

Approximations, ghosts and derived equivalences

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Approximation sequences and derived equivalences occur frequently in the research of mutation of tilting objects in representation theory, algebraic geometry and noncommutative geometry. In this paper, we introduce symmetric approximation sequences in additive categories and weakly n -angulated categories which include (higher) Auslander-Reiten sequences (triangles) and mutation sequences in algebra and geometry, and show that such sequences always give rise to derived equivalences between the quotient rings of endomorphism rings of objects in the sequences modulo some ghost and coghost ideals.

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1. Introduction

Derived categories and derived equivalences were first introduced by Grothendieck and Verdier in early 1960s. Derived equivalences between rings preserve many significant invariants such as Hochschild (co)homology, cyclic homology, centre and K -theory, and so on. In general, it is very hard to tell whether two given rings are derived equivalent or not, and it is also very difficult to describe the derived equivalence class of a given ring. One idea is to study derived equivalences ‘locally’, that is, to establish some elementary derived equivalences between certain nicely related rings, and hope that most derived equivalent rings can reach each other by a sequence of such elementary derived equivalences. Mutation of objects in categories provide many rings of such kind, where approximations play a central role. The mutation procedure reads as follows: let $T := M \oplus Y$ be an object in an abelian or a triangulated category, and let

$$X \xrightarrow{g} M' \xrightarrow{f} Y$$

be a short exact sequence or a triangle such that f is a right $\text{add}(M)$ -approximation. The object $T' := M \oplus X$ is called the right mutation of T at M . Dually, one has

left mutations. In many cases, the morphism g is also a left $\text{add}(M)$ -approximation, and the objects T and T' are left and right mutations of each other at the direct summand M . The endomorphism rings $\text{End}(T)$ and $\text{End}(T')$ are then related by this mutation procedure. This occurs in many aspects: mutations of exceptional sequences in the coherent sheaf category of varieties [21], mutations of tilting modules [10], mutations of cluster tilting objects [5], mutations of silting objects [1], and mutations of modifying modules [15] in the study of the NCCR conjecture ([22, Conjecture 4.6]). Auslander-Reiten sequences over artin algebras can also be viewed as mutation sequences.

One can ask whether the rings $\text{End}(T)$ and $\text{End}(T')$ are always derived equivalent or not. For an Auslander-Reiten sequence $0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0$ over an artin algebra, the endomorphism algebras $\text{End}(M \oplus X)$ and $\text{End}(M \oplus Y)$ are derived equivalent [12]. Mutation of tilting modules and modifying modules also provide examples where $\text{End}(T)$ and $\text{End}(T')$ are derived equivalent. However, this is not always true. For instance, the endomorphism rings of cluster tilting objects related by mutation are not always derived equivalent. Also, in general, Auslander-Reiten triangles do not give rise to derived equivalences. In [14], for certain triangles, it was proved that the quotient algebras of $\text{End}(T)$ and $\text{End}(T')$ modulo some particularly defined ideals are still derived equivalent. It remains unclear why these ideals naturally occur.

The aim of this paper is to find the general statement behind this phenomenon. In the mutation procedure, the symmetric (left and right) approximation property plays a central role. We shall introduce the notion of *symmetric approximation sequence* (definition 3.1), which can be viewed as a higher mutation sequence and covers all the known mutation sequences and triangles.

Another main ingredient of our results are ghost and coghost ideals. Let \mathcal{C} be an additive category, and let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory of \mathcal{C} . The \mathcal{D} -ghost ideal, denoted by $\text{gh}_{\mathcal{D}}$, is the ideal of \mathcal{C} consisting of all morphisms f in \mathcal{C} with $\mathcal{C}(D, f) = 0$ for all $D \in \mathcal{D}$. Dually, the \mathcal{D} -coghost ideal, denoted by $\text{cogh}_{\mathcal{D}}$ is the ideal of \mathcal{C} consisting of morphisms f in \mathcal{C} such that $\mathcal{C}(f, D) = 0$ for all $D \in \mathcal{D}$. If $\mathcal{D} = \text{add}(M)$ for some object $M \in \mathcal{C}$, then we simply write gh_M (respectively, cogh_M) for $\text{gh}_{\text{add}(M)}$ (respectively, $\text{cogh}_{\text{add}(M)}$).

Our main result can be described as the following theorem.

THEOREM 1.1 (=theorem 3.2). *Let \mathcal{C} be an additive category and let $M \in \mathcal{C}$ be an object. Suppose that*

$$X \xrightarrow{f_0} M_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} Y$$

is a symmetric $\text{add}(M)$ -approximation sequence (see definition 3.1 below). Then the quotient rings

$$\frac{\text{End}_{\mathcal{C}}(M \oplus X)}{\text{cogh}_M(M \oplus X)} \quad \text{and} \quad \frac{\text{End}_{\mathcal{C}}(M \oplus Y)}{\text{gh}_M(M \oplus Y)}$$

are derived equivalent.

Let us explain the generality of theorem 1.1. The notion of symmetric approximation sequences covers several notions in the literature:

- Auslander-Reiten sequences ([4, Chapter 5]) and mutation sequences of tilting modules over artin algebras ([9, §§ 1 and 2]), and modifying modules over certain commutative rings (see [15, § 6]).
- \mathcal{D} -split sequences in additive categories (see [12, definition 3.1]).
- Mutation triangles of exceptional sequences ([21]), cluster tilting objects ([5]) and silting objects ([1]) in triangulated categories.
- Higher Auslander-Reiten sequences over finite dimensional algebras and Auslander-Reiten n -angles in Krull-Schmidt n -angulated categories ($n \geq 3$), under the condition that the starting term does not occur as a direct summand of middle terms. Note that, by the proof of [12, proposition 3.15], the starting term of a higher Auslander-Reiten sequence appears as a direct summand of some middle term if and only if so does the ending term, in this case the endomorphism rings in theorem 1.2 are Morita equivalent. A similar proof applies to the case of Auslander-Reiten n -angles.

As a consequence, theorem 1.1 can be used to give a unified proof of several known results:

- [12, theorem 1.1 and proposition 3.15], where the sequences are \mathcal{D} -split sequences and higher Auslander-Reiten sequences respectively;
- [19, theorem 5.3], using mutation triangles of cluster tilting objects. In [19, theorem 5.3], the BB-tilting modules are defined if and only if the ghost and coghost ideals of the endomorphism algebras in theorem 1.1 are zero (cf. [19, theorem 4.2]);
- [15, theorem 6.8] via mutation sequences of modifying modules.

Let us remark that the ghost ideal and the coghost ideal occurring in theorem 1.1 are zero when f_0 is monic and f_n is epic. In this case, one gets a derived equivalence between the endomorphism rings. One can also apply theorem 1.1 to get derived equivalences from Auslander-Reiten n -angles (This was discussed in [12, proposition 5.1] for the special case $n = 3$).

Next, we show that the ideals occurring in theorem 1.1 can be chosen to be some smaller ideals when the category \mathcal{C} has a weak n -angulated structure which we shall define below.

As a generalization of triangulated categories, Geiss et al [7] introduced n -angulated categories, which occur widely in cluster tilting theory and are closely related to algebraic geometry and string theory. An n -angulated category is an additive category \mathcal{C} together with an automorphism Σ and a class of n -angles satisfying four axioms (F1), (F2), (F3) and (F4) (see [7] for details). The weak n -angulated structure we define in this paper can be obtained by dropping the axiom (F4) (the pushout axiom) and the condition (F1)(c) which says that every morphism in \mathcal{C} can be extended to an n -angle. An additive category \mathcal{C} with this weak

n -angulated structure will be called a *weakly n -angulated category* (see definition 4.1 below). Roughly speaking, the relationship between a weakly n -angulated category and an n -angulated category is like that between an additive category and an abelian category. In a weakly n -angulated category, we do not have the pushout axiom (Octahedral axiom when $n = 3$) and we do not require every morphism to be embedded into an n -angle. In an additive category, in general, pushouts/pullbacks do not exist, and a morphism does not necessarily have a kernel or cokernel.

Let \mathcal{C} be an additive category, and let \mathcal{D} be an additive subcategory of \mathcal{C} . We denote by $\mathcal{F}_{\mathcal{D}}$ the ideal of \mathcal{C} consisting of morphisms factorizing through an object in \mathcal{D} . The intersection $\text{gh}_{\mathcal{D}} \cap \mathcal{F}_{\mathcal{D}}$ is called the ideal of *factorizable \mathcal{D} -ghosts* of \mathcal{C} , denoted by $\text{Fgh}_{\mathcal{D}}$. Similarly, the intersection $\text{cogh}_{\mathcal{D}} \cap \mathcal{F}_{\mathcal{D}}$ is called the ideal of *factorizable \mathcal{D} -coghosts* of \mathcal{C} , denoted by $\text{Fcogh}_{\mathcal{D}}$. It turns out that the ideals defined in [14] are actually factorizable ghost and coghost ideals. So the following theorem generalizes [14, theorem 3.1], and shows that it fits into our general framework.

THEOREM 1.2 (=theorem 4.3). *Let $(\mathcal{C}, \Sigma, \diamond)$ be a weakly n -angulated category (see definition 4.1 below), and let M be an object in \mathcal{C} . Suppose that*

$$X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_{n-2} \xrightarrow{g} Y \xrightarrow{\eta} \Sigma X$$

is an n -angle in \mathcal{C} with $M_i \in \text{add}(M)$ for $1 \leq i \leq n - 2$ such that f and g are left and right $\text{add}(M)$ -approximations, respectively. Then the quotient rings

$$\frac{\text{End}_{\mathcal{C}}(M \oplus X)}{\text{Fcoh}_M(M \oplus X)} \quad \text{and} \quad \frac{\text{End}_{\mathcal{C}}(M \oplus Y)}{\text{Fgh}_M(M \oplus Y)}$$

are derived equivalent.

Let us remark that, in theorem 1.2, the sequence $X \xrightarrow{f} M_1 \rightarrow \cdots \rightarrow M_{n-2} \xrightarrow{g} Y$ is also a symmetric $\text{add}(M)$ -approximation sequence, and thus theorem 1.1 applies.

This paper is organized as follows. In § 2, we make some preparations, including the Φ -orbit construction, ghosts and coghosts. Sections 3 and 4 are devoted to proving theorem 1.1 and theorem 1.2, respectively. Some examples will be given in the final section.

2. Preliminary results

In this section, we shall recall basic definitions and facts which are needed in our proofs.

2.1. Conventions

Let \mathcal{C} be an additive category. For two objects X, Y in \mathcal{C} , we denote by $\mathcal{C}(X, Y)$ the set of morphisms from X to Y . The endomorphism ring $\mathcal{C}(X, X)$ of an object X is denoted by $\text{End}_{\mathcal{C}}(X)$. We write $\text{add}_{\mathcal{C}}(X)$ for the full subcategory of \mathcal{C} consisting of all direct summands of finite direct sums of copies of X . If there is no confusion, we just write $\text{add}(X)$ for $\text{add}_{\mathcal{C}}(X)$. For two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , we write fg for their composite. But for two functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G :$

$\mathcal{D} \rightarrow \mathcal{E}$ between categories, we write GF for their composite instead of FG . For a morphism $f : X \rightarrow Y$ and an object Z in \mathcal{C} the natural morphism $\mathcal{C}(Z, X) \rightarrow \mathcal{C}(Z, Y)$ sending g to gf is denoted by $\mathcal{C}(Z, f)$, and the morphism $\mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)$ sending h to fh is denoted by $\mathcal{C}(f, Z)$. Suppose that \mathcal{D} is a full subcategory of \mathcal{C} . The notion $\mathcal{C}(\mathcal{D}, f) = 0$ means that $\mathcal{C}(D, f) = 0$ for all objects $D \in \mathcal{D}$. Similarly $\mathcal{C}(f, \mathcal{D}) = 0$ means that $\mathcal{C}(f, D) = 0$ for all objects $D \in \mathcal{D}$.

All categories in this paper are additive categories, and all functors are additive functors. Let $\eta : F \rightarrow G$ be a natural transformation between two functors from \mathcal{C} to \mathcal{D} . For an object $X \in \mathcal{C}$, we denote by η_X the morphism from $F(X)$ to $G(X)$ induced by η . For functors $H : \mathcal{A} \rightarrow \mathcal{C}$ and $L : \mathcal{D} \rightarrow \mathcal{E}$, we have a natural transformation $L\eta H : LFH \rightarrow LGH$ induced by η .

Let \mathcal{C} be a category. A functor F from \mathcal{C} to itself is called an *endo-functor* of \mathcal{C} . If there is another endo-functor G of \mathcal{C} such that $FG = GF = id_{\mathcal{C}}$, where $id_{\mathcal{C}}$ is the identity functor on \mathcal{C} , then F is called an *automorphism* of \mathcal{C} . A endo-functor F is called an *auto-equivalence* provided that there is another endo-functor G of \mathcal{C} such that both FG and GF are naturally isomorphic to $id_{\mathcal{C}}$.

2.2. Complexes and derived equivalences

Let \mathcal{C} be an additive category. A complex X^\bullet over \mathcal{C} is a sequence of morphisms $\dots \rightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \dots$ between objects in \mathcal{C} such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$. The category of complexes over \mathcal{C} with morphisms being chain maps is denoted by $\mathcal{C}(\mathcal{C})$. The homotopy category of complexes over \mathcal{C} is denoted by $\mathcal{K}(\mathcal{C})$. If \mathcal{C} is an abelian category, then the derived category of complexes over \mathcal{C} is denoted by $\mathcal{D}(\mathcal{C})$. We write $\mathcal{C}^b(\mathcal{C})$, $\mathcal{K}^b(\mathcal{C})$ and $\mathcal{D}^b(\mathcal{C})$, respectively, for the full subcategories of $\mathcal{C}(\mathcal{C})$, $\mathcal{K}(\mathcal{C})$ and $\mathcal{D}(\mathcal{C})$ consisting of bounded complexes.

It is well known that the categories $\mathcal{K}(\mathcal{C})$ and $\mathcal{D}(\mathcal{C})$ are triangulated categories with $\mathcal{K}^b(\mathcal{C})$ and $\mathcal{D}^b(\mathcal{C})$ being their full triangulated subcategories, respectively. For basic results on triangulated categories, we refer to Happel’s book [8]. However, the shift functor in a triangulated category is written as Σ in this paper.

For two complexes X^\bullet and Y^\bullet over \mathcal{C} , we write $\text{Hom}_{\mathcal{C}}^\bullet(X^\bullet, Y^\bullet)$ for the total complex of the double complex with the (i, j) -term $\mathcal{C}(X^{-j}, Y^i)$.

Let Λ be a ring with identity. The category of left Λ -modules, denoted by $\Lambda\text{-Mod}$, is an abelian category. The full subcategory of $\Lambda\text{-Mod}$ consisting of finitely generated projective Λ -modules is denoted by $\Lambda\text{-proj}$. Following [20], two rings Λ and Γ are said to be *derived equivalent* provided that the derived categories $\mathcal{D}^b(\Lambda\text{-Mod})$ and $\mathcal{D}^b(\Gamma\text{-Mod})$ of bounded complexes are equivalent as triangulated categories. Due to the work of Rickard [20] (see also [16]), two rings Λ and Γ are derived equivalent if and only if there is a bounded complex T^\bullet of finitely generated projective Λ -modules satisfying the following two conditions,

- (a) T^\bullet is self-orthogonal, that is, $\mathcal{K}^b(\Lambda\text{-proj})(T^\bullet, \Sigma^i T^\bullet) = 0$ for all $i \neq 0$;
- (b) $\text{add}(T^\bullet)$ generates $\mathcal{K}^b(\Lambda\text{-proj})$ as a triangulated category,

such that $\text{End}_{\mathcal{K}^b(\Lambda\text{-proj})}(T^\bullet)$ is isomorphic to Γ as rings. A complex T^\bullet in $\mathcal{K}^b(\Lambda\text{-proj})$ satisfying the above two conditions is called a *tilting complex* over Λ .

2.3. Admissible sets and Φ -orbit categories

Let us recall from [13] and [14] the definition of admissible subsets. A subset Φ of \mathbb{Z} containing 0 is called an *admissible subset* provided that the following property holds:

If $i + j + k \in \Phi$ for three elements i, j, k in Φ , then $i + j \in \Phi$ if and only if $j + k \in \Phi$.

Typical examples of admissible subsets of \mathbb{Z} include $n\mathbb{Z}$ and $\{0, 1, \dots, n\}$. Suppose that Φ is an admissible subset of \mathbb{Z} . Then $-\Phi := \{-i \mid i \in \Phi\}$, $\Phi^{\geq 0} := \{i \in \Phi \mid i \geq 0\}$ and $\Phi^{\leq 0} := \{i \in \Phi \mid i \leq 0\}$ are all admissible. Let m be an integer. The set $m\Phi := \{mi \mid i \in \Phi\}$ is admissible. Moreover, if $m \geq 3$, then the set $\Phi^m := \{i^m \mid i \in \Phi\}$ is admissible. Nevertheless, not all subsets of \mathbb{Z} containing zero are admissible. For instance, the set $\{0, 1, 2, 4\}$ is not admissible. One can refer to [13, § 3.1] for more details.

Now let \mathcal{T} be an additive category, and let F be an endo-functor of \mathcal{T} . If F is not an equivalence, we set $F^i = 0$ for all $i < 0$. If F is an equivalence, we fix a quasi-inverse F^{-1} of F , and set $F^i := (F^{-1})^{-i}$ for $i < 0$. The functor F^0 is defined to be the identity functor on \mathcal{T} . Let Φ be an admissible subset of \mathbb{Z} . We can define a category $\mathcal{T}^{F, \Phi}$ as follows. The objects in $\mathcal{T}^{F, \Phi}$ are the same as \mathcal{T} , and the morphism space $\mathcal{T}^{F, \Phi}(X, Y)$ for two objects X, Y is defined to be

$$\bigoplus_{i \in \Phi} \mathcal{T}(X, F^i Y).$$

In [14], for each pair of integers u and v , a natural transformation $\chi(u, v)$ from $F^u F^v$ to F^{u+v} is defined, and it is proved that the composition

$$\mathcal{T}(X, F^u Y) \times \mathcal{T}(Y, F^v Z) \longrightarrow \mathcal{T}(X, F^{u+v} Z),$$

sending (f_u, g_v) to $f_u * g_v := f_u F^u(g_v)\chi(u, v)_Z$, is associative. We refer to [14, § 2.3] for the details of the natural transformations $\chi(u, v) : F^u F^v \longrightarrow F^{u+v}$. As a result, for morphisms $f = (f_i)_{i \in \Phi} \in \mathcal{T}^{F, \Phi}(X, Y)$ and $g = (g_i)_{i \in \Phi} \in \mathcal{T}^{F, \Phi}(Y, Z)$, the composition

$$(f, g) \mapsto fg := \left(\sum_{\substack{u, v \in \Phi \\ u+v=i}} f_u * g_v \right)_{i \in \Phi}$$

is associative. Thus $\mathcal{T}^{F, \Phi}$ is indeed an additive category, and is called the *Φ -orbit category* of \mathcal{T} under the functor F .

For each $X, Y \in \mathcal{T}$, the morphism space $\mathcal{T}^{F, \Phi}(X, Y) = \bigoplus_{i \in \Phi} \mathcal{T}(X, F^i Y)$ is Φ -graded. Every morphism $\alpha \in \mathcal{T}(X, F^i Y)$ can be viewed as a homogeneous morphism in $\mathcal{T}^{F, \Phi}(X, Y)$ of degree i .

Suppose that F is an auto-equivalence of \mathcal{T} . If both i and $-i$ are in the admissible subset Φ , then X and $F^i X$ are isomorphic in the Φ -orbit category $\mathcal{T}^{F, \Phi}$. Specifically, let $f := \chi(-i, i)_X^{-1} : X \longrightarrow F^{-i}(F^i X)$ and $g := 1_{F^i X} : F^i X \longrightarrow F^i X$. Considering

f as a homogeneous morphism in $\mathcal{T}^{F,\Phi}$ of degree $-i$, and g as a homogeneous morphism in $\mathcal{T}^{F,\Phi}$ of degree i , we have $f * g = 1_X$ and $g * f = 1_{F^i X}$. Here $g * f = 1_{F^i X}$ follows from the property:

$$F^j(\chi(k, l))\chi(j, k + l) = \chi(j, k)_{F^i}\chi(j + k, l)$$

for all integers j, k and l (see [14, § 2.3]) by taking $j = i, k = -i$ and $l = i$.

2.4. Approximations and cohomological approximations

Now we recall some definitions from [3].

Let \mathcal{C} be a category, and let \mathcal{D} be a full subcategory of \mathcal{C} , and X an object in \mathcal{C} . A morphism $f : D \rightarrow X$ in \mathcal{C} is called a *right \mathcal{D} -approximation* of X if $D \in \mathcal{D}$ and the induced map $\mathcal{C}(D', f) : \mathcal{C}(D', D) \rightarrow \mathcal{C}(D', X)$ is surjective for all $D' \in \mathcal{D}$. Dually, there is the notion of a *left \mathcal{D} -approximation*.

Cohomological approximations were introduced in [14]. Let \mathcal{T} be an additive category, and let F be a functor from \mathcal{T} to itself. Suppose that Φ is a non-empty subset of \mathbb{Z} , and that \mathcal{D} is a full additive subcategory of \mathcal{T} . A morphism $f : X \rightarrow D^X$ in \mathcal{T} with $D^X \in \mathcal{D}$ is called a *left (\mathcal{D}, F, Φ) -approximation* if every morphism $X \rightarrow F^i D$, where $D \in \mathcal{D}$ and $i \in \Phi$, factorizes through f . In the case that Φ is an admissible subset, we have the Φ -orbit category $\mathcal{T}^{F,\Phi}$, and that f is a left (\mathcal{D}, F, Φ) -approximation is equivalent to saying that $\mathcal{T}^{F,\Phi}(D^X, D) \rightarrow \mathcal{T}^{F,\Phi}(X, D)$ is surjective for all $D \in \mathcal{D}$, i.e., the morphism f , as a homogeneous morphism of degree zero, is a left \mathcal{D} -approximation in the Φ -orbit category $\mathcal{T}^{F,\Phi}$.

In [14], a right (\mathcal{D}, F, Φ) -approximation is defined to be a morphism $g : D_Y \rightarrow Y$ in \mathcal{T} with $D_Y \in \mathcal{D}$ such that every morphism from $F^i D$ to Y with $i \in \Phi$ and $D \in \mathcal{D}$ factorizes through g . Unfortunately, this does not fit the Φ -orbit category well: g is NOT a right \mathcal{D} -approximation in the Φ -orbit category $\mathcal{T}^{F,\Phi}$ in general. However, when F is an auto-equivalence with a quasi-inverse F^{-1} , a right $(\mathcal{D}, F, -\Phi)$ -approximation is still a right \mathcal{D} -approximation in $\mathcal{T}^{F,\Phi}$. Here we **re-define** a right (\mathcal{D}, F, Φ) -approximation as follows.

A *morphism $g : D_Y \rightarrow Y$ in \mathcal{T} with $D_Y \in \mathcal{D}$ is called a right (\mathcal{D}, F, Φ) -approximation if every morphism from D to $F^i Y$ with $i \in \Phi$ and $D \in \mathcal{D}$ factorizes through $F^i g$.*

Suppose that Φ is an admissible subset of \mathbb{Z} . Then, a morphism $g : D_Y \rightarrow Y$ in \mathcal{T} is a right (\mathcal{T}, F, Φ) -approximation if and only if g , as a homogenous morphism in degree zero, is a right \mathcal{D} -approximation in the Φ -orbit category $\mathcal{T}^{F,\Phi}$, no matter whether F is an equivalence or not.

2.5. Ghosts and factorizable ghosts

Let \mathcal{C} be an additive category. By an *ideal \mathcal{I}* of \mathcal{C} we mean additive subgroups $\mathcal{I}(A, B) \subseteq \mathcal{C}(A, B)$ for all A and B in \mathcal{C} , such that the composite $\alpha\beta$ of morphisms $\alpha, \beta \in \mathcal{C}$ belongs to \mathcal{I} provided either α or β is in \mathcal{I} . We denote $\mathcal{I}(A, A)$ simply by $\mathcal{I}(A)$. The quotient category \mathcal{C}/\mathcal{I} of \mathcal{C} modulo an ideal \mathcal{I} has the same objects as \mathcal{C} and has morphism space $(\mathcal{C}/\mathcal{I})(A, B) := \mathcal{C}(A, B)/\mathcal{I}(A, B)$ for two objects A and B . The quotient category \mathcal{C}/\mathcal{I} is also an additive category, and the projection functor $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{I}$ is full and dense [2, p. 421].

Let \mathcal{D} be a full additive subcategory of \mathcal{C} , a morphism f in \mathcal{C} is called a \mathcal{D} -ghost provided that $\mathcal{C}(\mathcal{D}, f) = 0$. All \mathcal{D} -ghosts in \mathcal{C} form an ideal of \mathcal{C} , called the *ideal of \mathcal{D} -ghosts* and denoted by $\text{gh}_{\mathcal{D}}$. Dually, a morphism g in \mathcal{C} is called a \mathcal{D} -coghost if $\mathcal{C}(g, \mathcal{D}) = 0$, and the ideal consisting of all \mathcal{D} -coghosts is denoted by $\text{cogh}_{\mathcal{D}}$.

Let $\mathcal{F}_{\mathcal{D}}$ be the ideal of morphisms in \mathcal{C} factorizing through an object in \mathcal{D} . The intersection $\text{gh}_{\mathcal{D}} \cap \mathcal{F}_{\mathcal{D}}$ is called the *ideal of factorizable \mathcal{D} -ghosts* of \mathcal{C} , denoted by $\text{Fgh}_{\mathcal{D}}$. Similarly, the intersection $\text{cogh}_{\mathcal{D}} \cap \mathcal{F}_{\mathcal{D}}$ is called the *ideal of factorizable \mathcal{D} -coghosts* of \mathcal{C} , denoted by $\text{Fcogh}_{\mathcal{D}}$.

LEMMA 2.1. *Keeping the notations above, we have the following:*

- (1) *If $A \in \mathcal{C}$ admits a right \mathcal{D} -approximation $f_A : D_A \rightarrow A$, then*

$$\text{gh}_{\mathcal{D}}(A, B) = \{g \in \mathcal{C}(A, B) \mid f_A g = 0\}.$$

- (2) *If $B \in \mathcal{C}$ admits a left \mathcal{D} -approximation $f^B : B \rightarrow D^B$, then*

$$\text{cogh}_{\mathcal{D}}(A, B) = \{g \in \mathcal{C}(A, B) \mid g f^B = 0\}.$$

- (3) *If $A \in \mathcal{D}$, then $\text{gh}_{\mathcal{D}}(A, B) = 0$ and $\text{cogh}_{\mathcal{D}}(A, B) = \text{Fcogh}_{\mathcal{D}}(A, B)$.*

- (4) *If $B \in \mathcal{D}$, then $\text{cogh}_{\mathcal{D}}(A, B) = 0$ and $\text{gh}_{\mathcal{D}}(A, B) = \text{Fgh}_{\mathcal{D}}(A, B)$.*

Proof. (1). Let $g : A \rightarrow B$ be in $\text{gh}_{\mathcal{D}}(A, B)$. Then $\mathcal{C}(\mathcal{D}, g) = 0$, and particularly $\mathcal{C}(D_A, g) = 0$. Consequently $f_A g = 0$. Conversely, let g be in $\mathcal{C}(A, B)$ such that $f_A g = 0$. It follows that

$$0 = \mathcal{C}(D', f_A g) = \mathcal{C}(D', f_A) \cdot \mathcal{C}(D', g)$$

for all $D' \in \mathcal{D}$. Moreover, since f_A is a right \mathcal{D} -approximation, the morphism $\mathcal{C}(D', f_A)$ is surjective. Hence $\mathcal{C}(D', g) = 0$ for all $D' \in \mathcal{D}$, that is, $g \in \text{gh}_{\mathcal{D}}(A, B)$.

The proof of (2) is dual to that of (1).

(3). Suppose that $A \in \mathcal{D}$. The identity map $1_A : A \rightarrow A$ is a right \mathcal{D} -approximation. It follows from (1) that $\text{gh}_{\mathcal{D}}(A, B) = 0$. Clearly, all the morphisms in $\mathcal{C}(A, B)$ factorize through the object A , which is in \mathcal{D} . This implies that $\text{Fcogh}_{\mathcal{D}}(A, B) = \text{cogh}_{\mathcal{D}}(A, B)$. Similarly, we can prove (4). □

3. Symmetric approximation sequences and derived equivalences

In this section, we introduce symmetric approximation sequences in additive categories and show that such a sequence always gives rise to a derived equivalence between the quotient rings of certain endomorphism rings modulo ghosts or coghosts (theorem 3.2). Examples of symmetric approximation sequences range from \mathcal{D} -split sequences and \mathcal{D} -split triangles to mutation triangles in cluster tilting theory, and to higher Auslander-Reiten sequences and higher Auslander-Reiten triangles.

Let \mathcal{C} be an additive category, and let \mathcal{D} be a full additive subcategory of \mathcal{C} . A right \mathcal{D} -approximation sequence in \mathcal{C} is a sequence

$$D_m \longrightarrow D_{m-1} \longrightarrow \cdots \longrightarrow D_0 \rightarrow Y$$

with $D_i \in \mathcal{D}$ for all $i = 0, \dots, m$ such that applying $\mathcal{C}(D, -)$ to the sequence results in an exact sequence

$$\mathcal{C}(D, D_m) \longrightarrow \mathcal{C}(D, D_{m-1}) \longrightarrow \cdots \longrightarrow \mathcal{C}(D, D_0) \rightarrow \mathcal{C}(D, Y) \longrightarrow 0$$

for all $D \in \mathcal{D}$. One can define left \mathcal{D} -approximation sequences dually. Recall that a pseudo-kernel of a morphism $u : X \rightarrow Y$ is a morphism $v : Z \rightarrow X$ such that

$$\mathcal{C}(C, Z) \xrightarrow{\mathcal{C}(C,v)} \mathcal{C}(C, X) \xrightarrow{\mathcal{C}(C,u)} \mathcal{C}(C, Y)$$

is exact for all $C \in \mathcal{C}$. The pseudo-cokernel is defined dually.

DEFINITION 3.1. Let \mathcal{C} be an additive category and let \mathcal{D} be a full additive subcategory of \mathcal{C} . A sequence

$$X \xrightarrow{f_0} D_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} D_n \xrightarrow{f_n} Y \tag{\dagger}$$

in \mathcal{C} is called a symmetric \mathcal{D} -approximation sequence if the following three conditions are satisfied.

- (1) The sequence $D_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} D_n \xrightarrow{f_n} Y$ is a right \mathcal{D} -approximation sequence;
- (2) The sequence $X \xrightarrow{f_0} D_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} D_n$ is a left \mathcal{D} -approximation sequence;
- (3) The morphism f_0 is a pseudo-kernel of f_1 , and f_n is a pseudo-cokernel of f_{n-1} .

In definition 3.1, if we replace the condition (3) with the following condition

$$(3') \text{ The morphism } f_0 \text{ is a kernel of } f_1, \text{ and } f_n \text{ is a cokernel of } f_{n-1},$$

then the sequence (\dagger) is called a higher \mathcal{D} -split sequence. Comparing with the definition of \mathcal{D} -split sequence [12, definition 3.1], a \mathcal{D} -split sequence is precisely a sequence (\dagger) with $n = 1$ satisfying the conditions (1), (2) and (3') above.

The main result in this section is the following theorem.

THEOREM 3.2. Let \mathcal{C} be an additive category and let $M \in \mathcal{C}$ be an object. Suppose that

$$X \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} Y$$

is a symmetric $\text{add}(M)$ -approximation sequence. Then the quotient rings

$$\frac{\text{End}_{\mathcal{C}}(M \oplus X)}{\text{cogh}_M(M \oplus X)} \quad \text{and} \quad \frac{\text{End}_{\mathcal{C}}(M \oplus Y)}{\text{gh}_M(M \oplus Y)}$$

are derived equivalent.

The following lemma and its corollary will be useful in the proof of theorem 3.2.

LEMMA 3.3. *Let \mathcal{C} be an additive category, and let M be an object in \mathcal{C} . Suppose that P^\bullet :*

$$0 \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \dots \longrightarrow P^{n-1} \xrightarrow{d^{n-1}} P^n \longrightarrow 0$$

is a complex over \mathcal{C} such that $P^i \in \text{add}(M)$ for all $i > 0$, and that the following two conditions are satisfied:

- (1) $H^i(\text{Hom}_{\mathcal{C}}^\bullet(M, P^\bullet)) = 0$ for all $i \neq 0, n$;
- (2) $H^i(\text{Hom}_{\mathcal{C}}^\bullet(P^\bullet, M)) = 0$ for all $i \neq -n$.

Then P^\bullet is self-orthogonal as a complex both in $\mathcal{K}^b(\mathcal{C}/\text{cogh}_M)$ and in $\mathcal{K}^b(\mathcal{C}/\text{Fcogh}_M)$.

Proof. For simplicity, we denote by $\bar{\mathcal{C}}$ the category $\mathcal{C}/\text{cogh}_M$, and denote by $\bar{\bar{\mathcal{C}}}$ the category $\mathcal{C}/\text{Fcogh}_M$.

If $n = 0$, then the problem is trivial. Now we assume that $n > 0$.

It follows from our assumption (2) that $H^0(\text{Hom}_{\mathcal{C}}^\bullet(P^\bullet, M)) = 0$, and consequently the map $\mathcal{C}(d^0, M) : \mathcal{C}(P^1, M) \rightarrow \mathcal{C}(P^0, M)$ is surjective. Thus, the morphism d^0 is a left $\text{add}(M)$ -approximation. By lemma 2.1 (2), one has $\text{cogh}_M(M, P^0) = \{f \in \mathcal{C}(M, P^0) \mid f d^0 = 0\} = \text{Ker } \mathcal{C}(M, d^0)$. Moreover, it follows from lemma 2.1 (3) that $\text{cogh}_M(M, P^0) = \text{Fcogh}_M(M, P^0)$. Hence the canonical functors $\mathcal{C} \rightarrow \bar{\mathcal{C}} \rightarrow \bar{\bar{\mathcal{C}}}$ induce isomorphisms

$$\mathcal{C}(M, P^0)/\text{Ker } \mathcal{C}(M, d^0) \longrightarrow \bar{\mathcal{C}}(M, P^0) \longrightarrow \bar{\bar{\mathcal{C}}}(M, P^0).$$

Note that for each $i > 0$, by lemma 2.1 (4), we have $\text{cogh}_M(M, P^i) = 0 = \text{Fcogh}_M(M, P^i)$ since $P^i \in \text{add}(M)$. Thus, for each $i > 0$, the canonical functors $\mathcal{C} \rightarrow \bar{\mathcal{C}} \rightarrow \bar{\bar{\mathcal{C}}}$ also induce isomorphisms

$$\mathcal{C}(M, P^i) \longrightarrow \bar{\mathcal{C}}(M, P^i) \longrightarrow \bar{\bar{\mathcal{C}}}(M, P^i).$$

In this way, we see that the complexes $\text{Hom}_{\bar{\mathcal{C}}}^\bullet(M, P^\bullet)$ and $\text{Hom}_{\bar{\bar{\mathcal{C}}}}^\bullet(M, P^\bullet)$ are both isomorphic to the complex

$$0 \longrightarrow \mathcal{C}(M, P^0)/\text{Ker } \mathcal{C}(M, d^0) \longrightarrow \mathcal{C}(M, P^1) \longrightarrow \dots \longrightarrow \mathcal{C}(M, P^n) \longrightarrow 0.$$

By assumption (1), the above complex has zero homology for all degrees not equal to n . Hence

$$H^i(\text{Hom}_{\bar{\mathcal{C}}}^\bullet(M, P^\bullet)) = 0 = H^i(\text{Hom}_{\bar{\bar{\mathcal{C}}}}^\bullet(M, P^\bullet))$$

for all $i \neq n$. By lemma 2.1 (4), we have $\text{cogh}_M(P^i, M) = 0$ for all i , and therefore $\text{Fcogh}_M(P^i, M) = 0$ for all i . Hence the complexes $\text{Hom}_{\bar{\mathcal{C}}}^\bullet(P^\bullet, M)$, $\text{Hom}_{\bar{\bar{\mathcal{C}}}}^\bullet(P^\bullet, M)$ and $\text{Hom}_{\mathcal{C}}^\bullet(P^\bullet, M)$ are all isomorphic. Hence $H^i(\text{Hom}_{\bar{\mathcal{C}}}^\bullet(P^\bullet, M)) = 0 = H^i(\text{Hom}_{\bar{\bar{\mathcal{C}}}}^\bullet(P^\bullet, M))$ for all $i \neq -n$ by assumption (2). The lemma then follows from the dual version of the result [11, lemma 2.1]. □

COROLLARY 3.4. Let \mathcal{C} be an additive category and let M be an object in \mathcal{C} . Suppose that

$$X \xrightarrow{f_0} M_1 \longrightarrow \cdots \longrightarrow M_n \xrightarrow{f_n} Y$$

is a symmetric $\text{add}(M)$ -approximation sequence in \mathcal{C} . Then the complex

$$0 \longrightarrow X \xrightarrow{f_0} M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

is self-orthogonal as a complex both in $\mathcal{K}^b(\mathcal{C}/\text{cogh}_M)$ and in $\mathcal{K}^b(\mathcal{C}/\text{Fcogh}_M)$.

Proof. It follows from definition 3.1 (2) that the sequence

$$0 \rightarrow X \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \rightarrow 0$$

is indeed a complex. We put X in degree zero and denote this complex by P^\bullet . By the conditions (1) and (3) of definition 3.1, applying $\mathcal{C}(M, -)$ to P^\bullet results in an exact sequence

$$\mathcal{C}(M, X) \longrightarrow \mathcal{C}(M, M_1) \longrightarrow \cdots \longrightarrow \mathcal{C}(M, M_n).$$

This implies that P^\bullet satisfies lemma 3.3(1). Since $X \rightarrow M_1 \rightarrow \cdots \rightarrow M_n$ is a left $\text{add}(M)$ -approximation sequence by definition, the complex P^\bullet satisfies lemma 3.3 (2), and the corollary then follows. \square

Proof of theorem 3.2. Note that the quotient rings in the theorem are precisely the endomorphism rings of $M \oplus X$ and $M \oplus Y$ in $\mathcal{C}/\text{cogh}_M$ and \mathcal{C}/gh_M respectively.

By the definition of symmetric approximation sequences, the sequence

$$X \longrightarrow M_1 \longrightarrow \cdots \xrightarrow{[f_{n-1}, 0]} M_n \oplus M \xrightarrow{\begin{bmatrix} f_n & 0 \\ 0 & 1 \end{bmatrix}} Y \oplus M$$

is again a symmetric $\text{add}(M)$ -approximation sequence. Let T^\bullet be the complex

$$0 \longrightarrow X \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_n \oplus M \longrightarrow 0$$

with X in degree zero. Then it follows from corollary 3.4 that T^\bullet is self-orthogonal in $\mathcal{K}^b(\mathcal{C}/\text{cogh}_M)$. Note that, for an additive category \mathcal{A} and an object U in \mathcal{A} , there is a fully faithful functor (see, for example, [18, proposition 2.3])

$$\mathcal{A}(U, -) : \text{add}(U) \longrightarrow \text{End}_{\mathcal{A}}(U)\text{-proj},$$

which further induces a fully faithful triangle functor

$$\text{Hom}_{\mathcal{A}}^\bullet(U, -) : \mathcal{K}^b(\text{add}(U)) \longrightarrow \mathcal{K}^b(\text{End}_{\mathcal{A}}(U)\text{-proj}).$$

Thus, we get a full triangle embedding

$$\text{Hom}_{\mathcal{C}/\text{cogh}_M}^\bullet(M \oplus X, -) : \mathcal{K}^b(\text{add}_{\mathcal{C}/\text{cogh}_M}(M \oplus X)) \longrightarrow \mathcal{K}^b(\text{End}_{\mathcal{C}/\text{cogh}_M}(M \oplus X)\text{-proj}),$$

and we see that $\tilde{T}^\bullet := \text{Hom}_{\mathcal{C}/\text{cogh}_M}^\bullet(M \oplus X, T^\bullet)$ is self-orthogonal in $\mathcal{K}^b(\text{End}_{\mathcal{C}/\text{cogh}_M}(M \oplus X)\text{-proj})$. Moreover, $\text{add}(\tilde{T}^\bullet)$ clearly generates $\mathcal{K}^b(\text{End}_{\mathcal{C}/\text{cogh}_M}(M \oplus X)\text{-proj})$.

X)-proj) as a triangulated category. Hence \tilde{T}^\bullet is a tilting complex over $\text{End}_{\mathcal{C}/\text{cogh}_M}(M \oplus X)$ with the endomorphism ring isomorphic to $\text{End}_{\mathcal{X}^b(\mathcal{C}/\text{cogh}_M)}(T^\bullet)$.

By [20, theorem 6.4], it remains to prove that $\text{End}_{\mathcal{X}^b(\mathcal{C}/\text{cogh}_M)}(T^\bullet)$ and $\text{End}_{\mathcal{C}/\text{gh}_M}(Y \oplus M)$ are isomorphic. Instead of giving a ring isomorphism directly, we prove that there is a surjective ring homomorphism from $\text{End}_{\mathcal{E}^b(\mathcal{C})}(T^\bullet)$ to each of the rings, and show that these two ring homomorphisms have the same kernel.

We denote the differentials of T^\bullet by d^i and denote by \tilde{d}^n the map $M_n \oplus M \xrightarrow{[f_n \atop 1]} Y \oplus M$.

Firstly, we show that there is a surjective ring homomorphism

$$\theta : \text{End}_{\mathcal{E}^b(\mathcal{C})}(T^\bullet) \longrightarrow \text{End}_{\mathcal{C}/\text{gh}_M}(Y \oplus M).$$

For each chain map $g^\bullet : T^\bullet \rightarrow T^\bullet$ in $\mathcal{E}^b(\mathcal{C})$, since \tilde{d}^n is a pseudo-cokernel of d^{n-1} , there is a morphism $g \in \text{End}_{\mathcal{C}}(Y \oplus M)$ such that the following diagram is commutative

$$\begin{array}{ccccccc} X & \xrightarrow{d^0} & M_1 & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & M_n \oplus M & \xrightarrow{\tilde{d}^n} & Y \oplus M \\ g^0 \downarrow & & g^1 \downarrow & & & & g^n \downarrow & & g \downarrow \\ X & \xrightarrow{d^0} & M_1 & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & M_n \oplus M & \xrightarrow{\tilde{d}^n} & Y \oplus M. \end{array} \quad (\clubsuit)$$

Moreover, if g' is another morphism in $\text{End}_{\mathcal{C}}(Y \oplus M)$ such that $\tilde{d}^n g' = g^n \tilde{d}^n$, then $\tilde{d}^n(g - g') = 0$. By definition \tilde{d}^n is a right $\text{add}(M)$ -approximation of $Y \oplus M$. Thus, by lemma 2.1 (1), the morphism $g - g'$ belongs to $\text{gh}_M(Y \oplus M)$. We denote by \bar{g} the corresponding morphism of g in \mathcal{C}/gh_M . Defining $\theta(g^\bullet) := \bar{g}$ gives rise to a ring homomorphism θ from $\text{End}_{\mathcal{E}^b(\mathcal{C})}(T^\bullet)$ to $\text{End}_{\mathcal{C}/\text{gh}_M}(Y \oplus M)$. We claim that θ is surjective. Actually, for each $g \in \text{End}_{\mathcal{C}}(Y \oplus M)$, it follows from definition 3.1(1) that there are morphisms $g^i : T^i \rightarrow T^i, i = 1, \dots, n$ such that $g^n \tilde{d}^n = \tilde{d}^n g$ and $g^k d^k = d^k g^{k+1}$ for all $k = 1, \dots, n - 1$. Since d^0 is a pseudo-kernel of d^1 by definition, we get a morphism $g^0 : X \rightarrow X$ such that $g^0 d^0 = d^0 g^1$. Thus we get a chain map g^\bullet in $\text{End}_{\mathcal{E}^b(\mathcal{C})}(T^\bullet)$ such that $\theta(g^\bullet) = \bar{g}$.

Secondly, we claim that there is a surjective ring homomorphism

$$\varphi : \text{End}_{\mathcal{E}^b(\mathcal{C})}(T^\bullet) \longrightarrow \text{End}_{\mathcal{X}^b(\mathcal{C}/\text{cogh}_M)}(T^\bullet).$$

Actually, we can define φ to be the composite of the ring homomorphism

$$\text{End}_{\mathcal{E}^b(\mathcal{C})}(T^\bullet) \longrightarrow \text{End}_{\mathcal{E}^b(\mathcal{C}/\text{cogh}_M)}(T^\bullet),$$

induced by the canonical functor $\mathcal{C} \rightarrow \mathcal{C}/\text{cogh}_M$, and the canonical surjective ring homomorphism

$$\text{End}_{\mathcal{E}^b(\mathcal{C}/\text{cogh}_M)}(T^\bullet) \longrightarrow \text{End}_{\mathcal{X}^b(\mathcal{C}/\text{cogh}_M)}(T^\bullet).$$

Let $g^i : T^i \rightarrow T^i, i = 0, 1, \dots, n$ be morphisms in \mathcal{C} such that g^\bullet is a chain map in $\mathcal{E}^b(\mathcal{C}/\text{cogh}_M)$. Then $g^i d^i - d^i g^{i+1} : T^i \rightarrow T^{i+1}$ is in cogh_M for all $i = 0, 1, \dots, n -$

1. Since $T^i \in \text{add}(M)$ for all $i > 0$ and by lemma 2.1 (4), we get $g^i d^i - d^i g^{i+1} = 0$ for all $i = 0, 1, \dots, n - 1$. Hence g^\bullet is a chain map in $\mathcal{C}^b(\mathcal{C})$, and the canonical map from $\text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet)$ to $\text{End}_{\mathcal{C}^b(\mathcal{C}/\text{cogh}_M)}(T^\bullet)$ is surjective. Consequently, φ is a surjective ring homomorphism.

Finally, we show that θ and φ have the same kernel, which would result in an isomorphism between $\text{End}_{\mathcal{C}^b(\mathcal{C}/\text{cogh}_M)}(T^\bullet)$ and $\text{End}_{\mathcal{C}/\text{gh}_M}(Y \oplus M)$. By definition, a chain map $g^\bullet : T^\bullet \rightarrow T^\bullet$ is in $\text{Ker } \varphi$ if and only if there exist $h^i : T^i \rightarrow T^{i-1}, i = 1, \dots, n$ in \mathcal{C} such that $g^n - h^n d^{n-1}, g^0 - d^0 h^1$ and $g^i - h^i d^{i-1} - d^i h^{i+1}$ are all in cogh_M for $i = 1, \dots, n - 1$. Using the fact that $T^i \in \text{add}(M)$ for all $i > 0$ and that d^0 is a left $\text{add}(M)$ -approximation of X , one can show, by lemma 2.1, that this is equivalent to saying that $g^n - h^n d^{n-1} = 0, (g^0 - d^0 h^1)d^0 = 0$ and $g^i = h^i d^{i-1} + d^i h^{i+1}$ for all $i = 1, \dots, n - 1$. Now suppose that g^\bullet is in $\text{Ker } \varphi$, and let $g : Y \oplus M \rightarrow Y \oplus M$ in \mathcal{C} be the map induced by the commutative diagram (\clubsuit) above, that is, $\theta(g^\bullet) = \bar{g}$. Then $\bar{d}^n g = g^n \bar{d}^n = h^n d^{n-1} \bar{d}^n = 0$, and consequently $g \in \text{gh}_M$ by lemma 2.1 (1), and $\bar{g} = 0$. Hence $\text{Ker } \varphi \subseteq \text{Ker } \theta$. Conversely, suppose that g^\bullet is a chain map in $\text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet)$ such that $\theta(g^\bullet) = \bar{g} = 0$. Then $g^n \bar{d}^n = \bar{d}^n g = 0$ by lemma 2.1(1). By definition 3.1 (1) and (3), one can inductively construct morphisms $h^i : T^i \rightarrow T^{i-1}$ for i from n down to 1 such that $g^n = h^n d^{n-1}$ and $g^i = h^i d^{i-1} + d^i h^{i+1}$ for all $i = n - 1, \dots, 1$. Finally, $(g^0 - d^0 h^1)d^0 = g^0 d^0 - d^0 h^1 d^0 = d^0 g^1 - d^0 (g^1 - d^0 h^2) = 0$. Hence g^\bullet is in $\text{Ker } \varphi$, and consequently $\text{Ker } \theta \subseteq \text{Ker } \varphi$.

Altogether, we have shown that θ and φ are surjective ring homomorphisms with the same kernel. Hence $\text{End}_{\mathcal{C}/\text{gh}_M}(Y \oplus M)$ and $\text{End}_{\mathcal{C}^b(\mathcal{C}/\text{cogh}_M)}(T^\bullet)$ are isomorphic, and the theorem is proved. \square

COROLLARY 3.5. *Let \mathcal{C} be an additive category, and let M be an object in \mathcal{C} . Suppose that*

$$X \xrightarrow{f_0} M_1 \longrightarrow \dots \longrightarrow M_n \xrightarrow{f_n} Y$$

is a higher $\text{add}(M)$ -split sequence. Then $\text{End}_{\mathcal{C}}(M \oplus X)$ and $\text{End}_{\mathcal{C}}(M \oplus Y)$ are derived equivalent.

Proof. This is an immediate consequence of theorem 3.2. Note that the ghost and coghost ideals vanish in this case since f_0 is monic and f_n is epic. \square

Let A be a finite dimensional algebra, and let P be a projective A -module with $\nu_A P \cong P$, where ν_A is the Nakayama functor $D \text{Hom}_A(-, A)$. Suppose that Y is an A -module admitting an $\text{add}(P)$ -presentation, namely, there is an exact sequence $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} Y \rightarrow 0$ in $A\text{-mod}$ with $P_i \in \text{add}(P)$ for $i = 0, 1$. Let $P_2 \rightarrow \text{Ker } f_1$ be a right $\text{add}(P)$ -approximation of $\text{Ker } f_1$, we get a sequence $P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} Y$. Continuing this process by taking a right $\text{add}(P)$ -approximation $P_i \rightarrow \text{Ker } f_{i-1}$ for $2 \leq i \leq n$, we get a sequence

$$(\ddagger) \quad X \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \dots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} Y,$$

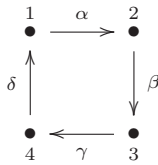
where f_{n+1} is the kernel of f_n .

COROLLARY 3.6. *In the above sequence (\ddagger) , the algebras $\text{End}_A(P \oplus X)$ and $\text{End}_A(P \oplus Y)$ are derived equivalent.*

Proof. By definition, the sequence $P_n \xrightarrow{f_n} \dots \rightarrow P_0 \xrightarrow{f_0} Y$ is a right $\text{add}(P)$ -approximation sequence. The assumption $P \cong \nu_A P$, together with the natural isomorphism $D \text{Hom}_A(P, -) \cong \text{Hom}_A(-, \nu_A P)$, implies that $X \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} \dots \xrightarrow{f_1} P_0$ is a left $\text{add}(P)$ -approximation sequence. Together with the fact that f_0 is a cokernel of f_1 and f_{n+1} is a kernel of f_n , we see that the sequence (\ddagger) is a higher $\text{add}(P)$ -split sequence. The corollary then follows from corollary 3.5. \square

Corollary 3.6 provides an easy construction of derived equivalences, as illustrated by the following example.

EXAMPLE. Let A be the Nakayama algebra given by the quiver



with relations $\alpha\beta\gamma\delta\alpha = \beta\gamma\delta\alpha\beta = \gamma\delta\alpha\beta\gamma = \delta\alpha\beta\gamma\delta = 0$. We denote by P_i the indecomposable projective A -module corresponding to the vertex i . Then $\nu_A P_i \cong P_i$ for all $1 \leq i \leq 4$. Let $P = P_1 \oplus P_3$, and let Y be the module $\frac{1}{2}$, which admits an $\text{add}(P)$ -presentation $P_3 \rightarrow P_1 \rightarrow Y \rightarrow 0$. Using the method in corollary 3.6, we can construct a sequence

$$X \xrightarrow{f_5} P_1 \xrightarrow{f_4} P_1 \xrightarrow{f_3} P_3 \xrightarrow{f_2} P_3 \xrightarrow{f_1} P_1 \xrightarrow{f_0} Y,$$

where f_i is a right $\text{add}(P)$ -approximation of $\text{Ker } f_{i-1}$ for $i = 2, 3, 4$, and $X = \frac{2}{4}$ is the kernel of f_4 . Note that both $\text{Hom}_A(X, P_3)$ and $\text{Hom}_A(X, P_1)$ are one-dimensional vector spaces and the induced maps $\text{Hom}_A(X, f_2)$ and $\text{Hom}_A(X, f_1)$ are both zero. Hence the sequence

$$\text{Hom}_A(X, P_3) \xrightarrow{\text{Hom}_A(X, f_2)} \text{Hom}_A(X, P_3) \xrightarrow{\text{Hom}_A(X, f_1)} \text{Hom}_A(X, P_1)$$

cannot be exact at the middle term. This shows that the condition (2) of [12, lemma 3.4] fails for this sequence. However, by corollary 3.6, we can deduce that $\text{End}_A(P_1 \oplus P_3 \oplus Y)$ and $\text{End}_A(P_1 \oplus P_3 \oplus X)$ are derived equivalent.

4. Symmetric approximation sequences in weakly n -angulated categories

Theorem 3.2 tells us that symmetric approximation sequences in arbitrary additive categories give rise to derived equivalences between quotient rings of endomorphism rings modulo ghost ideals and coghost ideals. In this section, we will see that, if the category \mathcal{C} in theorem 3.2 has some ‘weak’ n -angulated structure and the symmetric approximation sequence is an n -angle in \mathcal{C} , then the ideals can be chosen to be ideals of factorizable ghosts and coghosts, respectively.

The notion of n -angulated category is given in [7] as a generalization of triangulated categories (in this case $n = 3$). Typical examples of n -angulated categories include certain $(n - 2)$ -cluster tilting subcategories in a triangulated category, which appear in recent cluster tilting theory. The ‘weak’ n -angulated structure we need in this section is obtained from the definition of n -angulated categories [7] by dropping some axioms.

DEFINITION 4.1. Let $n \geq 3$ be an integer. A weakly n -angulated category is an additive category \mathcal{C} together with an automorphism Σ of \mathcal{C} , and a class \diamond of sequences of morphisms in \mathcal{C}

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1,$$

called n -angles, satisfying the following axioms:

- (F1') For each $X \in \mathcal{C}$, the sequence $X \xrightarrow{1_X} X \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \Sigma X$ belongs to \diamond . The class \diamond is closed under taking direct sums and direct summands.
- (F2) A sequence $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$ of morphisms in \mathcal{C} is in \diamond if and only if so is $X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2$.
- (F3) For each commutative diagram

$$\begin{array}{ccccccc}
 X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \dots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
 \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & & & \downarrow h_n & & \downarrow \Sigma h_1 \\
 Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \dots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1
 \end{array}$$

with rows in \diamond , there exists maps $h_i : X_i \rightarrow Y_i$ for $3 \leq i \leq n$ making the diagram commute. In other words, any such diagram can be completed to a morphism of n -angles.

The axioms in definition 4.1 are obtained from the axioms (F1), (F2), (F3) and (F4) in the definition of n -angulated categories in [7] by dropping the pushout axiom (F4) and by dropping the condition (F1)(c) (each morphism can be embedded into an n -angle) to obtain (F1').

REMARK.

- (a) The relationship between weakly n -angulated categories and n -angulated categories is like the relationship between additive categories and abelian categories. In an abelian category, pullbacks and pushouts always exist, and every morphism has a kernel and a cokernel, while additive categories do not have these properties in general. Correspondingly, an n -angulated category has a pushout axiom (F4), and every morphism can be embedded into an n -angle. However, a weakly n -angulated category does not necessarily have these properties.

(b) Just like additive categories, the axioms of definition 4.1 can be easily satisfied by many full subcategories of n -angulated categories. Suppose that $(\mathcal{C}, \Sigma, \diamond)$ is a weakly n -angulated category, and that \mathcal{C}' is a full additive subcategory of \mathcal{C} such that $\Sigma\mathcal{C}' = \mathcal{C}'$. Denote by \diamond' the class of n -angles in \diamond with all terms in \mathcal{C}' . Then it is easy to see that $(\mathcal{C}', \Sigma, \diamond')$ is again a weakly n -angulated category. Namely, every full additive subcategory of an n -angulated category closed under Σ and Σ^{-1} is weakly n -angulated.

An additive covariant functor H from a weakly n -angulated category $(\mathcal{C}, \Sigma, \diamond)$ to $\mathbb{Z}\text{-Mod}$ is called *cohomological*, if whenever

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

is an n -angle, the long sequence

$$\begin{aligned} \dots \longrightarrow H(\Sigma^i X_1) \xrightarrow{H(\Sigma^i f_1)} H(\Sigma^i X_2) \xrightarrow{H(\Sigma^i f_2)} \dots \xrightarrow{H(\Sigma^i f_{n-1})} H(\Sigma^i X_n) \\ \xrightarrow{H(\Sigma^i f_n)} H(\Sigma^{i+1} X_1) \longrightarrow \dots \end{aligned}$$

is exact. Dually we have *contravariant cohomological functors*.

LEMMA 4.2. *Let $(\mathcal{C}, \Sigma, \diamond)$ be a weakly n -angulated category, and let*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

be an n -angle in \mathcal{C} . Then we have the following:

- (1) $f_i f_{i+1} = 0$ for all $i = 1, 2, \dots, n - 1$;
- (2) $\mathcal{C}(X, -)$ and $\mathcal{C}(-, X)$ are cohomological for all $X \in \mathcal{C}$;
- (3) Suppose that $2 \leq m < n$. Each commutative diagram

$$\begin{array}{ccccccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \dots & \xrightarrow{f_{m-1}} & X_m & \xrightarrow{f_m} & X_{m+1} & \xrightarrow{f_{m+1}} & \dots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow h_1 & & \downarrow h_2 & & & & \downarrow h_m & & \downarrow h_{m+1} & & & & \downarrow h_n & & \downarrow \Sigma h_1 \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \dots & \xrightarrow{g_{m-1}} & Y_m & \xrightarrow{g_m} & Y_{m+1} & \xrightarrow{g_{m+1}} & \dots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

with rows in \diamond can be completed in \mathcal{C} to a morphism of n -angles.

Proof. The proofs of (1) and (2) are similar to that of [8, proposition 1.2] for triangulated categories.

(3). By (F3), there exist morphisms $h'_i : X_i \rightarrow Y_i, 3 \leq i \leq n$ such that $h_2 g_2 = f_2 h'_3, h'_n g_n = f_n \Sigma h_1$ and $h'_i g_i = f_i h'_{i+1}$ for all $3 \leq i \leq n - 1$. By (2), we can inductively construct morphisms $s_i : X_i \rightarrow Y_{i-1}$ for i from 4 to $m + 1$ such that $h_3 - h'_3 = f_3 s_4$ and $h_i - h'_i = s_i g_{i-1} + f_i s_{i+1}$ for all $4 \leq i \leq m$. Now define $h_{m+1} := h'_{m+1} + s_{m+1} g_m$ and $h_i := h'_i$ for all $m + 1 < i \leq n$. It is easy to check that the sequence of morphisms h_1, \dots, h_n is a morphism of n -angles. \square

THEOREM 4.3. Let $(\mathcal{C}, \Sigma, \triangleleft)$ be a weakly n -angulated category ($n \geq 3$), and let M be an object in \mathcal{C} . Suppose that

$$X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_{n-2} \xrightarrow{g} Y \xrightarrow{\eta} \Sigma X$$

is an n -angle in \mathcal{C} with $M_i \in \text{add}(M)$ for all $1 \leq i \leq n - 2$ such that f and g are left and right $\text{add}(M)$ -approximations, respectively. Then the quotient rings

$$\frac{\text{End}_{\mathcal{C}}(M \oplus X)}{\text{Fcogh}_M(M \oplus X)} \quad \text{and} \quad \frac{\text{End}_{\mathcal{C}}(M \oplus Y)}{\text{Fgh}_M(M \oplus Y)}$$

are derived equivalent.

Proof. The proof is similar to that of theorem 3.2. By definition 4.1, the sequence

$$X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_{n-2} \oplus M \xrightarrow{\tilde{g}} Y \oplus M \xrightarrow{\tilde{\eta}} \Sigma X,$$

where $\tilde{g} = \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}$ and $\tilde{\eta} := \begin{bmatrix} \eta \\ 0 \end{bmatrix}$, is still an n -angle. Moreover, the morphism \tilde{g} is still a right $\text{add}(M)$ -approximation. Thus, by our assumptions together with lemma 4.2, it is easy to check that

$$X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_{n-2} \oplus M \xrightarrow{\tilde{g}} Y \oplus M$$

is a symmetric $\text{add}(M)$ -approximation sequence.

We denote by T^\bullet the complex

$$0 \longrightarrow X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_{n-2} \oplus M \longrightarrow 0$$

with X in degree zero. Then by corollary 3.4, the complex T^\bullet is self-orthogonal in $\mathcal{K}^b(\mathcal{C}/\text{Fcogh}_M)$. As we have done similarly in the proof of theorem 3.2, it is easy to prove that $\tilde{T}^\bullet := \text{Hom}_{\mathcal{C}/\text{Fcogh}_M}^\bullet(M \oplus X, T^\bullet)$ is a tilting complex over $\text{End}_{\mathcal{C}/\text{Fcogh}_M}(M \oplus X)$. It remains to show that the endomorphism ring of \tilde{T}^\bullet , which is isomorphic to $\text{End}_{\mathcal{K}^b(\mathcal{C}/\text{Fcogh}_M)}(\tilde{T}^\bullet)$, is isomorphic to $\text{End}_{\mathcal{C}/\text{Fgh}_M}(Y \oplus M)$.

Firstly, for each chain map u^\bullet in $\text{End}_{\mathcal{K}^b(\mathcal{C})}(T^\bullet)$, by lemma 4.2 (3), there is a morphism $u \in \text{End}_{\mathcal{C}}(Y \oplus M)$ such that the diagram

$$\begin{array}{ccccccc} T^0 & \xrightarrow{d_T^0} & T^1 & \xrightarrow{d_T^1} & \cdots & \xrightarrow{d_T^{n-3}} & T^{n-2} & \xrightarrow{\tilde{g}} & Y \oplus M & \xrightarrow{\tilde{\eta}} & \Sigma T^0 \\ & \downarrow u^0 & & \downarrow u^1 & & & \downarrow u^{n-2} & & \downarrow u & & \downarrow \Sigma u^0 \\ T^0 & \xrightarrow{d_T^0} & T^1 & \xrightarrow{d_T^1} & \cdots & \xrightarrow{d_T^{n-3}} & T^{n-2} & \xrightarrow{\tilde{g}} & Y \oplus M & \xrightarrow{\tilde{\eta}} & \Sigma T^0 \end{array} \quad (\star)$$

is commutative. If u' is another morphism in $\text{End}_{\mathcal{C}}(Y \oplus M)$ making the above diagram commutative, then $\tilde{g}(u - u') = 0 = (u - u')\tilde{\eta}$. Since \tilde{g} is a right $\text{add}(M)$ -approximation by our assumption, the morphism $(u - u')$ belongs to $\text{gh}_M(Y \oplus M)$ by lemma 2.1 (1). It follows from $(u - u')\tilde{\eta} = 0$ that $u - u'$ factorizes through T^{n-2} ,

which is in $\text{add}(M)$. Hence $u - u'$ is in $\text{Fgh}_M(Y \oplus M)$. Denote by \bar{u} the morphism in \mathcal{C}/Fgh_M corresponding to u . Thus, we get a map

$$\theta : \text{End}_{\mathcal{E}^b(\mathcal{C})}(T^\bullet) \longrightarrow \text{End}_{\mathcal{C}/\text{Fgh}_M}(Y \oplus M)$$

sending u^\bullet to \bar{u} , which is clearly a ring homomorphism. For each $u \in \text{End}_{\mathcal{C}}(Y \oplus M)$, since \tilde{g} is a right $\text{add}(M)$ -approximation, there is $u^{n-2} : T^{n-2} \rightarrow T^{n-2}$ such that $\tilde{g}u = u^{n-2}\tilde{g}$. Thus, by the axioms (F2) and (F3) in definition 4.1, we get morphisms $u^i : T^i \rightarrow T^i, i = 0, \dots, n - 3$, making the above diagram (★) commutative. This shows that θ is a surjective ring homomorphism.

Secondly, similar to the proof of theorem 3.2, one can prove that there is a surjective ring homomorphism

$$\varphi : \text{End}_{\mathcal{E}^b(\mathcal{C})}(T^\bullet) \longrightarrow \text{End}_{\mathcal{X}^b(\mathcal{C}/\text{Fcogh}_M)}(T^\bullet),$$

which is the composite of the ring homomorphism

$$\text{End}_{\mathcal{E}^b(\mathcal{C})}(T^\bullet) \rightarrow \text{End}_{\mathcal{E}^b(\mathcal{C}/\text{Fcogh}_M)}(T^\bullet)$$

induced by the canonical functor $\mathcal{C} \rightarrow \mathcal{C}/\text{Fcogh}_M$ and the canonical surjective ring homomorphism

$$\text{End}_{\mathcal{E}^b(\mathcal{C}/\text{Fcogh}_M)}(T^\bullet) \longrightarrow \text{End}_{\mathcal{X}^b(\mathcal{C}/\text{Fcogh}_M)}(T^\bullet).$$

We have to show that θ and φ have the same kernel. A chain map u^\bullet is in $\text{Ker } \varphi$ if and only if there exist $h^i : T^i \rightarrow T^{i-1}, i = 1, \dots, n - 2$ in \mathcal{C} such that $u^0 - d_T^0 h^1, u^i - h^i d_T^{i-1} - d_T^i h^{i+1}, i = 1, \dots, n - 3$, and $u^{n-2} - h^{n-2} d_T^{n-3}$ are all in Fcogh_M . Using the fact that $T^i \in \text{add}(M)$ for all $i > 0$, one can see, by lemma 2.1, that this is equivalent to saying that $u^{n-2} - h^{n-2} d_T^{n-3} = 0, u^i = h^i d_T^{i-1} + d_T^i h^{i+1}$ for $i = 1, \dots, n - 3$, and $u^0 - d_T^0 h^1 \in \text{Fcogh}_M(T^0)$.

Let u^\bullet be in $\text{Ker } \varphi$, and suppose that $u \in \text{End}_{\mathcal{C}}(Y \oplus M)$ fits the commutative diagram (★) above. Then $\theta(u^\bullet) = \bar{u}$. We have $u^{n-2} = h^{n-2} d_T^{n-3}$, and consequently $\tilde{g}u = u^{n-2}\tilde{g} = h^{n-2} d_T^{n-3}\tilde{g}$, which is zero by lemma 4.2 (1). It follows from lemma 2.1 (1) that $u \in \text{gh}_M(Y \oplus M)$. The fact $u^\bullet \in \text{Ker } \varphi$ also implies that $u^0 - d_T^0 h^1 \in \text{Fcogh}_M(T^0)$. In particular, the morphism $u^0 - d_T^0 h^1$ factorizes through an object in $\text{add}(M)$. Assume that $u^0 - d_T^0 h^1 = ab$ for some $a \in \mathcal{C}(T^0, M')$ and $b \in \mathcal{C}(M', T^0)$ with $M' \in \text{add}(M)$. Since d_T^0 is a left $\text{add}(M)$ -approximation, we see that a factorizes through d_T^0 , and hence $u^0 - d_T^0 h^1$ factorizes through d_T^0 . Consequently, the morphism u^0 also factorizes through d_T^0 , say, $u^0 = d_T^0 \alpha$. Thus $\tilde{\eta}(\Sigma u^0) = \tilde{\eta}(\Sigma d_T^0)(\Sigma \alpha)$, which must be zero by the axiom (F2) in definition 4.1 and lemma 4.2 (1). Hence $u\tilde{\eta} = \tilde{\eta}(\Sigma u^0) = 0$, and consequently u factorizes through $T^{n-2} \in \text{add}(M)$ by lemma 4.2 (2). Altogether, we have shown that u belongs to $\text{Fgh}_M(Y \oplus M)$. It follows that $\bar{u} = 0$ and $u^\bullet \in \text{Ker } \theta$. Hence $\text{Ker } \varphi \subseteq \text{Ker } \theta$.

Conversely, suppose that $u^\bullet \in \text{Ker } \theta$ and $u \in \text{End}_{\mathcal{C}}(Y \oplus M)$ fits the commutative diagram (★). Then $\theta(u^\bullet) = \bar{u} = 0$, that is, $u \in \text{Fgh}_M(Y \oplus M)$. Since \tilde{g} is a right $\text{add}(M)$ -approximation, by lemma 2.1 (1), we have $\tilde{g}u = 0$. Thus $u^{n-2}\tilde{g} = 0$. By lemma 4.2 (2), there is a morphism $h^{n-2} : T^{n-2} \rightarrow T^{n-3}$ such that $u^{n-2} = h^{n-2} d_T^{n-3}$. Now $(u^{n-3} - d_T^{n-3} h^{n-2})d_T^{n-3} = u^{n-3} d_T^{n-3} - d_T^{n-3} u^{n-2} = 0$. If $n \geq 4$, then, by lemma 4.2 (2), there is a morphism $h^{n-3} : T^{n-3} \rightarrow T^{n-4}$ such that

$u^{n-3} - d_T^{n-3}h^{n-2} = h^{n-3}d_T^{n-4}$. Moreover, $(u^{n-4} - d_T^{n-4}h^{n-3})d_T^{n-4} = d_T^{n-4}u^{n-3} - d_T^{n-4}h^{n-3}d_T^{n-4} = d_T^{n-4}d_T^{n-3}h^{n-2} = 0$. Repeating this process, we get $h^i : T^i \rightarrow T^{i-1}, i = 1, \dots, n-2$ such that $u^{n-2} = h^{n-2}d_T^{n-3}$, $u^i = h^i d_T^{i-1} + d_T^i h^{i+1}$ for $i = 1, \dots, n-3$, and $(u^0 - d_T^0 h^1)d_T^0 = 0$. Since d_T^0 is a left $\text{add}(M)$ -approximation, we deduce from lemma 2.1 (2) that $u^0 - d_T^0 h^1 \in \text{cogh}_M(T^0)$. Since u factorizes through an object in $\text{add}(M)$ and \tilde{g} is a right $\text{add}(M)$ -approximation, it is easy to see that u factorizes through \tilde{g} , and thus $\tilde{\eta}(\Sigma u^0) = u\tilde{\eta} = 0$. By lemma 4.2 (2) and axiom (F2) in definition 4.1, the morphism Σu^0 factorizes through Σd_T^0 , or equivalently, u^0 factorizes through d_T^0 . Hence $u^0 - d_T^0 h^1$ factorizes through an object in $\text{add}(M)$, and consequently belongs to $\text{Fcogh}_M(T^0)$. Thus we have shown that $u^\bullet \in \text{Ker } \varphi$, and $\text{Ker } \theta \subseteq \text{Ker } \varphi$.

Thus θ and φ have the same kernel, and the rings $\text{End}_{\mathcal{K}^b(\mathcal{C}/\text{Fcogh}_M)}(T^\bullet)$ and $\text{End}_{\mathcal{C}/\text{Fgh}_M}(Y \oplus M)$ are isomorphic, and the theorem then follows. \square

We devoted the remainder of this section to the main application of theorem 4.3, namely, to recover the results [14, theorem 3.1] and [6, theorem 1.1] where Φ -orbit categories and certain auto-functors are involved.

Let $(\mathcal{T}, \Sigma, \diamond)$ and $(\mathcal{T}', \Sigma', \diamond')$ be weakly n -angulated categories. An additive functor F from \mathcal{T} to \mathcal{T}' is called an n -angle functor if there is a natural isomorphism $\psi : \Sigma' F \rightarrow F \Sigma$ and

$$F(X_1) \xrightarrow{F(f_1)} F(X_2) \xrightarrow{F(f_2)} \dots \xrightarrow{F(f_{n-1})} F(X_n) \xrightarrow{F(f_n)\psi_{X_1}^{-1}} \Sigma' F(X_1)$$

is in \diamond' whenever

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

is in \diamond .

Now let $(\mathcal{T}, \Sigma, \diamond)$ be a weakly n -angulated category, and let F be an n -angle functor from \mathcal{T} to itself. Suppose that Φ is an admissible subset of \mathbb{Z} , and $\mathcal{T}^{F, \Phi}$ is the Φ -orbit category of \mathcal{T} . Then one may ask whether the Φ -orbit category is again naturally weakly n -angulated. The answer is yes, as we shall prove in the following.

We fix a natural isomorphism $\psi(1) : \Sigma F \rightarrow F \Sigma$, and set $\psi(0) := id_\Sigma : \Sigma \rightarrow \Sigma$. For each positive integer u , we define $\psi(u) : \Sigma F^u \rightarrow F^u \Sigma$ to be the composite

$$\Sigma F^u \xrightarrow{\psi(1)F^{u-1}} F \Sigma F^{u-1} \xrightarrow{F\psi(1)F^{u-2}} F^2 \Sigma F^{u-2} \rightarrow \dots \xrightarrow{F^{u-1}\psi(1)} F^u \Sigma.$$

If F is not an equivalence, then, for each negative integer u , we set $F^u = 0$ and define $\psi(u) : \Sigma F^u \rightarrow F^u \Sigma$ to be zero. If F is an equivalence, and F^{-1} is a quasi-inverse of F , then (F, F^{-1}) is an adjoint pair. Let $\epsilon : id_{\mathcal{T}} \rightarrow F^{-1}F$ be the unit and let $\eta : FF^{-1} \rightarrow id_{\mathcal{T}}$ be the counit. We define $\psi(-1)$ to be the composite

$$\Sigma F^{-1} \xrightarrow{\epsilon \Sigma F^{-1}} F^{-1} \Sigma F^{-1} \xrightarrow{F^{-1}\psi(1)^{-1}F^{-1}} F^{-1} \Sigma F^{-1} \xrightarrow{F^{-1}\Sigma \eta} F^{-1} \Sigma,$$

and define $\psi(u)$, for each integer $u < 0$, to be the composite

$$\Sigma F^u \xrightarrow{\psi(-1)F^{u+1}} F^{-1} \Sigma F^{u+1} \xrightarrow{F^{-1}\psi(-1)F^{u+2}} F^{-2} \Sigma F^{u+2} \rightarrow \dots \rightarrow F^u \Sigma.$$

With these natural transformations in hand, we can define an automorphism Σ^Φ of $\mathcal{T}^{F, \Phi}$ as follows. $\Sigma^\Phi X$ is just ΣX for each object X . For each homogeneous

morphism $f_u : X \rightarrow F^u Y$ in $\mathcal{T}^{F,\Phi}(X, Y)$ of degree u , we define $\Sigma^\Phi(f_u)$ to be the composite

$$\Sigma X \xrightarrow{\Sigma f_u} \Sigma F^u Y \xrightarrow{\psi^{(u)}_Y} F^u \Sigma Y$$

which is a homogenous morphism in $\mathcal{T}^{F,\Phi}(\Sigma X, \Sigma Y)$ of degree u . One can check that Σ^Φ is indeed an automorphism of the Φ -orbit category $\mathcal{T}^{F,\Phi}$. Let \diamond^Φ be the closure class of \diamond in $\mathcal{T}^{F,\Phi}$ under taking direct summands. Note that the existence of non-homogenous maps means that there may be direct summands of \diamond in $\mathcal{T}^{F,\Phi}$ which are not direct summands of \diamond in \mathcal{T} and thus, to satisfy Axiom (F1') of definition 4.1, we must take \diamond^Φ to be the closure class of \diamond in $\mathcal{T}^{F,\Phi}$ under taking direct summands.

PROPOSITION 4.4. *Keeping the notations above, the Φ -orbit category $\mathcal{T}^{F,\Phi}$, together with Σ^Φ and \diamond^Φ , is a weakly n -angulated category.*

Proof. First, we remark that the canonical functor from \mathcal{T} to $\mathcal{T}^{F,\Phi}$ preserves direct sums. It follows easily that \diamond^Φ and Σ^Φ satisfy the axioms (F1') and (F2) of definition 4.1. Now take a commutative diagram

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma^\Phi X_1 \\ \downarrow h^1 & & \downarrow h^2 & & & & & & & & \downarrow \Sigma^\Phi(h^1) \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma^\Phi Y_1 \end{array}$$

in $\mathcal{T}^{F,\Phi}$ with rows in \diamond^Φ . Clearly, we can assume that the rows are in \diamond , and all the morphisms f_i are homogeneous morphisms of degree zero for all $i = 1, \dots, n$. Let $h^1 = (h^1_u)_{u \in \Phi}$ and $h^2 = (h^2_u)_{u \in \Phi}$. Then $h^1_u * g_1 = f_1 * h^2_u$ for all $u \in \Phi$. Thus, we get a commutative diagram

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow h^1_u & & \downarrow h^2_u & & & & & & & & \downarrow \Sigma(h^1_u) \\ F^u(Y_1) & \xrightarrow{F^u(g_1)} & F^u(Y_2) & \xrightarrow{F^u(g_2)} & F^u(Y_3) & \xrightarrow{F^u(g_3)} & \cdots & \xrightarrow{F^u(g_{n-1})} & F^u(Y_n) & \xrightarrow{F^u(g_n)\psi^{(u)}-1} & \Sigma F^u(Y_1) \end{array}$$

in \mathcal{T} with rows in \diamond . Thus, in the weakly n -angulated category \mathcal{T} , we get a commutative diagram

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow h^1_u & & \downarrow h^2_u & & \downarrow h^3_u & & & & \downarrow h^n_u & & \downarrow \Sigma(h^1_u) \\ F^u(Y_1) & \xrightarrow{F^u(g_1)} & F^u(Y_2) & \xrightarrow{F^u(g_2)} & F^u(Y_3) & \xrightarrow{F^u(g_3)} & \cdots & \xrightarrow{F^u(g_{n-1})} & F^u(Y_n) & \xrightarrow{F^u(g_n)\psi^{(u)}-1} & \Sigma F^u(Y_1) \end{array}$$

for all $u \in \Phi$. Defining $h^i := (h^i_u)_{u \in \Phi}$, we obtain a commutative diagram

$$\begin{array}{ccccccccccc}
 X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma^\Phi X_1 \\
 \downarrow h^1 & & \downarrow h^2 & & \downarrow h^3 & & & & \downarrow h^n & & \downarrow \Sigma^\Phi(h^1) \\
 Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma^\Phi Y_1
 \end{array}$$

in $\mathcal{T}^{F,\Phi}$. Thus $\mathcal{T}^{F,\Phi}$ satisfies the axiom (F3) of definition 4.1. Hence $\mathcal{T}^{F,\Phi}$, together with Σ^Φ and \diamond^Φ is a weakly n -angulated category. \square

REMARK. For an n -angulated category \mathcal{T} , the Φ -orbit category $\mathcal{T}^{F,\Phi}$ is not an n -angulated category in general (see [17, § 3]).

Following the terminology in [14], the endomorphism ring of an object X in the orbit category $\mathcal{T}^{F,\Phi}$, denoted by $E_{\mathcal{T}}^{F,\Phi}(X)$, is called the Φ -Yoneda algebra of X with respect to F . Φ -Yoneda algebras were first defined in [13] for the special case $F = \Sigma$ under the name ‘ Φ -Auslander-Yoneda algebras’.

In [14] and [6], the authors start from an n -angle

$$X \xrightarrow{f} M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_{n-2} \xrightarrow{g} Y \xrightarrow{w} \Sigma X$$

in an n -angulated category \mathcal{T} with $M_i \in \text{add}(M)$ for some $M \in \mathcal{T}$. Let F be an n -angle auto-equivalence of \mathcal{T} . The main results [14, theorem 1.1] and [6, theorem 1.1] state that, for each admissible subset Φ of \mathbb{Z} , there is a derived equivalence between the quotient rings $E_{\mathcal{T}}^{F,\Phi}(M \oplus X)/I$ and $E_{\mathcal{T}}^{F,\Phi}(M \oplus Y)/J$ of the Φ -Yoneda algebras provided that f and g are, respectively, left and right $\text{add}_{\mathcal{T}^{F,\Phi}}(M)$ -approximations in the Φ -orbit category $\mathcal{T}^{F,\Phi}$ and $\mathcal{T}(M, F^i X) = 0 = \mathcal{T}(Y, F^i M)$ for all $0 \neq i \in \Phi$. The ideals I and J are defined as follows. Set $\bar{w} := \begin{bmatrix} w \\ 0 \end{bmatrix} : Y \oplus M \rightarrow \Sigma X$ and $\tilde{w} := [w, 0] : Y \rightarrow \Sigma(X \oplus M)$. Then

$$\begin{aligned}
 I := & \left\{ (x_i) \in E_{\mathcal{T}}^{F,\Phi}(X \oplus M) \mid x_i = 0, \forall 0 \neq i \in \Phi, x_0 \right. \\
 & \left. \text{factorizes through } \text{add}(M) \text{ and } \Sigma^{-1}(\tilde{w}) \right\},
 \end{aligned}$$

$$\begin{aligned}
 J := & \left\{ (x_i) \in E_{\mathcal{T}}^{F,\Phi}(Y \oplus M) \mid x_i = 0, \forall 0 \neq i \in \Phi, x_0 \right. \\
 & \left. \text{factorizes through } \text{add}(M) \text{ and } \bar{w} \right\}.
 \end{aligned}$$

This looks quite artificially defined. However, we shall see that the ideals I and J are actually factorizable coghosts and ghosts in the Φ -orbit category, and then one can use theorem 4.3 and proposition 4.4 to give alternative proofs of the results in [6, 14]. Here we consider a more general situation: \mathcal{T} is a weakly n -angulated category, and F is just an n -angle endo-functor of \mathcal{T} , not necessarily an auto-equivalence.

LEMMA 4.5. *Keep the notations above and set $\mathcal{D} := \text{add}_{\mathcal{T}^{F,\Phi}}(M)$. We have the following.*

- (1) If g is a right \mathcal{D} -approximation in $\mathcal{T}^{F,\Phi}$, and $\mathcal{T}(Y, F^i M) = 0$ for all $0 \neq i \in \Phi$, then $J = \text{Fgh}_{\mathcal{D}}(Y \oplus M)$;
- (2) If f is a left \mathcal{D} -approximation in $\mathcal{T}^{F,\Phi}$, and $\mathcal{T}(M, F^i X) = 0$ for all $0 \neq i \in \Phi$, then $I = \text{Fcogh}_{\mathcal{D}}(X \oplus M)$.

Proof. (1) Set $\tilde{g} = \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} : M_{n-2} \oplus M \longrightarrow Y \oplus M$. By lemma 2.1(3), we have $\text{Fgh}_{\mathcal{D}}(M, Y \oplus M) = 0$. Now we consider $\text{Fgh}_{\mathcal{D}}(Y, Y \oplus M)$. Since g is a right \mathcal{D} -approximation, by lemma 2.1 (1), $\text{gh}_{\mathcal{D}}(Y, Y \oplus M)$ consists of morphisms $x := (x_i) \in \mathcal{T}^{F,\Phi}(Y, Y \oplus M)$ such that the composite gx in $\mathcal{T}^{F,\Phi}$ vanishes, or equivalently $g * x_i = gx_i = 0$ in \mathcal{T} for all $i \in \Phi$. Since \tilde{g} is also a right \mathcal{D} -approximation, a morphism $x := (x_i) \in \mathcal{T}^{F,\Phi}(Y, Y \oplus M)$ factorizes through an object in \mathcal{D} if and only if it factorizes through \tilde{g} . Thus $\text{Fgh}_{\mathcal{D}}(Y, Y \oplus M)$ consists of precisely those morphisms $(x_i) \in \mathcal{T}^{F,\Phi}(Y, Y \oplus M)$ satisfying the conditions:

- (a) $gx_i = 0$ for all $i \in \Phi$;
- (b) There is some $(y_i) \in \mathcal{T}^{F,\Phi}(Y, M_{n-2} \oplus M)$ such that $y_i * \tilde{g} = x_i$ for all $i \in \Phi$.

Note that, by our assumption that $\mathcal{T}(Y, F^i M) = 0$ for all $0 \neq i \in \Phi$, the morphism y_i in condition (b) above is zero for all $0 \neq i \in \Phi$, and correspondingly $x_i = 0$ for all $0 \neq i \in \Phi$. Then $\text{Fgh}_{\mathcal{D}}(Y, Y \oplus M)$ actually consists of morphisms $(x_i) \in \mathcal{T}^{F,\Phi}(Y, Y \oplus M)$ with $x_i = 0$ for all $0 \neq i \in \Phi$ such that $gx_0 = 0$ and $x_0 = y_0 \tilde{g}$ for some $y_0 : Y \rightarrow M_{n-2} \oplus M$ in \mathcal{T} . This is equivalent to saying that x_0 factorizes through w and $\text{add}(M)$ in \mathcal{T} . Hence $\text{Fgh}_{\mathcal{D}}(Y \oplus M, Y \oplus M) = J$ and the statement (1) is proved. The proof of (2) is dual. □

Combining theorem 4.3, proposition 4.4 and lemma 4.5, we get the following corollary.

COROLLARY 4.6. *Let $(\mathcal{T}, \Sigma, \diamond)$ be a weakly n -angulated category ($n \geq 3$) with an n -angle endo-functor F , and let M be an object in \mathcal{T} . Suppose that Φ is an admissible subset of \mathbb{Z} . Let*

$$X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_{n-2} \xrightarrow{g} Y \longrightarrow \Sigma X$$

be an n -angle in \mathcal{T} such that $M_i \in \text{add}(M)$ for all $i = 1, \dots, n - 2$, and that f and g are left and right $\text{add}(M)$ -approximations in the Φ -orbit category $\mathcal{T}^{F,\Phi}$, respectively. Suppose that $\mathcal{T}(Y, F^i M) = 0 = \mathcal{T}(M, F^i X)$ for all $0 \neq i \in \Phi$. Then the rings $E_{\mathcal{T}}^{F,\Phi}(M \oplus X)/I$ and $E_{\mathcal{T}}^{F,\Phi}(M \oplus Y)/J$ are derived equivalent.

This corollary generalizes the results [14, theorem 3.1] and [6, theorem 1.1]: the functor F here is not necessarily an auto-equivalence, while this is required in both [14] and [6].

5. Examples

In this section, we give some examples to illustrate our main results.

Throughout this section, we assume that A is a self-injective artin algebra, and write, for $n \in \mathbb{N}$,

$$\mathcal{D}_n := \text{add} \left(\bigoplus_{i=0}^n \Sigma^{-i} A \right)$$

in $\mathcal{K}(A\text{-mod})$. For simplicity, we will write \mathcal{K} for $\mathcal{K}(A\text{-mod})$. The i -th cohomology of a complex C^\bullet in \mathcal{K} is denoted by $H^i(C^\bullet)$. For an A -module X , we denote by X^* the right A -module $\text{Hom}_A(X, A)$. Let D be the usual duality, and let $\nu_A := D \text{Hom}_A(-, A)$ be the Nakayama functor. When P is a finitely generated projective A -module, there is a natural isomorphism $\text{Hom}_A(P, -) \cong D \text{Hom}_A(-, \nu_A P)$ which can be obtained by applying D to the isomorphism in [4, p. 41, proposition 4.4(b)]. This further induces an isomorphism $\mathcal{K}(P^\bullet, -) \cong D\mathcal{K}(-, \nu_A P^\bullet)$ for all bounded complexes P^\bullet of finitely generated projective A -modules. This will be frequently used in this section.

LEMMA 5.1. *With notation as above, the ideals $\text{cogh}_{\mathcal{D}_n}$ and $\text{gh}_{\mathcal{D}_n}$ in \mathcal{K} are equal, and both of them consist of morphisms α^\bullet such that $H^i(\alpha^\bullet) = 0$ for all $0 \leq i \leq n$.*

Proof. By definition, a chain map $\alpha^\bullet : X^\bullet \rightarrow Y^\bullet$ lies in $\text{gh}_{\mathcal{D}_n}$ if and only if $\mathcal{K}(\Sigma^{-i} A, \alpha^\bullet) = 0$, or equivalently $\mathcal{K}(A, \Sigma^i \alpha^\bullet) = 0$, for all $0 \leq i \leq n$. This is further equivalent to saying that $H^i(\alpha^\bullet) = 0$ for all $0 \leq i \leq n$, by the isomorphism $\mathcal{K}(A, \Sigma^i(-)) \cong H^i$.

Since A is self-injective, we have $\text{add}(A) = \text{add}(\nu_A A)$ and thus $\mathcal{D}_n = \text{add}(\bigoplus_{i=0}^n \Sigma^{-i} \nu_A A)$. A chain map α^\bullet belongs to $\text{cogh}_{\mathcal{D}_n}$ if and only if $\mathcal{K}(\alpha^\bullet, \Sigma^{-i} \nu_A A) = 0$ for all $0 \leq i \leq n$. From the isomorphisms

$$\mathcal{K}(-, \Sigma^{-i} \nu_A A) \cong \mathcal{K}(\Sigma^i(-), \nu_A A) \cong D\mathcal{K}(A, \Sigma^i(-)) \cong DH^i,$$

we conclude that $\alpha^\bullet \in \text{cogh}_{\mathcal{D}_n}$ if and only if $H^i(\alpha^\bullet) = 0$ for all $0 \leq i \leq n$. □

REMARK. This lemma implies $\text{Fcogh}_{\mathcal{D}_n}$ and $\text{Fgh}_{\mathcal{D}_n}$ also coincide. Moreover, if M^\bullet is a complex over $A\text{-mod}$ with zero homology in all degrees not in $\{0, 1, \dots, n\}$, then $\text{cogh}_{\mathcal{D}_n}(M^\bullet)$ consists of *ghost maps*, that is, chain maps $\alpha^\bullet : M^\bullet \rightarrow M^\bullet$ such that $H^i(\alpha^\bullet) = 0$ for all $i \in \mathbb{Z}$. We denote by \mathcal{G} the ideal of \mathcal{K} consisting of ghost maps. Then $\text{cogh}_{\mathcal{D}_n}(M^\bullet) = \text{gh}_{\mathcal{D}_n}(M^\bullet) = \mathcal{G}(M^\bullet)$. Let $\mathcal{G}_{\mathcal{D}_n} := \mathcal{G} \cap \mathcal{F}_{\mathcal{D}_n}$. Then $\mathcal{G}_{\mathcal{D}_n}(M^\bullet) = \text{Fcogh}_{\mathcal{D}_n}(M^\bullet) = \text{Fgh}_{\mathcal{D}_n}(M^\bullet)$.

Let \mathcal{T} be a triangulated category and let \mathcal{D} be a full subcategory of \mathcal{T} . Recall from [14, Subsection 2.4] that a triangle $X \xrightarrow{f} D \xrightarrow{g} Y \rightarrow \Sigma X$ in \mathcal{T} is called a \mathcal{D} -split triangle provided that f is a left \mathcal{D} -approximation and g is a right \mathcal{D} -approximation.

Let X^\bullet be a bounded complex over $A\text{-mod}$, and let i be an integer. Suppose that $\pi^i : P_X^i \rightarrow H^i(X^\bullet)$ is a projective cover of the i -th homology of X^\bullet . Then π^i can be lifted to a morphism $h^i : P_X^i \rightarrow \text{Ker } d_X^i$ along the canonical epimorphism $\text{Ker } d_X^i \rightarrow H^i(X^\bullet)$. Let $f^i : P_X^i \rightarrow X^i$ be the composite of h^i and the inclusion $\text{Ker } d_X^i \hookrightarrow X^i$. Then $f^i d_X^i = 0$, and f^i gives rise to a chain map from $\Sigma^{-i} P_X^i \rightarrow$

X^\bullet . Define

$$P_X^\bullet := \coprod_{i \in \mathbb{Z}} \Sigma^{-i} P_X^i.$$

Then we get a chain map $f^\bullet : P_X^\bullet \rightarrow X^\bullet$. Form a triangle

$$Y^\bullet \xrightarrow{g^\bullet} P_X^\bullet \xrightarrow{f^\bullet} X^\bullet \rightarrow \Sigma Y^\bullet \quad (*)$$

in \mathcal{K} and set $\mathcal{D}_\infty := \text{add}\{\Sigma^{-i}A \mid i \in \mathbb{Z}\}$. We claim that $(*)$ is actually a \mathcal{D}_∞ -split triangle in \mathcal{K} . Indeed, P_X^\bullet belongs to \mathcal{D}_∞ since there are only finitely many nonzero P_X^i , and for each $i \in \mathbb{Z}$, there is a commutative diagram

$$\begin{CD} \mathcal{K}(\Sigma^{-i}A, P_X^\bullet) @>\mathcal{K}(\Sigma^{-i}A, f^\bullet)>> \mathcal{K}(\Sigma^{-i}A, X^\bullet) \\ @V \cong VV @VV \cong V \\ P_X^i @>\pi^i>> H^i(X^\bullet). \end{CD}$$

Hence $\mathcal{K}(\Sigma^{-i}A, f^\bullet)$ is surjective for all $i \in \mathbb{Z}$. It follows from the isomorphism $\mathcal{K}(\Sigma^{-i}A, -) \cong D\mathcal{K}(-, \Sigma^{-i}\nu_A A)$ that $\mathcal{K}(f^\bullet, \Sigma^{-i}(\nu_A A))$ is injective for all $i \in \mathbb{Z}$. Since A is self-injective, we have $\text{add}(\nu_A A) = \text{add}({}_A A)$. Hence $\mathcal{K}(f^\bullet, \Sigma^{-i}A)$ is injective for all $i \in \mathbb{Z}$. Using the long exact sequences obtained by applying $\mathcal{K}(A, -)$ and $\mathcal{K}(-, A)$ to the triangle $(*)$, we deduce that the sequences

$$0 \rightarrow \mathcal{K}(\Sigma^{-i}A, Y^\bullet) \rightarrow \mathcal{K}(\Sigma^{-i}A, P_X^\bullet) \rightarrow \mathcal{K}(\Sigma^{-i}A, X^\bullet) \rightarrow 0, \quad (*)$$

$$0 \rightarrow \mathcal{K}(X^\bullet, \Sigma^{-i}A) \rightarrow \mathcal{K}(P_X^\bullet, \Sigma^{-i}A) \rightarrow \mathcal{K}(Y^\bullet, \Sigma^{-i}A) \rightarrow 0 \quad (**)$$

are exact for all $i \in \mathbb{Z}$. Particularly, f^\bullet is a right \mathcal{D}_∞ -approximation and g^\bullet is a left \mathcal{D}_∞ -approximation.

Note that the exact sequence $(*)$ is isomorphic to the sequence

$$0 \rightarrow H^i(Y^\bullet) \rightarrow P_X^i \rightarrow H^i(X^\bullet) \rightarrow 0.$$

This shows that $H^i(Y^\bullet) = \Omega(H^i(X^\bullet))$. For simplicity, we write H_X^i for $H^i(X^\bullet)$.

Now assume that X^\bullet is a bounded complex

$$0 \rightarrow X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} X^n \rightarrow 0.$$

Then $P_X^i = 0$ for all $i \notin \{0, \dots, n\}$. Thus P_X^\bullet lies in \mathcal{D}_n , and the triangle $(*)$ is a \mathcal{D}_n -split triangle. By theorem 4.3 and the remark after lemma 5.1, the algebras

$$\text{End}_{\mathcal{K}/\mathcal{G}_{\mathcal{D}_n}} \left(Y^\bullet \oplus \bigoplus_{i=0}^n \Sigma^{-i}A \right) \text{ and } \text{End}_{\mathcal{K}/\mathcal{G}_{\mathcal{D}_n}} \left(X^\bullet \oplus \bigoplus_{i=0}^n \Sigma^{-i}A \right)$$

are derived equivalent. Similarly, by theorem 3.2 and the remark after lemma 5.1, the algebras

$$\text{End}_{\mathcal{K}/\mathcal{G}} \left(Y^\bullet \oplus \bigoplus_{i=0}^n \Sigma^{-i}A \right) \text{ and } \text{End}_{\mathcal{K}/\mathcal{G}} \left(X^\bullet \oplus \bigoplus_{i=0}^n \Sigma^{-i}A \right)$$

are also derived equivalent. In the following, we shall see that these algebras have a very nice form.

Since A is self-injective, the regular module ${}_A A$ is injective and the contravariant functor $\text{Hom}_A(-, A)$ is exact. It follows that $H^{-i} \text{Hom}_A(C^\bullet, A)$ and $\text{Hom}_A(H^i(C^\bullet), A)$ are naturally isomorphic for all integers i and $C^\bullet \in \mathcal{K}$, which is equivalent to saying that $\mathcal{K}(C^\bullet, \Sigma^{-i}A) \cong \text{Hom}_A(H^i(C^\bullet), A) = (H^i(C^\bullet))^*$ for all complexes C^\bullet of A -modules and for all integers i . Particularly, one has $\mathcal{K}(X^\bullet, \Sigma^{-i}A) \cong (H_X^i)^*$ for all integers i .

With the preparations above, we can write

$$\text{End}_{\mathcal{K}/\mathcal{G}} \left(X^\bullet \oplus \bigoplus_{i=0}^n \Sigma^{-i}A \right) = \begin{pmatrix} \text{End}_{\mathcal{K}}(X^\bullet)/\mathcal{G}(X^\bullet) & (H_X^0)^* & \cdots & (H_X^n)^* \\ H_X^0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_X^n & 0 & \cdots & A \end{pmatrix},$$

$$\begin{aligned} & \text{End}_{\mathcal{K}/\mathcal{G}} \left(Y^\bullet \oplus \bigoplus_{i=0}^n \Sigma^{-i}A \right) \\ &= \begin{pmatrix} \text{End}_{\mathcal{K}}(Y^\bullet)/\mathcal{G}(Y^\bullet) & (\Omega(H_X^0))^* & \cdots & (\Omega(H_X^n))^* \\ \Omega(H_X^0) & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega(H_X^n) & 0 & \cdots & A \end{pmatrix}. \end{aligned}$$

The algebras $\text{End}_{\mathcal{K}/\mathcal{G}_{\mathcal{D}_n}}(X^\bullet \oplus \bigoplus_{i=0}^n \Sigma^{-i}A)$ and $\text{End}_{\mathcal{K}/\mathcal{G}_{\mathcal{D}_n}}(Y^\bullet \oplus \bigoplus_{i=0}^n \Sigma^{-i}A)$ have similar forms: just replacing \mathcal{G} with $\mathcal{G}_{\mathcal{D}_n}$ in the above matrices.

In the following, we give a concrete example.

EXAMPLE. Let k be a field, and let $A = k[x]/(x^n)$, $n \geq 2$. Suppose that $1 \leq m \leq n - 1$ and that X^\bullet is the complex

$$0 \longrightarrow A \xrightarrow{x^m} A \longrightarrow 0$$

with the left A in degree zero. The endomorphism algebra $\text{End}_{\mathcal{K}/\mathcal{G}}(X^\bullet \oplus A \oplus \Sigma^{-1}A)$ is denoted by $\Lambda(n, m)$. The construction above gives a \mathcal{D}_1 -split triangle

$$Y^\bullet \longrightarrow A \oplus \Sigma^{-1}A \longrightarrow X^\bullet \longrightarrow \Sigma Y^\bullet$$

in \mathcal{K} . An easy calculation shows that Y^\bullet is isomorphic in $\mathcal{K}(A\text{-mod})$ to the complex

$$0 \longrightarrow A \xrightarrow{x^{n-m}} A \longrightarrow 0.$$

Then the algebras $\Lambda(n, m) = \text{End}_{\mathcal{K}/\mathcal{G}}(X^\bullet \oplus A \oplus \Sigma^{-1}A)$ and $\text{End}_{\mathcal{K}/\mathcal{G}}(Y^\bullet \oplus A \oplus \Sigma^{-1}A)$ are derived equivalent. Note that $\text{End}_{\mathcal{K}/\mathcal{G}}(Y^\bullet \oplus A \oplus \Sigma^{-1}A)$ is just $\Lambda(n, n - m)$. That is, the algebra $\Lambda(n, m)$ is derived equivalent to $\Lambda(n, n - m)$ for all $1 \leq m \leq n - 1$.

To describe $\Lambda(n, m)$ in terms of quivers with relations, we give some morphisms in $\Lambda(n, m)$.

$$\begin{array}{ccc}
 \alpha_1 : 0 \longrightarrow A & \alpha_2 : A \longrightarrow 0 & , \quad \alpha_3 : A \xrightarrow{\cdot x^m} A & , \\
 \downarrow & \downarrow \cdot x & \downarrow \cdot x & \downarrow \cdot x \\
 0 \longrightarrow A & A \longrightarrow 0 & A \xrightarrow{\cdot x^m} A & \\
 \\
 \beta_1 : 0 \longrightarrow A & , \quad \beta_2 : A \xrightarrow{\cdot x^m} A & , \\
 \downarrow & \downarrow id & \downarrow \cdot x^{n-m} \\
 A \xrightarrow{\cdot x^m} A & & 0 \longrightarrow A \\
 \\
 \beta_3 : A \xrightarrow{\cdot x^m} A & , \quad \beta_4 : A \longrightarrow 0 & . \\
 \downarrow id & \downarrow \cdot x^{n-m} & \downarrow \cdot x^m \\
 A \longrightarrow 0 & & A \longrightarrow A
 \end{array}$$

It is easy to see that the above morphisms generate the Jacobson radical of $\Lambda(n, m)$.

Case I: $1 < m < n - 1$. In this case, the above morphisms are irreducible in $\text{add}(X^\bullet \oplus A \oplus \Sigma^{-1}A)$ and the algebra $\Lambda(n, m)$ is given by the following quiver with relations.

$$\begin{array}{c}
 \alpha_3 \\
 \curvearrowright \\
 \alpha_1 \quad \bullet_1 \xrightleftharpoons[\beta_2]{\beta_1} \bullet_2 \xrightleftharpoons[\beta_4]{\beta_3} \bullet_3 \quad \alpha_2 \quad \curvearrowleft \\
 \\
 \alpha_3^m = \beta_1\beta_3 = \beta_4\beta_2 = 0 \\
 \alpha_1^{n-m} = \beta_1\beta_2, \alpha_2^{n-m} = \beta_4\beta_3, \alpha_3^{n-m} = \beta_3\beta_4 + \beta_2\beta_1 \\
 \alpha_1\beta_1 = \beta_1\alpha_3, \alpha_3\beta_2 = \beta_2\alpha_1 \\
 \alpha_3\beta_3 = \beta_3\alpha_2, \alpha_2\beta_4 = \beta_4\alpha_3 \\
 \alpha_3^{i-n+m}\beta_2\beta_1 = 0 \quad (i := \max\{m, n - m\})
 \end{array}$$

Case II: $m = n - 1$. In this case, the morphisms $\alpha_1 = \beta_1\beta_2$, $\alpha_2 = \beta_4\beta_3$ and $\alpha_3 = \beta_3\beta_4 + \beta_2\beta_1$ are not irreducible in $\text{add}(X^\bullet \oplus A \oplus \Sigma^{-1}A)$ any more. However, the morphisms $\beta_i, i = 1, 2, 3, 4$ are still irreducible in $\text{add}(X^\bullet \oplus A \oplus \Sigma^{-1}A)$. The algebra $\Lambda(n, n - 1)$ is given by the following quiver with relations.

$$\begin{array}{c}
 \bullet_1 \xrightleftharpoons[\beta_2]{\beta_1} \bullet_2 \xrightleftharpoons[\beta_4]{\beta_3} \bullet_3 \\
 \\
 \beta_1\beta_3 = \beta_4\beta_2 = 0 \\
 (\beta_2\beta_1)^{n-1} = (\beta_3\beta_4)^{n-1} = 0
 \end{array}$$

Case III: $m = 1$. In this case, $\alpha_3 = 0$ and the other morphisms above are irreducible in $\text{add}(X^\bullet \oplus A \oplus \Sigma^{-1}A)$. The algebra $\Lambda(n, 1)$ is given by the following

quiver with relations.

$$\begin{array}{c}
 \alpha_1 \curvearrowright \bullet_1 \xrightleftharpoons[\beta_2]{\beta_1} \bullet_2 \xrightleftharpoons[\beta_4]{\beta_3} \bullet_3 \curvearrowright \alpha_2 \\
 \beta_1\beta_3 = \beta_4\beta_2 = \alpha_1\beta_1 = \beta_2\alpha_1 = \alpha_2\beta_4 = \beta_3\alpha_2 = 0 \\
 \beta_2\beta_1 = \beta_3\beta_4 = 0 \\
 \alpha_1^{n-1} = \beta_1\beta_2, \alpha_2^{n-1} = \beta_4\beta_3
 \end{array}$$

Let us explain how to get these quivers and relations. For all cases, let Q denote the quiver and let ρ denote the relations. The obvious map gives an algebra homomorphism from kQ to $\Lambda(n, m)$ which is surjective since the maps given above generate the Jacobson radical of $\Lambda(n, m)$. It is straightforward to check the the morphisms in $\Lambda(n, m)$ given above do satisfy the relations. This gives rise to a surjective algebra homomorphism from $kQ/\langle\rho\rangle$ to $\Lambda(n, m)$. Finally, one can verify that $kQ/\langle\rho\rangle$ and $\Lambda(n, m)$ have the same Cartan matrix and hence the same dimension. This forces that $kQ/\langle\rho\rangle$ and $\Lambda(n, m)$ are isomorphic algebras.

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References

- 1 T. Aihara and O. Iyama. Silting mutation in triangulated categories. *J. Lond. Math. Soc.* (2) **85** (2012), 633–668.
- 2 I. Assem, D. Simson and A. Skowroński. *Elements of the representation theory of associative algebras*. London Mathematical Society Student Texts 65, vol. 1 (Cambridge: Cambridge University Press, 2006).
- 3 M. Auslander and S. O. Smalø. Preprojective modules over artin algebras. *J. Algebra* **66** (1980), 61–122.
- 4 M. Auslander, I. Reiten and S. O. Smalø. *Representation theory of Artin algebras*. Cambridge Studies in Advanced Mathematics, vol. 36 (Cambridge: Cambridge University Press, 1995).
- 5 A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov. Tilting theory and cluster combinatorics. *Adv. Math.* **204** (2006), 572–618.
- 6 Y. Chen. Derived equivalences in n -angulated categories. *Algebr. Represent. Theory* **16** (2013), 1661–1684.
- 7 C. Geiss, B. Keller and S. Oppermann. n -angulated categories. *J. Reine Angew. Math.* **675** (2013), 101–120.
- 8 D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*. London Mathematical Society Lecture Note Series, vol. 119 (Cambridge: Cambridge University Press, 1988).
- 9 D. Happel and L. Unger. Almost complete tilting modules. *Proc. Amer. Math. Soc.* **107** (1989), 603–610.
- 10 D. Happel and L. Unger. Complements and the generalized Nakayama conjecture. In *Algebras and modules, II (Geiranger, 1996)*, CMS Conf. Proc., vol. 24, pp. 293–310 (Providence, RI: Amer. Math. Soc., 1998).

- 11 M. Hoshino and Y. Kato. An elementary construction of tilting complexes. *J. Pure Appl. Algebra* **177** (2003), 159–175.
- 12 W. Hu and C. C. Xi. \mathcal{D} -split sequences and derived equivalences. *Adv. Math.* **227** (2011), 292–318.
- 13 W. Hu and C. C. Xi. Derived equivalences for Φ -Auslander-Yoneda algebras. *Trans. Amer. Math. Soc.* **365** (2013), 5681–5711.
- 14 W. Hu, S. Koenig and C. C. Xi. Derived equivalences from cohomological approximations and mutations of Φ -Yoneda algebras. *Proc. Roy. Soc. Edinburgh Sect. A* **143** (2013), 589–629.
- 15 O. Iyama and M. Wemyss. Maximal modifications and Auslander-Reiten duality for non-isolated singularities. *Invent. Math.* **197** (2014), 521–586.
- 16 B. Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup.* **27** (1994), 63–102.
- 17 B. Keller. On triangulated orbit categories. *Doc. Math.* **10** (2005), 551–581.
- 18 H. Krause. Krull-Schmidt categories and projective covers. *Expo. Math.* **33** (2015), 535–549.
- 19 S. Ladkani. Perverse equivalences, BB-tilting, mutations and applications, Preprint (2010).
- 20 J. Rickard. Morita theory for derived categories. *J. London Math. Soc.* **39** (1989), 436–456.
- 21 A. N. Rudakov. Exceptional collections, mutations and helices. In *Helices and vector bundles*, London Math. Soc. Lecture Note Ser., vol. 148, pp. 1–6 (Cambridge: Cambridge Univ. Press, 1990).
- 22 M. Van den Bergh. Non-commutative crepant resolutions. In *The legacy of Niels Henrik Abel*, pp. 749–770 (Berlin: Springer-Verlag, 2004).