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Large Deviations for the Super-Brownian Motion with Super-Brownian Immigration

Wenming Hong[1]

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Local large deviation principles are established in dimensions $d \geqslant 3$ for the super Brownian motion with random immigration X_t^ϱ, where the immigration rate is governed by the trajectory of another super-Brownian motion ϱ. The speed function is t for $d \geqslant 4$ and $t^{1/2}$ for $d=3$, compared with the existing results, the interesting phenomenon happened in $d=4$ with speed t (although only the upper large deviation bound is derived here) is just because the structure of this new model: the random immigration "smooth" the critical dimension in some sense. The rate function are characterized by an evolution equation.

KEY WORDS: Large deviation; super-Brownian motion; random immigration; evolution equation.

1. INTRODUCTION AND STATEMENT OF RESULTS

Measure-valued branching processes, or superprocesses, have been studied extensively in recent years, for their rich mathematical structures and as the theoretical basis for studies of particle populations appearing in a number of applications. For the general theory, we refer to Dawson,[(2)] Dynkin,[(8)] etc. Immigration structure associated with the superprocesses has been studied by Dynkin,[(8)] Li,[(23, 24)] Li and Shiga,[(25)] etc., where the immigration rate is governed by a determined measure. By randomizing the immigration rate, Hong and Li[(15)] constructed the super-Brownian motion with super-Brownian immigration (SBMSBI, for short), where the immigration rate is governed by the trajectory of another super-Brownian motion, see also Hong.[(12, 13)] The study of such model (SBMSBI) is also motivated by the work of Dawson and Fleishmann,[(3)] where the branching mechanism was

[1] Department of Mathematics, Beijing Normal University, Beijing 100875, People's Republic of China. E-mail: wmhong@bnu.edu.cn

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randomized and got to the catalytic super-Brownian motion. Superprocesses in random medium has received much attention in recent years, see also Evans and Perkins,(10) Mytnik,(27) etc.

Large deviation principles (LDP) for the occupation time of the super-Brownian motion have been considered by many authors, see, e.g., Refs. 6, 19–21, etc. Iscoe and Lee consider for the dimension $d=3,4$ in Ref. 19 and Lee for $d\geqslant 5$ in Ref. 20, where they proved the speed function is $t^{1/2}$ for $d=3$, t for $d\geqslant 5$ and $\log t/t$ for $d=4$.

In the present paper, we will discuss the large deviation principles for SBMSBI *itself*. As pointed out by Hong and Li,(15) the SBMSBI *itself* presents the ergodicity with norming t because of the (random) immigration. We will prove a local large deviation principles for SBMSBI *itself* with the speed function t for $d\geqslant 4$ and $t^{1/2}$ for $d=3$, interesting phenomena is happened at the critical dimension ($d=4$) which reveals that the random immigration "smooth" the critical dimension in the sense that it does not appear the "log" term; Whereas for the super-Brownian motion (without immigration), the LDP were proved for the *occupation time process* not itself; For the occupation time of the SBMSBI, Hong(13) have consider the longtime behavior of it and, recently Hong and Zhao(16) have proved the LDP in higher dimensions. After this paper has submitted, Hong(14) got the moderate deviation for the SBMSBI, which fill in the gap of the central limit behavior and the large deviation.

1.1. The Model SBMSBI

We first recall the concept of SBMSBI briefly. Let $C(\mathbb{R}^d)$ denote the space of continuous bounded functions on $\mathbb{R}^d$. We fix a constant $p>d$ and let $\phi_p(x):=(1+|x|^2)^{-p/2}$ for $x\in\mathbb{R}^d$. Let $C_p(\mathbb{R}^d):=\{f\in C(\mathbb{R}^d):|f(x)|\leqslant \text{const}\cdot\phi_p(x)\}$. In duality, let $M_p(\mathbb{R}^d)$ be the space of Radon measures μ on $\mathbb{R}^d$ such that $\langle\mu,f\rangle:=\int f(x)\,\mu(dx)<\infty$ for all $f\in C_p(\mathbb{R}^d)$. We endow $M_p(\mathbb{R}^d)$ with the p-vague topology, that is, $\mu_k\to\mu$ if and only if $\langle\mu_k,f\rangle\to\langle\mu,f\rangle$ for all $f\in C_p(\mathbb{R}^d)$. Then $M_p(\mathbb{R}^d)$ is metrizable. Throughout this paper, λ denotes the Lebesgue measure on $\mathbb{R}^d$.

Suppose that $(w_t, t\geqslant 0)$ is a standard Brownian motion in $\mathbb{R}^d$ with semigroup $(P_t)_{t\geqslant 0}$. A *super-Brownian motion* $\varrho=(\varrho_t,P_\mu)$ is an $M_p(\mathbb{R}^d)$-valued Markov process with $\varrho_0=\mu$ and the transition probabilities given by

$$\mathbf{E}\exp\{-\langle\varrho_t,f\rangle\}=\exp\{-\langle\mu,v(t,\cdot)\rangle\},\qquad f\in C_p^+(\mathbb{R}^d),\tag{1.1}$$

where $v(\cdot,\cdot)$ is the unique mild solution of the evolution equation

$$\begin{cases}\dot v(t)=\Delta v(t)-v^2(t)\\ v(0)=f.\end{cases}\tag{1.2}$$

Let $\{g(t,\cdot): t \geqslant 0\}$ be a continuous $C_p^+(\mathbb{R}^d)$-valued path such that for each $a > 0$ there is a constant $C_a > 0$ such that $g(t) \leqslant C_a \phi_p$ for all $t \in [0, a]$. The weighted occupation time of the super Brownian motion may be determined by

$$\mathbf{E} \exp\left(-\int_0^t \langle \varrho_s, g(s)\rangle\, ds\right) = \exp\{-\langle \mu, u(0, t, \cdot)\rangle\}, \tag{1.3}$$

where $u(0,\cdot,\cdot)$ is the unique mild solution of

$$\begin{cases} \dot{u}(s) = \Delta u(s) - u^2(s) + g(t-s), & 0 \leqslant s \leqslant t \\ u(0) = 0. \end{cases} \tag{1.4}$$

See, e.g., Iscoe.[17]

Suppose that $\{\gamma_t, t \geqslant 0\}$ is an $M_p(\mathbb{R}^d)$-valued continuous path. A *super-Brownian motion with immigration* determined by $\{\gamma_t, t \geqslant 0\}$ is an $M_p(\mathbb{R}^d)$-valued Markov process $X^\gamma = (X_t^\gamma, Q_\mu^\gamma)$ with transition probabilities given by

$$\mathbf{E} \exp(-\langle X_t^\gamma, f\rangle) = \exp\left\{-\langle \mu, v(t,\cdot)\rangle - \int_0^t \langle \gamma_s, v(t-s,\cdot)\rangle\, ds\right\},$$
$$f \in C_p^+(\mathbb{R}^d), \tag{1.5}$$

where $v(\cdot,\cdot)$ is given by (1.2); see, e.g., Dawson[2] and Li and Wang.[26]

Based on (1.3) and (1.5) it is not difficult to construct a probability space $(\Omega, \mathscr{F}, \mathbf{Q})$ on which the processes $\{\varrho_t: t \geqslant 0\}$ and $\{X_t^\varrho: t \geqslant 0\}$ are defined, where $\{\varrho_t: t \geqslant 0\}$ is a super Brownian motion with $\varrho_0 = \lambda$ and, given $\{\varrho_t: t \geqslant 0\}$, the process $\{X_t^\varrho: t \geqslant 0\}$ is a super Brownian motion with immigration determined by $\{\varrho_t: t \geqslant 0\}$ with $X_0^\varrho = \lambda$. By (1.3) and (1.5) we have

$$\begin{aligned} \mathbf{E} \exp\{-\langle X_t^\varrho, f\rangle\} &= \mathbf{E}[\mathbf{E} \exp\{-\langle X_t^\varrho, f\rangle\} \mid \{\sigma(\varrho_s, s \leqslant t)\}] \\ &= \mathbf{E} \exp\left\{-\langle \lambda, v(t,\cdot)\rangle - \int_o^t \langle \varrho_s, v(t-s,\cdot)\rangle\, ds\right\} \\ &= \exp\{-\langle \lambda, v(t,\cdot)\rangle - \langle \lambda, u(t,\cdot)\rangle\} \end{aligned} \tag{1.6}$$

where $u(\cdot,\cdot)$ is the unique mild solution of the equation

$$\begin{cases} \dot{u}(s) = \Delta u(s) - u^2(s) + v(s), & 0 \leqslant s \leqslant t \\ u(0) = 0 \end{cases} \tag{1.7}$$

and $v(\cdot,\cdot)$ is the mild solution of Eq. (1.2).

The process $\{X_t^\varrho: t \geqslant 0, \mathbf{Q}\}$ is what we call *super-Brownian motion with super-Brownian immigration* (SBMSBI), for details, see Hong and Li[15] and Hong,[13] and it may be considered as one kind of multitype superprocesses, see also Dawson, Gorostiza, and Li,[4] Gorostiza and Lopez-Mimbela,[11] and Li.[22]

1.2. Statement of the Main Results

In this paragraph, we fix $f \in C_p^+(\mathbb{R}^d)$ satisfying $\langle \lambda, f \rangle = 1$ and let

$$\mathbf{W}(t) := \frac{1}{t} \langle X_t^\varrho, f \rangle,$$

and

$$\Lambda_d(t, \theta) := c_d^{-1}(t) \log \mathbf{E} \exp[\theta c_d(t) \mathbf{W}(t)], \tag{1.8}$$

where the speed function is defined by

$$c_d(t) = \begin{cases} t^{1/2}, & d = 3 \\ t, & d \geqslant 4. \end{cases}$$

The following estimation is useful in our proof, for any $f \in C_p^+(\mathbb{R}^d)$,

$$P_t f \leqslant c(1 \wedge t^{-d/2}). \tag{1.9}$$

where $c = \max\{(2\pi)^{-d/2}, \|f\|\}$ is a positive constant, and then $a := \int_0^\infty c(1 \wedge r^{-d/2})\, dr < \infty$ when $d \geqslant 3$.

To obtain the LDP, based on the Gärtner–Ellis Theorem,[5] the key step is to prove the existence of the limit function of $\Lambda_d(t, \theta)$ as $t \to \infty$ and some properties of the limit function.

For this purpose, it is proved below that for $d \geqslant 4$ the following equations

$$\begin{cases} \dfrac{\partial v(t, x; \theta)}{\partial t} = \Delta v(t, x; \theta) + v^2(t, x; \theta) \\[2ex] v(0, x; \theta) = \theta f \end{cases} \tag{1.10}$$

and

$$\begin{cases} \dfrac{\partial u(t, x; \theta)}{\partial t} = \Delta u(t, x; \theta) + u^2(t, x; \theta) + v(t, x; \theta) \\[2ex] u(0, x; \theta) = 0 \end{cases} \tag{1.11}$$

admit unique mild solutions $v(t,x;\theta)$ and $u(t,x;\theta)$ respectively when $|\theta|<\frac{1}{4a}$. Furthermore, for $d\geqslant 5$, there is $\delta>0$ such that

$$\Lambda(\theta):=\lim_{t\to\infty}\Lambda_d(t,\theta)=\theta+\int_0^\infty \langle\lambda,[v(s,\cdot;\theta)]^2\rangle\,ds, \tag{1.12}$$

exists and is strictly convex, continuously differentiable in $|\theta|<\delta<\frac{1}{4a}$ with $\Lambda'(0)=1$.

Let $I(\alpha)$ be the Legendre transform of $\Lambda(\theta)$, i.e.,

$$I(\alpha):=\sup_{|\theta|<\delta}[\alpha\theta-\Lambda(\theta)]. \tag{1.13}$$

Then we prove a local LDP for $d\geqslant 5$:

Theorem 1.1. For $d\geqslant 5$, the law of $\mathbf{W}_t$ under $\mathbf{Q}$ admit the LDP with speed function t and rate function $I(\alpha)$, i.e., there exists a neighborhood O of 1 such that if $U\subset O$ is open and $C\subset O$ is closed, then

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbf{Q}\{\mathbf{W}(t)\in U\}\geqslant-\inf_{\alpha\in U}I(\alpha),$$

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbf{Q}\{\mathbf{W}(t)\in C\}\leqslant-\inf_{\alpha\in C}I(\alpha).$$

For $d=4$, we have

$$\limsup_{t\to\infty}\Lambda_4(t,\theta)\leqslant\theta+\int_0^\infty\langle\lambda,[v(s,\cdot;\theta)]^2\rangle\,ds+c\beta(\theta)^2:=\Lambda_4(\theta), \tag{1.14}$$

where $\beta(\theta)$ is given below in Lemma 2.4, and $\Lambda_4(\theta)$ is finite, strictly convex, continuously differentiable in $|\theta|<\frac{1}{4a}$. Let $I_4(\alpha)$ be the Legendre transform of $\Lambda(\theta)$, we obtain an upper large deviation bound for $d=4$,

Theorem 1.2. For $d=4$, the law of $\mathbf{W}_t$ under $\mathbf{Q}$ admit the upper large deviation bound with speed function t and rate function $I_4(\alpha)$, i.e., for any closed C,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbf{Q}\{\mathbf{W}(t)\in C\}\leqslant-\inf_{\alpha\in C}I_4(\alpha).$$

Remark 1.1. At this moment, we only obtain the upper large deviation bound for $d=4$ because we can not get the exact limit of (1.14), but it is enough to ensure the speed function is right t. It is an interesting question to look for the lower bound for $d=4$.

For $d=3$, we will prove in Lemma 3.8 that the equation

$$\begin{cases} \dfrac{\partial \bar{u}(t)}{\partial t} = \Delta \bar{u}(t) + \bar{u}^2(t) + \theta p(t) & 0 \leqslant t \leqslant 1 \\ u(0) = 0 \end{cases} \tag{1.15}$$

admit unique mild solutions $\bar{u}(t, \cdot; \theta) \in C([0,1], L^2(R^3))$ for $|\theta| < \frac{3}{16c_3}$, where $c_3 = (2\pi)^{-3/2}$, $p(t) = p(t,x)$ is the transition density function of the Brownian motion. Moreover we will prove that there is $\delta_3 > 0$ such that

$$\Lambda_3(\theta) := \lim_{t \to \infty} \Lambda_d(t, \theta) = \langle \lambda, \bar{u}(1, \cdot; \theta) \rangle,$$

which is continuous differential and strictly convex in $|\theta| < \delta_3 < \frac{3}{16c_3}$ with $\Lambda_3'(0) = 1$. Let $I_3(\alpha)$ be the Legendre transform of $\Lambda_3(\theta)$, i.e.,

$$I_3(\alpha) := \sup_{|\theta| < \delta_3} [\alpha\theta - \Lambda_3(\theta)]. \tag{1.16}$$

Then we have

Theorem 1.3. For $d=3$, the law of $\mathbf{W}_t$ under $\mathbf{Q}$ admit the LDP with speed function $t^{1/2}$ and rate function $I_3(\alpha)$, i.e., there exists a neighborhood O of 1 such that if $U \subset O$ is open and $C \subset O$ is closed, then

$$\liminf_{t \to \infty} t^{-1/2} \log \mathbf{Q}\{\mathbf{W}(t) \in U\} \geqslant -\inf_{\alpha \in U} I_3(\alpha),$$

$$\limsup_{t \to \infty} t^{-1/2} \log \mathbf{Q}\{\mathbf{W}(t) \in C\} \leqslant -\inf_{\alpha \in C} I_3(\alpha).$$

We will prove Theorem 1.1 and 1.2 in Section 2, Theorem 1.3 in Section 3. Different from Lee[20] and Iscoe and Lee,[19] where they use the partial differential equation method to get the result, our technique is based on Dynkin's moment formula and the structure of this model to prove the existence of the solutions of the correspondence equations and to get some useful estimations for the solutions, which play a key role in the proofs. For $d=3$, to prove the L^2-convergence of the evolution equation, with some estimations in hand, the technique is adapted from Iscoe.[18]

Remark 1.2. (a) It should be pointed out that we obtained the LDP for the SBMSBI *itself*, whereas for the super-Brownian motion (without immigration), large deviations have been proved for the *occupation time* of it (see, e.g., Refs. 6, 19–21, etc.).

(b) The method of moment is effective in our proofs, but we can only obtain *local* large deviation principles because the convergence of the logarithmic generating function is only established for small parameters and we can verified the LDP only in a small neighborhood, which is the limitation of this method. In order to prove a full LDP, one would have to prove the convergence up to a critical parameter and then the steepness. This is rather challenging but would provide more insight, and we leave it as an open problem.

2. PROOFS OF THEOREM 1.1 AND 1.2

Firstly, for any functions $g(t,\cdot), h(t,\cdot) \in C_p(R^d)$, $\forall t \geqslant 0$, $p > 1$, we define the convolution

$$g(t,x) * h(t,x) := \int_0^t P_s[g(t-s,\cdot)\cdot h(t-s,\cdot)](x)\, ds. \tag{2.1}$$

Let

$$\begin{cases} g^{*1}(t,x) := g(t,x) \\ g(t,x)^{*n} := \sum_{k=1}^{n-1} g(t,x)^{*k} * g(t,x)^{*(n-k)}, \end{cases} \tag{2.2}$$

and $\{B_n, n \geqslant 1\}$ is a sequence of positive numbers determined by

$$\begin{cases} B_1 = B_2 = 1 \\ B_n = \sum_{k=1}^{n-1} B_k B_{n-k}, \end{cases} \tag{2.3}$$

see Dynkin[7] and Wang.[28] Recall (1.9) for the positive constant c.

Lemma 2.1. Let $d \geqslant 3$ and $F(t,x) = P_t f(x)$, then

$$F(t,x)^{*n} \leqslant B_n a^{n-1} \cdot P_t f(x) \tag{2.4}$$

where $a := \int_0^\infty c(1 \wedge t^{-d/2})\, dr < \infty$ when $d \geqslant 3$.

Proof. We will prove (2.4) by induction in n. It is trivial for $n = 1$. When $n = 2$, from the definition and (1.9), we have

$$F(t,x)^{*2} = \int_0^t P_s[P_{t-s}f]^2(x)\,ds$$
$$\leqslant P_t f(x) \int_0^t c(1 \wedge (t-s)^{-d/2})\,ds$$
$$= a \cdot P_t f(x),$$

as desired. If (2.4) is true for all $k < n$, by (2.2) and (2.3) we get

$$F(t,x)^{*n} \leqslant \sum_1^{n-1} B_k a^{k-1} \cdot P_t f(x) * B_{n-k} a^{n-k-1} \cdot P_t f(x)$$
$$= B_n a^{n-2} \cdot P_t f(x) * P_t f(x)$$
$$\leqslant B_n a^{n-1} \cdot P_t f(x),$$

and then the proof is complete by induction. □

Lemma 2.2. Let $d \geqslant 3$, $|\theta| < \frac{1}{4a}$, then Eq. (1.10) admits an unique mild solution $v(t, x; \theta)$, moreover it is analytic in $|\theta| < \frac{1}{4a}$ and

$$|v(t,x;\theta)| \leqslant b(\theta) \cdot P_t f(x), \tag{2.5}$$

where $b(\theta) = (2a)^{-1}\,[1-(1-4a\,|\theta|)^{1/2}]$.

Proof. The mild form of Eq. (1.10) is

$$v(t,x;\theta) = \theta P_t f(x) + \int_0^t P_s[v(t-s,\cdot;\theta)]^2(x)\,ds, \tag{2.6}$$

i.e.,

$$v(t,x;\theta) = \theta F(t,x) + v(t,x;\theta) * v(t,x;\theta). \tag{2.7}$$

Then

$$v(t,x;\theta) = \sum_{n=1}^{\infty} F(t,x)^{*n}\,\theta^n \tag{2.8}$$

by Dynkin[7] (see also Wang[28]) while we prove the convergence of the series on the right hand, where $F(t, x)$ is given in Lemma 2.1. By Lemma 2.1, the series is dominated by

$$|v(t,x;\theta)| \leqslant \sum_{n=1}^{\infty} B_n a^{n-1}\,|\theta|^n \cdot P_t f(x). \tag{2.9}$$

On the other hand, we know (see Dawson,[1] also Dynkin[7] and Wang[28]) that the function $g(z)=\frac{1}{2}[1-(1-4z)^{1/2}]$ can be expanded as a power series

$$g(z)=\tfrac{1}{2}[1-(1-4z)^{1/2}]=\sum_{n=1}^{\infty} B_n z^n,$$

when $|z|<1/4$, where B_n is given in (2.3). So the series (2.8) is absolutely convergence for $|\theta|<\frac{1}{4a}$, and from (2.9) we get

$$|v(t,x;\theta)|\leqslant (2a)^{-1}[1-(1-4a|\theta|)^{1/2}]\cdot P_t f(x),$$

as desired. □

The following two Lemmas can be proved by the same method, and they reflects the special structure properties of our model SBMSBI, but note that they are invalid for $d=3$.

Lemma 2.3. Let $d\geqslant 4$, $|\theta|<\frac{1}{4a}$, $v(t,x;\theta)$ be the mild solution of Eq. (1.10), and

$$G(t,x;\theta)=\int_0^t P_s v(t-s,\cdot;\theta)(x)\,ds,$$

then

$$G(t,x;\theta)^{*n}\leqslant B_n c^{n-1} b(\theta)^n\cdot tP_t f(x) \tag{2.10}$$

where c is given in (1.9) and $b(\theta)$ in Lemma 2.2.

Proof. By Lemma 2.2, it is trivial for $n=1$. For $n=2$,

$$\begin{aligned}
G(t,x;\theta)^{*2}&=\int_0^t P_s\left[\int_0^{t-s} P_r v(t-s-r,\cdot;\theta)\,dr\right]^2(x)\,ds\\
&\leqslant b(\theta)^2\cdot\int_0^t P_s\left[\int_0^{t-s} P_r(P_{t-s-r}f)\,dr\right]^2(x)\,ds\\
&=b(\theta)^2\cdot\int_0^t (t-s)^2\,P_s(P_{t-s}f)^2(x)\,ds\\
&\leqslant b(\theta)^2\,c\cdot\int_0^t (t-s)^2\,[1\wedge(t-s)^{-d/2}]\,ds\cdot P_t f(x)\\
&\leqslant b(\theta)^2\,c\cdot tP_t f(x),
\end{aligned}$$

we used (1.9) in the fourth step and note that $\int_0^t (t-s)^2 [1 \wedge (t-s)^{-d/2}]\, ds \leqslant t$ when $d \geqslant 4$. If (2.10) is true for all $k < n$, we get

$$G(t, x; \theta)^{*n} \leqslant \sum_{k=1}^{n-1} B_k c^{k-1} b(\theta)^k \cdot [tP_t f] * B_{n-k} c^{n-k-1} b(\theta)^{n-k} \cdot [tP_t f](x)$$

$$= B_n c^{n-2} b(\theta)^n \cdot \int_o^t P_s[(t-s)\, P_{t-s} f]^2\, (x)\, ds$$

$$\leqslant B_n c^{n-1} b(\theta)^n \cdot tP_t f(x)$$

as desired by induction. □

Lemma 2.4. Let $d \geqslant 4$, $|\theta| < \frac{1}{4a}$, $v(t, x; \theta)$ be the mild solution of Eq. (1.10), then Eq. (1.11) admits an unique mild solution $u(t, x; \theta)$, moreover it is analytic in $|\theta| < \frac{1}{4a}$ and

$$|u(t, x; \theta)| \leqslant \beta(\theta) \cdot tP_t f(x), \tag{2.11}$$

where $\beta(\theta) = (2c)^{-1} [1-(1-4b(\theta)\, c)^{1/2}]$.

Proof. The mild form of Eq. (1.11) is

$$u(t, x; \theta) = \int_0^t P_s v(t-s, \cdot; \theta)(x) + \int_0^t P_s[u(t-s, \cdot; \theta)]^2\, (x)\, ds, \tag{2.12}$$

i.e.,

$$u(t, x; \theta) = G(t, x; \theta) + u(t, x; \theta) * u(t, x; \theta). \tag{2.13}$$

Then

$$u(t, x; \theta) = \sum_{n=1}^{\infty} G(t, x; \theta)^{*n} \tag{2.14}$$

while we prove the convergence of the series on the right hand, where $G(t, x; \theta)$ is given in Lemma 2.3. By Lemma 2.3, the series is dominated by

$$|u(t, x; \theta)| \leqslant \sum_{n=1}^{\infty} B_n c^{n-1} b(\theta)^n \cdot tP_t f(x). \tag{2.15}$$

It is easy to check that $|4b(\theta)\, c| < 1$ whenever $|\theta| < \frac{1}{4a}$, then the series in (2.14) is uniform absolute convergence by the same method in Lemma 2.2, and

$$|u(t, x; \theta)| \leqslant (2c)^{-1} [1-(1-4b(\theta)\, c)^{1/2}] \cdot tP_t f(x), \tag{2.16}$$

which completes the proof. □

Lemma 2.5. Let $d \geqslant 4$, X_t^ϱ be the SBMSBI, then for $|\theta| < \frac{1}{4a}$, we have

$$\mathbf{E} \exp\{\langle X_t^\varrho, \theta f\rangle\} = \exp\{\langle \lambda, v(t, \cdot\,; \theta)\rangle + \langle \lambda, u(t, \cdot\,; \theta)\rangle\} \tag{2.17}$$

where $v(t, x; \theta)$ and $u(t, x; \theta)$ are the mild solutions of Eqs. (1.10) and (1.11) respectively.

Proof. From the introduction we know that the Laplace transition functional of the SBMSBI is given by (1.6), (1.2) (with f being replaced by $-\theta f$) and (1.7) for $\theta \leqslant 0$, i.e., (in which $-\theta \leftrightarrow \theta$, $-v \leftrightarrow v$, $-u \leftrightarrow u$).

$$\mathbf{E} \exp\{\langle X_t^\varrho, \theta f\rangle\} = \exp\{\langle \lambda, v(t, \cdot\,; \theta)\rangle + \langle \lambda, u(t, \cdot\,; \theta)\rangle\}, \tag{2.18}$$

Where $v(t, x; \theta)$ and $u(t, x; \theta)$ are the mild solutions of the following equations respectively,

$$\begin{cases} \dfrac{\partial v(t)}{\partial t} = \Delta v(t) + v^2(t) \\ v(0) = \theta f \end{cases} \tag{2.19}$$

and

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} = \Delta u(t) + u^2(t) + v(t) \\ u(0) = 0. \end{cases}$$

So (2.17) is true when $\theta \leqslant 0$. Note that $v(t, x; \theta)$ and $u(t, x; \theta)$ is analytic in θ when $|\theta| < \frac{1}{4a}$ by Lemma 2.2 and Lemma 2.4, then (2.17) also holds for $0 < \theta < \frac{1}{4a}$ by properties of Laplace transform of probability measure on $[0, \infty)$ (cf. Ref. 29). □

Lemma 2.6. Let $d \geqslant 5$, $|\theta| < \frac{1}{4a}$,

$$\Lambda(\theta) := \lim_{t \to \infty} t^{-1} \log \mathbf{E} \exp[\theta t \mathbf{W}(t)], \tag{2.20}$$

then

$$\Lambda(\theta) = \theta + \int_0^\infty \langle \lambda, [v(s, \cdot\,; \theta)]^2 \rangle \, ds. \tag{2.21}$$

Proof. Recall (1.8), then by Lemma 2.5 we have

$$\Lambda(\theta) = \lim_{t \to \infty} t^{-1} [\langle \lambda, v(t, \cdot\,; \theta) \rangle + \langle \lambda, u(t, \cdot\,; \theta) \rangle], \tag{2.22}$$

where $v(t, x; \theta)$ and $u(t, x; \theta)$ are the mild solutions of Eqs. (1.10) and (1.11) respectively, i.e.,

$$v(t, x; \theta) = \theta P_t f(x) + \int_0^t P_s [v(t-s, \cdot\,; \theta)]^2 (x) \, ds, \tag{2.23}$$

$$u(t, x; \theta) = \int_0^t P_s v(t-s, \cdot\,; \theta)(x) \, ds + \int_0^t P_s [v(t-s, \cdot\,; \theta)]^2 (x) \, ds. \tag{2.24}$$

Then (recall $\langle \lambda, f \rangle = 1$)

$$\langle \lambda, v(t, \cdot\,; \theta) \rangle = \theta + \int_0^t \langle \lambda, [v(s, \cdot\,; \theta)]^2 \rangle \, ds, \tag{2.25}$$

$$\langle \lambda, u(t, \cdot\,; \theta) \rangle = \int_0^t \langle \lambda, v(s, \cdot\,; \theta) \rangle \, ds + \int_0^t \langle \lambda, [u(s, \cdot\,; \theta)]^2 \rangle \, ds. \tag{2.26}$$

By (2.5) and (2.11), it is easy to check that as $t \to \infty$

$$\int_0^\infty \langle \lambda, [v(s, \cdot\,; \theta)]^2 \rangle \, ds \leqslant b(\theta)^2 \int_0^\infty (1 \wedge s^{-d/2}) \, ds < \infty,$$

$$t^{-1} \int_0^t \langle \lambda, [u(s, \cdot\,; \theta)]^2 \rangle \, ds \leqslant \beta(\theta)^2 \, t^{-1} \int_0^t \langle \lambda, [s P_s f]^2 \rangle \, ds$$

$$\leqslant c \beta(\theta)^2 \, t^{-1} \int_0^t s^2 [1 \wedge (s^{-d/2})] \, ds \to 0,$$

$$t^{-1} \langle \lambda, v(t, \cdot\,; \theta) \rangle \leqslant t^{-1} \langle \lambda, b(\theta) \, P_t f \rangle \to 0.$$

Combing all the above with (2.22), by l'Hospital's rule we get

$$\begin{aligned}\Lambda(\theta) &= \lim_{t\to\infty} t^{-1}[\langle \lambda, v(t,\cdot;\theta)\rangle + \langle \lambda, u(t,\cdot;\theta)\rangle]\\ &= \lim_{t\to\infty} t^{-1}[\langle \lambda, u(t,\cdot;\theta)\rangle]\\ &= \lim_{t\to\infty} t^{-1}\left[\int_0^t \langle \lambda, v(s,\cdot;\theta)\rangle\, ds + \int_0^t \langle \lambda, [u(s,\cdot;\theta)]^2\rangle\, ds\right]\\ &= \lim_{t\to\infty} \langle \lambda, v(t,x;\theta)\rangle = \theta + \int_0^\infty \langle \lambda, [v(s,\cdot;\theta)]^2\rangle\, ds,\end{aligned}$$

completes the proof. □

Lemma 2.7. Let $d \geqslant 5$ and $\Lambda(\theta)$ as in Lemma 2.6, then there is $\delta > 0$ such that $\Lambda(\theta)$ is strictly convex, continuous differentiable in $|\theta| < \delta < \frac{1}{4a}$ with $\Lambda'(0) = 1$.

Proof. From Lemma 2.2, $v(t, x; \theta)$ is analytic in θ, write $v'(t, x; \theta) := \frac{\partial v(t,x;\theta)}{\partial \theta}$, by (2.8) it is easy to check that for $|\theta| < \frac{1}{4a}$

$$|v'(t, x; \theta)| \leqslant \bar{b}(\theta)\, P_t f,$$

where $\bar{b}(\theta) = (1 - 4a\theta)^{-1/2}$. Then

$$\int_0^\infty \langle \lambda, |v(s,\cdot;\theta)|\, |v'(s,\cdot;\theta)|\rangle\, ds \leqslant cb(\theta)\, \bar{b}(\theta) \int_0^\infty (1 \wedge s^{-d/2})\, ds < \infty.$$

So by (2.21) we get

$$\Lambda'(\theta) = 1 + \int_0^\infty \langle \lambda, v(s,\cdot;\theta)\cdot v'(s,\cdot;\theta)\rangle\, ds,$$

for $|\theta| < \frac{1}{4a}$ and then $\Lambda'(0) = 1$.

Similarly,

$$\Lambda''(\theta) = \int_0^\infty \langle \lambda, v'(s,\cdot;\theta)^2 + v(s,\cdot;\theta)\cdot v''(s,\cdot;\theta)\rangle\, ds,$$

we get

$$\Lambda''(0) = \int_0^\infty \langle \lambda, v'(s,\cdot;0)^2\rangle\, ds = \int_0^\infty \int f P_{2s} f\, dx\, ds > 0,$$

and then there is $\delta > 0$ such that $\Lambda(\theta)$ is strictly convex in $|\theta| < \delta < \frac{1}{4a}$. □

Lemma 2.8. Let $d=4$, $|\theta|<\frac{1}{4a}$, then

$$\limsup_{t\to\infty} t^{-1}\log \mathbf{E}\exp[\theta t\mathbf{W}(t)]$$

$$\leqslant \theta+\int_0^\infty \langle\lambda, [v(s,\cdot;\theta)]^2\rangle\, ds + c\beta(\theta)^2 := \Lambda_4(\theta), \tag{2.27}$$

and $\Lambda_4(\theta)$ is finite, convex and differentiable in $|\theta|<\frac{1}{4a}$, where $\beta(\theta)$ is that in Lemma 2.4.

Proof. All the calculations carried out in Lemma 2.6 is valid here for $d=4$ except that

$$\begin{aligned} t^{-1}\int_0^t \langle\lambda, [u(s,\cdot;\theta)]^2\rangle\, ds &\leqslant \beta(\theta)^2\, t^{-1}\int_0^t \langle\lambda, [sP_s f]^2\rangle\, ds \\ &\leqslant c\beta(\theta)^2\, t^{-1}\int_0^t s^2[1\wedge(s^{-2})]\, ds \\ &\leqslant c\beta(\theta)^2<\infty. \end{aligned}$$

Then

$$\bar{u}(\theta) := \limsup_{t\to\infty} t^{-1}\int_0^t \langle\lambda, [u(s,\cdot;\theta)]^2\rangle\, ds, \tag{2.28}$$

is finite for $|\theta|<\frac{1}{4a}$. We get

$$\begin{aligned} &\limsup_{t\to\infty} t^{-1}\log \mathbf{E}\exp[\theta t\mathbf{W}(t)] \\ &\quad = \theta+\int_0^\infty \langle\lambda, [v(s,\cdot;\theta)]^2\rangle\, ds + \bar{u}(\theta) \\ &\quad \leqslant \theta+\int_0^\infty \langle\lambda, [v(s,\cdot;\theta)]^2\rangle\, ds + c\beta(\theta)^2 := \Lambda_4(\theta), \end{aligned}$$

and the properties of $\Lambda_4(\theta)$ can be proved similar to Lemma 2.7. □

Proof of Theorem 1.1. Based on Lemma 2.7, Theorem 1.1 followed from the general large deviation result Gärtner–Ellis Theorem [cf. Dembo and Zeitouni[5] or Ellis[9]]. The neighborhood O is that of $\{\Lambda'(\theta): |\theta|<\delta\}$. □

Proof of Theorem 1.2. The upper bound for the compact set is followed by Theorem 4.5.3 of Ref. 5. For the closed set, we need to establish exponential tightness of the distribution of $\mathbf{W}_t$, which can be proved by the fact that $\Lambda_4(\theta)$ is finite for $|\theta| < \frac{1}{4a}$ (see the proof of the Gärtner–Ellis Theorem[5]). □

3. PROOF OF THEOREM 1.3.

In this section, we will consider the large deviation for the SBMSBI in $d = 3$. Recall the Laplace transition functional of X_t^{ϱ}, (in which $-\theta \leftrightarrow \theta$, $-v \leftrightarrow v$, $-u \leftrightarrow u$, $\theta \leqslant 0$).

$$\mathbf{E}\exp\{\langle X_t^{\varrho}, \theta t^{-\frac{1}{2}} f\rangle\} = \exp\{\langle \lambda, v(t,\cdot;\theta)\rangle + \langle \lambda, u(t,\cdot;\theta)\rangle\}, \tag{3.1}$$

where $v(t, x; \theta)$ and $u(t, x; \theta)$ are the mild solutions of the following equations respectively,

$$\begin{cases} \dfrac{\partial v(s)}{\partial s} = \Delta v(s) + v^2(s), & 0 \leqslant s \leqslant t \\ v(0) = \theta t^{-\frac{1}{2}} f \end{cases} \tag{3.2}$$

and

$$\begin{cases} \dfrac{\partial u(s)}{\partial s} = \Delta u(s) + u^2(s) + v(s), & 0 \leqslant s \leqslant t \\ u(0) = 0. \end{cases} \tag{3.3}$$

Let $\bar{u}(s, x; t, \theta) := tu(ts, t^{\frac{1}{2}}x; \theta)$, then by (3.3) $\bar{u}(s, x; t, \theta)$ satisfy the following equation,

$$\begin{cases} \dfrac{\partial \bar{u}(s, x; t, \theta)}{\partial s} = \Delta \bar{u}(s, x; t, \theta) + \bar{u}^2(s, x; t, \theta) + t^2 v(ts, t^{\frac{1}{2}}x; \theta), & 0 \leqslant s \leqslant 1 \\ \bar{u}(0, x; \theta) = 0, \end{cases} \tag{3.4}$$

the mild form of (3.4) is,

$$\bar{u}(s, x; t, \theta) = t^2 \int_0^s P_{s-r} v(tr, t^{\frac{1}{2}}x; \theta)\, dr + \int_0^s P_{s-r} \bar{u}^2(r, x; t, \theta)\, dr, \qquad 0 \leqslant s \leqslant 1 \tag{3.5}$$

where $v(s, x; \theta)$ is the mild solution of (3.2), i.e.,

$$v(s, x; \theta) = \theta t^{-\frac{1}{2}} P_s f + \int_0^s P_{s-r} v^2(r, x; \theta)\, dr, \qquad 0 \leqslant s \leqslant t. \tag{3.6}$$

The key step in this section is to prove the mild solution of (3.4) converge to that of

$$\begin{cases} \dfrac{\partial \bar{u}(s, x; \theta)}{\partial s} = \Delta \bar{u}(s, x; \theta) + \bar{u}^2(s, x; \theta) + \theta p(s, x) & 0 \leqslant s \leqslant 1 \\ \bar{u}(0, x; \theta) = 0, \end{cases} \tag{3.7}$$

as $t \to \infty$, where $p(t, x)$ is the transition density function of Brownian motion.

The existence of the solutions of Eqs. (3.2), (3.4), and (3.7) is well known when $\theta \leqslant 0$. Here we need the existence of the solutions in $|\theta| < \delta$ for some $\delta > 0$, and we will also use the method as in Section 2 to get the result in the following three lemmas, which we will omit the details.

Lemma 3.1. Let $d = 3$, $|\theta| < \frac{t^{1/2}}{4a}$, $F_3(s, x) = t^{-1/2} P_s f(x)$, then Eq. (3.6) admits an unique mild solution $v(s, x; \theta)$,

$$v(s, x; \theta) = \sum_{n=1}^{\infty} F_3(s, x)^{*n}\, \theta^n, \tag{3.8}$$

moreover it is analytic in $|\theta| < \frac{t^{1/2}}{4a}$ and

$$|v(s, x; \theta)| \leqslant b_3(\theta, t) \cdot P_s f(x), \tag{3.9}$$

where $b_3(\theta, t) = (2a)^{-1} [1 - (1 - 4a\, |\theta|\, t^{-1/2})^{1/2}]$.

For the Eq. (3.5), we have,

Lemma 3.2. Let $d = 3$, $|\theta| < \frac{3}{16a}$, $G_3(s, x; \theta) = t^2 \int_0^s P_r [v(t(s-r), t^{1/2} \cdot ; \theta)] \times (x)\, ds$, $0 \leqslant s \leqslant 1$, $t > 1$, then

$$G(s, x; \theta)^{*n} \leqslant B_n c_3^{n-1} [t^{1/2} b_3(t, \theta)]^n \cdot s \int p(s, x, t^{-1/2} z)\, f(z)\, dz, \tag{3.10}$$

where $c_3 = (2\pi)^{-3/2}$, and Eq. (3.5) admits an unique mild solution $\bar{u}(s, x; t, \theta)$,

$$\bar{u}(s, x; t, \theta) = \sum_{n=1}^{\infty} G_3(s, x; \theta)^{*n} \tag{3.11}$$

moreover it is analytic in $|\theta| < \frac{3}{16a}$, and

$$|\bar{u}(s, x; t, \theta)| \leqslant \beta_3(\theta, t) \cdot s \int p(s, x, t^{-1/2}z)\, f(z)\, dz, \tag{3.12}$$

where $\beta_3(\theta, t) = (2c_3)^{-1} [1-(1-4c_3 t^{1/2} b_3(t, \theta))^{1/2}]$, and a as in Lemma 2.1.

Proof. From Lemma (3.1),

$$|t^2 v(ts, t^{1/2}x; \theta)| \leqslant t^2 b_3(t, \theta) \cdot P_{ts} f(t^{1/2}x),$$

and then

$$\begin{aligned} G_3(s, x; \theta) &= t^2 \int_0^s P_r[v(t(s-r), t^{1/2}\cdot; \theta)](x)\, ds \\ &\leqslant t^2 b_3(t, \theta) \int_0^s \int \int p(r, x, y)\, p(t(s-r), t^{1/2}y, z)\, f(z)\, dz\, dy\, dr \\ &\leqslant t^{1/2} b_3(t, \theta) \cdot s \int p(s, x, t^{-1/2}z)\, f(z)\, dz. \end{aligned}$$

If (3.10) is true for $k < n$, by (2.2), we get,

$$\begin{aligned} G_3(s, x; \theta)^{*n} &\leqslant \sum_{k=1}^{n-1} B_k c_3^{k-1} [t^{1/2} b_3(t, \theta)]^k \cdot s \int p(s, x, t^{-1/2}z)\, f(z)\, dz \\ &\quad * B_{n-k} c_3^{n-k-1} [t^{1/2} b_3(t, \theta)]^{n-k} \cdot s \int p(s, x, t^{-1/2}z)\, f(z)\, dz \\ &= B_n c_3^{n-2} [t^{1/2} b_3(t, \theta)]^n \\ &\quad \cdot \int_0^s P_r \left[(s-r) \int p(s-r, \cdot, t^{-1/2}z)\, f(z)\, dz \right]^2 (x)\, dr \\ &\leqslant B_n c_3^{n-2} [t^{1/2} b_3(t, \theta)]^n \\ &\quad \cdot \int_0^s (s-r)^2 \int p(r, x, y) \int [p(s-r, y, t^{-1/2}z)]^2\, f(z)\, dz\, dy\, dr \\ &\leqslant B_n c_3^{n-1} [t^{1/2} b_3(t, \theta)]^n \cdot \int_0^s (s-r)^{1/2}\, dr \int p(s, x, t^{-1/2}z)\, f(z)\, dz \\ &\leqslant B_n c_3^{n-1} [t^{1/2} b_3(t, \theta)]^n \cdot s \int p(s, x, t^{-1/2}z)\, f(z)\, dz, \end{aligned}$$

and (3.10) is proved by induction, in the third step we used Jensen's inequality. The remaining proof could carry out as Lemma 2.4, note that $|4c_3 t^{1/2} b_3(t, \theta)| < 1$ whenever $|\theta| < \frac{3}{16a}$. □

Lemma 3.3. Let $d = 3$, $|\theta| < \frac{1}{4c_3}$, then the equation

$$\begin{cases} \dfrac{\partial \bar{u}(s)}{\partial s} = \Delta \bar{u}(s) + \bar{u}^2(s) + \theta p(s) \qquad 0 \leqslant s \leqslant 1 \\ \bar{u}(0) = 0 \end{cases} \tag{3.13}$$

admits an unique mild solution $\bar{u}(s, x; \theta)$, moreover it is nondecreasing and analytic in $|\theta| < \frac{1}{4c_3}$, and

$$|\bar{u}(s, x; \theta)| \leqslant \bar{\beta}_3(\theta) \cdot sp(s, x), \tag{3.14}$$

where $\bar{\beta}_3(\theta) = (2c_3)^{-1} [1 - (1 - 4c_3\theta)^{1/2}]$, c_3 as in Lemma 3.2. Furthermore, we have

$$\bar{u}(s, x; t, \theta) \to \bar{u}(s, x; \theta), \qquad 0 \leqslant s \leqslant 1 \tag{3.15}$$

pointwise and in $L^2(R^3, \lambda)$ for $|\theta| < \frac{3}{16a}$ as $t \to \infty$, where $\bar{u}(s, x; t, \theta)$ is the mild solution of (3.5).

Now we will prove that the mild solution of Eq. (3.4) convergence to that of Eq. (3.7).

Lemma 3.4. Let $d = 3$, then

$$\lim_{t \to \infty} \theta t^{3/2} \int_0^s P_{s-r}[(P_{tr} f)(t^{1/2} \cdot)](x) \, dr = \theta sp(s, x), \tag{3.16}$$

pointwise and in $L^2(R^3, \lambda)$ for $0 \leqslant s \leqslant 1$.

Proof. It is easy to check that, as $t \to \infty$,

$$\begin{aligned} &\theta t^{3/2} \int_0^s P_{s-r}[(P_{tr} f)(t^{1/2} \cdot)](x) \, dr \\ &= \theta t^{3/2} \int_0^s dr \int_{R^3} dy \int_{R^3} p(s-r, x, y) \, p(tr, t^{1/2} y, z) \, f(z) \, dz \\ &= \theta \int_0^s dr \int_{R^3} dy \int_{R^3} p(s-r, x, y) \, p(r, y, t^{-1/2} z) \, f(z) \, dz \\ &= \theta s \int_{R^3} p(s, x, t^{-1/2} z) \, f(z) \, dz \\ &\to \theta sp(s, x), \end{aligned}$$

by Lebesgue's dominated convergence theorem. Recall $\langle \lambda, f \rangle = 1$, we have

$$\left\| \theta t^{3/2} \int_0^s P_{s-r}[(P_{tr} f)(t^{1/2} \cdot)](x)\, dr - \theta s p(s, x) \right\|_{L^2}^2$$

$$= \left\| \theta s \int_{R^3} [p(s, x, t^{-1/2} z) - p(s, x)]\, f(z)\, dz \right\|_{L^2}^2$$

$$= \theta^2 s^2 \int_{R^3} \int_{R^3} [p(2s, t^{-1/2} z, t^{-1/2} z') - p(2s, t^{-1/2} z, 0)$$

$$- p(2s, t^{-1/2} z', 0) + p(2s, 0)]\, f(z)\, f(z')\, dz\, dz',$$

which goes to zero as $t \to \infty$ by Lebesgue's dominated convergence theorem, because the integrand is dominated by $4p(2s, 0)\, f(z)\, f(z')$, which is integrable. □

Lemma 3.5. Let $d = 3$, $|\theta| < \frac{3}{16a}$, $v(s, x; \theta)$ is the mild solution of Eq. (3.6), then

$$\lim_{t \to \infty} t^2 \int_0^s P_{s-r} \left[\int_0^{tr} P_{tr-h}[v^2(h, \cdot; \theta)](t^{1/2} \cdot)\, dh \right](x)\, dr = 0 \tag{3.17}$$

pointwise and in $L^2(R^3, \lambda)$ for $0 \leqslant s \leqslant 1$.

Proof. From (3.9), we know that $|v(s, x; \theta)| \leqslant b_3(\theta, t) \cdot P_s f(x)$, note that $b_3(\theta, t) \sim t^{-1/2}$ as $t \to \infty$ we have,

$$t^2 \int_0^s P_{s-r} \left[\int_0^{tr} P_{tr-h}[v^2(h, \cdot; \theta)](t^{1/2} \cdot)\, dh \right](x)\, dr$$

$$= t^2 \int_0^s dr \int_0^{tr} dh \int_{R^3} \int_{R^3} p(s-r, x, y)\, p(tr-h, t^{1/2} y, z)\, v^2(h, z; \theta)\, dy\, dz$$

$$\leqslant b_3(\theta, t)^2\, t^2 \int_0^s dr \int_0^{tr} dh \int_{R^3} \int_{R^3} p(s-r, x, y)$$

$$\times p(tr-h, t^{1/2} y, z) (P_h f(z))^2\, dy\, dz$$

$$\leqslant b_3(\theta, t)^2\, t^{1/2} \int_0^s dr \int_0^{tr} dh \int_{R^3} \int_{R^3} p(s-r, x, y)\, p(r, y, t^{-1/2} z')\, f(z')$$

$$\cdot c(1 \wedge h^{-3/2})\, dy\, dz'$$

$$\leqslant a b_3(\theta, t)^2\, t^{1/2} s \int_{R^3} p(s, x, t^{-1/2} z')\, f(z')\, dz'$$

$$\to 0,$$

by Lebesgue's dominated convergence theorem, we used (1.9) in the third step, and the L^2-convergence can be proved easily. □

Lemma 3.6. Let $d=3$, $|\theta|<\frac{3}{16a}$, $v(s,x;\theta)$ as in (3.6), then

$$\lim_{t\to\infty} t^2\int_0^s P_{s-r}v(tr, t^{\frac{1}{2}}x;\theta)\,dr = \theta s p(s,x), \tag{3.18}$$

pointwise and in $L^2(R^3,\lambda)$ for $0\leqslant s\leqslant 1$.

Proof. It is enough to note that, from (3.6) we have,

$$t^2\int_0^s P_{s-r}v(tr, t^{\frac{1}{2}}x;\theta)\,dr = \theta t^{3/2}\int_0^s P_{s-r}[(P_{tr}f)(t^{1/2}\cdot)](x)\,dr$$
$$+t^2\int_0^s P_{s-r}\left[\int_0^{tr} P_{tr-h}[v^2(h,\cdot;\theta)](t^{1/2}\cdot)\,dh\right](x)\,dr,$$

then (3.18) followed by Lemmas 3.4 and 3.5. □

Lemma 3.7. Let $d=3$, $|\theta|<\frac{3}{16a}$, $v(s,x;\theta)$ as in (3.6), then

$$\bar{u}(s,x;t,\theta)\to\bar{u}(s,x;\theta), \tag{3.19}$$

pointwise and in $L^2(R^3,\lambda)$ uniformly as $t\to\infty$ for $0\leqslant s\leqslant 1$, where $\bar{u}(s,x;t,\theta)$ and $\bar{u}(s,x;t,\theta)$ are the mild solutions of Eq. (3.4) and that of (3.7) respectively. Moreover,

$$\lim_{t\to\infty}\langle\lambda,\bar{u}(s,\cdot;t,\theta)\rangle = \langle\lambda,\bar{u}(s,\cdot;\theta)\rangle. \tag{3.20}$$

Proof. Note that, from (3.12),

$$|\bar{u}(s,x;t,\theta)|\leqslant\beta_3(\theta,t)\cdot s\int p(s,x,t^{-1/2}z)\,f(z)\,dz.$$

And then for t large enough

$$\|\bar{u}(s,x;t,\theta)\|_{L^2}^2\leqslant\langle\lambda,(\beta_3(\theta,t)\cdot s\int p(s,x,t^{-1/2}z)\,f(z)\,dz)^2\rangle$$
$$\leqslant\beta_3(\theta,t)^2\,s^2p(2s,0)\leqslant Cs^2p(2s,0),$$

because $\beta_3(\theta, t) \to C_1$, a constant, as $t \to \infty$. Based on Lemma 3.6 and this estimation, (3.19) and (3.20) can be proved similar to that of Proposition 3.9 and Corollary 3.12 in Iscoe[18] respectively, we omit the details here. □

Lemma 3.8. Let $d = 3$, $|\theta| < \frac{3}{16a}$,

$$\Lambda_3(\theta) := \lim_{t \to \infty} t^{-1/2} \log \mathbf{E} \exp[\theta t^{1/2} \mathbf{W}(t)], \tag{3.21}$$

then

$$\Lambda_3(\theta) = \langle \lambda, \bar{u}(1, \cdot\,; \theta) \rangle \tag{3.22}$$

where $\bar{u}(s, x; \theta)$ is the mild solution of Eq. (3.13). And there is $\delta_3 > 0$ such that $\Lambda_3(\theta)$ is strictly convex, continuous differentiable in $|\theta| < \delta_3 < \frac{3}{16a}$ with $\Lambda'(0) = 1$.

Proof. Recall the Laplace transition functional of the SBMSBI is given by (in which $-\theta \leftrightarrow \theta$, $-v \leftrightarrow v$, $-u \leftrightarrow u$).

$$\mathbf{E} \exp\{\langle X_t^{\varrho}, \theta t^{-\frac{1}{2}} f \rangle\} = \exp\{\langle \lambda, v(t, \cdot\,; \theta) \rangle + \langle \lambda, u(t, \cdot\,; \theta) \rangle\}, \tag{3.23}$$

for $\theta \leqslant 0$, where $v(t, x; \theta)$ and $u(t, x; \theta)$ are the mild solutions of (3.2) and (3.3) respectively. Note that $\bar{u}(s, x; t, \theta) := t^2 u(ts, t^{1/2} x; \theta)$, where $\bar{u}(s, x; t, \theta)$ is the mild solution of (3.5), one gets,

$$\langle \lambda, u(t, \cdot\,; \theta) \rangle = t^{1/2} \langle \lambda, \bar{u}(1, \cdot\,; t, \theta) \rangle, \tag{3.24}$$

then from (3.23) we have,

$$\mathbf{E} \exp\{\langle X_t^{\varrho}, \theta t^{-\frac{1}{2}} f \rangle\} = \exp\{\langle \lambda, v(t, \cdot\,; \theta) \rangle + t^{1/2} \langle \lambda, \bar{u}(1, \cdot\,; t, \theta) \rangle\}, \tag{3.25}$$

by Lemma 3.1 and Lemma 3.2, both $v(t, x; \theta)$ and $\bar{u}(s, x; t, \theta)$ are analytic in $|\theta| < \frac{3}{16a}$ when t is large enough, so (3.25) is also valid for $0 < \theta < \frac{3}{16a}$ by an analytic extension similar as Lemma 2.5.

From (3.9), we know that $|v(s, x; \theta)| \leqslant b_3(\theta, t) \cdot P_s f(x)$, it is easy to verify that

$$\lim_{t \to \infty} t^{-1/2} \langle \lambda, v(t, \cdot\,; \theta) \rangle = 0. \tag{3.26}$$

Then

$$\begin{aligned}\Lambda_3(\theta) &= \lim_{t\to\infty} t^{-1/2} \log \mathbf{E} \exp[\theta t^{1/2} \mathbf{W}(t)] \\ &= \lim_{t\to\infty} t^{-1/2} \log \mathbf{E} \exp\{\langle X_t^{\varrho}, \theta t^{-\frac{1}{2}} f\rangle\} \\ &= \lim_{t\to\infty} t^{-1/2} [\langle \lambda, v(t, \cdot; \theta)\rangle + t^{1/2} \langle \lambda, \bar{u}(1, \cdot; t, \theta)\rangle] \\ &= \langle \lambda, \bar{u}(1, \cdot; \theta)\rangle,\end{aligned}$$

where $\bar{u}(1, \cdot; \theta)$ is the mild solution of (3.7), the second step is by the definition of $\mathbf{W}$ in (1.8), the last step is followed by (3.26) and Lemma 3.7.

The mild solution of equation of (3.7) is

$$\bar{u}(s, x; \theta) = \theta s p(s, x) + \int_0^s P_{s-r} \bar{u}(r, \cdot; \theta)^2 (x)\, dr.$$

Then

$$\Lambda_3(\theta) = \theta s + \int_0^1 \langle \lambda, \bar{u}(r, \cdot; \theta)^2\rangle\, dr.$$

We get $\Lambda'(0) = 1$ and $\Lambda''(0) = (4\pi)^{-3/2}/3$, and then there is $\delta_3 > 0$ such that $\Lambda_3(\theta)$ is strictly convex in $|\theta| < \delta_3 < \frac{3}{16a}$. □

Proof of Theorem 1.3. Based on Lemma 3.8, Theorem 1.3 followed from the general large deviation result Gärtner–Ellis Theorem [cf. Dembo and Zeitouni,(5) or Ellis(9)]. The neighborhood O is that of $\{\Lambda_3'(\theta) : |\theta| < \delta_3\}$. □

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REFERENCES

1. Dawson, D. A. (1977). The critical measure diffusion process. *Z. Wahrsch. verw. Geb.* **40**, 125–145.

2. Dawson, D. A. (1993). Measure-valued Markov processes. In *Lect. Notes. Math.*, Vol. 1541, Springer-Verlag, Berlin, pp. 1–260.
3. Dawson, D. A., and Fleischmann, K. (1997). A continuous super-Brownian motion in a super–Brownian medium. *J. Theor. Probab.* **10**, 213–276.
4. Dawson, D. A., Gorostiza, L. G., and Li, Z. H. (2002). Non-local branching superprocesses and some related models. *Acta Appl. Math.* **74**, 93–112.
5. Dembo, A., and Zeitouni, O. (1998). *Large Deviations Techniques and Applications*, Springer.
6. Deuschel, J.-D., and Rosen, J. (1998). Occupation time lage deviations for critical branching Brownian motion, super-Brownian motion and related processes. *Ann. Probab.* **28**, 602–643.
7. Dynkin, E. B. (1989). Superprocesses and their linear additive functionals. *Trans. Amer. Math. Soc.* **314**, 255–282.
8. Dynkin, E. B. (1994). *An Introduction to Branching Measure-Valued Processes*, Amer. Math. Soc., Providence, RI.
9. Ellis, R. S. (1985). *Entropy, Large Deviations, and Statistical Mechanics*, Springer, New York.
10. Evans, S. N., and Perkins, P. A. (1994). Measure-valued branching diffusions with singular interactions. *Can. J. Math.* **46**, 120–168.
11. Gorostiza, L. G., and Lopez-Mimbela, J. A. (1990). The multitype measure branching process. *Adv. Appl. Probab.* **22**, 49–67.
12. Hong, W. M. (2000). Ergodic theorem for the two-dimensional super-Brownian motion with super-Brownian immigration, *Progress in Natural Science* **10**, 111–116.
13. Hong, W. M. (2002). Longtime behavior for the occupation time of super-Brownian motion with random immigration, *Stoch. Proc. Appl.* **102**, 43–62.
14. Hong, W. M. (2002). Moderate deviation for super-Brownian motion with super-Brownian immigration. *J. Appl. Probab.* **39**, 829–838.
15. Hong, W. M., and Li, Z. H. (1999). A central limit theorem for the super-Brownian motion with super-Brownian immigration. *J. Appl. Probab.* **36**, 1218–1224.
16. Hong, W. M., and Zhao, X. L. (2003). Occupation time large deviations for the super-Brownian motion with random immigration, *Chinese Ann. Math.*, to appear.
17. Iscoe, I. (1986). A weighted occupation time for a class of measure-valued critical branching Brownian motion. *Probab. Th. Rel. Fields* **71**, 85–116.
18. Iscoe, I. (1986). Ergodic theory and a local occupation time for measure-valued critical branching Brownian motion. *Stochastics* **18**, 197–243.
19. Iscoe, I., and Lee, T. Y. (1993). Large deviations for occupation times of measure-valued branching Brownian motions. *Stoch. Stoch. Rep.* **45**, 177–209.
20. Lee, T. Y. (1993). Some limit theorems for super-Brownian motion and semilinear differential equations. *Ann. Probab.* **21**, 979–995.
21. Lee, T. Y., and Remillard, B. (1995). Large deviation for three dimensional super-Brownian motion. *Ann. Probab.* **23**, 1755–1771.
22. Li, Z. H. (1992). A note on the multitype measure branching process. *Adv. Appl. Probab.* **24**, 496–498.
23. Li, Z. H. (1992). Measure-valued branching processes with immigration. *Stoch. Proc. Appl.* **43**, 249–264.
24. Li, Z. H. (1996). Immigration structures associated with Dawson–Watanabe superprocesses. *Stoch. Proc. Appl.* **62**, 73–86.
25. Li, Z. H., and Shiga, T. (1995). Measure-valued branching diffusions: Immigrations, excursions, and limit theorems. *J. Math. Kyoto Univ.* **35**, 233–274.

26. Li, Z. H., and Wang, Z. K. (1999). Measure-valued branching processes and immigration processes. *Adv. Math. (China)* **28**, 105–134.
27. Mytnik, L. (1996). Superprocesses in random environments. *Ann. Probab.* **24**, 1953–1978.
28. Wang, Z. K. (1990). Power series expansion for superprocesses. *Math. Acta. Scientia.* (in chinese), **10**, 361–364.
29. Widder, D. V. (1941). *The Laplace Transform*, Princeton University Press.