

Branching structure for the transient random walk on a strip in a random environment

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Abstract An intrinsic branching structure within the transient random walk on a strip in a random environment is revealed. As applications, which enables us to express the hitting time explicitly, and specifies the density of the absolutely continuous invariant measure for the “environments viewed from the particle”.

Keywords: Branching structure; random walk on a strip; random environment; hitting time, invariant measure, environments viewed from the particle.

Mathematics Subject Classification: Primary 60J80; secondary 60G50.

1 Introduction

Let $d \geq 1$ be any integer and denote $\mathcal{D} = \{1, 2, \dots, d\}$, we consider random walks in a random environment on the strip $S = \mathbb{Z} \times \{1, 2, \dots, d\}$. This model was introduced by Bolthausen and Goldsheid ([1], 2000), where the conditions for recurrent and transient has been obtained. After that, Goldsheid ([4], 2008) considered the hitting time of the walk by the method of “enlarged random environments”; Bolthausen and Goldsheid ([2], 2008) obtained the $(\log t)^2$ asymptotic behaviour and Roitershtein ([10], 2008) proved a strong law of large numbers and an annealed central limit theorem for the walk in a suitable environment situation; etc..

The aim of this paper is to reveal the intrinsic branching structure within the transient random walk on a strip in a random environment, which enables us to express the hitting time explicitly. Roitershtein (Theorem 2.3, [10], 2008) figured out the stationary distribution for the Markov chain of “environments viewed from the particle” which is equivalent to the original distribution. To specify the density of the absolutely continuous invariant measure is another application of our branching structure. And as a by product, the rate of the LLN can be obtained.

For the nearest random walk in random environment (RWRE, for short) on the line, as we known, the branching structure is a powerful tool in the proof of the famous result about “stable law” by Kesten et al ([8], 1975), and is also used by Dembo et al ([3], 1996), Ganterta and Shi in ([5], 2002), etc. The branching structure for the one dimensional RWRE with bounded jumps has been considered by Key ([9], 1987), Hong & Wang ([6], 2009) and Hong & Zhang ([7], 2010).

1.1 Description of the model.

We adapt the description of the model as that of [1]. Let $(P_n, Q_n, R_n), -\infty < n < \infty$, be a strictly stationary ergodic sequence of triples of $m \times m$ matrices with non-negative elements such that for all n the sum $P_n + Q_n + R_n$ is a stochastic matrix, i.e., $(P_n + Q_n + R_n)\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is a column vector whose components are all equal to 1. We write the components of P_n as $P_n(i, j), 1 \leq i, j \leq m$, and similarly for Q_n and R_n . Let $(\Omega, \mathcal{F}, P, \theta)$ be the corresponding dynamical system with Ω denoting the space of all sequences $\omega := (\omega_n) = ((P_n, Q_n, R_n))$ of triples described above, \mathcal{F} being the corresponding natural σ -algebra, P denoting the probability measure on (Ω, \mathcal{F}) , and the shift operator on Ω defined by $\theta: (\theta\omega)_n = \omega_{n+1}, n \in \mathbb{Z}$. The random walk on the strip $S = \mathbb{Z} \times \mathcal{D} := \mathbb{Z} \times \{1, 2, \dots, d\}$ is denoted by $X = \{X_n, n \in \mathbb{Z}\}$,

$$X_n = (\xi_n, Y_n), \quad \xi_n \in \mathbb{Z}, Y_n \in \mathcal{D}.$$

ξ_n is the \mathbb{Z} -coordinate of the walk and Y_n takes values in $\mathcal{D} := \{1, 2, \dots, d\}$.

For describing the initial distribution, we introduce \mathcal{M}_d ,

$$\mathcal{M}_d = \left\{ (\mu_\omega)_{\omega \in \Omega} : \mu_\omega \text{ is a probability measure vector on } \mathcal{D} = \{1, 2, \dots, d\} \right\}.$$

Given an environment $\omega \in \Omega$ and an $\mu = (\mu_\omega) \in \mathcal{M}_d$, one can define the random walk X_n on the strip $S = \mathbb{Z} \times \mathcal{D}$ to be a time-homogeneous Markov chain taking values in $\mathbb{Z} \times \{1, 2, \dots, d\}$, which is determined by its transition probabilities $\mathfrak{Q}_\omega(z, z_1)$:

$$\mathfrak{Q}_\omega(z, z_1) = \begin{cases} P_n(i, j) & \text{if } z = (n, i), z_1 = (n+1, j), \\ R_n(i, j) & \text{if } z = (n, i), z_1 = (n, j), \\ Q_n(i, j) & \text{if } z = (n, i), z_1 = (n-1, j), \\ 0 & \text{otherwise,} \end{cases}$$

and initial distribution

$$P_\omega^\mu(\xi_0 = 0, Y_0 = z_0) = \mu_\omega(z_0) \quad \text{for any } z_0 \in \mathcal{D}. \quad (1.1)$$

This defines for any starting point $x_0 = (0, y_0) \in S$ and for any $\omega \in (\Omega, \mathcal{F}, P)$, the quenched law P_ω^μ for the Markov chain by

$$P_\omega^\mu(X_0 = x_0, X_1 = x_1, \dots, X_n = x_n) := \mu_\omega(y_0) \mathfrak{Q}_\omega(x, x_1) \mathfrak{Q}_\omega(x_1, x_2) \cdots \mathfrak{Q}_\omega(x_{n-1}, x_n). \quad (1.2)$$

Then we define an annealed law $\mathbb{P}^\mu = P \otimes P_\omega^\mu$ on $(\Omega \times (\mathbb{Z} \times \mathcal{D})^\mathbb{N}, \mathcal{F} \times \mathcal{G})$ by

$$\mathbb{P}^\mu(F \times G) = \int_F P_\omega^\mu(G) P(d\omega) \quad F \in \mathcal{F}, G \in \mathcal{G}, \quad (1.3)$$

and the expectation with respect to \mathbb{P}^μ defined by \mathbb{E}^μ . Statements involving P_ω^μ and \mathbb{P}^μ are called quenched and annealed, respectively.

Notations and assumption. Throughout the paper we use the notation $\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^d$, $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^d$, and denote $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$, ($i = 1, 2, \dots, d$) as the canonical basis of \mathbb{R}^d . For the vector $\mathbf{x} = (x_j)$ and matrix $A = (a(i, j))$, define

$$\|\mathbf{x}\| := \max_j |x_j| \quad \text{and} \quad \|A\| := \max_i \sum_j |a(i, j)|.$$

We say that A is strictly positive (denoted by $A > 0$) if all its components satisfy $a(i, j) > 0$, and A is non-negative (which is denoted by $A \geq 0$) if all $a(i, j)$ are negative. If a $d \times d$ real matrix A is non-negative, $\|A\| := \|A\mathbf{1}\|$. Finally, we use the notation $\mathbf{1}_A$ for the indicator function of the set A . For the random walk $X_n = (\xi_n, Y_n)$, we often use the expressions like $\lim_{n \rightarrow \infty} X_n = +\infty$ which simply means ξ_n tends to $+\infty$ as $n \rightarrow \infty$.

The *hitting time* T_n is defined as the the first time when the walk reaches layer n , $L_n := \{(n, j), 1 \leq j \leq m\}$ starting from a point $z \in L_0 := \{(0, j), 1 \leq j \leq m\}$. Let $T_o = 0$, and for $n \geq 1$,

$$T_n := \inf\{t : X(t) \in L_n\} \quad \text{and} \quad \tau_n := T_n - T_{n-1}, \quad (1.4)$$

with the usual convention that the infimum over an empty set is ∞ and $\infty - \infty = \infty$.

The following Condition C is from Bolthausen and Goldshied [1].

Condition C.

C1 The dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{T})$ is ergodic.

C2

$$\mathbb{E} \log(1 - \|R_n + P_n\|)^{-1} < \infty \quad \text{and} \quad \mathbb{E} \log(1 - \|R_n + Q_n\|)^{-1} < \infty. \quad (1.5)$$

C3 For all $j \in \{1, 2, \dots, m\}$ and all n ,

$$\sum_{i=1}^m Q_n(i, j) > 0, \quad \sum_{i=1}^m P_n(i, j) > 0 \quad \mathbb{P}\text{-almost surely.} \quad (1.6)$$

C4 With positive \mathbb{P} -probability, the layer 0 is in one communication class.

Known results. Let us first review some known results about the random walk in a random environment on the strip.

1. *recurrence and transience.* If Condition C is satisfied, Theorem 1 in [1] proved ζ_n , $n \in \mathbb{Z}$ of $m \times m$ matrices is the unique sequence of stochastic matrices which satisfies the following system of equations:

$$\zeta_n = (I - Q_n \zeta_{n-1} - R_n)^{-1} P_n, \quad P - a.s. \quad n \in \mathbb{Z}, \quad (1.7)$$

and the enlarged sequence (P_n, Q_n, R_n, ζ_n) , $-\infty < n < \infty$, is stationary and ergodic.

Let

$$A_n := (I - Q_n \zeta_{n-1} - R_n)^{-1} Q_n \quad \text{and} \quad u_n := (I - Q_n \zeta_{n-1} - R_n)^{-1} \mathbf{1} \quad (1.8)$$

and

$$\lambda^+ := \lim_{n \rightarrow \infty} \frac{1}{n} \log \| A_n A_{n-1} \cdots A_1 \|, \quad (1.9)$$

Theorem 2 in [1] gave the criterion of recurrent and transient behavior for $X_n = (\xi_n, Y_n)$. One of the cases is

$$\lim_{t \rightarrow \infty} \xi(t) = \infty, \quad \mathbb{P} - a.e. \quad \text{if and only if} \quad \lambda^+ < 0. \quad (1.10)$$

2. *exit probability.* Let $\eta_n(i, j)$ be the probability of a random walk starting in (n, i) reaches the layer $n + 1$ at point (n, j) finally, we usually called it the exiting probability. If the random walk is transient to the right, we have $\eta_n = \zeta_n$, $P - a.e.$ (see [4], (1.15)). And if Condition C is satisfied, $\zeta_n > 0$ for $P - a.s.$ ω .

We only concentrate on random walks which are transient to the right in our paper.

3. *stationary sequence of probability vectors y_n .*

If Condition C is satisfied then following limit exists for $P - a.s.$ ω (Lemma 1, [4]):

$$\mathbf{y}_n := \lim_{a \rightarrow -\infty} \mathbf{u}_a \zeta_a(\omega) \zeta_{a+1}(\omega) \cdots \zeta_n(\omega). \quad (1.11)$$

where \mathbf{u}_a is any sequence of row-vectors with non-negative components $u_a(i)$, and $\sum_{i=1}^d u_a(i) = 1$. Note that the sequence $\{\mathbf{y}_n\}$ is the unique solution of $\mathbf{y}_n = \mathbf{y}_{n-1} \zeta_n$ in the class of probability vectors and it has the property $y_n > 0$, which is a probability measure on $\mathcal{D} = \{1, 2, \dots, d\}$ whose support is the whole \mathcal{D} . It is clear that vectors $\mathbf{y}_n := \mathbf{y}(\omega_{\leq n})$ form a stationary sequence.

1.2 Statement of main results.

We assume the walk $X_n = (\xi_n, Y_n)$ starts from layer 0, the initial distribution $P_\omega^\mu(\xi_0 = 0, Y_0 = i) = \mu_\omega(i)$, $P - a.s.$ ω , for any $i \in \mathcal{D}$ with $\mu_\omega \in \mathcal{M}_d$. In what follows, suppose Condition C is satisfied and $\lambda^+ < 0$, i.e., we concentrate on random walks $X_n = (\xi_n, Y_n)$ transient to the right $X_n \rightarrow +\infty$, $\mathbb{P} - a.s.$. In this case, suppose $T_0 = 0$ and we have $T_k < \infty$, $\mathbb{P} - a.s.$ for any positive integer $k \geq 1$. The aim of this paper is to calculate the hitting time $T_1 = \inf\{i : \xi(i) = 1\}$ accurately in terms of the intrinsic branching structure within the walk. For $n \leq 1$, define

$\mathbf{U}_n = (U_n^1, U_n^2, \dots, U_n^d)$, where U_n^i ($1 \leq i \leq d$) is the number of steps from layer n to layer $n - 1$ at the site $(n - 1, i)$ before time T_1 .

$\mathbf{Z}_n = (Z_n^1, Z_n^2, \dots, Z_n^d)$, where Z_n^i ($1 \leq i \leq d$) is the number of steps from layer n to the same layer at the site (n, i) before time T_1 .

And

$$|\mathbf{U}_n| = \sum_{i=1}^d U_n^i = \mathbf{U}_n \mathbf{1} \quad \text{and} \quad |\mathbf{Z}_n| = \sum_{i=1}^d Z_n^i = \mathbf{Z}_n \mathbf{1}. \quad (1.12)$$

All steps before T_1 can be recorded by \mathbf{U}_n and \mathbf{Z}_n . Since $X_n \rightarrow +\infty$, $\mathbb{P} - a.s.$, if the random walk takes a step to the left from any layer n ($n \leq 0$), it must come back finally

from layer $n - 1$ to layer n , so

$$T_1 = 1 + \sum_{n \leq 0} (2|\mathbf{U}_n| + |\mathbf{Z}_n|),$$

and the following theorem tells us that $\{|\mathbf{U}_n|, |\mathbf{Z}_n|, n \leq 1\}$ is an inhomogeneous branching process with immigration. The exit probability η_n plays an important role, when $X_n \rightarrow +\infty$, $\eta_n = \zeta_n$, $P - a.e.$, (see [4], (1.15)), which is given by (1.7).

Theorem 1.1 *Assume Condition C is satisfied and $X_n \rightarrow +\infty$, $\mathbb{P} - a.s.$, the initial distribution $P_\omega^\mu(\xi_0 = 0, Y_0 = i) = \mu_\omega(i)$, $P - a.s.$. Then*

(1) *for $P - a.s.$ ω , $\{|\mathbf{U}_n|, n \leq 1\}$ and $\{|\mathbf{Z}_n|, n \leq 1\}$ are inhomogeneous branching processes with immigration. The offspring distribution is given by for $n \leq 0$*

$$P_\omega^\mu\left(|\mathbf{U}_n| = m \mid \mathbf{U}_{n+1} = \mathbf{e}_i\right) = \mathbf{e}_i[(I - R_n)^{-1}Q_n\zeta_{n-1}]^m(I - R_n)^{-1}P_n\mathbf{1}, \quad (1.13)$$

$$P_\omega^\mu\left(|\mathbf{Z}_n| = K \mid \mathbf{U}_{n+1} = \mathbf{e}_i\right) = \mathbf{e}_i[(I - Q_n\zeta_{n-1})^{-1}R_n]^K(I - Q_n\zeta_{n-1})^{-1}P_n\mathbf{1}, \quad (1.14)$$

with immigration

$$P_\omega^\mu\left(\mathbf{U}_1 = \mathbf{e}_i\right) = \mu_\omega(i), \quad i \in \mathcal{D}, \quad (1.15)$$

where $\zeta_n = \eta_n$ (see [4], (1.15)) is exit probability, which is given by (1.7).

(2) *The first hitting time T_1 is given by*

$$T_1 = 1 + \sum_{n \leq 0} (2|\mathbf{U}_n| + |\mathbf{Z}_n|). \quad (1.16)$$

□

Remark (1) In Theorem 1.1, we restrict ourselves only to the trajectory of the walk X_t for $t \in [0, T_1]$, and all the steps have been counted in $\{|\mathbf{U}_n|, |\mathbf{Z}_n|, n \leq 1\}$ which formulate a branching structure as (1.13) and (1.14) with immigration (1.15). After that, the trajectory of the walk X_t follows the same structure. For example, the trajectory of the walk X_t for $t \in [T_1, T_2]$, all the steps have been counted in $\{|\mathbf{U}_n|, |\mathbf{Z}_n|, n \leq 2\}$ which formulate a branching structure as (1.13) and (1.14) with immigration $P_\omega^\mu(\mathbf{U}_2 = \mathbf{e}_i) = Y_{T_1}(i)$, and so on.

(2) Note that it is “unsymmetrical” in the branching structure (1.13) and (1.14) between the “father” and “children”. It can be explained as that we focus on the number of the “children” but the individual of the “father” (determine the probability). □

As an immediate application of the branching structure, we can calculate the mean of the hitting time explicitly.

Theorem 1.2 *Assume Condition C is satisfied and $X_n \rightarrow +\infty$, $\mathbb{P} - a.s.$, and the initial distribution $P_\omega^\mu(\xi_0 = 0, Y_0 = i) = \mu_\omega(i)$, $P - a.s.$. Then*

$$ET_1 = E(\vec{\mu}_\omega^\rightarrow(u_0 + A_0u_{-1} + \cdots + A_0A_{-1} \cdots A_{-k}u_{-k-1} + \cdots)),$$

where A_n, u_n is given in (1.8). □

Another application of the branching structure is to specify the density of the absolutely continuous invariant measure for the “environments viewed from the particle”. Let us review the process discussed in Section 4 of [10]. Let $\bar{\omega}_n = \theta^{\xi_n} w$, for $n \geq 0$, and consider the process $Z_n := (\bar{\omega}_n, Y_n)$, defined in $(\Omega \times \mathcal{D}, \mathcal{F} \otimes \mathcal{H})$, where \mathcal{H} as the set of all subsets of \mathcal{D} , and the initial distribution $P_\omega^\mu(\xi_0 = 0, Y_0 = i) = \mu_\omega(i) = \mathbf{y}_{-1}(i)$ given by (1.11). $(Z_n)_{n \geq 0}$ is a Markov chain under \mathbb{P}^μ with transition kernel

$$K(\omega, i; B, j) = P_0(i, j)I_B(\theta w) + R_0(i, j)I_B(w) + Q_0(i, j)I_B(\theta^{-1}w). \quad (1.17)$$

Usually, $Z_n = (\bar{\omega}_n, Y_n)$ be called as auxiliary Markov chain.

Let $v_p = \frac{1}{\mathbb{E}T_1}$, whenever $\mathbb{E}T_1 < \infty$. For $B \in \mathcal{F}$, $i \in \mathcal{D}$, define a probability measure Q on $(\Omega \times \mathcal{D}, \mathcal{F} \otimes \mathcal{H})$:

$$Q(B, i) := v_p \mathbb{E} \left(\sum_{n=0}^{T_1-1} I_B(\bar{\omega}_n) I_{Y_n}(i) \right). \quad (1.18)$$

$Q(\cdot)$ is a invariant measure under the Markov kernel K (Proposition 4.1, [10]).

Define a probability measure $\bar{Q}(\cdot)$ on (Ω, \mathcal{F}) by setting

$$\bar{Q}(B) := Q(B, \mathcal{D}), \quad B \in \mathcal{F}. \quad (1.19)$$

and let $Q_i(B) := Q(B, i)$ for $B \in \mathcal{F}$. Then both $Q_i(\cdot)$ and $\bar{Q}(\cdot)$ are absolutely continuous with regard to P (Proposition 4.1, [10]), but where only the up bound of the density have been proved. The branching structure enable us to specify the density completely in the following theorem.

Theorem 1.3 *Assume Condition C is satisfied and $X_n \rightarrow +\infty$, \mathbb{P} - a.s., the initial distribution $P_\omega^\mu(\xi_0 = 0, Y_0 = i) = \mu_\omega(i) = \mathbf{y}_{-1}(i)$, for P - a.s. ω , and assume in addition that $v_p > 0$. Then $Q_i(\cdot)$ is absolutely continuous with regard to P , and so is $\bar{Q}(\cdot)$. The density is given by*

$$\frac{dQ_i}{dP} = \Lambda_\omega^{(i)}, \quad (1.20)$$

where

$$\Lambda_\omega^{(i)} = v_p [\mu_\omega(\tilde{u}_0 + \zeta_0 A_1 \tilde{u}_0 + \zeta_0 \zeta_1 A_2 A_1 \tilde{u}_0 + \cdots)](i). \quad (1.21)$$

and

$$\frac{d\bar{Q}}{dP} = \Lambda_\omega, \quad (1.22)$$

where

$$\Lambda_\omega = v_p [\mu_\omega(\tilde{u}_0 + \zeta_0 A_1 \tilde{u}_0 + \zeta_0 \zeta_1 A_2 A_1 \tilde{u}_0 + \cdots)] \mathbf{1}, \quad (1.23)$$

where $\tilde{u}_n := (I - Q_n \zeta_{n-1} - R_n)^{-1}$. \square

Remark (1) The first part of the Theorem 1.3 is obtained in Proposition 4.1 of [10]. We will focus on the “density” part only.

(2) As a by product, we can prove the LLN from two different method as the situation for the nearest RWRE on the line ([13]). On the one hand, If $\vec{\mu}_\omega = \mathbf{y}_{-1}$, then $\{\tau_i : i \in N\}$ in (1.4) is a stationary and ergodic sequence variables (Lemma 3.2, [10]), so the LLN can be obtained from the hitting time decomposition; on the other hand, with the “density” in hand, it is easy to obtain the LLN again from the point of view “environments viewed from the particle”. We omit the details of the proof.

Corollary 1.4 *Assume Condition C is satisfied and $X_n \rightarrow +\infty$, $\mathbb{P} - a.s.$, the initial distribution $P_\omega^\mu(\xi_0 = 0, Y_0 = i) = \mu_\omega(i) = \mathbf{y}_{-1}(i)$, for $P - a.s.$ ω , and assume in addition that $v_p > 0$. Then $\mathbb{P} - a.s.$,*

$$\lim_{n \rightarrow \infty} \frac{\xi(n)}{n} = \frac{1}{E\left(\mathbf{y}_{-1}(u_0 + A_0 u_{-1} + \cdots + A_0 A_{-1} \cdots A_{-k} u_{-k-1} + \cdots)\right)}. \quad (1.24)$$

□

2 Proofs

2.1 Intrinsic branching structure—Proof of Theorem 1.1.

Assume Condition C is satisfied and $X_n \rightarrow +\infty$, $\mathbb{P} - a.s.$, the initial distribution $P_\omega^\mu(\xi_0 = 0, Y_0 = i) = \mu_\omega(i)$, $P - a.s.$. Note that $T_k < \infty$, $\mathbb{P} - a.s.$ for any positive integer $k \geq 1$. We will analyze the trajectory of the walk, and restrict to the first excursion between lay 0 to lay 1, i.e., the path of X_k for $k \in [0, T_1]$. Define for $n \leq 0$,

$$\begin{aligned} \alpha_{n,0} &= \min\{k \leq T_1 : X_k \in L_n\}, \\ \beta_{n,0} &= \min\{\alpha_{n,0} < k \leq T_1 : X_{k-1} \in L_n, X_k \in L_{n-1}\}. \end{aligned}$$

And for $b \geq 1$,

$$\begin{aligned} \alpha_{n,b} &= \min\{\beta_{n,b-1} < k \leq T_1 : X_k \in L_n\}, \\ \beta_{n,b} &= \min\{\alpha_{n,b} < k \leq T_1 : X_{k-1} \in L_n, X_k \in L_{n-1}\}. \end{aligned}$$

(with the usual convention that the minimum over an empty set is $+\infty$).

We refer to the time interval $[\beta_{n,b-1}, \alpha_{n,b}]$ as the b -th excursion from $n-1$ layer to n layer.

For any $b \geq 0$, any $n \leq 0$, and $i \in \{1, 2, \dots, d\}$, define

$$U_{n,b}^i := \#\{k \geq 0 : X_{k-1} \in L_n, X_k = (n-1, i), \beta_{n+1,b} < k < \alpha_{n+1,b+1}\}, \quad (2.1)$$

$$Z_{n,b}^i := \#\{k \geq 0 : X_{k-1} \in L_n, X_k = (n, i), \beta_{n+1,b} < k < \alpha_{n+1,b+1}\}. \quad (2.2)$$

Note that $U_{n,b}^i$ is the number of steps from layer n to $(n-1, i)$ during the $b+1$ -th excursion from layer n to layer $n+1$, whereas $Z_{n,b}^i$ is the number of steps from layer n to (n, i) during the same excursion.

Define for $n \leq 0$ and $i \in \{1, 2, \dots, d\}$, $U_n^i := \sum_{b \geq 0} U_{n,b}^i$, then U_n^i is the number of steps from layer n to $(n-1, i)$ before time T_1 . Similarly define $Z_n^i := \sum_{b \geq 0} Z_{n,b}^i$. $\mathbf{U}_n = (U_n^1, U_n^2, \dots, U_n^d)$, and $|\mathbf{U}_n| = \sum_{i=1}^d U_n^i = \mathbf{U}_n \mathbf{1}$; $\mathbf{Z}_n = (Z_n^1, Z_n^2, \dots, Z_n^d)$, and $|\mathbf{Z}_n| = \sum_{i=1}^d Z_n^i = \mathbf{Z}_n \mathbf{1}$ which have been defined in (1.12).

By Markov property, we obtain

$$\begin{aligned} & P_\omega^\mu \left(|\mathbf{U}_n| = m, |\mathbf{Z}_n| = K \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right) \\ &= \mathbf{e}_i \sum_{k_0+k_1+\dots+k_m=K} R_n^{k_0} Q_n \zeta_{n-1} R_n^{k_1} \cdots Q_n \zeta_{n-1} R_n^{k_m} P_n \mathbf{1}. \end{aligned} \quad (2.3)$$

where $\zeta_n = \eta_n$ is the exiting probability matrix (see (1.7)).

In (2.3), the path of an excursion is considered: the particle start from layer n (given by $\mathbf{U}_{n+1} = \mathbf{e}_i$), moves at layer n by $|\mathbf{Z}_n| = K$ steps (each step with probability R_n) and $|\mathbf{U}_n| = m$ steps from layer n to layer $n-1$ (but in the trajectory point, each “down” step with probability Q_n must connect with a path “from layer $n-1$ finally goes back to layer n ” with probability ζ_{n-1}), the last step of the excursion is from layer n to layer $n+1$ with probability P_n .

The idea of (2.3) is that we only care the number of the “children”, which lead to the “unsymmetrical”. Note that only the “U” type particles produce “children”. With a similar consideration, the branching mechanism can also be expressed as

$$\begin{aligned} & P_\omega^\mu \left(|\mathbf{U}_n| = m, |\mathbf{Z}_n| = K \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right) \\ &= \mathbf{e}_i \sum_{m_0+m_1+\dots+m_K=m} Q_n \zeta_{n-1}^{m_0} R_n Q_n \zeta_{n-1}^{m_1} \cdots R_n Q_n \zeta_{n-1}^{m_K} P_n \mathbf{1}. \end{aligned} \quad (2.4)$$

In what follows, we will derive the marginal distribution of $|\mathbf{U}_n|$ and $|\mathbf{Z}_n|$ respectively. Let's discuss the marginal distribution of $|\mathbf{U}_n|$ first, summarize over K in (2.3),

$$\begin{aligned} & P_\omega^\mu \left(|\mathbf{U}_n| = m \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right) \\ &= \sum_{K=0}^{+\infty} P_\omega^\mu \left(|\mathbf{U}_n| = m, |\mathbf{Z}_n| = K \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right) \\ &= \mathbf{e}_i \left[\sum_{K=0}^{+\infty} \sum_{k_0+k_1+\dots+k_m=K} R_n^{k_0} Q_n \zeta_{n-1} R_n^{k_1} \cdots Q_n \zeta_{n-1} R_n^{k_m} P_n \mathbf{1} \right]. \end{aligned} \quad (2.5)$$

It's not hard to see

$$\begin{aligned} & \sum_{K=0}^{+\infty} \sum_{k_0+k_1+\dots+k_m=K} R_n^{k_0} Q_n \zeta_{n-1} R_n^{k_1} \cdots Q_n \zeta_{n-1} R_n^{k_m} \\ &= (I - R_n)^{-1} Q_n \zeta_{n-1} (I - R_n)^{-1} \cdots Q_n \zeta_{n-1} (I - R_n)^{-1} \\ &= [(I - R_n)^{-1} Q_n \zeta_{n-1}]^m (I - R_n)^{-1}. \end{aligned} \quad (2.6)$$

Taking together (2.5) and (2.6), derives the marginal distribution of $|\mathbf{U}_n|$,

$$P_\omega^\mu \left(|\mathbf{U}_n| = m \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right) = \mathbf{e}_i [(I - R_n)^{-1} Q_n \zeta_{n-1}]^m (I - R_n)^{-1} P_n \mathbf{1}. \quad (2.7)$$

For the marginal distribution of $|\mathbf{Z}_n|$, summarize over m in (2.4), we have

$$\begin{aligned}
& P_\omega^\mu \left(|\mathbf{Z}_n| = K \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right) \\
&= \mathbf{e}_i \sum_{m=0}^{+\infty} \sum_{m_0+m_1+\dots+m_K=m} Q_n \zeta_{n-1}^{m_0} R_n Q_n \zeta_{n-1}^{m_1} \cdots R_n Q_n \zeta_{n-1}^{m_K} P_n \mathbf{1} \\
&= \mathbf{e}_i (I - Q_n \zeta_{n-1})^{-1} R_n (I - Q_n \zeta_{n-1})^{-1} \cdots R_n (I - Q_n \zeta_{n-1})^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i [(I - Q_n \zeta_{n-1})^{-1} R_n]^K (I - Q_n \zeta_{n-1})^{-1} P_n \mathbf{1}. \tag{2.8}
\end{aligned}$$

Complete the proof of part (1) of Theorem (1.1); and part (2) is immediate. \square

Remark. From the marginal distribution, we also can test of the validity of the branching structure. In fact,

$$\begin{aligned}
\sum_{m=0}^{+\infty} P_\omega^\mu \left(|\mathbf{U}_n| = m \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right) &= \mathbf{e}_i \left[\sum_{m=0}^{+\infty} [(I - R_n) Q_n \zeta_{n-1}]^m (I - R_n)^{-1} P_n \mathbf{1} \right. \\
&= \mathbf{e}_i [I - (I - R_n) Q_n \zeta_{n-1}]^{-1} (I - R_n)^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i [(I - R_n) - Q_n \zeta_{n-1}]^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i \zeta_n \mathbf{1} = 1.
\end{aligned}$$

2.2 ET_1 —Proof of Theorem 1.2

The random walk $X_n = (\xi_n, Y_n)$ starts from layer 0 with the initial distribution μ_ω . With the branching structure in hand, we can calculate the mean of the first hitting time T_1 . We discuss it by four steps as follows.

Step 1. $E_\omega^\mu \left(|\mathbf{U}_n| \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right)$ and $E_\omega^\mu \left(|\mathbf{Z}_n| \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right)$.

From (1.13) of Theorem 1.1,

$$\begin{aligned}
E_\omega^\mu \left(|\mathbf{U}_n| \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right) &= \sum_{m=0}^{+\infty} m P_\omega^\mu \left(|\mathbf{U}_n| = m \mid \mathbf{U}_{n+1} = \mathbf{e}_i \right) \\
&= \mathbf{e}_i \sum_{m=1}^{+\infty} m [(I - R_n)^{-1} Q_n \zeta_{n-1}]^m (I - R_n)^{-1} P_n \mathbf{1}. \tag{2.9}
\end{aligned}$$

To process the calculation, we need the following

lemma 2.1 For matrix B , $I - B$ is non-degenerate, then $\sum_{m=1}^{+\infty} m B^m = B(I - B)^{-2}$.

Proof.

$$\sum_{m=1}^{+\infty} m B^m = (B + 2B^2 + 3B^3 + \cdots) = B(I + 2B + 3B^2 + \cdots),$$

and

$$(I - B)^{-2} = ((I - B)^{-1})^2 = \left(\sum_{m=1}^{+\infty} B^m\right)^2 = \left(\sum_{m=1}^{+\infty} B^m\right)\left(\sum_{m=1}^{+\infty} B^m\right) = (I + 2B + 3B^2 + 4B^3 \dots).$$

Thus $\sum_{m=1}^{+\infty} mB^m = B(I - B)^{-2}$. \square

Let $B = (I - R_n)^{-1}Q_n\zeta_{n-1}$, (2.9) can be continued as

$$\begin{aligned} E_\omega^\mu\left(|\mathbf{U}_n| \middle| \mathbf{U}_{n+1} = \mathbf{e}_i\right) &= \mathbf{e}_i(I - R_n)^{-1}Q_n\zeta_{n-1}[I - (I - R_n)^{-1}Q_n\zeta_{n-1}]^{-2}(I - R_n)^{-1}P_n\mathbf{1} \\ &= \mathbf{e}_i(I - Q_n\zeta_{n-1} - R_n)^{-1}Q_n\zeta_{n-1}\zeta_n\mathbf{1} \end{aligned} \quad (2.10)$$

$$= \mathbf{e}_i A_n \mathbf{1}. \quad (2.11)$$

The second equality (2.10) need a series calculations about the matrix which we leave it as Appendix, where A_n is given in (1.8). Similarly,

$$\begin{aligned} E_\omega^\mu\left(|\mathbf{Z}_n| \middle| \mathbf{U}_{n+1} = \mathbf{e}_i\right) &= \sum_{K=0}^{+\infty} \mathbf{e}_i K [(I - Q_n\zeta_{n-1})^{-1}R_n]^K (I - Q_n\zeta_{n-1})^{-1}P_n\mathbf{1} \\ &= \mathbf{e}_i (I - Q_n\zeta_{n-1} - R_n)^{-1}R_n\zeta_n\mathbf{1} \\ &= \mathbf{e}_i (I - Q_n\zeta_{n-1} - R_n)^{-1}R_n\mathbf{1}. \end{aligned} \quad (2.12)$$

As a result, we have

$$E_\omega^\mu\left(|\mathbf{U}_n| \middle| \mathbf{U}_{n+1}\right) = \mathbf{U}_{n+1} A_n \mathbf{1}, \quad (2.13)$$

$$E_\omega^\mu\left(|\mathbf{Z}_n| \middle| \mathbf{U}_{n+1}\right) = \mathbf{U}_{n+1} (I - Q_n\zeta_{n-1} - R_n)^{-1}R_n\mathbf{1}. \quad (2.14)$$

Step 2. Steps visited on layer n .

For any $n \leq 0$, define

$$N_n^i = \#\{k \in [0, T_1) : X_k = (n, i)\}. \quad (2.15)$$

Note that N_n^i is the number of steps visited (n, i) before time T_1 . Let

$$\mathbf{N}_n = (N_n^1, N_n^2, \dots, N_n^d) \quad \text{and} \quad |\mathbf{N}_n| = \sum_{i=1}^d N_n^i = \mathbf{N}_n \mathbf{1}.$$

Define a vector valued random variable \mathbf{U}'_n where \mathbf{U}'_n^i , $1 \leq i \leq d$ is the number of steps from layer $n - 1$ to (n, i) . Then

$$|\mathbf{N}_n| = |\mathbf{U}'_n| + |\mathbf{Z}_n| + |\mathbf{U}_{n+1}|, \quad \mathbb{P} - a.s.. \quad (2.16)$$

For another perspective, $T_1 = \sum_{n \leq 0} (|\mathbf{N}_n|)$, $\mathbb{P} - a.s..$ Since $X_n \rightarrow +\infty$, $\mathbb{P} - a.s..$, if the random walk takes a step to the left from any layer n ($n \leq 0$) to layer $n - 1$, it must come back finally from layer $n - 1$ to layer n , therefore $|\mathbf{U}_n| = |\mathbf{U}'_n|$, $\mathbb{P} - a.s..$

Together with (2.16), we have

$$\begin{aligned} E_\omega^\mu(|\mathbf{N}_n|) &= E_\omega^\mu(|\mathbf{U}_n| + |\mathbf{Z}_n| + |\mathbf{U}_{n+1}|) \\ &= E_\omega^\mu \left[E_\omega^\mu(|\mathbf{U}_n| \mid \mathbf{U}_{n+1}) + E_\omega^\mu(|\mathbf{Z}_n| \mid \mathbf{U}_{n+1}) + E_\omega^\mu(|\mathbf{U}_{n+1}| \mid \mathbf{U}_{n+1}) \right]. \end{aligned}$$

By using (2.13), one can calculate the quenched expectation of $|\mathbf{N}_n|$ as

$$\begin{aligned} E_\omega^\mu(|\mathbf{N}_n|) &= E_\omega^\mu[\mathbf{U}_{n+1}A_n\mathbf{1} + \mathbf{U}_{n+1}(I - Q_n\zeta_{n-1} - R_n)^{-1}R_n\mathbf{1} + \mathbf{U}_{n+1}\mathbf{1}] \\ &= E_\omega^\mu[\mathbf{U}_{n+1}(I - Q_n\zeta_{n-1} - R_n)^{-1}(Q_n\zeta_{n-1} + R_n + I - Q_n\zeta_{n-1} - R_n)\mathbf{1}] \\ &= E_\omega^\mu(\mathbf{U}_{n+1})(I - Q_n\zeta_{n-1} - R_n)^{-1}\mathbf{1}. \end{aligned} \quad (2.17)$$

Step 3. The next object is to discuss $E_\omega^\mu(\mathbf{U}_{n+1})$.

Define a probability matrix B_m , where $B_m(i, j)$ is the probability of a particle starting from $(n+1, i)$, takes more than m steps to the layer n , and the m -th step located at (n, j) . $B_m(i, j)$ can be expressed by our branching structure

$$\begin{aligned} B_m(i, j) &= \mathbf{e}_i \left[\sum_{\bar{K}=0}^{+\infty} \sum_{k_0+k_1+\dots+k_{m-1}=\bar{K}} R_{n+1}^{k_0} Q_{n+1} \zeta_n R_{n+1}^{k_1} Q_{n+1} \zeta_n R_{n+1}^{k_2} \cdots Q_{n+1} \zeta_n R_{n+1}^{k_{m-1}} Q_{n+1} \right] \mathbf{e}_j \\ &= \mathbf{e}_i [(I - R_{n+1})Q_{n+1}\zeta_n]^{m-1} (I - R_{n+1})^{-1} Q_{n+1} \mathbf{e}_j. \end{aligned} \quad (2.18)$$

Let

$$\tilde{P}_{i,j}^m := B_m(i, j) - B_{m+1}(i, j),$$

be the probability of a particle starts from $(n+1, i)$, the m -th step takes to the left and located at (n, j) . We have

$$\begin{aligned} E_\omega^\mu(U_{n+1}^j \mid \mathbf{U}_{n+2} = \mathbf{e}_i) &= \sum_{m=1}^{+\infty} m \tilde{P}_{i,j}^m = \mathbf{e}_i \sum_{m=1}^{+\infty} m (B_m - B_{m+1}) \mathbf{e}_j \\ &= \mathbf{e}_i \sum_{m=1}^{+\infty} B_m \mathbf{e}_j. \end{aligned} \quad (2.19)$$

Combine with (2.18),

$$\begin{aligned} E_\omega^\mu(\mathbf{U}_{n+1} \mid \mathbf{U}_{n+2}) &= \mathbf{U}_{n+2} \sum_{m=1}^{+\infty} B_m \\ &= \mathbf{U}_{n+2} \sum_{m=1}^{+\infty} [(I - R_{n+1})Q_{n+1}\zeta_n]^{m-1} (I - R_{n+1})^{-1} Q_{n+1} \\ &= \mathbf{U}_{n+2} (I - Q_{n+1}\zeta_n - R_{n+1})^{-1} Q_{n+1} \\ &= \mathbf{U}_{n+2} A_{n+1}. \end{aligned} \quad (2.20)$$

Then

$$E_\omega^\mu(\mathbf{U}_{n+1}) = E_\omega^\mu[E_\omega^\mu(\mathbf{U}_{n+1} \mid \mathbf{U}_{n+2})] = E_\omega^\mu(\mathbf{U}_{n+2}) A_{n+1}, \quad (2.21)$$

where A_n is given in (1.8). By recursive argument, we obtain

$$\begin{aligned} E_\omega^\mu(\mathbf{U}_{n+1}) &= E_\omega^\mu(\mathbf{U}_{n+3})A_{n+2}A_{n+1} \\ &= \dots \\ &= E_\omega^\mu(\mathbf{U}_1)A_0A_{-1}A_{-2}\cdots A_{n+2}A_{n+1}. \end{aligned} \quad (2.22)$$

Step 4. Calculate $\mathbb{E}(T_1)$. It follows from (2.17) and (2.22) that,

$$\begin{aligned} \mathbb{E}(T_1) &= E(E_\omega(T_1)) = E\left(\sum_{n \leq 0} E_\omega(|\mathbf{N}_n|)\right) \\ &= E\left(\sum_{n \leq 0} E_\omega(\mathbf{U}_{n+1})(I - Q_n \zeta_{n-1} - R_n)^{-1} \mathbf{1}\right) \\ &= E\left(\sum_{n \leq 0} E_\omega(\mathbf{U}_1)A_0A_{-1}A_{-2}\cdots A_{n+2}A_{n+1}(I - Q_n \zeta_{n-1} - R_n)^{-1} \mathbf{1}\right) \\ &= E(E_\omega^\mu(\mathbf{U}_1)(u_0 + A_0u_{-1} + \cdots + A_0A_{-1}\cdots A_{-k}u_{-k-1} + \cdots)) \\ &= E(\vec{\mu}_\omega^\rightarrow(u_0 + A_0u_{-1} + \cdots + A_0A_{-1}\cdots A_{-k}u_{-k-1} + \cdots)). \end{aligned} \quad (2.23)$$

where A_n, u_n is given in (1.8). □

2.3 Density of the absolutely continuous invariant measure— Proof of Theorem 1.3

Let us review the process discussed in Section 4 of [10]. From the point of view “environments viewed from the particle”, let $\bar{\omega}_n = \theta^{\xi_n} \omega$, for $n \geq 0$, and consider the process $Z_n := (\bar{\omega}_n, Y_n)$, defined in $(\Omega \times \mathcal{D}, \mathcal{F} \otimes \mathcal{H})$, where \mathcal{H} as the set of all subsets of \mathcal{D} , and the initial distribution $P_\omega^\mu(\xi_0 = 0, Y_0 = i) = \mu_\omega(i) = \mathbf{y}_{-1}(i)$ given by (1.11). $(Z_n)_{n \geq 0}$ is a Markov chain under \mathbb{P}^μ with transition kernel

$$K(\omega, i; B, j) = P_0(i, j)I_B(\theta \omega) + R_0(i, j)I_B(\omega) + Q_0(i, j)I_B(\theta^{-1} \omega). \quad (2.24)$$

Let $v_p = \frac{1}{\mathbb{E}T_1}$, whenever $\mathbb{E}T_1 < \infty$. For $B \in \mathcal{F}$, $i \in \mathcal{D}$, define a probability measure Q on $(\Omega \times \mathcal{D}, \mathcal{F} \otimes \mathcal{H})$:

$$Q(B, i) := v_p \mathbb{E} \left(\sum_{n=0}^{T_1-1} I_B(\bar{\omega}_n) I_{Y_n}(i) \right) = v_p \sum_{j \in \mathcal{D}} E_p \left(\mu_\omega(j) E_\omega^j \left(\sum_{n=0}^{T_1-1} I_B(\theta^{\xi_n} \omega) I_{Y_n}(i) \right) \right)$$

$Q(\cdot)$ is a invariant measure under the Markov kernel K (Proposition 4.1, [10]).

Define a probability measure $\bar{Q}(\cdot)$ on (Ω, \mathcal{F}) by setting

$$\bar{Q}(B) := Q(B, \mathcal{D}), \quad B \in \mathcal{F}. \quad (2.25)$$

and let $Q_i(B) := Q(B, i)$ for $B \in \mathcal{F}$. Then both $Q_i(\cdot)$ and $\bar{Q}(\cdot)$ are absolutely continuous with regard to P (Proposition 4.1, [10]), but where only the up bound of the density have been proved.

For $m \leq 0$, $i \in \mathcal{D}$, define N_m^i as in (2.15):

$$N_m^i := \{\#\!n \in [0, T_1) : \xi_n = m, Y_n = i\}.$$

Note that for any bounded measurable function $f : \Omega \rightarrow \mathbb{R}$ and $\forall i \in \mathcal{D}$,

$$\begin{aligned} \int_{\Omega} f(\omega) Q(d\omega, i) &= v_p \sum_{n=0}^{+\infty} \mathbb{E}^{\mu}(f(\bar{w}_n); Y_n = i, T_1 > n) \\ &= v_p \sum_{m \leq 0} \mathbb{E}^{\mu}(f(\theta^m \omega) N_m^i) \\ &= v_p E_p \left(\sum_{k \in \mathcal{D}} \mu_{\omega}(k) \sum_{m \leq 0} (f(\theta^m \omega) E_{\omega}^k N_m^i) \right) \\ &= v_p E_p \left(f(\omega) \sum_{k \in \mathcal{D}} \sum_{m \leq 0} \mu_{\theta^{-m} \omega}(k) E_{\theta^{-m} \omega}^k N_m^i \right). \end{aligned} \quad (2.26)$$

Therefore, Q_i is absolutely continuous with respect to P , and also \bar{Q} is absolutely continuous with respect to P . And the density is

$$\Lambda_{\omega}^{(i)} := \frac{dQ_i}{dP} = v_p \sum_{k \in \mathcal{D}} \sum_{m \leq 0} \mu_{\theta^{-m} \omega}(k) E_{\theta^{-m} \omega}^k(N_m^i) \quad (2.27)$$

$$\Lambda_{\omega} := \frac{d\bar{Q}}{dP} = v_p \sum_{k \in \mathcal{D}} \sum_{m \leq 0} \mu_{\theta^{-m} \omega}(k) E_{\theta^{-m} \omega}^k(\mathbf{N}_m \mathbf{1}). \quad (2.28)$$

We intend to specify the density $\Lambda_{\omega}^{(i)}$ and Λ_{ω} by branching structure. Note that $\mu_{\omega}(i) = \mathbf{y}_{-1}(i)$ given by (1.11), and $\zeta_{-n} = \eta_{-n}$ is the exit probability,

$$\begin{aligned} \mu_{\theta \omega} &= \lim_{n \rightarrow \infty} \mathbf{e}_i \zeta_{-n}(\theta \omega) \cdots \zeta_{-2}(\theta \omega) \zeta_{-1}(\theta \omega) \\ &= \lim_{n \rightarrow \infty} \mathbf{e}_i \zeta_{-n+1}(\omega) \cdots \zeta_{-1}(\omega) \zeta_0(\omega) \\ &= \mu_{\omega} \zeta_0(\omega). \end{aligned}$$

Thus

$$\sum_{k \in \mathcal{D}} \mu_{\theta \omega}(k) E_{\theta \omega}^k(N_{-1}^i) = \sum_{k \in \mathcal{D}} \mu_{\theta \omega}(k) E_{\omega}^k(N_0^i) = \sum_{k \in \mathcal{D}} \mu_{\omega} \zeta_0(k) E_{\omega}^k(N_0^i).$$

Similarly, for $m \leq 0$ and $i \in \mathcal{D}$

$$\sum_{k \in \mathcal{D}} \mu_{\theta^{-m} \omega}(k) E_{\theta^{-m} \omega}^k(N_m^i) = \sum_{k \in \mathcal{D}} \mu_{\omega} \zeta_0 \zeta_1 \cdots \zeta_{-m-1}(k) E_{\omega}^k(N_0^i). \quad (2.29)$$

The following lemma is closely related to branching structure.

lemma 2.2 For $n < 0$,

$$E_\omega(\mathbf{N}_n) = \mu_\omega A_0 A_{-1} \cdots A_{n+1} \tilde{u}_n. \quad (2.30)$$

Proof. Due to the definition of \mathbf{N}_n , \mathbf{U}'_n , \mathbf{Z}_n and \mathbf{U}_{n+1} ,

$$\mathbf{N}_n = \mathbf{U}'_n + \mathbf{Z}_n + \mathbf{U}_{n+1}.$$

Recall the branching structure and by similarly argument as in the proof of Theorem 1.2, we obtain that

$$\begin{aligned} E_\omega(\mathbf{U}'_n) &= \mathbf{U}_{n+1} \sum_{m=1}^{+\infty} ((I - R_n)^{-1} Q_n \zeta_{n-1})^{m-1} (I - R_n)^{-1} Q_n \zeta_{n-1} \\ &= \mathbf{U}_{n+1} ((I - Q_n \zeta_{n-1} - R_n)^{-1} Q_n \zeta_{n-1} = \mathbf{U}_{n+1} A_n \zeta_{n-1}, \end{aligned}$$

and

$$\begin{aligned} E_\omega(\mathbf{Z}_n) &= \mathbf{U}_{n+1} \sum_{K=0}^{+\infty} ((I - Q_n \zeta_{n-1})^{-1} R_n)^K (I - Q_n \zeta_{n-1})^{-1} R_n \\ &= \mathbf{U}_{n+1} (I - Q_n \zeta_{n-1} - R_n)^{-1} R_n. \end{aligned}$$

Thus

$$\begin{aligned} E_\omega(\mathbf{N}_n | \mathbf{U}_{n+1}) &= E_\omega(\mathbf{N}_n | \mathbf{U}_{n+1}) \\ &= E_\omega(\mathbf{U}'_n | \mathbf{U}_{n+1}) + E_\omega(\mathbf{Z}_n | \mathbf{U}_{n+1}) + E_\omega(\mathbf{U}_{n+1} | \mathbf{U}_{n+1}) \\ &= E_\omega(\mathbf{U}_{n+1} [(I - Q_n \zeta_{n-1} - R_n)^{-1} Q_n \zeta_{n-1} + (I - Q_n \zeta_{n-1} - R_n)^{-1} R_n + I]) \\ &= E_\omega(\mathbf{U}_{n+1}) (I - Q_n \zeta_{n-1} - R_n)^{-1}. \end{aligned}$$

Together with the fact

$$E_\omega(\mathbf{U}_{n+1}) = \mathbf{U}_{n+2} A_{n+1},$$

we have

$$\begin{aligned} E_\omega(\mathbf{N}_n) &= \mu_\omega A_0 A_{-1} \cdots A_{n+1} (I - Q_n \zeta_{n-1} - R_n)^{-1} \\ &= \mu_\omega A_0 A_{-1} \cdots A_{n+1} \tilde{u}_n. \end{aligned}$$

Then Lemma 2.2 follows. \square

It follows from equation (2.29) and lemma 2.2 that

$$\begin{aligned} \sum_{k \in \mathcal{D}} \mu_{\theta-m\omega}(k) E_{\theta-m\omega}^k(N_m^i) &= \sum_{k \in \mathcal{D}} \mu_\omega \zeta_0 \zeta_1 \cdots \zeta_{-m-1}(k) E_\omega^k(N_0^i) \\ &= \mu_\omega \zeta_0 \zeta_1 \cdots \zeta_{-m-1} A_{-m} A_{-m-1} \cdots A_2 A_1 \tilde{u}_0(i). \end{aligned} \quad (2.31)$$

Thus

$$\begin{aligned} \Lambda_\omega^{(i)} &= \frac{dQ_i}{dP} = v_p \sum_{k \in \mathcal{D}} \sum_{m \leq 0} \mu_{\theta-m\omega}(k) E_{\theta-m\omega}^k(N_m^i) \\ &= v_p \sum_{m \leq 0} [\mu_\omega \zeta_0 \zeta_1 \cdots \zeta_{-m-1} A_{-m} A_{-m-1} \cdots A_2 A_1 \tilde{u}_0](i) \\ &= v_p [\mu_\omega (\tilde{u}_0 + \zeta_0 A_1 \tilde{u}_0 + \zeta_0 \zeta_1 A_2 A_1 \tilde{u}_0 + \cdots)](i). \end{aligned}$$

and similarly

$$\begin{aligned}
\frac{d\bar{Q}}{dP} = \Lambda_\omega &= v_p \sum_{k \in \mathcal{D}} \sum_{m \leq 0} \mu_{\theta - m\omega}(k) E_{\theta - m\omega}^k(\mathbf{N}_m \mathbf{1}) \\
&= v_p \sum_{m \leq 0} [\mu_\omega \zeta_0 \zeta_1 \cdots \zeta_{-m-1} A_{-m} A_{-m-1} \cdots A_2 A_1 \tilde{u}_0] \mathbf{1} \\
&= v_p [\mu_\omega (\tilde{u}_0 + \zeta_0 A_1 \tilde{u}_0 + \zeta_0 \zeta_1 A_2 A_1 \tilde{u}_0 + \cdots)] \mathbf{1}.
\end{aligned}$$

□

Appendix

The following calculation is needed in (2.10), it is a details calculations on the matrix.

$$\begin{aligned}
&E_w^\mu(|\mathbf{U}_n| \mid \mathbf{U}_{n+1} = \mathbf{e}_i) \\
&= \sum_{m=0}^{+\infty} m P_w^\mu(|\mathbf{U}_n| = m \mid \mathbf{U}_{n+1} = \mathbf{e}_i) \\
&= \sum_{m=0}^{+\infty} \mathbf{e}_i m [(I - R_n)^{-1} Q_n \zeta_{n-1}]^m (I - R_n)^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i \sum_{m=1}^{+\infty} m [(I - R_n)^{-1} Q_n \zeta_{n-1}]^m (I - R_n)^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i (I - R_n)^{-1} Q_n \zeta_{n-1} [I - (I - R_n)^{-1} Q_n \zeta_{n-1}]^{-2} (I - R_n)^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i (I - R_n)^{-1} Q_n \zeta_{n-1} [I - (I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1} [I - (I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1} (I - R_n)^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i \{[(I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1}\}^{-1} [I - (I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1} [I - (I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1} (I - R_n)^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i \{[I - (I - R_n)^{-1} Q_n \zeta_{n-1}][(I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1}\}^{-1} [I - (I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1} (I - R_n)^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i \{[(I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1} - I\}^{-1} [I - (I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1} (I - R_n)^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i \{[I - (I - R_n)^{-1} Q_n \zeta_{n-1}][[(I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1} - I]\}^{-1} (I - R_n)^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i \{(I - R_n)[I - (I - R_n)^{-1} Q_n \zeta_{n-1}][[(I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1} - I]\}^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i \{[(I - R_n) - Q_n \zeta_{n-1}][[(I - R_n)^{-1} Q_n \zeta_{n-1}]^{-1} - I]\}^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i \{[(I - R_n) - Q_n \zeta_{n-1}][(Q_n \zeta_{n-1})^{-1} (I - R_n) - I]\}^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i \{[(I - R_n) - Q_n \zeta_{n-1}][(Q_n \zeta_{n-1})^{-1} (I - R_n - Q_n \zeta_{n-1})]\}^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i [(I - Q_n \zeta_{n-1} - R_n)(Q_n \zeta_{n-1})^{-1} (I - Q_n \zeta_{n-1} - R_n)]^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i (I - Q_n \zeta_{n-1} - R_n)^{-1} Q_n \zeta_{n-1} (I - Q_n \zeta_{n-1} - R_n)^{-1} P_n \mathbf{1} \\
&= \mathbf{e}_i (I - Q_n \zeta_{n-1} - R_n)^{-1} Q_n \zeta_{n-1} \zeta_n \mathbf{1} \\
&= \mathbf{e}_i A_n \zeta_{n-1} \zeta_n \mathbf{1} \\
&= \mathbf{e}_i A_n \mathbf{1}.
\end{aligned}$$

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