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Tilting modules and representation dimensions

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ABSTRACT

In this paper, we compare the representation dimensions of two algebras linked by certain tilting modules. Our main results can be stated as follows: Suppose T is a tilting module over A and $B = \mathrm{End}_A(T)$. Then: (1) If T is separating and splitting, then $\mathrm{rep.dim}(A) = \mathrm{rep.dim}(B)$; (2) If $T = P \oplus \tau^{-1}S$ is an APR-tilting module and the injective dimension of S is at most 2, then $\mathrm{rep.dim}(B) \leqslant \mathrm{rep.dim}(A) + 1$.

1. Introduction

The representation dimension of an Artin algebra A, denoted by $\mathrm{rep.dim}(A)$, was introduced by Auslander [4] as a way of measuring homologically how far an algebra is from being representation-finite. Auslander [4] proved that an algebra A is representation-finite if and only if $\mathrm{rep.dim}(A) \leqslant 2$. For three decades, it was unclear whether Auslander's philosophy works. The situation has changed dramatically in the last few years. Iyama [8] proved that the representation dimension of an Artin algebra is always finite, and Rouquier [9] proved that the representation dimensions of Artin algebras can be arbitrary integers $\geqslant 2$ by showing that the exterior algebra of an n-dimensional vector space has representation dimension $n + 1$. So we get a new division of algebras according to the size of their representation dimensions.

The precise value of a given algebra is not known in general, and is very hard to compute even for small examples. One possible method is to study the relationship between the representation dimensions of "nicely" related algebras. For instance, it is proved in [6] that representation dimension is invariant under stable equivalences (for stable equivalence of Morita type, this was already shown by Xi [10]).

Recently, in a paper of Assem, Platzeck and Trepode [2], they proved that a tilted algebra, which is the endomorphism algebra of a tilting module over a hereditary algebra, has representation dimension

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at most 3. In [11], Xi proved that the endomorphism algebra of certain APR-tilting module T over an algebra A has the same representation dimension as A. This motivates the following question.

Question. Suppose B is the endomorphism algebra of a tilting module T over an algebra A. What is the relationship between the representation dimensions of A and B?

Note that in general A and B do not have the same representation dimension, since there are examples where A is representation-finite while B is representation-infinite. It is well known that T induces two torsion pairs $(\mathcal{T}_T, \mathcal{F}_T)$ and $(\mathcal{X}_T, \mathcal{Y}_T)$ in A-mod and B-mod, respectively. The tilting module T is called *separating and splitting* if the torsion pairs $(\mathcal{T}_T, \mathcal{F}_T)$ and $(\mathcal{X}_T, \mathcal{Y}_T)$ are both splitting. Our first result in this paper is the following theorem.

Theorem 1.1. *Let T be a tilting module over an Artin algebra A and $B = \mathrm{End}_A(T)$. If T is separating and splitting, then* $\mathrm{rep.dim}(A) = \mathrm{rep.dim}(B)$.

Let A be an Artin algebra with a projective non-injective simple module S, and let P be the direct sum of all the pairwise non-isomorphic indecomposable projective A-modules other than S. Then the A-module $T := P \oplus \tau^{-1}S$ is a tilting module, and is called an APR-tilting module. This module was first studied in [5], which is the starting point of tilting theory. So, it is natural to consider our question for APR-tilting modules. Let B be the endomorphism algebra of the APR-tilting module T. In [11], it is proved that if the injective dimension of S is 1, then A and B have the same representation dimension. Note that in this case, T is separating and splitting. So, the above theorem also generalizes the result [11, Theorem 6.5]. For APR-tilting modules, the next step is: when the simple module S has injective dimension 2, what can we say about the representation dimensions of A and B. In this case, we have the following theorem.

Theorem 1.2. *Assume that A is an Artin algebra with a projective non-injective simple module S. Let P be the direct sum of all the pairwise non-isomorphic indecomposable projective A-modules other than S, and let $T = P \oplus \tau^{-1}S$ be the APR-tilting module. Set $B = \mathrm{End}_A(T)$. If the injective dimension of S is at most 2, then* $\mathrm{rep.dim}(B) \leqslant \mathrm{rep.dim}(A) + 1$. *In particular, if the global dimension of A is at most 2, then* $\mathrm{rep.dim}(B) \leqslant \mathrm{rep.dim}(A) + 1$.

This paper is organized as follows. In Section 2, we shall recall some basic definitions and facts needed in our proofs. Our main results Theorem 1.1 and Theorem 1.2 will be proved in Section 3 and Section 4, respectively. In the final section, we shall give two examples to illustrate our results.

2. Preliminaries

In this section, we recall some basic definitions and facts on representation dimension and tilting theory for Artin algebras. We refer to [4,1] for relevant literature.

Throughout this paper, all algebras are Artin algebras. Let A be an Artin algebra. Unless specified otherwise, an A-module always means a finitely generated left A-module. We use A-mod to denote the category of finitely generated left A-modules. The global dimension of A is denoted by $\mathrm{gl.dim}(A)$. For each A-module X, its projective dimension and injective dimension are denoted by $\mathrm{pd}_A X$ and $\mathrm{id}_A X$, respectively. We use $\mathrm{add}(X)$ to denote the full subcategory of A-mod consisting of direct summands of finite direct sums of copies of X. The usual duality is denoted by D and the Auslander–Reiten translation $D\,\mathrm{Tr}$ is denoted by τ.

Let A be an Artin algebra. An A-module M is called a *generator* (respectively, *cogenerator*) for A if ${}_AA \in \mathrm{add}(M)$ (respectively, $D(A) \in \mathrm{add}(M)$). The *representation dimension* of A is defined to be

$$\mathrm{rep.dim}(A) := \inf\big\{\mathrm{gl.dim}\big(\mathrm{End}_A(M)\big) \bigm| M \text{ is a generator-cogenerator for } A\big\}.$$

The following lemma is well known (see [4]).

Lemma 2.1. *Let A be an Artin algebra, n be a non-negative integer and M be a generator-cogenerator for A-mod. Then $\mathrm{gl.dim}(\mathrm{End}_A(M)) \leqslant n+2$ if and only if for each A-module X, there exists an exact sequence*

$$0 \to M_n \to \cdots \to M_0 \to X \to 0$$

with M_i in $\mathrm{add}(M)$ for all i, such that the induced sequence

$$0 \to \mathrm{Hom}_A(M, M_n) \to \cdots \to \mathrm{Hom}_A(M, M_0) \to \mathrm{Hom}_A(M, X) \to 0$$

is exact.

Let A be an Artin algebra. A *torsion pair* in A-mod is a pair $(\mathcal{T}, \mathcal{F})$ of classes of A-modules such that

(1) $\mathrm{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}$ and $N \in \mathcal{F}$;
(2) if $\mathrm{Hom}_A(M, -)|_{\mathcal{F}} = 0$, then $M \in \mathcal{T}$; and
(3) if $\mathrm{Hom}_A(-, N)|_{\mathcal{T}} = 0$, then $N \in \mathcal{F}$.

The class $\mathcal{T}$ (respectively, $\mathcal{F}$) is called the *torsion class* (respectively, the *torsion-free class*). It is known (for example, see [1, Proposition 1.1]) that a class $\mathcal{T}$ of A-modules is a torsion class of a torsion pair in A-mod if and only if $\mathcal{T}$ is closed under images, direct sums and extensions. Similarly, a class $\mathcal{F}$ of A-module is a torsion-free class of a torsion pair in A-mod if and only if $\mathcal{F}$ is closed under submodules, direct sums and extensions.

We collect some basic facts on torsion pairs in the following lemma. One may refer to [1] for the proof.

Lemma 2.2. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in A-mod. Then we have the following.*

(1) *For each A-module M, there is a unique submodule $t(M)$ in the torsion class $\mathcal{T}$ such that the quotient module $M/t(M)$ is in the torsion-free class $\mathcal{F}$;*
(2) *An indecomposable $M \in \mathcal{T}$ is Ext-projective in $\mathcal{T}$ (that is, $\mathrm{Ext}^1_A(M, -)|_{\mathcal{T}} = 0$) if and only if $\tau M \in \mathcal{F}$;*
(3) *An indecomposable $N \in \mathcal{F}$ is Ext-injective in $\mathcal{F}$ if and only if $\tau^{-1} N \in \mathcal{T}$.*

A torsion pair $(\mathcal{T}, \mathcal{F})$ in A-mod is *splitting* if for each A-module M, the exact sequence $0 \to t(M) \to M \to M/t(M) \to 0$ splits, or equivalently, each indecomposable A-module is either in $\mathcal{T}$ or in $\mathcal{F}$.

An A-module T is called a *tilting module* if (1) $\mathrm{pd}_A T \leqslant 1$, (2) $\mathrm{Ext}^1_A(T, T) = 0$, and (3) there exists a short exact sequence $0 \to {}_A A \to T^0 \to T^1 \to 0$ with $T^0, T^1 \in \mathrm{add}(T)$ (or, (3′) the number of non-isomorphic indecomposable direct summands of T is equal to the number of non-isomorphic simple A-modules). It is well known that a tilting module ${}_A T$ induces a torsion pair $(\mathcal{T}_T, \mathcal{F}_T)$ in A-mod and a torsion pair $(\mathcal{X}_T, \mathcal{Y}_T)$ in $\mathrm{End}_A(T)$-mod, where $\mathcal{T}_T = \mathrm{Gen}\, T$ (the A-modules which are homomorphic images of T^n for some n) and $\mathcal{Y}_T = \{\mathrm{Hom}_A(T, X) \mid X \in \mathcal{T}_T\}$. A tilting module ${}_A T$ is called *separating* (respectively, *splitting*) if the torsion pair $(\mathcal{T}_T, \mathcal{F}_T)$ (respectively, $(\mathcal{X}_T, \mathcal{Y}_T)$) is splitting.

Lemma 2.3. *Let ${}_A T$ be a tilting module over an Artin algebra A and let $B = \mathrm{End}_A(T)$. Then we have the following.*

(1) $\mathcal{T}_T = \{{}_A X \mid \mathrm{Ext}^1_A(T, X) = 0\}$ *and* $D(A) \in \mathcal{T}_T$;
(2) $\mathrm{Hom}_A(T, -) : \mathcal{T}_T \to \mathcal{Y}_T$ *is an equivalence*;
(3) $\mathrm{Ext}^1_A(T, -) : \mathcal{F}_T \to \mathcal{X}_T$ *is an equivalence*;

(4) *The right* B*-module* T_B *is a tilting module and* $\mathrm{End}(T_B) \simeq A^{\mathrm{op}}$;
(5) ${}_AT$ *is separating if and only if* $\mathrm{pd}_B X = 1$ *for all* ${}_BX \in \mathcal{X}_T$;
(6) ${}_AT$ *is splitting if and only if* $\mathrm{id}_A M = 1$ *for all* ${}_AM \in \mathcal{F}_T$;
(7) $\mathrm{id}_B \mathrm{Hom}_A(T, X) \leqslant \mathrm{id}_A X + 1$ *for all* $X \in \mathcal{T}_T$;
(8) $\mathrm{id}_A X \leqslant \max\{\mathrm{id}_B \mathrm{Ext}^1_A(T, X), 1\} + 1$ *for all* $X \in \mathcal{F}_T$.

Proof. The statements (1)–(4) of are well known (see, for example, [1, Theorem 2.1, p. 20]). For the proof of (5) and (6), we refer to [1, Theorem 3.6, p. 49]. For the proof of (7) and (8), we refer to [1, p. 27] and the references therein. □

For a tilting module ${}_AT$, throughout this paper, we use F (respectively, E) to denote the functor $\mathrm{Hom}_A(T, -)$ (respectively, $\mathrm{Ext}^1_A(T, -)$). The triple (A, T, B) is called a *tilting triple* if ${}_AT$ is a tilting module over A and $B = \mathrm{End}_A(T)$.

3. Separating splitting tilting modules and representation dimensions

Let (A, T, B) be a tilting triple. In this section, we shall consider the case when ${}_AT$ is separating and splitting and prove the following theorem.

Theorem 3.1. *Let* (A, T, B) *be a tilting triple. If* ${}_AT$ *is separating and splitting, then* $\mathrm{rep.dim}(A) = \mathrm{rep.dim}(B)$.

Remark.

(1) This generalizes the result [11, Theorem 6.5]. In [11, Theorem 6.5], the tilting modules considered are separating and splitting.
(2) Note that under the assumption of the above theorem, the algebras A and B are not stably equivalent in general. For a counterexample, see Example 1 in Section 5.

Before giving the proof of the theorem, we give some useful lemmas.

Lemma 3.2. *Suppose* (A, T, B) *is a tilting triple such that* ${}_AT$ *is separating. If* T_0 *is a non-projective indecomposable direct summand of* T *and* $0 \to \tau T_0 \to U \to T_0 \to 0$ *is the AR-sequence ending at* T_0, *then* $U \in \mathrm{add}(T \oplus \tau T)$.

Proof. Since ${}_AT$ is separating, we can assume that $U \simeq K \oplus L$, where $K \in \mathcal{T}_T$ and $L \in \mathcal{F}_T$. Let K' be an indecomposable direct summand of K. We claim K' is Ext-projective in $\mathcal{T}_T$. If K' is projective, then clearly K' is Ext-projective in $\mathcal{T}_T$. If K' is not projective, then there is an irreducible map $\tau K' \to \tau T_0$. Since $\tau T_0 \in \mathcal{F}_T$, we have $\tau K' \in \mathcal{F}_T$. Thus K' is Ext-projective in $\mathcal{T}_T$ by Lemma 2.2(2), and consequently we get $K' \in \mathrm{add}(T)$ by [1, Corollary 1.8]. This shows that $K \in \mathrm{add}(T)$. Let L' be an indecomposable direct summand of L. Then L' cannot be injective since $DA \in \mathcal{T}_T$ by Lemma 2.3(1). Thus there is an irreducible map from T_0 to $\tau^{-1}L'$, and it follows that $\tau^{-1}L'$ is in $\mathcal{T}_T$. Hence L' is Ext-injective in $\mathcal{F}_T$ by Lemma 2.2(3), and therefore $L' \in \mathrm{add}(\tau T)$. Hence $U \simeq K \oplus L \in \mathrm{add}(T \oplus \tau T)$. □

Lemma 3.3. *Suppose* (A, T, B) *is a tilting triple such that* ${}_AT$ *is separating and splitting. Then* $\mathrm{Hom}_B(E(\tau T), X) = 0$ *for any indecomposable* B*-module* $X \notin \mathrm{add}(E(\tau T))$.

Proof. Since τT is Ext-injective in $\mathcal{F}$, the B-module $E(\tau T)$ is Ext-injective in $\mathcal{X}_T$. Since T is splitting, the torsion pair $(\mathcal{X}_T, \mathcal{Y}_T)$ splits. Let X be an indecomposable B-module. If $X \in \mathcal{X}_T$, then obviously $\mathrm{Ext}^1_B(X, E(\tau T)) = 0$ since $E(\tau T)$ is Ext-injective in $\mathcal{X}_T$. If $X \in \mathcal{Y}_T$, then it follows from [3, Proposition VI.1.7(c)] that $\mathrm{Ext}^1_B(X, E(\tau T)) = 0$. Hence $E(\tau T)$ is Ext-injective in B-mod, which means that $E(\tau T)$ is an injective B-module. Note that each indecomposable direct summand of τT is of the form τT_0, where T_0 is an indecomposable non-projective direct summand of T. By Lemma 3.2, the AR-sequence ending at T_0 is of the following form

$$0 \to \tau T_0 \xrightarrow{[f,g]} T_1 \oplus \tau T_2 \to T_0 \to 0,$$

where $T_1, T_2 \in \operatorname{add}(T)$. Now let W be an indecomposable B-module. Suppose $h : E(\tau T_0) \to W$ is not split monic. We claim that h factors through $E(g)$. Clearly, we can assume h is nonzero. Then $W \in \mathcal{X}_T$, and hence $W \simeq E(V)$ for some indecomposable $V \in \mathcal{F}_T$. Thus, there is some map $h' : \tau T_0 \to V$ which is not a split monomorphism such that $E(h') = h$. Thus, h' factors through $[f, g]$. Note that $\operatorname{Hom}_A(T_1, V) = 0$. It follows that h' actually factors through g, and consequently $h = E(h')$ factors through $E(g)$. In addition, g is a left minimal map, so is $E(g)$. Hence $E(g) : E(\tau T_0) \to E(\tau T_2)$ is a left minimal almost split morphism, and consequently, we have $E(\tau T_0)/\operatorname{soc}(E(\tau T_0)) \simeq E(\tau T_2) \in \operatorname{add} E(\tau T)$.

Now let T_0 be an indecomposable module in $\operatorname{add}(E(\tau T))$, and let X be an indecomposable B-module not in $\operatorname{add}(E(\tau T))$. Suppose that $u : E(\tau T_0) \to X$ is a B-module homomorphism. Since $X \notin \operatorname{add}(E(\tau T))$ and $E(\tau T)$ is injective, the map u cannot be a split monomorphism. Therefore, the map u factors through $E(\tau T_0)/\operatorname{soc}(E(\tau T_0))$ which is still in $\operatorname{add}(E(\tau T))$ but has smaller length than $E(\tau T_0)$. Thus, using induction on the length of $E(\tau T_0)$, we have $\operatorname{Hom}_B(E(\tau T_0), X) = 0$ and the lemma is proved. $\square$

We also have another lemma.

Lemma 3.4. *Let (A, T, B) be a tilting triple such that ${}_AT$ is a separating tilting module. Suppose that M is a generator for A and $N = F(M) \oplus B \oplus X$ with $X \in \mathcal{X}_T$. Then*

$$\operatorname{pd}_{\operatorname{End}_B(N)} \operatorname{Hom}_B\big(N, F(U)\big) \leqslant \operatorname{pd}_{\operatorname{End}_A(M)} \operatorname{Hom}_A(M, U)$$

for each A-module $U \in \mathcal{T}_T$.

Proof. Set $\Lambda = \operatorname{End}_A(M)$ and $\Gamma = \operatorname{End}_B(N)$. If $\operatorname{pd}_\Lambda \operatorname{Hom}_A(M, U) = \infty$, then there is nothing to prove. Assume $\operatorname{pd}_\Lambda \operatorname{Hom}_A(M, U) = n < \infty$. Since M is a generator for A, there is an exact sequence

$$0 \to M_n \to \cdots \to M_0 \to U \to 0$$

with all $M_i \in \operatorname{add}(M)$ such that

$$0 \to \operatorname{Hom}_A(M, M_n) \to \cdots \to \operatorname{Hom}_A(M, M_0) \to \operatorname{Hom}_A(M, U) \to 0$$

is exact. Equivalently, there is an exact sequence $0 \to K_{i+1} \to M_i \xrightarrow{f_i} K_i \to 0$ for each $i = 0, \ldots, n-1$, such that f_i is a right $\operatorname{add}(M)$-approximation, where $K_0 = U$ and $K_n = M_n$. Applying $F = \operatorname{Hom}_A(T, -)$, for each i, we have an exact sequence

$$0 \to F(K_{i+1}) \to F(M_i) \xrightarrow{F(f_i)} F(K_i) \to E(K_{i+1}) \to E(M_i) \to E(K_i) \to 0.$$

Let $C_i = \operatorname{Coker} F(f_i)$. Then we have an exact sequence

$$0 \to C_i \to E(K_{i+1}) \to E(M_i) \to E(K_i) \to 0.$$

Since T is separating, by Lemma 2.3(5), all the modules in $\mathcal{X}_T$ have projective dimension 1. Note that the modules of the above exact sequence, except C_i, are all in $\mathcal{X}_T$, and therefore have projective dimension 1. It follows that $\operatorname{pd}_B C_i \leqslant 1$. Let $0 \to F(T_i'') \to F(T_i') \to C_i \to 0$ be a projective resolution of C_i. We have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(K_{i+1}) & \longrightarrow & F(K_{i+1}) \oplus F(T''_i) & \longrightarrow & F(T''_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(M_i) & \longrightarrow & F(M_i) \oplus F(T'_i) & \longrightarrow & F(T'_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Im} F(f_i) & \longrightarrow & F(K_i) & \longrightarrow & C_i & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Thus we have a short exact sequence

$$0 \to F(K_{i+1}) \oplus F\big(T''_i\big) \to F(M_i) \oplus F\big(T'_i\big) \xrightarrow{g_i} F(K_i) \to 0.$$

Now let N' be an indecomposable direct summand of N. We claim that every map from N' to $F(K_i)$ factors through g_i. In fact, this is clearly true when N' is projective. If $N' \in \operatorname{add}(X)$, then $\operatorname{Hom}_B(N', F(K_i)) = 0$. Now assume that N' is isomorphic to $F(M')$ for some $M' \in \operatorname{add}(M) \cap \mathcal{T}_T$. In this case, we have a natural isomorphism induced by F:

$$\operatorname{Hom}_B\big(F\big(M'\big), F(K_i)\big) \simeq \operatorname{Hom}_A\big(M', t(K_i)\big) \simeq \operatorname{Hom}_A\big(M', K_i\big).$$

Since every map from M' to K_i factors through f_i, we know that every map from $F(M')$ to $F(K_i)$ factors through $F(f_i)$ which clearly factors through g_i. This proves the claim: every map from N' to $F(K_i)$ factors through g_i. Define $N_i = F(M_i) \oplus F(T'_i) \oplus F(T''_{i-1})$, $\overline{K}_i = F(K_i) \oplus F(T''_{i-1})$ for $1 \leqslant i \leqslant n$ and $\overline{K}_0 = F(U)$, $N_0 = F(M_0) \oplus F(T'_0)$. Then for each $i = 0, \dots, n-1$ we have an exact sequence

$$0 \to \overline{K}_{i+1} \to N_i \to \overline{K}_i \to 0$$

such that the following sequence is exact

$$0 \to \operatorname{Hom}_B(N, \overline{K}_{i+1}) \to \operatorname{Hom}_B(N, N_i) \to \operatorname{Hom}_B(N, \overline{K}_i) \to 0.$$

Moreover, $\overline{K}_n = F(K_n) \oplus F(T''_{n-1})$ is in $\operatorname{add}(N)$. Hence $\operatorname{pd}_\Gamma \operatorname{Hom}_B(N, F(U)) \leqslant n = \operatorname{pd}_\Lambda \operatorname{Hom}_A(M, U)$. $\square$

Now we are ready to give a proof of Theorem 3.1.

Proof of Theorem 3.1. Since ${}_AT$ is separating and splitting, the torsion pairs $(\mathcal{T}_T, \mathcal{F}_T)$ and $(\mathcal{X}_T, \mathcal{Y}_T)$ are both splitting. Let $A = P \oplus Q$ such that $P \in \mathcal{T}_T$ and $Q \in \mathcal{F}_T$. Suppose $M = DA \oplus A \oplus K \oplus L$ is a generator-cogenerator for A such that $\operatorname{gl.dim}(\operatorname{End}_A(M)) = \operatorname{rep.dim}(A) = n + 2$, where $n \geqslant 0$, $K \in \mathcal{T}_T$ and $L \in \mathcal{F}_T$. Now define

$$N = B \oplus F(DA) \oplus F(K) \oplus E(Q) \oplus E(L) \oplus E(\tau T).$$

Let $T = P \oplus T'$. Then $D(B) = D\operatorname{Hom}_A(P \oplus T', T) \simeq D(P^* \otimes_A T) \oplus \operatorname{Ext}^1_A(T, \tau T') \simeq F(D(P^*)) \oplus E(\tau T')$ which is in $\operatorname{add}({}_BN)$, where $P^* = \operatorname{Hom}_A(P, A)$. Hence N is a generator-cogenerator for B. Set $\Lambda = \operatorname{End}_A(M)$ and $\Gamma = \operatorname{End}_B(N)$. We claim that $\operatorname{gl.dim}(\Gamma) \leqslant n + 2$. By Lemma 2.1, it is equivalent to say $\operatorname{pd}_\Gamma \operatorname{Hom}_B(N, X) \leqslant n$ for all B-modules X. Clearly, we can assume X is indecomposable. Since the torsion pair $(\mathcal{X}_T, \mathcal{Y}_T)$ is splitting, X is either in $\mathcal{X}_T$ or $\mathcal{Y}_T$. If $X \in \mathcal{Y}_T$, then $X \simeq F(U)$ for some indecomposable A-module $U \in \mathcal{T}_T$. By Lemma 3.4, we have $\mathrm{pd}_\Gamma \mathrm{Hom}_B(N, X) \leqslant \mathrm{pd}_\Lambda \mathrm{Hom}_A(M, U) \leqslant n$. If $X \in \mathcal{X}_T$, then $X \simeq E(V)$ for some indecomposable A-module $V \in \mathcal{F}_T$. If $V \in \mathrm{add}(\tau T)$, then $\mathrm{pd}_\Gamma \mathrm{Hom}_B(N, X) = 0 \leqslant n$. Now suppose $V \notin \mathrm{add}(\tau T)$. Then, for each $0 \leqslant i \leqslant n-1$, we have an exact sequence

$$0 \to K_{i+1} \to M_i \xrightarrow{f_i} K_i \to 0$$

with f_i being a right minimal $\mathrm{add}(M)$-approximation of K_i, where $K_0 = V$ and $K_n \in \mathrm{add}(M)$. Since the torsion pair $(\mathcal{T}_T, \mathcal{F}_T)$ is splitting, we have all K_i and M_i are in the torsion-free class $\mathcal{F}_T$. Thus, the module K_i does not have direct summands in $\mathrm{add}(\tau T)$ since τT is Ext-injective in $\mathcal{F}_T$. Applying E, we have short exact sequences

$$0 \to E(K_{i+1}) \to E(M_i) \xrightarrow{E(f_i)} E(K_i) \to 0.$$

Let N_1 be an indecomposable direct summand of N. If $N_1 \in \mathcal{Y}_T$, then, applying $\mathrm{Hom}_B(N_1, -)$, we have exact sequences

$$\mathrm{Hom}_B\big(N_1, E(M_i)\big) \to \mathrm{Hom}_B\big(N_1, E(K_i)\big) \to \mathrm{Ext}^1_B\big(N_1, E(K_{i+1})\big).$$

Since the torsion pair $(\mathcal{X}_T, \mathcal{Y}_T)$ is splitting, by [3, Proposition VI.1.7], we have $\mathrm{Ext}^1_B(N_1, E(K_{i+1})) = 0$, and consequently every map from N_1 to $E(K_i)$ factors through $E(f_i)$. Now assume $N_1 \in \mathcal{X}_T$. If $N_1 \notin \mathrm{add}\, E(\tau T)$, then $N_1 \simeq E(M')$ for some $M' \in \mathrm{add}(M) \cap \mathcal{F}_T$. There is a natural isomorphism $\mathrm{Hom}_B(E(M'), E(K_i)) \simeq \mathrm{Hom}_A(M', K_i)$ induced by E. Since every map from M' to K_i factors through f_i, we know that every map from N_1 to $E(K_i)$ factors through $E(f_i)$. If $N_1 \in \mathrm{add}(E(\tau T))$, then by Lemma 3.3 we have $\mathrm{Hom}_B(N_1, E(K_i)) = 0$ since each K_i has no direct summands in $\mathrm{add}(\tau T)$. Hence we get exact sequences

$$0 \to \mathrm{Hom}_B\big(N, E(K_{i+1})\big) \to \mathrm{Hom}_B\big(N, E(M_i)\big) \to \mathrm{Hom}_B\big(N, E(K_i)\big) \to 0$$

for $i = 0, \dots, n-1$. Moreover, $E(M_i) \in \mathrm{add}(N)$ and $E(K_n) \in \mathrm{add}(N)$. This implies that $\mathrm{pd}_\Gamma \mathrm{Hom}_B(N, X) \leqslant n$.

Thus, we have $\mathrm{gl.dim}(\mathrm{End}_B(N)) \leqslant n+2 = \mathrm{gl.dim}(\mathrm{End}_A(M)) = \mathrm{rep.dim}(A)$. Hence $\mathrm{rep.dim}(B) \leqslant \mathrm{rep.dim}(A)$. Dually, T_B is a separating and splitting tilting module over B^{op} with endomorphism algebra isomorphic to A^{op}. This implies that $\mathrm{rep.dim}(A^{\mathrm{op}}) \leqslant \mathrm{rep.dim}(B^{\mathrm{op}})$. Since the representation dimensions of an algebra and its opposite algebra are equal, we have $\mathrm{rep.dim}(A) \leqslant \mathrm{rep.dim}(B)$. Hence finally $\mathrm{rep.dim}(A) = \mathrm{rep.dim}(B)$. □

Let A be an Artin algebra. A projective A-module ${}_AQ$ is called *strongly hereditary projective* if $\mathrm{add}({}_AQ)$ is closed under submodules. The following lemma shows that there are a lot of separating and splitting tilting modules over Artin algebras.

Lemma 3.5. *Let A be a basic Artin algebra, and let ${}_AA = P \oplus Q$ be a decomposition such that ${}_AQ$ has no injective direct summands. If ${}_AQ$ is strongly hereditary projective and* $\mathrm{id}_A Q \leqslant 1$, *then $T := P \oplus \tau^{-1}Q$ is a separating and splitting tilting module over A.*

Proof. Since ${}_AQ$ is strongly hereditary projective, we have $\mathrm{Hom}_A(X, Q) = 0$ for all indecomposable A-modules X not in $\mathrm{add}({}_AQ)$. By assumption, ${}_AQ$ has no injective direct summands. Hence we have $\mathrm{Hom}_A(D(A), \tau T) = \mathrm{Hom}_A(D(A), {}_AQ) = 0$, and therefore $\mathrm{pd}_A T \leqslant 1$. By Auslander–Reiten Formula, we have $\mathrm{Ext}^1_A(T, T) \simeq D\,\mathrm{Hom}_A(T, \tau T)$ since $\mathrm{pd}_A T \leqslant 1$. Since $\tau^{-1}Q$ has no projective direct summands, we can easily see that T has no direct summands in $\mathrm{add}({}_AQ)$, and consequently we have

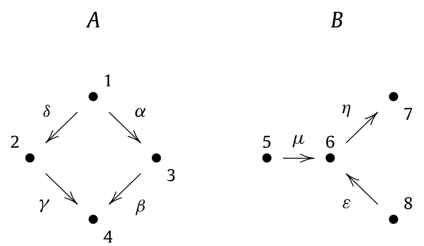
$\mathrm{Hom}_A(T, \tau T) = \mathrm{Hom}_A(T, Q) = 0$. Hence $\mathrm{Ext}^1_A(T, T) = 0$. Finally, since Q has no injective direct summands, $\tau^{-1}Q$ has the same number of indecomposable direct summands as Q. It follows that T and A have the same number of non-isomorphic indecomposable direct summands. Hence T is a tilting module.

Now we consider the torsion pair $(\mathcal{T}_T, \mathcal{F}_T)$. For any indecomposable module X not in $\mathrm{add}(Q)$, we have $\mathrm{Ext}^1_A(T, X) \simeq D\overline{\mathrm{Hom}}_A(X, \tau T) = D\overline{\mathrm{Hom}}_A(X, Q) = 0$, and therefore $X \in \mathcal{T}_T$, where $\overline{\mathrm{Hom}}_A(-,-)$ stands for the Hom-space in the quotient category $A\text{-}\overline{\mathrm{mod}}$ of A-mod modulo the ideal generated by morphisms factoring through injective modules. Clearly, $\mathrm{add}({}_AQ)$ is contained in $\mathcal{F}_T$ since $\mathrm{Hom}_A(T, Q) = 0$. Hence $\mathcal{F}_T = \mathrm{add}(Q)$, and consequently T is separating. Since $\mathrm{id}_A\, Q \leqslant 1$, we have $\mathrm{id}_A\, X \leqslant 1$ for all $X \in \mathcal{F}_T$. By assumption, the class $\mathcal{F}_T = \mathrm{add}({}_AQ)$ has no injective modules. Hence $\mathrm{id}_A\, X = 1$ for all $X \in \mathcal{F}_T$. It follows from Lemma 2.3(6) that T is also splitting. $\square$

In the following, we give a method to construct separating and splitting tilting modules. Let k be a field, and let $A = kQ/\langle\rho\rangle$ be a finite-dimensional algebra over k given by a quivers with relations. Let $B = k\Gamma$ be a path algebra of a connected quiver Γ without oriented cycles. Then Γ must contain a source vertex (that is, a vertex which is not the end of any arrow in Γ). Let p be a source vertex of Γ, and let q be any vertex of Q. Denote by $\Delta(\Gamma, Q, p, q)$ the quiver obtained by gluing p to q, and denote by $\Sigma(A, B, p, q)$ the k-algebra given by the quiver $\Delta(\Gamma, Q, p, q)$ with relations $\langle\rho\rangle$. Let e_p be the trivial path at vertex p and let $U = B/(Be_pB)$. Then it is easy to see that U is a projective $\Sigma(A, B, p, q)$-module. Let P be a projective $\Sigma(A, B, p, q)$-module such that $P \oplus U \simeq \Sigma(A, B, p, q)$. We have the following proposition.

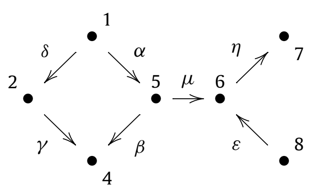
Proposition 3.6. *Keeping the notations above, $P \oplus \tau^{-1}U$ is a separating and splitting tilting module over $\Sigma(A, B, p, q)$.*

Before giving the proof of the above proposition, we illustrate our construction by an example. Let A and B be the k-algebras given by the quivers with relations.



$\alpha\beta - \delta\gamma$

Let p be the source vertex 5 and let q be the vertex 3. Then $\Sigma(A, B, p, q)$ is given by the following quiver



with relations $\alpha\beta - \delta\gamma$, and $U = P(6) \oplus P(7) \oplus P(8)$, where $P(i)$ denotes the projective module corresponding to the vertex i.

Now we give a proof of Proposition 3.6.

Proof of Proposition 3.6. For simplicity, we denote by Σ the algebra $\Sigma(A, B, p, q)$. By our construction, since p is a source vertex, all projective $B/(Be_pB)$-modules are projective as Σ-modules. Thus, it is easy to see that U is strongly hereditary projective as a Σ-module.

We claim that ${}_\Sigma U$ has no injective direct summands. If ${}_\Sigma V$ is an injective direct summand of ${}_\Sigma U$, then V is injective as a $B/(Be_pB)$-module. For a path algebra kQ' of a connected quiver without oriented cycles, it is well known that kQ' has projective–injective modules if and only if Q' is of the form $\bullet \to \bullet \to \cdots \to \bullet$. Thus, the sub-quiver of Γ obtained by deleting the vertex p has a component θ of this form. If there is no arrow from p to a vertex in θ, then Γ is not connected. This is a contradiction. Hence there is some arrow from p to a vertex in θ. But then the module V is not injective, a contradiction.

For each simple module S corresponding to the vertices of Γ other than p, one can easily check that $\mathrm{id}_\Sigma S \leqslant 1$. Hence we have $\mathrm{id}_\Sigma U \leqslant 1$. Now the proposition follows from Lemma 3.5. $\square$

4. APR-tilting modules and representation dimensions

Let A be an Artin algebra with a projective non-injective simple module S, and let P be the direct sum of all the pairwise non-isomorphic indecomposable projective A-modules other than S. Then $T := P \oplus \tau^{-1}S$ is a tilting module, and is called an *APR-tilting module*. This kind of tilting module is the first example of tilting modules [5], and tilting theory started from the study of such modules. So, to consider the relationship between representation dimensions and tilting modules, it is natural to consider APR-tilting modules first. Our result on this direction is the following theorem.

Theorem 4.1. *Suppose ${}_AT = P \oplus \tau^{-1}S$ is an* APR-*tilting module over an algebra A and $B = \mathrm{End}_A(T)$. If $\mathrm{id}_A S \leqslant 2$, then* $\mathrm{rep.dim}(B) \leqslant \mathrm{rep.dim}(A) + 1$.

Note that for an APR-tilting module $T = P \oplus \tau^{-1}S$, it is well known that $\mathcal{F}_T = \mathrm{add}({}_AS)$, and $\mathcal{X}_T = \mathrm{add}({}_BE(S))$. Moreover, $E(S)$ is a simple B-module.

Lemma 4.2. *Let S be a projective non-injective simple A-module and let ${}_AT$ be the corresponding* APR-*tilting module with endomorphism algebra B. Then $\mathrm{id}_B E(S) = \mathrm{id}_A S - 1$.*

Proof. Let $0 \to S \to I_0 \xrightarrow{f_0} I_1 \to \cdots$ be a minimal injective resolution of S. Set $C = \mathrm{Im}\, f_0$. Then we have a short exact sequence $0 \to S \to I_0 \to C \to 0$, which cannot be split since S is not injective. Applying $F = \mathrm{Hom}_A(T, -)$, we have an exact sequence $0 \to F(I_0) \to F(C) \to E(S) \to 0$. By Lemma 2.3, $\mathrm{id}_B F(I_0) \leqslant \mathrm{id}_A I_0 + 1 = 1$, and consequently we have $\mathrm{id}_B E(S) = \mathrm{id}_B F(C)$. Since S is projective, we have $\mathrm{Ext}^i_A(S, S) = 0$ for all $i > 0$, and consequently the injective envelope $I(S)$ of S does not occur as a direct summand of I_i for any $i > 0$. Thus, by [1, Lemma 2.3], $F(I_i)$ is injective for all $i > 0$. Since $\mathrm{Im}\, f_i$ is in $\mathcal{T}_T$ for all $i \geqslant 0$, we have $\mathrm{Ext}^1_A(T, \mathrm{Im}\, f_i) = 0$ for all $i \geqslant 0$, and consequently we get an exact sequence $0 \to F(C) \to F(I_1) \to F(I_2) \to \cdots$, which is a minimal injective resolution of $F(C)$ since all $F(I_i)$ are injective and all $F(f_i)$ are radical maps for $i > 0$. It follows that $\mathrm{id}_A C = \mathrm{id}_B F(C)$. Therefore, $\mathrm{id}_B E(S) = \mathrm{id}_B F(C) = \mathrm{id}_A C = \mathrm{id}_A S - 1$. $\square$

Now we come to the place to prove Theorem 4.1.

Proof of Theorem 4.1. Let M be a generator-cogenerator for A such that $\mathrm{rep.dim}(A) = \mathrm{gl.dim}(\mathrm{End}_A(M)) = n + 2$ with $n \geqslant 0$. Let I be an injective envelope of $E(S)$. Set $N = F(M \oplus \tau^{-1}S) \oplus E(S) \oplus I$ and $\Gamma = \mathrm{End}_B(N)$. Since $T = P \oplus \tau^{-1}S$ is in $\mathrm{add}(M \oplus \tau^{-1}S)$, we have ${}_BB = F(T) \in \mathrm{add}(F(M \oplus \tau^{-1}S)) \subseteq \mathrm{add}({}_BN)$. For each indecomposable injective B-module J not isomorphic to I, by [1, Lemma 2.3], we have $J \simeq \mathrm{Hom}_A(T, V)$ for some injective A-module V. Hence $J \in \mathrm{add}(F(M))$, and consequently N is a generator-cogenerator for B. We claim that $\mathrm{gl.dim}(\mathrm{End}_B(N)) \leqslant \mathrm{gl.dim}(\mathrm{End}_A(M)) + 1$. Actually, for each B-module U not in $\mathrm{add}(N)$, there is a right minimal $\operatorname{add}(F(M \oplus \tau^{-1}S))$-approximation $\alpha : N_1 \to U$. Since $T \in \operatorname{add}(M \oplus \tau^{-1}S)$, we have ${}_BB = F(T) \in \operatorname{add}(F(M \oplus \tau^{-1}S))$, and consequently α is a surjective map. Let $t(U)$ be the maximal torsion submodule of U with respect to the torsion pair $(\mathcal{X}_T, \mathcal{Y}_T)$, and let $\beta : t(U) \to U$ be the inclusion map. Note that $t(U) \in \mathcal{X}_T = \operatorname{add}({}_BE(S))$. It follows that $f := \left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right] : N_1 \oplus t(U) \to U$ is a right $\operatorname{add}(F(M \oplus \tau^{-1}S) \oplus E(S))$-approximation. Since $\operatorname{id}_A S \leqslant 2$, by Lemma 4.2, $\operatorname{id}_B E(S) \leqslant 1$. Let $0 \to E(S) \to I \to I_1 \to 0$ be a minimal injective resolution of $E(S)$. Clearly, the B-module I is not a direct summand of I_1. Hence $I_1 \in \operatorname{add} F(M)$ and the map $I \to I_1$ is a left almost split morphism. For each map $g : I \to U$, g cannot be a split monomorphism since $U \notin \operatorname{add}({}_BN)$. Thus, g factors through I_1. Since $I_1 \in \operatorname{add}(F(M))$, every map from I_1 to U factors through f, and consequently g factors through f. Hence f is a right $\operatorname{add}({}_BN)$-approximation. Let $K = \operatorname{Ker} f$. There is an exact sequence

$$0 \to \operatorname{Hom}_B(N, K) \to \operatorname{Hom}_B\big(N, N_1 \oplus t(U)\big) \to \operatorname{Hom}_B(N, U) \to 0.$$

We also have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & K & \longrightarrow & t(U) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & L & \longrightarrow & N_1 & \xrightarrow{\alpha} & U & \longrightarrow & 0.
\end{array}$$

It follows that λ is monic, and therefore K is in $\mathcal{Y}_T$ since $\mathcal{Y}_T$ is closed under submodules. Suppose $K = F(X)$ for some $X \in \mathcal{T}_T$. Since ${}_AT$ is an APR-tilting module, the torsion pair $(\mathcal{T}_T, \mathcal{F}_T)$ is splitting. Set $N' = F(M \oplus \tau^{-1}S) \oplus E(S)$, $\Lambda = \operatorname{End}_A(M)$ and $\Gamma' = \operatorname{End}_B(N')$. By Lemma 3.4, $\operatorname{pd}_{\Gamma'} \operatorname{Hom}_B(N', K) \leqslant \operatorname{pd}_\Lambda \operatorname{Hom}_A(M, X)$. Since $I \to I_1$ is a left almost split morphism with $I_1 \in \operatorname{add}(N')$, by the proof of [7, Theorem 3.1], we have $\operatorname{pd}_\Gamma \operatorname{Hom}_B(N, K) \leqslant \operatorname{pd}_{\Gamma'} \operatorname{Hom}_B(N', K)$. Hence we have

$$\operatorname{pd}_\Gamma \operatorname{Hom}_B(N, U) \leqslant \operatorname{pd}_\Gamma \operatorname{Hom}_B(N, K) + 1 \leqslant \operatorname{pd}_\Lambda \operatorname{Hom}_A(M, X) + 1 \leqslant n + 1.$$

Hence $\operatorname{rep.dim}(B) \leqslant \operatorname{gl.dim}(\Gamma) \leqslant n + 3 = \operatorname{rep.dim}(A) + 1$. $\square$

As an immediate consequence, we have the following corollary.

Corollary 4.3. *Let ${}_AT$ be an* APR-*tilting module over an algebra A with* $\operatorname{gl.dim}(A) \leqslant 2$. *Then*

$$\operatorname{rep.dim}\big(\operatorname{End}_A(T)\big) \leqslant \operatorname{rep.dim}(A) + 1.$$

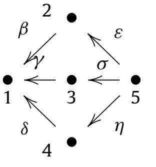
5. Examples

In this section, we give some examples to illustrate our results. The following example shows that in Theorem 3.1, the algebras A and B are not stably equivalent in general.

Example 1. Let A be the path algebra of the quiver $1 \to 2 \to 3$, and let T be the APR-tilting module corresponding to the simple projective A-module at the vertex 3. Then $B = \operatorname{End}_A(T)$ is isomorphic to the path algebra of the quiver $1 \to 2 \leftarrow 3$. Clearly, T is separating. It is also splitting since the simple module 3 has injective dimension 1. We use A-$\underline{\mathrm{mod}}$ to denote the stable module category of A, and use $\underline{\operatorname{Hom}}_A(-,-)$ to denote the Hom-space in A-$\underline{\mathrm{mod}}$. Then both A-$\underline{\mathrm{mod}}$ and B-$\underline{\mathrm{mod}}$ have three non-isomorphic indecomposable objects. There is an indecomposable object $X = {}^{1}{}_{2}{}^{3}$ in B-$\underline{\mathrm{mod}}$ such that $\underline{\operatorname{Hom}}_B(X, Y) \neq 0$ for all nonzero objects Y in B-$\underline{\mathrm{mod}}$, but A-$\underline{\mathrm{mod}}$ has no such object. Hence A and B

cannot be stably equivalent. This shows that under the assumptions of Theorem 3.1, the algebras A and B are not stably equivalent in general.

Example 2. Let A be the algebra (over a field k) given by the quiver



with relations $\varepsilon\beta = \sigma\gamma = \eta\delta$. This example was also given in [11]. A has finite representation type. The endomorphism algebra B of the only APR-tilting module over A corresponding to the vertex 1 has infinite representation type. Let $S(1)$ be the simple projective A-module corresponding to the vertex 1. The injective dimension of $S(1)$ is 2. Thus by Theorem 4.1, we have $\text{rep.dim}(B) \leqslant \text{rep.dim}(A) + 1 = 2 + 1 = 3$. Hence we have $\text{rep.dim}(B) = 3$.

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