Functional central limit theorem for super α-stable processes

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Received November 17, 2003; revised February 9, 2004

Abstract A functional central limit theorem is proved for the centered occupation time process of the super α-stable processes in the finite dimensional distribution sense. For the intermediate dimensions $\alpha < d < 2\alpha$ $(0 < \alpha \leqslant 2)$, the limiting process is a Gaussian process, whose covariance is specified; for the critical dimension $d = 2\alpha$ and higher dimensions $d > 2\alpha$, the limiting process is Brownian motion.

Keywords: super α-stable processes, occupation time, central limit theorem, evolution equation.

DOI: 10.1360/03ys0339

1 Introduction and statement of results

The asymptotic behavior of superprocesses has been investigated over the past years[1−6]. A general central limit theorem was proved by Iscoe[5] for the (α, d, β)-superprocess in higher dimensions $d > \alpha(1 + \beta)$, whereas for the intermediate and critical dimensions, the CLT was only obtained in the situation that the underlying motion is Brownian motion and the branching is of finite variance, i.e. $\alpha = 2$, $\beta = 1$. Hong et al.[2−4] have considered the limiting behavior of the super-Brownian motion with super-Brownian immigration, and some new and interesting phenomena were revealed for this new model. Recently, Dawson et al.[7] considered the occupation time fluctuations of branching particle system in 3 levels, and complete results were obtained for the situation of finite variance branching and α-stable motion.

In this paper, we will focus on the path-valued limit behavior, i.e. the functional central limit theorem for the occupation time process of super α-stable processes with finite variance branching $(\beta = 1)$. For the intermediate dimensions $\alpha < d < 2\alpha$ $(0 < \alpha \leqslant 2)$, the norming is $T^{(\frac{3}{2}-\frac{d}{2\alpha})}$ and the limiting process is a non-Brownian Gaussian process, whose covariance is given explicitly; for the critical dimension $d = 2\alpha$ and higher dimensions $d > 2\alpha$, the limiting process (in the finite dimensional distribution sense) is Brownian motion when the norming is $(T \log T)^{\frac{1}{2}}$ and $T^{\frac{1}{2}}$ respectively. Similar results appeared for the zero-range process[8] and branching random walk[9]. For the super-Brownian motion, Iscoe[5] has got a result for $d = 3$, and proved a functional ergodic theorem in ref. [6]; recently, Zhang1) has considered the path-valued limit behavior for

1) Zhang Mei, Functional central limit theorem for the super-brownian motion with super-Brownian immigration, J. Theoret. Probab., to appear.

the super-Brownian motion with super-Brownian immigration.

We will recall the concept of the superprocess briefly, for the general background we refer the readers to Dawson[10]. Let $C(\mathbb{R}^d)$ denote the space of bounded continuous functions on $\mathbb{R}^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1+|x|^2)^{-p/2}$ for $x \in \mathbb{R}^d$. Let $C_p(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : |f(x)| \leqslant \text{const}\cdot\phi_p(x)\}$. In duality, let $M_p(\mathbb{R}^d)$ be the space of Radon measures μ on $\mathbb{R}^d$ such that $\langle \mu, f\rangle := \int f(x)\mu(dx) < \infty$ for all $f \in C_p(\mathbb{R}^d)$. We endow $M_p(\mathbb{R}^d)$ with the p-vague topology, that is, $\mu_k \to \mu$ if and only if $\langle \mu_k, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_p(\mathbb{R}^d)$. Then $M_p(\mathbb{R}^d)$ is metrizable. Throughout this paper, λ denotes the Lebesgue measure on $\mathbb{R}^d$.

Suppose that $\xi = (\xi_t, t \geqslant 0)$ is an α-stable process ($0 < \alpha \leqslant 2$) in $\mathbb{R}^d$ with semigroup $(P_t^\alpha)_{t\geqslant 0}$. Its infinitesimal generator is a fractional power of the Laplacian, $\Delta_\alpha = -(-\Delta)^{\alpha/2}$. Let $p^\alpha(t,x,y)$ be the transition density function of the α-stable process, it is smooth, symmetric and unimodal with the self-similar property (cf. ref. [5]),

$$p^\alpha(t,x,y) = p^\alpha(t, x-y) = t^{-d/\alpha}p^\alpha(1, t^{-1/\alpha}(x-y)).$$

A *super α-stable process* $X = (X_t, Q_\mu)$ is an $M_p(\mathbb{R}^d)$-valued Markov process with $X_0 = \mu$ and the transition probabilities given by the following Laplace functional

$$\mathbf{E}\exp\{-\langle X_t, f\rangle\} = \exp\{-\langle \mu, n(t,\cdot)\rangle\}, \quad f \in C_p^+(\mathbb{R}^d), \tag{1.1}$$

where $n(\cdot,\cdot)$ is the unique mild solution of the evolution equation

$$\begin{cases} \dot{n}(t) = \Delta_\alpha n(t) - n^2(t), \\ n(0) = f. \end{cases} \tag{1.2}$$

Let $\{g(t,\cdot) : t \geqslant 0\}$ be a continuous $C_p^+(\mathbb{R}^d)$-valued path such that for each $a > 0$ there is a constant $C_a > 0$ such that $g(t) \leqslant C_a\phi_p$ for all $t \in [0,a]$. The weighted occupation time of the super α-stable process may be determined by

$$\mathbf{E}\exp\Big(-\int_0^t \langle X_s, g(s)\rangle ds\Big) = \exp\{-\langle \mu, m(0,t,\cdot)\rangle\}, \tag{1.3}$$

where $m(0,\cdot,\cdot)$ is the unique mild solution of

$$\begin{cases} \dot{m}(s) = \Delta_\alpha m(s) - m^2(s) + g(t-s), \quad 0 \leqslant s \leqslant t, \\ m(0) = 0. \end{cases} \tag{1.4}$$

See e.g. Iscoe[5].

For the path-valued setting, we consider the numerical centered occupation time process $Z_t^T(f)$,

$$Z_t^T(f) := a_d^{-1}(T)\int_0^{Tt} [\langle X_s, f\rangle - \mathbf{E}\langle X_s, f\rangle] ds, \quad f \in C_p(\mathbb{R}^d)^+, \tag{1.5}$$

where

$$a_d(T) = \begin{cases} T^{(\frac{3}{2}-\frac{d}{2\alpha})}, & d/2 < \alpha < d, \\ (T\log T)^{\frac{1}{2}}, & d = 2\alpha, \\ T^{\frac{1}{2}}, & d > 2\alpha. \end{cases} \tag{1.6}$$

Theorem 1.1. For $d > \alpha$, as $T \to \infty$, in finite dimensional distribution sense

$$Z_t^T(f) \longrightarrow Z_t(f),$$

in $C([0,\infty), R)$, and the finite dimensional distribution of $(Z_t(f))_{t\geqslant 0}$ is characterized by the Laplace transformation given by

$$\mathbf{E}\exp\Big\{-\sum_{i=1}^{k}\theta_i Z_{t_i}(f)\Big\} = \exp\{\hat\theta\mathbf{A}\hat\theta'\},$$

where $0 \leqslant t_1 \leqslant t_2 \cdots \leqslant t_k, 0 \leqslant \theta_1, \theta_2, \cdots, \theta_k$, $\hat\theta = (\theta_1, \theta_2, \cdots, \theta_k)$, $\mathbf{A} = (a_{ij})_{k\times k}$,

(i) When $\alpha < d < 2\alpha$ $(0 < \alpha \leqslant 2)$,

$$\begin{aligned} a_{ij} &= \mathbf{E}\left[Z_{t_i}(f)Z_{t_j}(f)\right] \\ &= \frac{p^\alpha(1,0)}{C_{d,\alpha}}\langle\lambda, f\rangle^2\left[t_i^{3-d/\alpha} + t_j^{3-d/\alpha} - \frac{1}{2}(t_i+t_j)^{3-d/\alpha} - \frac{1}{2}|t_i - t_j|^{3-d/\alpha}\right], \end{aligned}$$

i.e. $Z_t(f)$ is a Gaussian process with covariance

$$\begin{aligned} &\mathbf{E}\left[Z_s(f)Z_t(f)\right] \\ &= \frac{2p^\alpha(1,0)}{C_{d,\alpha}}\langle\lambda, f\rangle^2\left[s^{3-d/\alpha} + t^{3-d/\alpha} - \frac{1}{2}(s+t)^{3-d/\alpha} - \frac{1}{2}|s-t|^{3-d/\alpha}\right], \end{aligned}$$

and

$$C_{d,\alpha} = \Big(\frac{d}{\alpha} - 1\Big)\Big(2 - \frac{d}{\alpha}\Big)\Big(3 - \frac{d}{\alpha}\Big).$$

(ii) When $d \geqslant 2\alpha$, $a_{ij} = C_d \min(t_i, t_j)$, i.e. $Z_t(f)$ is Brownian motion, with covariance $2C_d\min(s,t)$, and

$$C_d = \begin{cases} p^\alpha(1,0)\langle\lambda, f\rangle^2, & d = 2\alpha, \\ \int_0^\infty dr \int_0^\infty dr' \int f(y)P^\alpha_{r+r'}f(y)dy, & d > 2\alpha. \end{cases}$$

Remark 1.1. For $d > \alpha$, the limiting process $(Z_t(f))_{t\geqslant 0}$ is something like fractional Brownian motion, but there is some difference in the covariance. (Recall that the fractional Brownian motion with Hurst parameter H has covariance $\frac{1}{2}\{|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\}$ (see for example, ref. [11])).

Remark 1.2. We only obtain a limit in the finite dimensional distribution sense, the tightness has not been proved yet. Actually, we are uncertain whether tightness holds for this situation, and we leave it as an open problem.

As a byproduct, we get a central limit theorem in the intermediate and critical dimensions for the super α-stable processes; whereas for the higher dimensions (i.e. $d > 2\alpha$), the result is well known by Iscoe (Theorem 5.4 in ref. [5]). Consider the centered occupation time process $Y_T(f)$,

$$Y_T(f) := a_d^{-1}(T)\int_0^T[\langle X_s, f\rangle - \mathbf{E}\langle X_s, f\rangle]ds, \quad f \in C_p(\mathbb{R}^3)^+. \tag{1.7}$$

Proposition 1.1. As $T \to \infty$,

$$Y_T(f) \longrightarrow Y_\infty(f),$$

where $Y_\infty(f)$ is a centered Gaussian random variable and

(i) for $\alpha < d < 2\alpha$ $(0 < \alpha \leqslant 2)$, $a_d(T) = T^{(\frac{3}{2}-\frac{d}{2\alpha})}$, and the variance of $Y_\infty(f)$ is

$$\mathbf{E}\,(Y_\infty^2(f)) = \frac{2}{C_{d,\alpha}}(1-2^{1-d/\alpha})p^\alpha(1,0)\langle\lambda, f\rangle^2;$$

(ii) for $d = 2\alpha$ $(0 < \alpha \leqslant 2)$, $a_d(T) = (T\log T)^{1/2}$, and the variance of $Y_\infty(f)$ is

$$\mathbf{E}\,(Y_\infty^2(f)) = \frac{1}{2}p^\alpha(1,0)\langle\lambda, f\rangle^2.$$

The proof is similar as in Theorem 1.1, we omit the details.

2 Proofs

To simplify the notation, we shall consider $k = 2$ and let $f_T := a_d^{-1}(T)f$. From (1.3), (1.4) and (1.5) the Laplace transformation of $(Z_{t_1}^T(f), Z_{t_2}^T(f))$, $(0 \leqslant t_1 \leqslant t_2, 0 \leqslant \theta_1, \theta_2)$ is given by

$$\begin{aligned}
&\mathbf{E}\,\exp\{-\theta_1 Z_{t_1}^T(f) - \theta_2 Z_{t_2}^T(f)\}\\
=\ &\mathbf{E}\,\exp\Big\{-\int_0^{Tt_2}\Big[\langle X_s, \theta_1 f_T 1_{[0,Tt_1]}(s) + \theta_2 f_T\rangle\\
&\qquad - \mathbf{E}\,(\langle X_s, \theta_1 f_T 1_{[0,Tt_1]}(s) + \theta_2 f_T\rangle)\Big]ds\Big\}\\
=\ &\exp\left\{\int_0^{Tt_2}\langle\lambda, u_T^2(s,\cdot)\rangle ds\right\},
\end{aligned} \tag{2.1}$$

where $u_T(s,x)$ is the mild solution of the equation

$$\begin{cases} \dot{u}_T(s) = \Delta_\alpha u_T(s) - u_T^2(s) + \theta_1 f_T 1_{[0,Tt_1]}(Tt_2 - s) + \theta_2 f_T, & 0 \leqslant s \leqslant Tt_2,\\ u_T(0) = 0, \end{cases} \tag{2.2}$$

i.e.

$$\begin{cases} \dot{u}_T(s) = \Delta_\alpha u_T(s) - u_T^2(s) + \theta_1 f_T 1_{[T(t_2-t_1),Tt_2]}(s) + \theta_2 f_T, & 0 \leqslant s \leqslant Tt_2,\\ u_T(0) = 0, \end{cases} \tag{2.3}$$

and with mild form

$$\begin{aligned}
u_T(s,x) = &\int_0^s P_{s-r}^\alpha[\theta_1 f_T 1_{[T(t_2-t_1),Tt_2]}(r) + \theta_2 f_T](x)dr\\
&- \int_0^s P_{s-r}^\alpha u_T(r,\cdot)^2(x)dr.
\end{aligned} \tag{2.4}$$

Let

$$A_T(\theta_1,\theta_2;t_1,t_2) = \int_0^{Tt_2}\langle\lambda, [\int_0^s P_{s-r}^\alpha(\theta_1 f_T 1_{[T(t_2-t_1),Tt_2]}(r) + \theta_2 f_T)dr]^2\rangle ds. \tag{2.5}$$

We first note that (let $\delta = t_2 - t_1$)

$$\begin{aligned} & A_T(\theta_1, \theta_2; t_1, t_2) \\ &= a_d^{-2}(T) \int_0^{Tt_2} ds \int_0^s dr \int_0^s dr' \int\int p^\alpha(2s - r - r', y, z) \\ &\quad \times [\theta_1 f(y) 1_{[T\delta, Tt_2]}(r) + \theta_2 f(y)][\theta_1 f(z) 1_{[T\delta, Tt_2]}(r') + \theta_2 f(z)] dydz \\ &=: A_T^{(1)} + A_T^{(2)} + A_T^{(3)} + A_T^{(4)}, \end{aligned}$$

where

$$A_T^{(1)} = \theta_1^2 a_d^{-2}(T) \int_{T\delta}^{Tt_2} ds \int_0^{s-T\delta} dr \int_0^{s-T\delta} dr' \int\int p^\alpha(r + r', y, z) f(y) f(z) dydz,$$

$$A_T^{(2)} = \theta_1\theta_2 a_d^{-2}(T) \int_{T\delta}^{Tt_2} ds \int_0^{s} dr \int_0^{s-T\delta} dr' \int\int p^\alpha(r + r', y, z) f(y) f(z) dydz,$$

$$A_T^{(3)} = \theta_1\theta_2 a_d^{-2}(T) \int_{T\delta}^{Tt_2} ds \int_0^{s-T\delta} dr \int_0^{s} dr' \int\int p^\alpha(r + r', y, z) f(y) f(z) dydz,$$

$$A_T^{(4)} = \theta_2^2 a_d^{-2}(T) \int_{0}^{Tt_2} ds \int_0^{s} dr \int_0^{s} dr' \int\int p^\alpha(r + r', y, z) f(y) f(z) dydz.$$

We will consider $\alpha < d < 2\alpha$ ($0 < \alpha \leqslant 2$) at first by the following two lemmas.

Lemma 2.1. Let $\alpha < d < 2\alpha$ ($0 < \alpha \leqslant 2$), then

$$\lim_{T\to\infty} A_T(\theta_1, \theta_2; t_1, t_2) = \hat{\theta}\mathbf{A}\hat{\theta}',$$

where $\hat{\theta}, \mathbf{A}$ are as in Theorem 1.1 (with $k = 2$).

Proof. Recall $a_d(T)$ in (1.6) and the self-similar property of the transition density function of the α-stable process, when $\alpha < d < 2\alpha$,

$$\begin{aligned} A_T^{(1)} &= \theta_1^2 \int_\delta^{t_2} ds \int_0^{s-\delta} dr \int_0^{s-\delta} dr' \\ &\times \int\int (r + r')^{-\frac{d}{\alpha}} p^\alpha(1, (T(r + r'))^{-1/\alpha}(y - z)) f(y) f(z) dydz, \end{aligned}$$

by dominated convergence theorem

$$\begin{aligned} \lim_{T\to\infty} A_T^{(1)} &= \theta_1^2 p^\alpha(1, 0)\langle\lambda, f\rangle^2 \int_\delta^{t_2} ds \int_0^{s-\delta} dr \int_0^{s-\delta} (r + r')^{-\frac{d}{\alpha}} dr' \\ &= \frac{2}{C_{d,\alpha}}(1 - 2^{1-d/\alpha}) p^\alpha(1, 0)\langle\lambda, f\rangle^2 \theta_1^2 t_1^{3-d/\alpha}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \lim_{T\to\infty} A_T^{(2)} &= \lim_{T\to\infty} A_T^{(3)} \\ &= \frac{\langle\lambda, f\rangle^2}{C_{d,\alpha}} p^\alpha(1, 0)\theta_1\theta_2 \left[t_1^{3-d/\alpha} + t_2^{3-d/\alpha} - \frac{(t_1 + t_2)^{3-d/\alpha}}{2} - \frac{|t_1 - t_2|^{3-d/\alpha}}{2} \right], \end{aligned}$$

$$\lim_{T\to\infty} A_T^{(4)} = \frac{2}{C_{d,\alpha}}(1 - 2^{1-d/\alpha}) p^\alpha(1, 0)\langle\lambda, f\rangle^2 \theta_2^2 t_2^{3-d/\alpha},$$

where $C_{d,\alpha}$ is given in Theorem 1.1. Combining the above, we are done.

Lemma 2.2. Let $\alpha < d < 2\alpha$ $(0 < \alpha \leqslant 2)$,

$$\beta_T(\theta_1, \theta_2; t_1, t_2) := A_T(\theta_1, \theta_2; t_1, t_2) - \int_0^{Tt_2} \langle \lambda, u_T^2(s, \cdot)\rangle ds.$$

Then

$$\lim_{T\to\infty} \beta_T(\theta_1, \theta_2; t_1, t_2) = 0.$$

Proof. Let

$$G_T(s) = \int_0^s P_{s-r}^\alpha[\theta_1 f_T 1_{[T(t_2-t_1),Tt_2]}(r) + \theta_2 f_T]dr,$$

from (2.4) we know that

$$0 \leqslant u_T(s) \leqslant G_T(s) \leqslant C\int_0^s P_{s-r}^\alpha f_T dr,$$

where C is a positive constant (it can take different values in different lines), and

$$0 \leqslant G_T^2(s) - u_T^2(s) \leqslant 2G_T(s)\cdot\int_0^s P_{s-r}^\alpha u_T^2(r)dr.$$

Then

$$\begin{aligned}
\beta_T(\theta_1, \theta_2; t_1, t_2) &= \int_0^{Tt_2}\langle\lambda, G_T^2(s) - u_T^2(s)\rangle ds\\
&\leqslant 2\int_0^{Tt_2}\langle\lambda, G_T(s)\cdot\int_0^s P_{s-r}^\alpha u_T^2(r)dr\rangle ds\\
&\leqslant C\int_0^{Tt_2}\langle\lambda, \int_0^s P_{s-r'}^\alpha f_T dr'\cdot\int_0^s P_{s-r}^\alpha[\int_0^r P_{r-h}^\alpha f_T dh]^2 dr\rangle ds\\
&\leqslant Ca_d(T)^{-3}T^{3-d/\alpha}\int_0^{t_2} ds\int_0^s dr'\int_0^s p^\alpha(2s-r-r,0)dr\\
&\quad\cdot\int[\int_0^{Tr} P_h^\alpha f_T dh]^2(z)dz\\
&\leqslant Ca_d(T)^{-3}T^{5-2d/\alpha}\int_0^{t_2} ds\int_0^s dr'\int_0^s p^\alpha(2s-r-r,0)dr\\
&\quad\cdot\int_0^{t_2} dh\int_0^{t_2} p^\alpha(h+h',0)dh',
\end{aligned}$$

which goes to 0 as $T \to \infty$, because the integral at the right hand side is finite and $a_d(T)^{-3}T^{5-2d/\alpha} = T^{1/2-d/2\alpha} \to 0$ when $\alpha < d < 2\alpha$. This completes the proof.

Remark 2.1. To consider k-dimensional distribution of the process Z_t^T where $k > 2$, it is enough to replace $\theta_1 f_T 1_{[0,Tt_1]}(Tt_2 - s) + \theta_2 f_T$ by

$$\sum_{i=1}^{k-1}\theta_i f_T 1_{[0,Tt_i]}(Tt_k - s) + \theta_k f_T$$

in eq. (2.2), and we can prove the counterpart of Lemma 2.1 and Lemma 2.2 with a bit more complicated calculation.

Proof of part (i) of Theorem 1.1. Combining Lemma 2.1 and Lemma 2.2 with (2.1), we are done by the discussion in Theorem 5.4 of Iscoe[5] on bilateral Laplace transform. Q.E.D.

We can prove part (ii) of Theorem 1.1 by the same method, so we will only calculate the limit of A_T and the remaining proof is similar as that of part (i). Recall (2.5), and $t_1 < t_2$, $\delta = t_2 - t_1$. Firstly, note that

$$A_T^{(1)} = \theta_1^2 a_d^{-2}(T) \int_{T\delta}^{Tt_2} ds \int_0^{s-T\delta} dr \int_0^{s-T\delta} dr' \int\int p^\alpha(r+r', y, z) f(y) f(z) dydz$$
$$= \theta_1^2 a_d^{-2}(T) T \int_\delta^{t_2} ds \int_0^{(s-\delta)T} dr \int_0^{(s-\delta)T} dr' \int\int p^\alpha(r+r', y, z) f(y) f(z) dydz.$$

When $d > 2\alpha$, recall (1.6),

$$A_T^{(1)} = \theta_1^2 \int_\delta^{t_2} ds \int_0^{(s-\delta)T} dr \int_0^{(s-\delta)T} dr' \int\int p^\alpha(r+r', y, z) f(y) f(z) dydz,$$

and by monotone convergence theorem

$$\lim_{T\to\infty} A_T^{(1)} = \theta_1^2 \int_\delta^{t_2} ds \int_0^{\infty} dr \int_0^{\infty} dr' \int\int p^\alpha(r+r', y, z) f(y) f(z) dydz$$
$$= \theta_1^2 C_d t_1.$$

When $d = 2\alpha$,

$$\lim_{T\to\infty} A_T^{(1)} = \lim_{T\to\infty} \theta_1^2 (\log T)^{-1} \int_\delta^{t_2} ds \int_0^{(s-\delta)T} dr' \int_0^{(s-\delta)T} dr \int\int p^\alpha(r+r', y, z) f(y) f(z) dydz$$
$$= \lim_{T\to\infty} \theta_1^2 (\log T)^{-1} \int_\delta^{t_2} ds \int_a^{(s-\delta)T} dr' \int_a^{(s-\delta)T} dr \int\int p^\alpha(r+r', y, z) f(y) f(z) dydz$$
$$= \lim_{T\to\infty} \theta_1^2 (\log T)^{-1} \int_\delta^{t_2} ds \int_a^{(s-\delta)T} dr' \int_{a+r'}^{(s-\delta)T+r'} dr \int\int p^\alpha(r, y, z) f(y) f(z) dydz,$$

where a is a positive constant. Breaking up the integral over r at $(s-\delta)T + a$ and interchanging the order of integral, we get

$$\lim_{T\to\infty} A_T^{(1)} = \theta_1^2 \lim_{T\to\infty} \int_\delta^{t_2} [I_1(T,s) + I_2(T,s)] ds,$$

where

$$I_1(T,s) = (\log T)^{-1} \int_{2a}^{(s-\delta)T+a} (r-2a) r^{-2} dr \int\int p^\alpha(1, r^{-1/\alpha}(y-z)) f(y) f(z) dydz,$$
$$I_2(T,s) = (\log T)^{-1} \int_{(s-\delta)T+a}^{2(s-\delta)T} (2(s-\delta)T - r) r^{-2} dr \int\int p^\alpha(1, r^{-1/\alpha}(y-z)) f(y) f(z) dydz.$$

But by a simple calculation,

$$0 \leqslant I_2(T,s)$$

$$
\begin{aligned}
&\leqslant (\log T)^{-1} p^{\alpha}(1,0)\langle \lambda, f\rangle^2 \int_{(s-\delta)T+a}^{2(s-\delta)T} ((s-\delta)T-a)((s-\delta)T+a)^{-2} dr \\
&\leqslant (\log T)^{-1} p^{\alpha}(1,0)\langle \lambda, f\rangle^2,
\end{aligned}
$$

and then

$$\lim_{T\to\infty} \int_{\delta}^{t_2} I_2(T,s)ds = 0.$$

Similarly,

$$
\begin{aligned}
0 \leqslant I_1(T,s) &\leqslant (\log T)^{-1} p^{\alpha}(1,0)\langle \lambda, f\rangle^2 \int_{2a}^{(s-\delta)T+a} r^{-1} dr \\
&\leqslant 2p^{\alpha}(1,0)\langle \lambda, f\rangle^2,
\end{aligned}
$$

when T is large enough. Then by dominated convergence theorem and L'Hopsital's rule

$$\lim_{T\to\infty} A_T^{(1)} = \theta_1^2 \lim_{T\to\infty} \int_{\delta}^{t_2} I_1(T,s)ds = \theta_1^2 p^{\alpha}(1,0)\langle \lambda, f\rangle^2 t_1.$$

By the same method, we get

$$\lim_{T\to\infty} A_T^{(2)} = \lim_{T\to\infty} A_T^{(3)} = \theta_1\theta_2 C_d t_1, \quad \lim_{T\to\infty} A_T^{(4)} = \theta_2^2 C_d t_2,$$

where

$$
C_d = \begin{cases} p^{\alpha}(1,0)\langle \lambda, f\rangle^2, & d = 2\alpha \\ \int_0^{\infty} dr \int_0^{\infty} dr' \int f(y) P_{r+r'}^{\alpha} f(y) dy, & d > 2\alpha. \end{cases}
$$

Acknowledgements The author thanks Prof. D. Dawson for his valuable suggestions and comments, and is grateful for the financial support and the hospitality at Carleton University during his visiting period, where part of the work was done. He also thanks the referee for his/her valuable suggestions. This work was supported by the National Natural Science Foundation of China (Grant Nos.10101005 and 10121101), and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

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