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Longtime behavior for the occupation time process of a super-Brownian motion with random immigration

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Abstract

Longtime behavior for the occupation time of a super-Brownian motion with immigration governed by the trajectory of another super-Brownian motion is considered. Central limit theorems are obtained for dimensions $d \geqslant 3$ that lead to some Gaussian random fields: for $3 \leqslant d \leqslant 5$, the field is spatially uniform, which is caused by the randomness of the immigration branching; for $d \geqslant 7$, the covariance of the limit field is given by the potential operator of the Brownian motion, which is caused by the randomness of the underlying branching; and for $d = 6$, the limit field involves a mixture of the two kinds of fluctuations. Some extensions are made in higher dimensions. An ergodic theorem is proved as well for dimension $d = 2$, which is characterized by an evolution equation.

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Keywords: Super-Brownian motion; Random immigration; Central limit theorem; Ergodic theorem; Evolution equation

1. Introduction and main results

A variety of limit theorems have been proved for Dawson–Watanabe superprocesses. Dawson (1977) obtained a spatial central limit theorem for the stationary state of

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an (α, d, β)-superprocess with underlying dimension $d > \alpha/\beta$. Iscoe (1986a) proved central limit theorems for the associated weighted occupation time process in the same situation, and right norming $a_d(t)$ for the occupation time fluctuation is $t^{3/4}$ for $d = 3$, $(t \log t)^{1/2}$ for $d = 4$, and $t^{1/2}$ for $d \geqslant 5$, where $d = 4$ is the critical dimension. Immigration structures associated with Dawson–Watanabe superprocesses have been studied by several authors; see Gorostiza and Lopez-Mimbela (1990), Li (1992, 1996), Li and Wang (1999) and the references cited therein. Limit theorems for immigration processes were studied in Li and Shiga (1995), where the immigration is governed by a deterministic measure.

Superprocesses in random medium have received much attention in recent years, see, for examples, Dawson and Fleischmann (1997), Mytnik (1996), etc. Stimulated by the work of Dawson and Fleischmann (1997), who studied a super-Brownian motion with random branching mechanism governed by another super-Brownian motion, Hong (2000a) and Hong and Li (1999) considered a super-Brownian motion X with immigration governed by the trajectory of another super-Brownian ϱ motion (SBMSBI, for short), denoted it by X^{ϱ}.

In the present paper, we will consider the occupation time process of the super-Brownian motion with super-Brownian immigration (SBMSBI). A central limit theorem is obtained for $d \geqslant 3$ that leads to some Gaussian random fields: for $3 \leqslant d \leqslant 5$, the field is spatially uniform; for $d \geqslant 7$, the covariance of the limit field is given by the potential operator of the underlying Brownian motion; and for $d = 6$, the limit field involves a mixture of the two kinds of fluctuations, which exhibits a departure from the phenomena in the existing models. The right norming $a_d(t)$ is $t^{(10-d)/4}$ for $3 \leqslant d \leqslant 6$, and is t for $d \geqslant 7$, which reveal that the random immigration "smooth" the critical dimension in the sense that there is no log term.

In particles picture, there are two kind of particles in our model: one is the underlying particles governed by X, the other is the immigration particles governed by ϱ which undergo as the underlying particles when they immigrate into the system. With our chosen norming, only the immigration particles contribute to the limit behavior: in higher dimensions ($d \geqslant 7$), the contribution of the *randomized* immigration particles in the asymptotic behavior is the same as the deterministic immigration particles (Proposition 1.1), i.e., the randomness of the underlying (governed by X) contributes to the limit; in lower dimensions ($3 \leqslant d \leqslant 5$), the *randomized* immigration particles make the contributions, i.e., the randomness of the immigration (governed by ϱ) contributes to the limit (Proposition 1.2); and in dimension $d = 6$, it is interesting that both of the two kind of fluctuations contribute to the limiting behavior.

Although there is no non-degenerate central limit theorem for dimension $d = 2$, by analyzing the related evolution equation, we proved an ergodic theorem.

Let $C(\mathbb{R}^d)$ denote the space of continuous bounded functions on $\mathbb{R}^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1 + |x|^2)^{-p/2}$ for $x \in \mathbb{R}^d$. Let $C_p(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : |f(x)| \leqslant \mathrm{const}\,\phi_p(x)\}$. In duality, let $M_p(\mathbb{R}^d)$ be the space of Radon measures μ on $\mathbb{R}^d$ such that $\langle \mu, f \rangle := \int f(x)\mu(\mathrm{d}x) < \infty$ for all $f \in C_p(\mathbb{R}^d)$. We endow $M_p(\mathbb{R}^d)$ with the p-value topology, that is, $\mu_k \to \mu$ if and only if $\langle \mu_k, f \rangle \to \langle \mu, f \rangle$ for all $f \in C_p(\mathbb{R}^d)$. Then $M_p(\mathbb{R}^d)$ is metrizable. Throughout this paper, λ denotes the Lebesgue measure on $\mathbb{R}^d$.

Suppose that $W=(w_t, t\geqslant 0)$ is a standard Brownian motion in $\mathbb{R}^d$ with semigroup $(P_t)_{t\geqslant 0}$. A *super-Brownian motion* $X=(X_t, Q_\mu)$ is an $M_p(\mathbb{R}^d)$-valued Markov process with $X_0=\mu$ and the transition probabilities given by

$$\mathbf{E}\exp\{-\langle X_t, f\rangle\} = \exp\{-\langle \mu, n(t,\cdot)\rangle\}, \quad f\in C_p^+(\mathbb{R}^d), \tag{1.1}$$

where $n(\cdot,\cdot)$ is the unique mild solution of the evolution equation

$$\begin{aligned} &\dot{n}(t) = \Delta n(t) - n^2(t),\\ &n(0) = f. \end{aligned} \tag{1.2}$$

Let $\{g(t,\cdot)\colon t\geqslant 0\}$ be a continuous $C_p^+(\mathbb{R}^d)$-valued path such that for each $a>0$ there is a constant $C_a>0$ such that $g(t)\leqslant C_a\phi_p$ for all $t\in[0,a]$. The weighted occupation time of the super Brownian motion may be determined by

$$\mathbf{E}\exp\left(-\int_0^t \langle X_s, g(s)\rangle\,\mathrm{d}s\right) = \exp\{-\langle \mu, m(0,t,\cdot)\rangle\}, \quad f\in C_p^+(\mathbb{R}^d), \tag{1.3}$$

where $m(0,\cdot,\cdot)$ is the unique mild solution of

$$\begin{aligned} &\dot{m}(s) = \Delta m(s) - m^2(s) + g(t-s), \quad 0\leqslant s\leqslant t,\\ &m(0) = 0. \end{aligned} \tag{1.4}$$

See e.g. Iscoe (1986a).

Suppose that $\{\gamma_t, t\geqslant 0\}$ is an $M_p(\mathbb{R}^d)$-valued continuous path. A *super-Brownian motion with immigration* determined by $\{\gamma_t, t\geqslant 0\}$ is an $M_p(\mathbb{R}^d)$-valued Markov process $X^\gamma=(X_t^\gamma, Q_\mu^\gamma)$ with transition probabilities given by

$$\mathbf{E}\exp(-\langle X_t^\gamma, f\rangle) = \exp\left\{-\langle \mu, n(t,\cdot)\rangle - \int_0^t \langle \gamma_s, n(t-s,\cdot)\rangle\,\mathrm{d}s\right\}, \quad f\in C_p^+(\mathbb{R}^d), \tag{1.5}$$

where $n(\cdot,\cdot)$ is given by (1.2); see e.g. Dawson (1993), Dynkin (1991) and Li and Wang (1999).

Based on (1.3) and (1.5) it is not difficult to construct a probability space $(\Omega,\mathscr{F},\mathbf{Q})$ on which the processes $\{\varrho_t\colon t\geqslant 0\}$ and $\{X_t^\varrho\colon t\geqslant 0\}$ are defined, where $\{\varrho_t\colon t\geqslant 0\}$ is a super Brownian motion with $\varrho_0=\lambda$ and, given $\{\varrho_t\colon t\geqslant 0\}$, the process $\{X_t^\varrho\colon t\geqslant 0\}$ is a super Brownian motion with immigration determined by $\{\varrho_t : t\geqslant 0\}$ with $X_0^\varrho=\lambda$. By (1.3) and (1.5) we have

$$\begin{aligned} \mathbf{E}\exp\{-\langle X_t^\varrho, f\rangle\} &= \mathbf{E}[\mathbf{E}\exp\{-\langle X_t^\varrho, f\rangle\}|\{\sigma(\varrho_s, s\leqslant t)\}],\\ &= \mathbf{E}\exp\left\{-\langle\lambda, n(t,\cdot)\rangle - \int_0^t \langle \varrho_s, n(t-s,\cdot)\rangle\,\mathrm{d}s\right\},\\ &= \exp\{-\langle\lambda, n(t,\cdot)\rangle - \langle\lambda, m(t,\cdot)\rangle\}, \end{aligned} \tag{1.6}$$

where $m(\cdot,\cdot)$ is the unique mild solution of the equation

$$\begin{aligned} &\dot{m}(s) = \Delta m(s) - m^2(s) + n(s), \quad 0\leqslant s\leqslant t,\\ &m(0) = 0 \end{aligned} \tag{1.7}$$

and $n(\cdot,\cdot)$ is the mild solution of Eq. (1.2).

The process $\{X_t^\varrho\colon t \geqslant 0\}$ is what we call *super-Brownian motion with super-Brownian immigration* (SBMSBI), for details, see Hong and Li (1999). Let

$$Y_t^\varrho := \int_0^t X_s^\varrho \, \mathrm{d}s \tag{1.8}$$

be the occupation time process of SBMSBI in the sense that $\langle Y_t^\varrho, f\rangle := \int_0^t \langle X_s^\varrho, f\rangle \, \mathrm{d}s$, where $f \in C_p^+(R^d)$.

To distinguish the effect of the two kind of branching in the model SBMSBI, we will consider the situation which the branching rate of X and ϱ are the positive constant k_1 and k_2, i.e., the branching functional of X and ϱ are $\psi_X(z) = k_1 z^2$ and $\psi_\varrho(z) = k_2 z^2$ respectively. By (1.3) and (1.6), we know that the Laplace transition functional of Y_t^ϱ under $\mathbf{Q}$ is given by

$$\mathbf{E}\exp\{-\langle Y_t^\varrho, f\rangle\} = \exp\{-\langle \lambda, v(t,\cdot)\rangle - \langle \lambda, u(t,\cdot)\rangle\}, \tag{1.9}$$

where $u(\cdot,\cdot)$ is the mild solution of the equation

$$\begin{aligned} &\dot{u}(s) = \Delta u(s) - k_2 u^2(s) + v(s), \quad 0 \leqslant s \leqslant t, \\ &u(0) = 0 \end{aligned} \tag{1.10}$$

and $v(\cdot,\cdot)$ is the solution of the equation

$$\begin{aligned} &\dot{v}(t) = \Delta v(t) - k_1 v^2(t) + f, \\ &v(0) = 0. \end{aligned} \tag{1.11}$$

Here is the position to state our main results. Let $\mathscr{S}(\mathbb{R}^d)$ be the space of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$ whose all partial derivatives are also rapidly decreasing, and let $\mathscr{S}'(\mathbb{R}^d)$ be the dual space of $\mathscr{S}(\mathbb{R}^d)$. We define the $\mathscr{S}'(\mathbb{R}^d)$-valued process $\{Z_t^\varrho\colon t > 0\}$ by

$$\langle Z_t^\varrho, f\rangle := a_d(t)^{-1}[\langle Y_t^\varrho, f\rangle - \mathbf{E}\langle Y_t^\varrho, f\rangle], \quad f \in \mathscr{S}(\mathbb{R}^d), \tag{1.12}$$

where $a_d(t) = t^{(10-d)/4}$ for $3 \leqslant d \leqslant 6$ and $a_d(t) = t$ for $d \geqslant 7$. Then we have

Theorem 1.1. *For $d \geqslant 3$, Z_t^ϱ converges in distribution to a centered Gaussian random variable Z_∞ in $\mathscr{S}'(\mathbb{R}^d)$ with covariance*

$$\mathbf{Cov}(\langle Z_\infty, f\rangle, \langle Z_\infty, g\rangle) = \begin{cases} k_2 C_d \langle \lambda, f\rangle\langle \lambda, g\rangle, & 3 \leqslant d \leqslant 5, \\ k_1 \int_0^\infty \mathrm{d}h \int_0^\infty \mathrm{d}h' \int_{R^d} f(y) P_{h+h'} g(y)\,\mathrm{d}y + k_2 C_d \langle \lambda, f\rangle\langle \lambda, g\rangle, & d = 6, \\ k_1 \int_0^\infty \mathrm{d}h \int_0^\infty \mathrm{d}h' \int_{R^d} f(y) P_{h+h'} g(y)\,\mathrm{d}y, & d \geqslant 7. \end{cases}$$

where $f, g \in \mathscr{S}(\mathbb{R}^d)$, and

$$C_d = \int_0^1 s^{4-d/2}\,\mathrm{d}s \int_0^1 r\,\mathrm{d}r \int_0^1 r'\,\mathrm{d}r' \int_0^1 \mathrm{d}h \int_0^1 [2\pi(2 - hr - h'r')]^{-d/2}\,\mathrm{d}h'$$

is finite for $3 \leqslant d \leqslant 6$.

Remark 1.1. For $3 \leqslant d \leqslant 5$, we actually have $Z_\infty = \zeta\lambda$, where λ is the Lebesgue measure on R^d, and ζ is a centered Gaussian random variable in R with variance C_d.

In higher dimensions ($d \geqslant 7$), the contribution to the asymptotic behavior of the *randomized* immigration particles is the same as the deterministic immigration particles. To see this, let X^λ be the SBM with deterministic immigration governed by the Lebesgue measure λ, and Y^λ be the occupation time of X^λ. The Laplace transition functional of $Y^\lambda$is determined by

$$\mathbf{E}\exp\{-\langle Y_t^\lambda, f\rangle\} = \exp\left\{-\langle \lambda, v(t,\cdot)\rangle - \int_0^t \langle \lambda, v(s,\cdot)\rangle\,\mathrm{d}s\right\},$$

where $v(\cdot,\cdot)$ is the mild solution of Eq. (1.11). Let Z^λ be defined by (1.12) with ϱ replaced by λ, then we have

Proposition 1.1. *For $d \geqslant 7$, both Z_t^λ and Z_t^ϱ possess the same limiting field as $t \to \infty$.* □

In lower dimensions ($3 \leqslant d \leqslant 5$), the *randomized* immigration particles make the contributions, i.e., the randomness of the immigration (governed by ϱ) contributes to the asymptotic behavior. To this point, we can find a functional of ϱ, which acts as Z_t^ϱ in longtime behavior. Define Y_t by

$$\langle Y_t, f :=\rangle \int_0^t \left\langle \varrho_s, \int_0^{t-s} P_r f\,\mathrm{d}r \right\rangle \mathrm{d}s,$$

and let

$$\langle Z_t, f\rangle := a_d(t)^{-1}[\langle Y_t, f\rangle - \mathbf{E}\langle Y_t, f\rangle], \quad f \in \mathscr{S}(\mathbb{R}^d),$$

with the same norming $a_d(t) = t^{(10-d)/4}$ for $3 \leqslant d \leqslant 5$, Then we get that

Proposition 1.2. *For $3 \leqslant d \leqslant 5$, both Z_t and Z_t^ϱ possess the same limiting field as $t \to \infty$.*

Although there is no non-degenerate central limit theorem for $d < 3$, by analyzing the related evolution equation, we prove a weak ergodic theorem for $d = 2$.

Theorem 1.2. *Let $d = 2$, then as $T \to \infty$*

$$T^{-2} Y_T^\varrho \to \xi\lambda \quad (\textit{with respect to } \mathbf{Q}),$$

where ξ is a non-negative, infinitely divisible random variable whose Laplace transform is given by

$$\mathbf{E}\exp\{-\theta\xi\} = \exp\{-\langle \lambda, w(1,\cdot;\theta)\rangle\}, \tag{1.13}$$

where $w \equiv w(t,x;\theta)$ is the mild solution of the evolution equation

$$\dot{w}(t) = \Delta w(t) - k_2 w^2(t) + \theta q_t,$$

$$w(0) = 0, \tag{1.14}$$

where $q_t(\cdot)=\int_0^t p(s,\cdot)\,\mathrm{d}s$, *and* $p(s,x)$ *is the transition density function of the standard Brownian motion.*

Theorem 1.1, Propositions 1.1 and 1.2 will be proved in Section 2; Theorem 1.2 will be proved in Section 3; and some extension is made in Section 4.

2. Proof of Theorem 1.1

Now we proceed to the proof of Theorem 1.1. In what follows, C will denote a constant which may take different values in different lines. Let $f_t = a_d(t)^{-1} f$, $f \in \mathscr{S}(\mathbb{R}^d)^+$. Let $u_t(\cdot,\cdot)$ and $v_t(\cdot,\cdot)$ be the mild solutions of evolution Eqs. (1.10) and (1.11), respectively, with f being replaced by f_t, i.e., $u_t(\cdot,\cdot)$ and $v_t(\cdot,\cdot)$ satisfy

$$u_t(r,x) = \int_0^r P_{t-s} v_t(s,\cdot)(x)\,\mathrm{d}s - k_2 \int_0^r P_{r-s} u_t^2(s,\cdot)(x)\,\mathrm{d}s, \quad t \geqslant r \geqslant 0. \tag{2.1}$$

and

$$v_t(r,x) = \int_0^r P_s f_t(x)\,\mathrm{d}s - k_1 \int_0^r P_{r-s} v_t^2(s,\cdot)(x)\,\mathrm{d}s, \quad t \geqslant r \geqslant 0. \tag{2.2}$$

Lemma 2.1. $\mathbf{E}\langle Y_t^\varrho, f\rangle = t\langle \lambda, f\rangle + \frac{1}{2}t^2\langle \lambda, f\rangle$.

Proof. This is a simple calculation based on (1.9)–(1.11). □

Remark 2.1. By (1.9)–(1.12) and Lemma 2.1 we get the Laplace transition functional of Z_t, $t \geqslant 0$ under $\mathbf{Q}$

$$\mathbf{E}\exp\{-\langle Z_t^\varrho, f\rangle\} = \exp\left\{k_1 \int_0^t \langle \lambda, v_t(s,\cdot)^2\rangle\,\mathrm{d}s + k_1 \int_0^t \mathrm{d}s \int_0^s \langle \lambda, v_t(r,\cdot)^2\rangle\,\mathrm{d}r \right.$$
$$\left. + k_2 \int_0^t \langle \lambda, u_t(s,\cdot)^2\rangle\,\mathrm{d}s \right\}, \tag{2.3}$$

where $v_t(\cdot,\cdot)$ and $u_t(\cdot,\cdot)$ are the solutions of (2.2) and (2.1), respectively. The first term in the bracket of the right-hand side of (2.3) is caused by the underlying particles with branching of X_t, the second and the third terms are caused by the immigration particles. From Eq. (2.2) and by a simple calculation, we have

$$\int_0^t \langle \lambda, v_t(s,\cdot)^2\rangle\,\mathrm{d}s \leqslant a_d(t)^{-2} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^s \mathrm{d}r' \int_{R^d} f(y) P_{r+r'} f(y)\,\mathrm{d}y \to 0, \tag{2.4}$$

as $t \to \infty$ when $d \geqslant 3$. We could see the underlying particles with no contribution to the limit behavior. How do the immigration particles contribute to the limit behavior? We will see that by calculating the limit values of the last two terms in the bracket of the right-hand side of (2.3) as $t \to \infty$ by a series of lemmas.

Lemma 2.2. *Let*

$$A_d(t,f) := a_d(t)^{-2} \int_0^t \mathrm{d}s \int_0^s \left\langle \lambda, \left(\int_0^r P_h f \, \mathrm{d}h \right)^2 \right\rangle \mathrm{d}r.$$

Then we have

$$\lim_{t\to\infty} A_d(t,f) = \begin{cases} 0, & 3 \leqslant d \leqslant 5, \\ \int_0^\infty \mathrm{d}h \int_0^\infty \mathrm{d}h' \int_{R^d} f(y) P_{h+h'} f(y) \, \mathrm{d}y, & d \geqslant 6. \end{cases}$$

Proof. It is easy to check that for any $f \in \mathscr{S}(\mathbb{R}^d)^+$, we have

$$\|P_s f\| \leqslant C(1 \wedge s^{-d/2}). \tag{2.5}$$

When $3 \leqslant d \leqslant 5$, one has

$$\begin{aligned}
\lim_{t\to\infty} A_d(t,f) &= \lim_{t\to\infty} a_d(t)^{-2} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_{R^d} \mathrm{d}x \int_0^r P_h f(x) \, \mathrm{d}h \int_0^r P_{h'} f(x) \, \mathrm{d}h' \\
&= \lim_{t\to\infty} a_d(t)^{-2} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}h \int_0^r \mathrm{d}h' \\
&\quad \times \int_{R^d} \int_{R^d} p(h+h', y-y') f(y) f(y') \, \mathrm{d}y \, \mathrm{d}y' \\
&= \lim_{t\to\infty} a_d(t)^{-2} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}h \int_0^r \mathrm{d}h' \int_{R^d} f(y) P_{h+h'} f(y) \, \mathrm{d}y \\
&\leqslant \lim_{t\to\infty} C a_d(t)^{-2} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}h \int_0^r (1 \wedge (h+h')^{-d/2}) \, \mathrm{d}h' \langle \lambda, f \rangle \\
&= 0
\end{aligned}$$

as $t \to \infty$. Similarly, when $d \geqslant 6$, we use l'Hospital's rule to get

$$\begin{aligned}
\lim_{t\to\infty} A_d(t,f) &= \lim_{t\to\infty} a_d(t)^{-2} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}h \int_0^r \mathrm{d}h' \int_{R^d} f(y) P_{h+h'} f(y) \, \mathrm{d}y \\
&= \int_0^\infty \mathrm{d}h \int_0^\infty \mathrm{d}h' \int_{R^d} f(y) P_{h+h'} f(y) \, \mathrm{d}y,
\end{aligned}$$

is finite as desired. □

Lemma 2.3. *Let* $d \geqslant 3$,

$$\varepsilon_d(t) := A_d(t,f) - \int_0^t \mathrm{d}s \int_0^s \langle \lambda, v_t(r,\cdot)^2 \rangle \, \mathrm{d}r$$

then $\lim_{t\to\infty} \varepsilon_d(t) = 0$.

Proof. From Eq. (2.2), we know that

$$v_t(r,a) \leqslant \int_0^r P_s f_t(a) \, \mathrm{d}s, \quad 0 \leqslant r \leqslant t$$

and then $\varepsilon_d(t)$ is non-negative, applying (2.5), we have

$$\begin{aligned}
\varepsilon_d(t) &\leqslant 2\int_0^t \mathrm{d}s \int_0^s \left\langle \lambda, \left[\int_0^r P_h f_t \,\mathrm{d}h\right] \left[\int_0^r P_{r-h'} v_t(h',\cdot)^2 \,\mathrm{d}h'\right]\right\rangle \mathrm{d}r \\
&\leqslant 2\int_0^t \mathrm{d}s \int_0^s \left\langle \lambda, \left[\int_0^r P_h f_t \,\mathrm{d}h\right] \left[\int_0^r P_{r-h'} \left(\int_0^{h'} P_l f_t \,\mathrm{d}l\right)^2 \mathrm{d}h'\right]\right\rangle \mathrm{d}r \\
&= 2a_d(t)^{-3} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}h \int_0^r \mathrm{d}h' \int_{R^d}\int_{R^d} p(r+h-h', y-z) f(y) \\
&\quad \times \left(\int_0^{h'} P_l f \,\mathrm{d}l\right)^2 (z)\,\mathrm{d}y\,\mathrm{d}z \\
&\leqslant 2a_d(t)^{-3} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}h \int_0^r \mathrm{d}h' \int_{R^d} P_{r+h-h'} f(z) \left(\int_0^t P_l f \,\mathrm{d}l\right)^2 (z)\mathrm{d}z \\
&\leqslant 2Ca_d(t)^{-3} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}h \int_0^r [1 \wedge (h+h')^{-d/2}]\,\mathrm{d}h' \\
&\quad \times \int_0^t \mathrm{d}l \int_0^t \mathrm{d}l' \int_{R^d} f(z') P_{l+l'} f(z')\,\mathrm{d}z' \\
&\leqslant 2Ca_d(t)^{-3} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}h \int_0^r [1 \wedge (h+h')^{-d/2}]\,\mathrm{d}h' \\
&\quad \times \int_0^t \mathrm{d}l \int_0^t [1 \wedge (l+l')^{-d/2}]\,\mathrm{d}l' \\
&\to 0,
\end{aligned}$$

as $t \to \infty$. □

Remark 2.2. From Lemmas 2.2 and 2.3, we get

$$\lim_{t\to\infty} k_1 \int_0^t \mathrm{d}s \int_0^s \langle \lambda, v_t(r,\cdot)^2\rangle \,\mathrm{d}r$$
$$= \begin{cases} 0, & 3 \leqslant d \leqslant 5, \\ k_1 \int_0^\infty \mathrm{d}h \int_0^\infty \mathrm{d}h' \int_{R^d} f(y) P_{h+h'} f(y)\,\mathrm{d}y, & d \geqslant 6. \end{cases} \tag{2.6}$$

We could see that with our chosen norming $a_d(t)$, the second terms in the bracket of the right-hand side of (2.3) contributes to the limit behavior only in the higher dimension $d \geqslant 6$.

Lemma 2.4. *Let*

$$B_d(t,f) = a_d(t)^{-2} \int_0^t \left\langle \lambda, \left[\int_0^s P_{s-r} \int_0^r P_h f \,\mathrm{d}h\,\mathrm{d}r\right]^2 \right\rangle \mathrm{d}s.$$

Then we have

$$\lim_{t\to\infty} B_d(t,f) = \begin{cases} C_d\langle\lambda,f\rangle^2, & 3\leqslant d\leqslant 6,\\ 0, & d\geqslant 7. \end{cases}$$

where

$$C_d = \int_0^1 s^{4-d/2}\,\mathrm{d}s \int_0^1 r\,\mathrm{d}r \int_0^1 r'\,\mathrm{d}r' \int_0^1 \mathrm{d}h \int_0^1 [2\pi(2-hr-h'r')]^{-d/2}\,\mathrm{d}h'$$

is finite for $3\leqslant d\leqslant 6$.

Proof. We first observe that when $3\leqslant d\leqslant 6$,

$$\begin{aligned}
B_d(t,f) &= a_d(t)^{-2}\int_0^t \left\langle \lambda, \left[\int_0^s \mathrm{d}r \int_0^r P_{s-r+h} f\,\mathrm{d}h\right]^2\right\rangle \mathrm{d}s\\
&= a_d(t)^{-2}\int_0^t \mathrm{d}s \int_{R^d} \mathrm{d}x \left[\int_0^s \mathrm{d}r\int_0^r P_{s-r+h}f(x)\,\mathrm{d}h\right]\\
&\quad\times\left[\int_0^s \mathrm{d}r'\int_0^{r'} P_{s-r'+h'}f(x)\,\mathrm{d}h'\right]\\
&= a_d(t)^{-2}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\int_0^r \mathrm{d}h\int_0^s \mathrm{d}r'\int_0^{r'} \mathrm{d}h'\\
&\quad\times\int_{R^d}\int_{R^d} p(2s-h-h', y-y')f(y)f(y')\,\mathrm{d}y\,\mathrm{d}y'\\
&= a_d(t)^{-2}t^{5-d/2}\int_0^1 u^{4-d/2}\,\mathrm{d}u\int_0^1 v\,\mathrm{d}v\int_0^1 v'\,\mathrm{d}v'\int_0^1 \mathrm{d}w\int_0^1 \mathrm{d}w'\\
&\quad\times\int_{R^d}\int_{R^d}[2\pi(2-vw-v'w')]^{-d/2}\\
&\quad\times\exp\left\{-\frac{(y-y')^2}{2tu(2-vw-v'w')}\right\} f(y)f(y')\,\mathrm{d}y\,\mathrm{d}y'\\
&\quad\to C_d\langle\lambda,f\rangle^2,
\end{aligned}$$

as $t\to\infty$ by Lebesgue's dominated convergence theorem, where

$$C_d = \int_0^1 u^{4-d/2}\,\mathrm{d}u\int_0^1 v\,\mathrm{d}v\int_0^1 v'\,\mathrm{d}v'\int_0^1 \mathrm{d}w\int_0^1 [2\pi(2-vw-v'w')]^{-d/2}\,\mathrm{d}w'$$

is finite for $3\leqslant d\leqslant 6$, and in the fourth step we used the change of variables $s=tu,\ r=tuv,\ r'=tuv',\ h=tuvw,\ h'=tuv'w'$.

When $d \geqslant 7$, using (2.5) we have

$$B_d(t,f) = a_d(t)^{-2}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\int_0^r \mathrm{d}h\int_0^s \mathrm{d}r'\int_0^{r'} \mathrm{d}h'\int_{R^d} f(y)P_{2s-h-h'}f(y)\,\mathrm{d}y$$

$$\leqslant Ca_d(t)^{-2}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\int_0^r \mathrm{d}h\int_0^s \mathrm{d}r'\int_0^{r'} [1\wedge(2s-h-h')^{-d/2}]\mathrm{d}h'\langle\lambda,f\rangle,$$

which goes to zero as $t\to\infty$. □

Lemma 2.5. *Let* $d \geqslant 3$,

$$\eta_d(t) := B_d(t,f) - \int_0^t \langle\lambda, u_t(s,\cdot)^2\rangle\,\mathrm{d}s.$$

Then $\lim_{t\to\infty}\eta_d(t) = 0$.

Proof. We first note that

$$\eta_d(t) = \left\{\int_0^t \left\langle \lambda, \left[\int_0^s P_{s-r}\int_0^r P_h f_t\,\mathrm{d}h\,\mathrm{d}r\right]^2\right\rangle \mathrm{d}s - \int_0^t \left\langle \lambda, \left[\int_0^s P_{s-r}v_t(r,\cdot)\mathrm{d}r\right]^2\right\rangle \mathrm{d}s\right\}$$

$$+\left\{\int_0^t \left\langle \lambda, \left[\int_0^s P_{s-r}v_t(r,\cdot)\,\mathrm{d}r\right]^2\right\rangle \mathrm{d}s - \int_0^t \langle\lambda, u_t(s,\cdot)^2\rangle\,\mathrm{d}s\right\}$$

$$:= \eta_d^{(1)}(t) + \eta_d^{(2)}(t).$$

From Eq. (2.1),

$$u_t(r,a) \leqslant \int_0^r P_{t-s}v_t(s,\cdot)(a)\,\mathrm{d}s, \quad 0 \leqslant r \leqslant t$$

then both $\eta_d^{(1)}(t)$ and $\eta_d^{(2)}(t)$ are non-negative, by use of (2.5) we get

$$\eta_d^{(1)}(t) \leqslant 2\int_0^t \mathrm{d}s\left\langle \lambda, \left[\int_0^s \mathrm{d}r\int_0^r P_{s-r+h}f_t\,\mathrm{d}h\right]\left[\int_0^s \mathrm{d}r'\int_0^{r'} P_{s-h'}v_t(h',\cdot)^2\,\mathrm{d}h'\right]\right\rangle$$

$$= 2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\int_0^r \mathrm{d}h\int_0^s \mathrm{d}r'\int_0^{r'} \mathrm{d}h'$$

$$\times\int_{R^d}\int_{R^d} p(2s-r+h-h', y-z)f_t(y)v_t(h',z)^2\,\mathrm{d}y\,\mathrm{d}z$$

$$\leqslant 2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\int_0^r \mathrm{d}h\int_0^s \mathrm{d}'\int_0^{r'} \mathrm{d}h'\int_{R^d} P_{2s-r+h-h'}f_t(z)\left[\int_0^t P_l f_t\mathrm{d}l\right]^2(z)\mathrm{d}z$$

$$\leqslant 2Ca_d(t)^{-3}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\int_0^r \mathrm{d}h\int_0^s \mathrm{d}r'\int_0^{r'} [1\wedge(2s+h-h')^{-d/2}]\,\mathrm{d}h'$$

$$\times\int_0^t \mathrm{d}l\int_0^t [1\wedge(l+l')^{-d/2}]\,\mathrm{d}l'$$

$$\to 0,$$

as $t \to \infty$, and from Eqs. (2.1) and (2.2) we have

$$\eta_d^{(2)}(t) \leqslant 2\int_0^t \mathrm{d}s \left\langle \lambda, \left[\int_0^s P_{s-r} v_t(r,\cdot)\,\mathrm{d}r\right]\left[\int_0^s P_{s-r'} u_t(r',\cdot)^2\,\mathrm{d}r'\right]\right\rangle$$

$$\leqslant 2\int_0^t \mathrm{d}s \Bigg\langle \lambda, \left[\int_0^s P_{s-r}\left(\int_0^r P_h f_t\,\mathrm{d}h\right)\mathrm{d}r\right]$$

$$\times \left[\int_0^s P_{s-r'}\left(\int_0^{r'} P_{r'-h'} v_t(h',\cdot)\,\mathrm{d}h'\right)^2 \mathrm{d}r'\right]\Bigg\rangle$$

$$= 2\int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}h \int_0^s \mathrm{d}r' \int_{R^d} P_{2s-r-r'+h} f_t(z)\left[\int_0^{r'} \mathrm{d}h' \int_{h'}^{r'} P_l f_t\,\mathrm{d}l\right]^2 (z)\,\mathrm{d}z$$

$$\leqslant 2Ca_d(t)^{-3}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^s \mathrm{d}r' \int_0^r [1 \wedge (r+r'+h)^{-d/2}]\,\mathrm{d}h$$

$$\times \int_0^t \mathrm{d}h' \int_0^t \mathrm{d}h'' \int_{h'}^t \mathrm{d}l \int_{h''}^t [1 \wedge (l+l')^{-d/2}]\,\mathrm{d}l'$$

$$\to 0,$$

combining the above yields a proof. □

Remark 2.3. From Lemmas 2.4 and 2.5, we obtain

$$\lim_{t\to\infty} k_2 \int_0^t \langle \lambda, u_t(s,\cdot)^2\rangle\,\mathrm{d}s = \begin{cases} k_2 C_d \langle \lambda, f\rangle^2, & 3 \leqslant d \leqslant 6, \\ 0, & d \geqslant 7. \end{cases} \tag{2.7}$$

We could see that with our chosen norming $a_d(t)$, the third terms in the bracket of the right-hand side of (2.3) contributes to the limit behavior only in the lower dimension $3 \leqslant d \leqslant 6$.

Proof of Theorem 1.1. Combining (2.4), (2.6) and (2.7) with (2.3) and the discussions in Theorem 5.5 of Iscoe (1986a), we get the desired result. □

Proof of Proposition 1.1. By the definition of Z_t^λ, we have

$$\mathbf{E}\exp\{-\langle Z_t^\lambda, f\rangle\} = \exp\left\{k_1 \int_0^t \langle \lambda, v^2(s,\cdot)\rangle\,\mathrm{d}s + k_1\int_0^t \mathrm{d}s \int_0^s \langle \lambda, v^2(r,\cdot)\rangle\,\mathrm{d}r\right\},$$

where $v(\cdot,\cdot)$ is the mild solution of Eq. (1.11). For $d \geqslant 7$, the results followed from (2.4) and (2.6). □

Proof of Proposition 1.2. Let $F(s) = a_d(t)^{-1}\int_0^{t-s} P_r f\,\mathrm{d}r$, by the definition of Z_t, we have

$$\mathbf{E}\exp\{-\langle Z_t, f\rangle\} = \exp\left\{k_2 \int_0^t \langle \lambda, u^2(s,\cdot)\rangle\,\mathrm{d}s\rangle\,\mathrm{d}r\right\},$$

where $u(s,x)$ is the solution of the equation

$$u(s,x) = \int_0^t P_{t-s}F(t-s)\,\mathrm{d}s - k_2\int_0^t P_{t-s}u^2(s,\cdot)(x)\,\mathrm{d}s, \quad 0 \leqslant s \leqslant t.$$

Then for $3 \leqslant d \leqslant 6$, the results followed from Lemmas 2.4 and 2.5. $\square$

An immediate consequence of Theorem 1.1 and Lemma 2.1 is the following

Corollary 2.6. *For $d \geqslant 3$ we have $t^{-2}Y_t^{\varrho} \to \frac{1}{2}\lambda$ in probability.*

Remark 2.5. This ergodic theorem leads to investigate the large deviation principles in Hong (2000b).

3. Proof of Theorem 1.2

In this section, we proceed to the proof of Theorem 1.2. Let $u_T(t,x)$ and $v_T(t,x)$ be the mild solutions of the Eqs. (1.10) and (1.11), respectively, with f being replaced by $T^{-2}f$, i.e.,

$$\begin{aligned} &\dot{u}_T(t) = \Delta u_T(t) - k_2u_T^2(t) + v_T(t), \quad 0 \leqslant t \leqslant T,\\ &u(0) = 0 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\dot{v}_T(t) = \Delta v_T(t) - k_1v_T^2(t) + T^{-2}f, \quad 0 \leqslant t \leqslant T,\\ &v(0) = 0. \end{aligned} \tag{3.2}$$

Let $w_T(t,x) := Tu_T(Tt, T^{1/2}x)$ and $\bar{v}_T(t,x) := T^2v_T(Tt, T^{1/2}x)$. From (3.1) and (3.2) we could verify that $w_T(t,x)$ and $\bar{v}_T(t,x)$ satisfy the following equations, respectively:

$$\begin{aligned} &\dot{w}_T(t) = \Delta w_T(t) - k_2w_T^2(t) + \bar{v}_T(t), \quad 0 \leqslant t \leqslant T,\\ &w(0) = 0 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\dot{\bar{v}}_T(t) = \Delta\bar{v}_T(t) - k_1T^{-1}\bar{v}_T^2(t) + Tf(T^{1/2}\cdot), \quad 0 \leqslant t \leqslant T,\\ &\hat{v}(0) = 0. \end{aligned} \tag{3.4}$$

In the following lemmas, we firstly consider $f \in C_p(\mathbb{R}^d)^+$ such that $\langle\lambda, f\rangle = 1$.

Lemma 3.1. *Let $d = 2$, $\tau > 0$. Then we have*

$$\lim_{T\to\infty}\int_0^t \mathrm{d}sP_{t-s}\int_0^s P_r[Tf(T^{1/2}\cdot)](x)\,\mathrm{d}r = \int_0^t \mathrm{d}s\int_0^s p(t-r,x)\,\mathrm{d}r,$$

in $L^2(R^2,\lambda)$ and pointwise uniformly in $t \in [0,\tau]$, where $p(t,x)$ is the transition density function of the Brownian motion.

Proof. First of all, we note that

$$\int_0^t \mathrm{d}s P_{t-s} \int_0^s P_r[Tf(T^{1/2}\cdot)](x)\,\mathrm{d}r$$

$$= \int_0^t \mathrm{d}s \int_0^{t-s} \int_{R^2} p(s+r, x - T^{-1/2}y) f(y)\,\mathrm{d}y\,\mathrm{d}r, \tag{3.5}$$

which converges pointwise as $T \to \infty$ by Lebesgue's dominated convergence theorem, because $p(s+r, x - T^{-1/2}y)$ is dominated by $[2\pi(s+r)]^{-1}$ and $\int_0^\tau \mathrm{d}s \int_0^{\tau-s} [2\pi(s+r)]^{-1}\,\mathrm{d}r < \infty$.

Noting $\langle \lambda, f \rangle = 1$ we have

$$\left\| \int_0^t \mathrm{d}s P_{t-s} \int_0^s P_r[Tf(T^{1/2}\cdot)](x)\,\mathrm{d}r - \int_0^t \mathrm{d}s \int_0^s p(t-r, x)\,\mathrm{d}r \right\|_{L^2}^2$$

$$= \left\| \int_0^t \mathrm{d}s \int_0^s \int_{R^2} p(t-r, x - T^{-1/2}y) f(y)\,\mathrm{d}y\,\mathrm{d}r \right.$$

$$\left. - \int_0^t \mathrm{d}s \int_0^s \int_{R^2} p(t-r, x) f(y)\,\mathrm{d}y\,\mathrm{d}r \right\|_{L^2}^2$$

$$= \int_{R^2} \left[\int_0^t \mathrm{d}s \int_0^s \int_{R^2} [p(t-r, x - T^{-1/2}y) - p(t-r, x)] f(y)\,\mathrm{d}y\,\mathrm{d}r \right]^2 \mathrm{d}x$$

$$= \int_0^t \mathrm{d}s \int_0^{t-s} \mathrm{d}r \int_0^t \mathrm{d}s' \int_0^{t-s'} \mathrm{d}r' \int_{R^2} f(y)\,\mathrm{d}y \int_{R^2} f(y')\,\mathrm{d}y'$$

$$\times \int_{R^2} [p(s+r, x - T^{-1/2}y) - p(s+r, x)]$$

$$\times [p(s'+r', x - T^{-1/2}y') - p(s'+r', x)]\,\mathrm{d}x$$

$$\leqslant \int_0^\tau \mathrm{d}s \int_0^{\tau-s} \mathrm{d}r \int_0^\tau \mathrm{d}s' \int_0^{\tau-s'} \mathrm{d}r' \int_{R^2} f(y)\,\mathrm{d}y$$

$$\times \int_{R^2} f(y')[p(s+s'+r+r', T^{-1/2}y - T^{-1/2}y')$$

$$- p(s+s'+r+r', T^{-1/2}y) - p(s+s'+r+r', T^{-1/2}y')$$

$$+ p(s+s'+r+r', 0)]\,\mathrm{d}y' \to 0,$$

as $T \to \infty$ by Lebesgue's dominated convergence theorem, because the integrand is dominated by $4p(s+s'+r+r', 0)$ and

$$\int_0^\tau \mathrm{d}s \int_0^{\tau-s} \mathrm{d}r \int_0^\tau \mathrm{d}s' \int_0^{\tau-s'} [4p(s+s'+r+r', 0)]\,\mathrm{d}r'$$

is finite when $d = 2$. $\square$

Lemma 3.2. *Let $d=2$, $\tau>0$, then we have*

$$\lim_{T\to\infty}\int_0^t P_{t-s}\bar{v}(s,\cdot)(x)=\int_0^t \mathrm{d}s\int_0^s p(t-r,x)\,\mathrm{d}r,$$

in $L^2(R^2,\lambda)$ and pointwise uniformly in $t\in[0,\tau]$, where $p(t,x)$ is the transition density function of the Brownian motion.

Proof. The mild form of Eq. (3.4) is

$$\bar{v}_T(t,x)=\int_0^t P_s[Tf(T^{1/2}\cdot)](x)\,\mathrm{d}s-k_1T^{-1}\int_0^t P_s\bar{v}_T^2(t-s,\cdot)(x)\,\mathrm{d}s. \tag{3.6}$$

Then

$$\begin{aligned}
g_T(t,x)&:=\int_0^t P_{t-s}\bar{v}_T(s,\cdot)(x)-\int_0^t \mathrm{d}s\int_0^s p(t-r,x)\,\mathrm{d}r\\
&=\left[\int_0^t \mathrm{d}sP_{t-s}\int_0^s P_r[Tf(T^{1/2}\cdot)](x)\,\mathrm{d}r-\int_0^t\mathrm{d}s\int_0^s p(t-r,x)\,\mathrm{d}r\right]\\
&\quad-k_1T^{-1}\left[\int_0^t \mathrm{d}sP_{t-s}\int_0^s P_r\bar{v}_T^2(s-r,\cdot)(x)\,\mathrm{d}r\right]\\
&:=g_T^{(1)}(t,x)-k_1g_T^{(2)}(t,x).
\end{aligned}$$

Note that by (3.6), (3.5) and (2.5), one gets

$$\begin{aligned}
g_T^{(2)}(t,x)&\leqslant T^{-1}\int_0^t\mathrm{d}s\int_0^s P_{t-r}\left[\int_0^r P_h(Tf(T^{1/2}\cdot))\,\mathrm{d}h\right]^2(x)\,\mathrm{d}r\\
&\leqslant T^{-1}\int_0^t\mathrm{d}s\int_0^{t-s}[2\pi(s+r)]^{-1}\,\mathrm{d}r\int_0^t\mathrm{d}h\int_0^t\mathrm{d}h'\\
&\qquad\int_{R^2}\int_{R^2}p(h+h',T^{-1/2}y-T^{-1/2}z)f(y)f(z)\,\mathrm{d}y\,\mathrm{d}z\\
&\leqslant T^{-1}\int_0^\tau\mathrm{d}s\int_0^{\tau-s}[2\pi(s+r)]^{-1}\,\mathrm{d}r\int_0^\tau\mathrm{d}h\int_0^\tau[2\pi(h+h')]^{-1}\,\mathrm{d}h'\\
&\to 0,
\end{aligned}$$

as $T\to\infty$. Combining with Lemma 3.1 we get the pointwise convergence of $g_T(t,x)$.

One has

$$\|g_T(t,x)\|_{L^2}^2\leqslant C(\|g_T^{(1)}(t,x)\|_{L^2}^2+\|g_T^{(2)}(t,x)\|_{L^2}^2). \tag{3.7}$$

By (3.6) and calculations we have

$$\begin{aligned}
\|g_T^{(2)}(t,x)\|_{L^2}^2&\leqslant T^{-2}\int_{R^2}\left[\int_0^t\mathrm{d}s\int_0^s P_{t-r}\left[\int_0^r P_h(Tf(T^{1/2}\cdot))\,\mathrm{d}h\right]^2(x)\,\mathrm{d}r\right]^2\mathrm{d}x\\
&\leqslant T^{-2}\int_0^t\mathrm{d}s\int_0^{t-s}\mathrm{d}r\int_0^t\mathrm{d}s'\int_0^{t-s'}\mathrm{d}r'\int_{R^2}\int_{R^2}p(s+r+s'+r',y-z)
\end{aligned}$$

$$\times \left[\int_0^r P_h(Tf(T^{1/2}\cdot))\,\mathrm{d}h\right]^2(y)\left[\int_0^{r'} P_l(Tf(T^{1/2}\cdot))\,\mathrm{d}l\right]^2(z)\,\mathrm{d}y\,\mathrm{d}z$$

$$\leqslant T^{-2}\int_0^\tau \mathrm{d}s\int_0^{\tau-s}\mathrm{d}r\int_0^\tau \mathrm{d}s'\int_0^{\tau-s'}[2\pi(s+r+s'+r')]^{-1}\,\mathrm{d}r'$$

$$\times\left[\int_0^\tau \mathrm{d}h\int_0^\tau [2\pi(h+h')]^{-1}\,\mathrm{d}h'\right]\left[\int_0^\tau \mathrm{d}l\int_0^\tau [2\pi(l+l')]^{-1}\,\mathrm{d}l'\right]$$

$$\to 0,$$

as $T\to\infty$. Then from (3.7) and Lemma 3.1, we are done. □

Lemma 3.3. *Let* $d=2$, $w_T(t,x)$ *be the mild solution of Eq.* (3.3), i.e.,

$$\dot w_T(t,x)=\Delta w_T(t,x)-k_2w_T^2(t)+\bar v_T(t,x),$$
$$w_T(0)=0. \tag{3.8}$$

Then $w(t,x):=\lim_{T\to\infty} w_T(t,x)$ *exists in* $C([0,+\infty),L^2(\lambda))$ *and pointwise and,* $w(t,x)$ *is the mild solution of the equation*

$$\dot w(t,x)=\Delta w(t,x)-k_2w^2(t,x)+q_t(x),$$
$$w(0)=0, \tag{3.9}$$

where $q_t(x)=\int_0^t p(s,x)\,\mathrm{d}s$, *and* $p(s,x)$ *is the transition density function of the standard Brownian motion.*

Proof. The mild form of Eq. (3.8) is

$$w_T(t,x)=\int_0^t P_{t-s}\bar v_T(s,\cdot)(x)\,\mathrm{d}s-k_2\int_0^t P_{t-s}w_T^2(s,\cdot)(x). \tag{3.10}$$

Then

$$\|w_T(t,x)\|_{L^2}^2 \leqslant \int_{R^2}\left[\int_0^t P_{t-s}\bar v_T(s,\cdot)\,\mathrm{d}s\right]^2\mathrm{d}x$$

$$\leqslant \int_{R^2}\left[\int_0^t\int_0^s P_{t-r}Tf(T^{1/2}\cdot)\,\mathrm{d}r\,\mathrm{d}s\right]^2\mathrm{d}x$$

$$\leqslant \int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\int_0^t \mathrm{d}s'\int_0^{s'}[2\pi(2t-r-r')]^{-1}\,\mathrm{d}r'$$

$$<\infty$$

and with Lemma 3.2 in hand, the remaining proof is similar to that of Proposition 3.9 in Iscoe (1986b): firstly, we can prove that the limit $w(t,x)$ exists in $C([0,+\infty),L^2(\lambda))$. Then, the limit is taken in pointwise and satisfies (3.9). Finally, the mild solution of (3.9) is unique. We omit the details here. □

Lemma 3.4. $\lim_{T\to\infty}\langle\lambda, w_T(t,\cdot)\rangle = \langle\lambda, w(t,\cdot)\rangle$, *for* $t \geqslant 0$.

Proof. The mild form of (3.9) is

$$w(t,x) = \int_0^t \mathrm{d}s \int_0^s p(t-r,x)\,\mathrm{d}r - k_2 \int_0^t P_{t-s} w^2(s,\cdot)(x)\,\mathrm{d}s. \tag{3.11}$$

Then from Eqs. (3.10) and (3.11), we have

$$\langle\lambda, w_T(t,\cdot)\rangle = \int_0^t \langle\lambda, \bar{v}_T(s,\cdot)\rangle\,\mathrm{d}s - k_2 \int_0^t \langle\lambda, w_T^2(s,\cdot)\rangle\,\mathrm{d}s \tag{3.12}$$

and

$$\langle\lambda, w(t,\cdot)\rangle = \tfrac{1}{2}t^2 - k_2 \int_0^t \langle\lambda, w^2(s,\cdot)\rangle\,\mathrm{d}s. \tag{3.13}$$

But from (3.6) we know

$$\int_0^t \langle\lambda, \bar{v}_T(s,\cdot)\rangle\,\mathrm{d}s = \tfrac{1}{2}t^2 - k_1 T^{-1} \int_0^t \mathrm{d}s \int_0^s \langle\lambda, \bar{v}_T^2(r,\cdot)\rangle\,\mathrm{d}r.$$

By Lemmas 3.2 and 3.3, the two terms in the right-hand side of (3.12) converge to that of (3.13), respectively, proving the desired conclusion. □

Proof of Theorem 1.2. The Laplace functional of $T^{-2}Y_T$ is

$$\mathbf{E}\exp(-T^{-2}\langle Y_T^{\varrho}, f\rangle) = \exp(-\langle\lambda, v_T(T,\cdot)\rangle - \langle\lambda, u_T(T,\cdot)\rangle\,\mathrm{d}s), \quad f \in C_p^+(\mathbb{R}^d), \tag{3.14}$$

where $v_T(\cdot,\cdot)$ and $u_T(\cdot,\cdot)$ are the mild solutions of Eqs. (3.1) and (3.2), respectively.

By time and space transformation, let $w_T(t,x) := Tu_T(Tt, T^{1/2}x)$ and $\bar{v}_T(t,x) := T^2 v_T(Tt, T^{1/2}x)$. Then $w_T(t,X)$ and $\bar{v}_T(t,x)$ satisfy Eqs. (3.3) and (3.4), respectively. Furthermore, as $d = 2$, we have

$$\langle\lambda, w_T(1,\cdot)\rangle = \langle\lambda, u_T(T,\cdot)\rangle \tag{3.15}$$

and by Lemma 3.4, as $T \to \infty$, we get

$$\langle\lambda, u_T(T,\cdot)\rangle \to \langle\lambda, w(1,\cdot)\rangle, \tag{3.16}$$

where $w(\cdot,\cdot)$ is the mild solution of (3.9), i.e.,

$$\begin{aligned} &\dot{w}(t,x) = \Delta w(t,x) - k_2 w^2(t,x) + q_t(x),\\ &w(0) = 0. \end{aligned} \tag{3.17}$$

For the unnormalised case, we can replace f with θf, where $\theta > 0$ and $\langle\lambda, f\rangle = 1$, and arrive at

$$\begin{aligned} &\dot{w}(t,x) = \Delta w(t,x) - k_2 w^2(t,x) + \theta p_t(x),\\ &w(0) = 0. \end{aligned} \tag{3.18}$$

On the other hand, from Eq. (3.2), we have

$$\langle\lambda, v_T(T,\cdot)\rangle \leqslant \left\langle \lambda, \int_0^T P_s(T^{-2}f)\,\mathrm{d}s \right\rangle \to 0 \tag{3.19}$$

as $T \to \infty$.

Combining (3.16) and (3.19) with (3.14), we obtain that

$$\lim_{T\to\infty} \mathbf{E}\exp(-T^{-2}\langle Y_T^{\varrho}, f\rangle) = \exp(-\langle \lambda, w(1,\cdot;\theta)\rangle),$$

where $w(t,x;\theta)$ is given by (3.18), and the rest of the proof is similar to Iscoe (1986b). □

Furthermore, we have

Theorem 3.5. *Let $d = 2$. Then as $T \to \infty$*

$$T^{-2} Y_{Tt}^{\varrho} \to \xi_t \lambda, \quad \textit{weakly},$$

where ξ_t is a non-negative increasing stochastic process such that

$$\mathbf{E}\exp\{-\theta\xi_t\} = \exp\{-\langle \lambda, w(t,\cdot;\theta)\rangle\},$$

where $w(t,x;\theta)$ is the same as Theorem 1.2.

Combining Lemma 3.3, we can complete the proof which is similar to Theorem 3 in Iscoe (1986b). We omit the details.

4. Extension: higher dimensions

In Section 2, we have seen that the two kinds of random fluctuations in our model play different roles in different dimensions. We will consider more general branching mechanism is this section, but for more simplicity we take care of the case $k_1 = k_2 = 1$. Instead of the binary branching as considered above, let the branching mechanism of the underlying super-Brownian motion X be $\Psi_1(x) = x^{1+\beta_1}$, and that of the immigration process ϱ be $\Psi_2(x) = x^{1+\beta_2}$, where $0 < \beta_i \leqslant 1, i = 1,2$, and we get to the SBMSBI as in Section 1, also denote it as X^{ϱ}. Consider its occupation time process Y^{ϱ}, Lemma 2.1 is still valid. For fixed $f \in \mathscr{S}(\mathbb{R}^d)$, to the centered numerical process

$$z_t^{\varrho} := b_d(t)^{-1}[\langle Y_t^{\varrho}, f\rangle - \mathbf{E}\langle Y_t^{\varrho}, f\rangle],$$

where $b_d(t) := t^{2/(1+\beta_1)}$, we arrive at

Theorem 4.1. *Let $0 < \beta_i \leqslant 1,\ i = 1,2$. If*

$$d > \max\left\{\frac{2}{\beta_1} + 2,\ \frac{2}{\beta_2} + \frac{4\beta_1(1+\beta_2)}{\beta_2(1+\beta_1)}\right\} \tag{4.1}$$

then $z_t^{\varrho} \to z_\infty$ weakly as $t \to \infty$, where z_∞ is a stable random variable of index $1 + \beta_1$:

$$\mathbf{E}\exp\{-\theta z_\infty\} = \exp\left\{\theta^{1+\beta_1} \int \left[c_d \int \frac{f(y)}{|x-y|^{d-2}}\,\mathrm{d}y\right]^{1+\beta_1} \mathrm{d}x\right\},$$

where

$$c_d = \Gamma(d/2-1)[4(4\pi)^{d/2}]^{-1}.$$

Remark 4.1. If $\beta_1=\beta_2=1$, the result coincides with Theorem 1.1 in higher dimensions $d\geqslant 7$.

Proof of Theorem 4.1. As in Section 2, we get the Laplace transition functional of z_t^ϱ under **Q**,

$$\mathbf{E}\exp\{-z_t^\varrho\}=\exp\left\{\int_0^t\langle\lambda,\hat{v}_t(s,\cdot)^{1+\beta_1}\rangle\,\mathrm{d}s+\int_0^t\mathrm{d}s\int_0^s\langle\lambda,\hat{v}_t(r,\cdot)^{1+\beta_1}\rangle\,\mathrm{d}r\right.$$
$$\left.+\int_0^t\langle\lambda,\hat{u}_t(s,\cdot)^{1+\beta_2}\rangle\,\mathrm{d}s\right\} \tag{4.2}$$

where $\hat{u}_t(\cdot,\cdot)$ and $\hat{v}_t(\cdot,\cdot)$ are the mild solutions of the following equations respectively,

$$\hat{u}_t(r,x)=\int_0^r P_{t-s}\hat{v}_t(s,\cdot)(x)\,\mathrm{d}s-\int_0^r P_{r-s}\hat{u}_t^{1+\beta_2}(s,\cdot)(x)\,\mathrm{d}s,\quad t\geqslant r\geqslant 0 \tag{4.3}$$

and

$$\hat{v}_t(r,x)=\int_0^r P_s f_t(x)\,\mathrm{d}s-\int_0^r P_{r-s}\hat{v}_t^{1+\beta_1}(s,\cdot)(x)\,\mathrm{d}s,\quad t\geqslant r\geqslant 0 \tag{4.4}$$

with $f_t(x):=b_d(t)^{-1}f(x)$. In view of Theorem 5.4 of Iscoe (1986a), when $d>2/\beta_1+2$,

$$\lim_{t\to\infty}\int_0^t\mathrm{d}s\int_0^s\langle\lambda,\hat{v}_t(r,\cdot)^{1+\beta_1}\rangle\,\mathrm{d}r=\int\left[c_d\int\frac{f(y)}{|x-y|^{d-2}}\,\mathrm{d}y\right]^{1+\beta_1}\mathrm{d}x, \tag{4.5}$$

and

$$\lim_{t\to\infty}\int_0^t\langle\lambda,\hat{v}_t(s,\cdot)^{1+\beta_1}\rangle\,\mathrm{d}s=0. \tag{4.6}$$

In what follows, we will prove the third term in the bracket on the right-hand side of (4.2) goes to zero as $t\to\infty$.

By condition (4.1), let

$$0<2\varepsilon<\left[d-\frac{2}{\beta_2}-\frac{4\beta_1(1+\beta_2)}{\beta_2(1+\beta_1)}\right](1+\beta_1)\quad\text{and}\quad\alpha=\frac{(2-\varepsilon)\beta_2+(1-\beta_1)}{(1+\beta_1)(1+\beta_2)}.$$

From Eqs. (4.3) and (4.4),we get

$$\lim_{t\to\infty}\int_0^t\langle\lambda,\hat{u}_t(s,\cdot)^{1+\beta_2}\rangle\,\mathrm{d}s$$
$$\leqslant\lim_{t\to\infty}b_d(t)^{-(1+\beta_2)}\int_0^t\left\langle\lambda,\left[\int_0^s\mathrm{d}r\int_0^r P_{s-r+h}f\,\mathrm{d}h\right]^{1+\beta_2}\right\rangle\mathrm{d}s$$
$$=\lim_{t\to\infty}t^{-2(1+\beta_2)/1+\beta_1+1}\left\langle\lambda,\left[\int_0^t\mathrm{d}r\int_0^r P_{t-h}f\,\mathrm{d}h\right]^{1+\beta_2}\right\rangle\quad\text{(by l'Hospital's rule)}$$
$$=\lim_{t\to\infty}t^{-2(1+\beta_2)/1+\beta_1+1}\left\langle\lambda,\left[\int_0^t hP_hf\,\mathrm{d}h\right]^{1+\beta_2}\right\rangle\quad\text{(interchange the integration)}$$

$$\leqslant \lim_{t\to\infty} t^{-[2(1+\beta_2)/1+\beta_1-1-\alpha(1+\beta_2)]}\left\langle \lambda, \left[\int_0^t h^{1-\alpha}P_h f\,\mathrm{d}h\right]^{1+\beta_2}\right\rangle.$$

By simple calculations, we have

$$\left\langle \lambda, \left[\int_0^\infty h^{1-\alpha}P_h f\,\mathrm{d}h\right]^{1+\beta_2}\right\rangle = \int \left[C_\alpha \int \frac{f(y)}{|x-y|^{d-4+2\alpha}}\,\mathrm{d}y\right]^{1+\beta_2}\,\mathrm{d}x$$

(C_α is a positive constant) which is finite by Lemma 5.3 of Iscoe (1986a), because $d > ((4-2\alpha)/\beta_2 + 4 - 2\alpha)$ with the chosen α. and then

$$\lim_{t\to\infty} \int_0^t \langle \lambda, \hat{u}_t(s,\cdot)^{1+\beta_2}\rangle\,\mathrm{d}s = 0 \tag{4.7}$$

because

$$\frac{2(1+\beta_2)}{1+\beta_1} - 1 - \alpha(1+\beta_2) = \frac{\varepsilon\beta_2}{(1+\beta_1)} > 0.$$

Then combining (4.5)–(4.7) with (4.2), completes the proof. □

Remark 4.2. With β_1 and β_2 labeling on the two kinds of branching mechanism in our model make us to see more clearly which fluctuation contributes to the limit behavior, i.e., the immigration particles taking the underling branch contribute to the limit behavior in high dimension.

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References

Dawson, D.A., 1977. The critical measure diffusion process. Z. Wahrsch. verw. Geb. 40, 125–145.

Dawson, D.A., 1993. Measure-valued Markov processes. Lecture Notes in Mathematics, Vol. 1541, Springer, Berlin, pp. 1–260.

Dawson, D.A., Fleischmann, K., 1997. A continuous super-Brownian motion in a super–Brownian medium. J. Theoret. Probab. 10, 213–276.

Dynkin, E.B., 1991. Branching particle systems and superprocesses. Ann. Probab. 19, 1157–1194.

Gorostiza, L.G., Lopez-Mimbela, J.A., 1990. The multitype measure branching process. Adv. Appl. Probab. 22, 49–67.

Hong, W.M., 2000a. Ergodic theorem for the two-dimensional super-Brownian motion with super-Brownian immigration. Prog. Nat. Sci. 10, 111–116.

Hong, W.M., 2000b. Large deviations for the super-Brownian motion with super-Brownian immigration. Preprint.

Hong, W.M., Li, Z.H., 1999. A central limit theorem for the super-Brownian motion with super-Brownian immigration. J. Appl. Probab. 36, 1218–1224.

Iscoe, I., 1986a. A weighted occupation time for a class of measure-valued critical branching Brownian motion. Probab. Theory Related Fields 71, 85–116.
Iscoe, I., 1986b. Ergodic theory and a local occupation time for measure-valued critical branching Brownian motion. Stochastics 18, 197–243.
Li, Z.H., 1992. Measure-valued branching processes with immigration. Stochastic Proc. Appl. 43, 249–264.
Li, Z.H., 1996. Immigration structures associated with Dawson–Watanabe superprocesses. Stochastic Proc. Appl. 62, 73–86.
Li, Z.H., Shiga, T., 1995. Measure-valued branching diffusions: immigrations, excursions and limit theorems. J. Math. Kyoto Univ. 35, 233–274.
Li, Z.H., Wang, Z.K., 1999. Measure-valued branching processes and immigration processes. Adv. in Math. China 28, 105–134.
Mytnik, L., 1996. Superprocesses in random environments. Ann. Probab. 24, 1953–1978.