2, y^{2+n})$ and their factor algebras A_n are basic. The structure of the projective-indecomposable A_n-module can conveniently be described diagrammatically in Fig. 1.

Here, the vertices correspond to basis vectors of the module and the edges describe the action of the generators x and y of A_n.

Now, we use a Table 1 to introduce and illustrate the notations for several classes of A_n-modules which will be used throughout the paper.

Note that the projective A_n-module will be denoted by A_n^0. For any $n \geqslant 0$, we will prove that $\text{rep.dim}(A_n) \leqslant 3$ by constructing inductively a generator–cogenerator M_n for A_n such that $\text{gl.dim}(\text{End}_{A_n}(M_n)) \leqslant 3$.

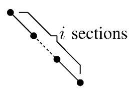
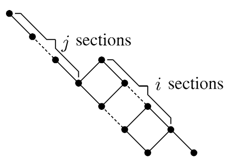
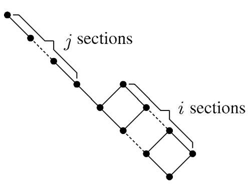
For $n = 0$ we set

$$M_0 = A_0^0 \oplus DA_0^0 \oplus U_0 \oplus U_1 \oplus X.$$

If n is positive, we set

$$M_n = \bigoplus_{i,j\geqslant 0,\, i+j\leqslant n} DA_i^j \oplus \bigoplus_{i\geqslant 0,\, j>0,\, i+j\leqslant n} A_i^j \oplus A_n^0 \oplus \bigoplus_{0\leqslant i\leqslant n+1} U_i \oplus X.$$

Table 1

Notation	Shape of the A_n-module
$U_i,\ 0 \leqslant i \leqslant n+1$	
X	
$A_i^j,\ i, j \geqslant 0,\ i+j \leqslant n$	
$DA_i^j,\ i, j \geqslant 0,\ i+j \leqslant n$	

Remark 3.2. In M_n we still have all summands of M_{n-1}, except the projective module A_{n-1}^0. The new summands in M_n are DA_i^{n-i} and A_i^{n-i} (for $0 \leqslant i \leqslant n$) and the uniserial module U_{n+1}. With the exception of A_n^0 and its dual, they can be obtained from summands of M_{n-1} as follows. For all summands of M_{n-1} we add an additional vertex on top (i.e. an additional basis vector of the module) whenever possible; but we only allow this when no new squares are created in the shape of the module. All modules obtained in this way are added as summands of M_n (but only once, to avoid multiplicities).

For instance, going from M_0 to M_1, the extension of DA_0^0 to the four-dimensional module having the shape of a square is not allowed; on the other hand, the extension of DA_0^0 to the four-dimensional module DA_0^1 is allowed and gives a new summand. With this procedure, for $i = 0, \ldots, n-1$, the new modules DA_i^{n-i} are obtained from the summands DA_i^{n-i-1} of M_{n-1}, the new summands A_i^{n-i} are obtained from A_i^{n-i-1}, and U_{n+1} is obtained from U_n. Note that the new projective module A_n^0 and the new injective module DA_n^0 cannot be obtained with this method from summands of M_{n-1}, so they have to be added separately.

We are now in the position to give the proof of our main result.

Proof of Theorem 1.1. We have to show that $\mathrm{gl.dim}(\mathrm{End}_{A_n}(M_n)) \leqslant 3$ for any $n \geqslant 0$.

(I) The case $n = 0$. We have $M_0 = A_0^0 \oplus DA_0^0 \oplus U_0 \oplus U_1 \oplus X$.

For the algebra $A_0 = k[x,y]/(x^2, y^2, xy)$ it is known that $\mathrm{rep.dim}(A_0) = 3$; in fact, A_0 is special biserial and we can apply Corollary 1.3 from [3]. There it is shown that the above generator–cogenerator M_0 has an endomorphism ring of global dimension at most 3. We refer to [3] for details.

(II) The case $n > 0$. We have

$$M_n = \bigoplus_{i,j \geqslant 0,\, i+j \leqslant n} DA_i^j \oplus \bigoplus_{i \geqslant 0,\, j>0,\, i+j \leqslant n} A_i^j \oplus A_n^0 \oplus \bigoplus_{0 \leqslant i \leqslant n+1} U_i \oplus X.$$

For any indecomposable summand N of M_n, we show that $\mathrm{proj.dim}\, E_N \leqslant 3$ by explicitly constructing a projective resolution for the corresponding simple $\mathrm{End}_{A_n}(M_n)$-module E_N. For the convenience of the reader, we list the indecomposable summands of M_n as follows:

$$\begin{array}{cccccc}
A_n^0 & & & & & X \\
& A_{n-1}^1 & A_{n-2}^1 & \cdots & A_1^1 & A_0^1 \\
& & A_{n-2}^2 & \cdots & A_1^2 & A_0^2 \\
& & & \ddots & \vdots & \vdots \\
& & & & A_1^{n-1} & A_0^{n-1} \\
& & & & & A_0^n
\end{array}$$

$$\begin{array}{cccccc}
DA_n^0 & DA_{n-1}^0 & DA_{n-2}^0 & \cdots & DA_1^0 & DA_0^0 \\
 & DA_{n-1}^1 & DA_{n-2}^1 & \cdots & DA_1^1 & DA_0^1 \\
 & & DA_{n-2}^2 & \cdots & DA_1^2 & DA_0^2 \\
 & & & \ddots & \vdots & \vdots \\
 & & & & DA_1^{n-1} & DA_0^{n-1} \\
 & & & & & DA_0^n
\end{array}$$

$$U_0, U_1, U_2, \ldots, U_n, U_{n+1}.$$

(1) The projective module A_n^0. Clearly, the exact sequence $0 \to \operatorname{rad} A_n^0 \to A_n^0 \to 0$ has the property $(*)$. Since $\operatorname{rad} A_n^0 = A_{n-1}^1 \in \operatorname{add} M_n$, applying the functor $(M_n, -)$ to the exact sequence gives a projective resolution of the form

$$0 \to Q_{A_{n-1}^1} \to Q_{A_n^0} \to E_{A_n^0} \to 0.$$

Hence $\operatorname{proj.dim} E_{A_n^0} = 1$.

(2) The module X. There is a short exact sequence $0 \to A_{n-1}^1 \to U_0 \oplus A_n^0 \to X \to 0$ with the property $(*)$. (Actually, the only epimorphisms, apart from the multiples of the identity on X, are from A_n^0, so clearly they factor through the middle term. All other maps factor through the radical $\operatorname{rad} X = U_0$.) By applying the functor $(M_n, -)$, we get a projective resolution

$$0 \to Q_{A_{n-1}^1} \to Q_{U_0} \oplus Q_{A_n^0} \to Q_X \to E_X \to 0.$$

Hence $\operatorname{proj.dim} E_X \leqslant 2$.

(3) The modules A_i^1. If $i = 0$, then there is a short exact sequence $0 \to DA_0^1 \to U_1 \oplus U_0 \oplus DA_1^0 \to A_0^1 \to 0$ with the property $(*)$. (Except for the multiples of the identity on A_0^1, the epimorphisms are from A_s^1 $(s > 0)$ and DA_t^0. They all factor through DA_1^0.) Hence we get a projective resolution

$$0 \to Q_{DA_0^1} \to Q_{U_1} \oplus Q_{U_0} \oplus Q_{DA_1^0} \to Q_{A_0^1} \to E_{A_0^1} \to 0.$$

Hence $\operatorname{proj.dim} E_{A_0^1} \leqslant 2$.

If $i > 0$, then there is a short exact sequence $0 \to DA_i^1 \to A_{i-1}^2 \oplus DA_{i+1}^0 \to A_i^1$ with the property $(*)$. (Apart from the multiples of the identity on A_i^1, the epimorphisms to A_i^1 are from A_s^1 $(s > i)$ and DA_t^1 $(t > i)$. They all factor through DA_i^1. The non-epimorphisms all factor through the middle term by the definition.) Applying the functor $(M_n, -)$, we get a projective resolution

$$0 \to Q_{DA_i^1} \to Q_{A_{i-1}^2} \oplus Q_{DA_{i+1}^0} \to Q_{A_i^1} \to E_{A_i^1} \to 0.$$

(4) The modules A_i^j $(j > 1, i > 0)$. For every such module, there is a short exact sequence $0 \to K \to A_{i-1}^{j+1} \oplus A_i^{j-1} \oplus DA_{i+1}^{j-1} \to A_i^j \to 0$ with the property $(*)$. (In fact, the

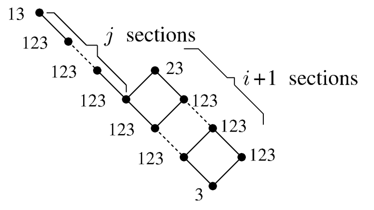


Fig. 2.

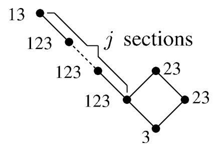


Fig. 3.

only epimorphisms from summands other than A_i^j are from A_s^j $(s > i)$ and DA_t^{j-1} $(t > i)$. They all factor through DA_{i+1}^{j-1}. Other maps factor through $A_{i-1}^{j+1} \oplus A_i^{j-1}$.) The kernel K has the shape of Fig. 2.

Clearly, K is not in $\operatorname{add} M_n$. There is a short exact sequence $0 \to DA_i^{j-1} \to A_{i-1}^j \oplus DA_i^j \oplus DA_{i+1}^{j-2} \to K \to 0$ with the property $(**)$. Applying the functor $(M_n, -)$ to the above two exact sequences and glueing them, we get a projective resolution

$$0 \to Q_{DA_i^{j-1}} \to Q_{A_{i-1}^j} \oplus Q_{DA_i^j} \oplus Q_{DA_{i+1}^{j-2}} \to Q_{A_{i-1}^{j+1}} \oplus Q_{A_i^{j-1}} \oplus Q_{DA_{i+1}^{j-1}} \to Q_{A_i^j} \to E_{A_i^j} \to 0.$$

Hence $\operatorname{proj.dim} E_{A_i^j} \leqslant 3$ for $j > 1$ and $i > 0$.

(5) The modules A_0^j $(j > 1)$. For any A_0^j with $j > 1$, there is a short exact sequence $0 \to K \to U_j \oplus A_0^{j-1} \oplus DA_1^{j-1} \to A_0^j \to 0$ with the property $(*)$. (The only epimorphisms not from A_0^j itself are from A_s^j $(s > 0)$ or DA_t^j $(t > 0)$. They all factor through DA_1^{j-1}.) The kernel K has the shape of Fig. 3.

Again, K is not in $\operatorname{add} M_n$, but there is a short exact sequence $0 \to U_j \to U_{j-1} \oplus U_{j+1} \oplus DA_1^{j-2} \to K \to 0$ with the property $(**)$. Applying the functor $(M_n, -)$ to the above two short exact sequences and glueing them, we get a projective resolution of the simple $\operatorname{End}_{A_n}(M_n)$-module $E_{A_0^j}$:

$$0 \to Q_{U_j} \to Q_{U_{j-1}} \oplus Q_{U_{j+1}} \oplus Q_{DA_1^{j-2}} \to Q_{U_j} \oplus Q_{A_0^{j-1}} \oplus Q_{DA_1^{j-1}} \to Q_{A_0^j} \to E_{A_0^j} \to 0.$$

Hence $\operatorname{proj.dim} E_{A_0^j} \leqslant 3$ for $j > 1$.

(6) The module DA_n^0. There is a short exact sequence $0 \to A_{n-1}^1 \to DA_{n-1}^1 \oplus A_n^0 \to DA_n^0 \to 0$ with the property $(*)$ and all terms of the short exact sequence are in $\operatorname{add} M_n$. This leads to the following projective resolution

$$0 \to Q_{A_{n-1}^1} \to Q_{DA_{n-1}^1} \oplus Q_{A_n^0} \to Q_{DA_n^0} \to E_{DA_n^0} \to 0.$$

Hence $\operatorname{proj.dim} DA_n^0 \leqslant 2$.

(7) The modules DA_i^0 $(0 < i < n)$. For any DA_i^0 with $0 < i < n$, there is a short exact sequence $0 \to A_{i-1}^2 \to DA_{i-1}^1 \oplus A_i^1 \to DA_i^0 \to 0$ with the property $(*)$. (Except for the multiples of the identity on DA_i^0, the only epimorphisms to DA_i^0 are from DA_s^0 $(s > i)$ and A_t^1 $(t \geqslant i)$. They all factor through A_i^1.) Applying the functor $(M_n, -)$, we get a projective resolution of $E_{DA_i^0}$:

$$0 \to Q_{A_{i-1}^2} \to Q_{DA_{i-1}^1} \oplus Q_{A_i^1} \to Q_{DA_i^0} \to E_{DA_i^0} \to 0.$$

Hence we have $\operatorname{proj.dim} E_{DA_i^0} \leqslant 2$ for $0 < i < n$.

(8) The module DA_0^0. There is a short exact sequence $0 \to K \to X \oplus A_0^1 \to DA_0^0 \to 0$ with the property $(*)$. The kernel K is isomorphic to A_0^0, which is not in $\operatorname{add} M_n$. There is a short exact sequence $0 \to A_{n-1}^1 \to U_0 \oplus U_0 \oplus A_n^0 \to K \to 0$ with the property $(**)$. Applying the functor $(M_n, -)$ to the above two exact sequences leads to the projective resolution

$$0 \to Q_{A_{n-1}^1} \to Q_{U_0} \oplus Q_{U_0} \oplus Q_{A_n^0} \to Q_X \oplus Q_{A_0^1} \to Q_{DA_0^0} \to E_{DA_0^0} \to 0.$$

Hence we have $\operatorname{proj.dim} E_{DA_0^0} \leqslant 3$.

(9) The modules DA_i^j $(i, j > 0, i + j = n)$. For each module DA_i^j there is a short exact sequence $0 \to DA_{i-1}^j \to DA_{i-1}^{j+1} \oplus DA_i^{j-1} \to DA_i^j$ with the property $(*)$. Applying the functor $(M_n, -)$, we get a projective resolution

$$0 \to Q_{DA_{i-1}^j} \to Q_{DA_{i-1}^{j+1}} \oplus Q_{DA_i^{j-1}} \to Q_{DA_i^j} \to E_{DA_i^j} \to 0.$$

Hence we have $\operatorname{proj.dim} E_{DA_i^j} \leqslant 2$ for $i, j > 0$ and $i + j = n$.

(10) The modules DA_0^j $(j > 0)$. First, we consider DA_0^n. There is a short exact sequence $0 \to U_n \to U_{n+1} \oplus DA_0^{n-1} \to DA_0^n$ with property $(*)$. We get a projective resolution

$$0 \to Q_{U_n} \to Q_{U_{n+1}} \oplus Q_{DA_0^{n-1}} \to Q_{DA_0^n} \to E_{DA_0^n} \to 0.$$

For DA_0^j with $0 < j < n$, there is a short exact sequence $0 \to A_0^j \to DA_0^{j-1} \oplus A_0^{j+1} \to DA_0^j \to 0$ with property $(*)$. The projective resolution takes the form

$$0 \to Q_{A_0^j} \to Q_{DA_0^{j-1}} \oplus Q_{A_0^{j+1}} \to Q_{DA_0^j} \to E_{DA_0^j} \to 0.$$

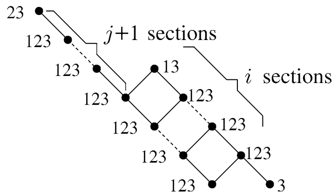


Fig. 4.

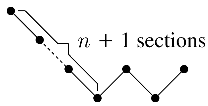


Fig. 5.

Hence in both cases we have proj.dim $E_{DA_0^j} \leqslant 2$.

(11) The modules DA_i^j $(i, j > 0, i + j < n)$. For each module DA_i^j with $i, j > 0$ and $i + j < n$, there is a short exact sequence $0 \to K \to DA_i^{j-1} \oplus DA_{i-1}^{j+1} \oplus A_i^{j+1} \to DA_i^j$ with the property $(*)$. (Except the multiples of the identity on DA_i^j, the only epimorphisms to DA_i^j are from DA_s^j $(s > i)$ and A_t^{j+1} $(t \geqslant i)$. They all factor through A_i^{j+1}.) The kernel K has the shape of Fig. 4.

Clearly, K is not in add M_n. However, there is a short exact sequence $0 \to A_{i-1}^{j+1} \to DA_{i-1}^j \oplus A_i^j \oplus A_{i-1}^{j+2} \to K \to 0$ with the property $(**)$. Applying the functor $(M_n, -)$ and glueing the resulting sequences together, we get a projective resolution of the form

$$0 \to Q_{A_{i-1}^{j+1}} \to Q_{DA_{i-1}^j} \oplus Q_{A_i^j} \oplus Q_{A_{i-1}^{j+2}} \to Q_{DA_i^{j-1}} \oplus Q_{DA_{i-1}^{j+1}} \oplus Q_{A_i^{j+1}} \to Q_{DA_i^j} \to E_{DA_i^j} \to 0.$$

Hence we have proj.dim $E_{DA_i^j} \leqslant 3$ for $i, j > 0$ and $i + j < n$.

(12) The module U_0. There is a short exact sequence $0 \to K \to DA_0^n \oplus DA_0 \to U_0 \to 0$ with the property $(*)$. The kernel K has the shape of Fig. 5.

Clearly, K is not in add M_n. However, there is a short exact sequence $0 \to U_n \oplus U_0 \to U_{n+1} \oplus A_0^n \oplus X \to K \to 0$ with the property $(**)$. Applying the functor $(M_n, -)$ and glueing the resulting sequences together, we get the projective resolution

$$0 \to Q_{U_n} \oplus Q_{U_0} \to Q_{U_{n+1}} \oplus Q_{A_0^n} \oplus Q_X \to Q_{DA_0^n} \oplus Q_{DA_0} \to Q_{U_0} \to E_{U_0} \to 0.$$

Hence we have proj.dim $E_{U_0} \leqslant 3$.

(13) The module U_1. There is a short exact sequence $0 \to K \to DA_0^1 \oplus A_0^n \to U_1 \to 0$ with the property $(*)$. (Note that the maps having image in the radical of U_1 factor through A_0^n.) The kernel K has the shape of Fig. 6.

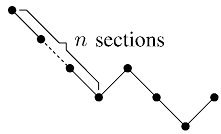


Fig. 6.

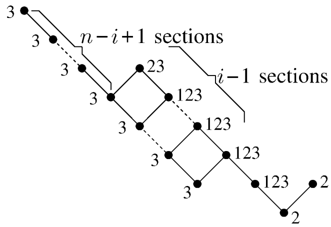


Fig. 7.

There is another short exact sequence $0 \to A_0^n \to U_n \oplus A_1^{n-1} \oplus DA_0^0 \to K \to 0$ with the property $(**)$. As before, we get a projective resolution

$$0 \to Q_{A_0^n} \to Q_{U_n} \oplus Q_{A_1^{n-1}} \oplus Q_{DA_0^0} \to Q_{DA_0^1} \oplus Q_{A_0^n} \to Q_{U_1} \to E_{U_1} \to 0.$$

Hence we have proj.dim $E_{U_1} \leqslant 3$.

(14) The modules U_i $(1 < i < n+1)$. For each module U_i with $1 < i < n+1$, there is a short exact sequence $0 \to K \to U_{i-1} \oplus DA_0^i \oplus A_{i-1}^{n-i+1} \to U_i \to 0$ with the property $(*)$. (In fact, the epimorphisms from an indecomposable summand of M_n to U_i, except the multiples of the identity on U_i, either factor through DA_0^i or A_{i-1}^{n-i+1}. Other maps factor through the U_{i-1}, which is the radical of U_i.) The kernel K has the shape of Fig. 7.

There is a short exact sequence $0 \to A_{i-1}^{n-i+1} \to A_i^{n-i} \oplus A_{i-2}^{n-i+2} \oplus DA_0^{i-1} \to K \to 0$ with the property $(**)$. Applying the functor $(M_n, -)$, we get a projective resolution of the simple $\mathrm{End}_{A_n}(M_n)$ module E_{U_i}:

$$0 \to Q_{A_{i-1}^{n-i+1}} \to Q_{A_i^{n-i}} \oplus Q_{A_{i-2}^{n-i+2}} \oplus Q_{DA_0^{i-1}} \to Q_{U_{i-1}} \oplus Q_{DA_0^i} \oplus Q_{A_{i-1}^{n-i+1}} \to Q_{U_i} \to E_{U_i} \to 0.$$

Hence proj.dim $E_{U_i} \leqslant 3$ for $1 < i < n+1$.

(15) The module U_{n+1}. There is a short exact sequence $0 \to A_{n-1}^1 \to U_n \oplus A_n^0 \to U_{n+1} \to 0$ with the property $(*)$. (In fact, except the multiples of the identity on U_{n+1}, the only epimorphisms are from A_n^0. Other maps factor through the radical U_n.) This leads to a projective resolution of the form

$$0 \to Q_{A_{n-1}^1} \to Q_{U_n} \oplus Q_{A_n^0} \to Q_{U_{n+1}} \to E_{U_{n+1}} \to 0.$$

Hence we have proj.dim $E_{U_{n+1}} \leqslant 2$.

Thus we have shown that $\mathrm{gl.dim}\,\mathrm{End}_{A_n}(M_n) \leqslant 3$ for any non-negative integer n. Hence we have $\mathrm{rep.dim}\,A_n \leqslant 3$ for all $n \geqslant 0$. On the other hand, the algebras A_n $(n \geqslant 0)$ are not representation finite, so we have $\mathrm{rep.dim}\,A_n > 2$ for all $n \geqslant 0$. Altogether, we come to the conclusion that $\mathrm{rep.dim}\,A_n = 3$ for all non-negative integers n and this completes the proof of Theorem 1.1. $\square$

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