Branching structure for the transient $(1, R)$–random walk in random environment and its applications *

Wenming Hong[†] and Lin Zhang [‡]

Beijing Normal University

Abstract

An intrinsic multitype branching structure within the transient $(1, R)$-RWRE is revealed. The branching structure enables us to specify the density of the absolutely continuous invariant measure for the environments seen from the particle and reprove the LLN with an drift *explicitly* in terms of the environment, comparing with the results in Brémont (2002).

Key words and phrases: random walk, random environment, multitype branching process, Wald's equality, law of large numbers

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1 Introduction and main results

A random walk in random environment (RWRE, for short) $\{X_n, n \in \mathbb{N}\}$ with bounded jumps on the line, written as (L, R)-RWRE, means that for each step the possible jump range to the left is bounded by L and to the right by R, where L and R are positive integers. The aim of this paper is to reveal the branching structure within the $(1, R)$-RWRE, which is a story different with but essentially complement for that of $(L, 1)$-RWRE (Hong and Wang, [10], 2009) when it transient to the right.

It is well-known that when the walk is transient, i.e., $X_n \to \infty$, the intrinsic branching structure within the $(1, 1)$-RWRE is a Galton-Watson branching process with geometric offspring distribution, which plays an important role in the proofs of the limiting stable law (Kesten et

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† School of Mathematical Sciences & Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R.China. Email: wmhong@bnu.edu.cn

‡ School of Mathematical Sciences & Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R.China. Email:zhanglin2008@mail.bnu.edu.cn

al [12], 1975), the renewal theorem (Kesten [11], 1977), and the law of large numbers (Alili [1], 1999) for this nearest random walk in random environment, see also Gantert and Shi ([9], 2002).

However, when L or $R > 1$, and the random walk is transient (we consider $X_n \to \infty$), It seems no result about the intrinsic branching structure within the (L, R)-RWRE up to our knowledge, even for random walks in non-random environment. Recently, partially progress has been made by Hong and Wang ([10], 2009), where a multitype branching structure within the $(L, 1)$-RWRE have been revealed when $X_n \to \infty$ and the Kesten's type stable law have been proved. In the present paper, we focus on the opposite direction which is more complicated: we will figure out the intrinsic branching structure within the $(1, R)$-RWRE when $X_n \to \infty$(note that the situation for the $(L, 1)$-RWRE transient to the left is equivalent to that for the $(1, R)$-RWRE transient to the right). We will discuss $R = 2$ in detail and extend to general R at the end.

RWRE has been studied extensively in recent years, especially for the nearest $(1, 1)$-RWRE on which many results have been obtained, see for example [18], [12], [11], [2], etc., we refer to Sznitman ([17], 2002) and Zeitouni ([19], 2004) as a general review. For the (L, R)-RWRE, Key ([13], 1984) discussed recurrence/transience criterion in terms of the sign of two intermediate Liapounov exponents of a random matrix. Brémont ([6], 2009) formulate a criterion for the existence of the absolutely continuous invariant measure for the environments seen from the particle and deduce a characterization of the non-zero-speed regime of the model. It should note that the (L, R)-RWRE can be treated as a special case of the random walks in random environments on a strip, see Bolthausen and Goldsheid ([3], 2000; [4], 2008) and Roitershtein ([16], 2008).

Brémont ([5], 2002) has proved a recurrence/transience criterion for $(L, 1)$-RWRE involving the greatest Lyapunov exponent with respect to a random matrix M, and the law of large numbers by assuming the (IM) condition related the existence of an invariant measure for the environments seen from the particle. As an important application of our branching structure within the $(1, R)$-RWRE, it enables us to specify the density of the absolutely continuous invariant measure *explicitly* and reprove the LLN with an drift *explicitly* in terms of the environment.

We now introduce the model of $(1, 2)$-RWRE. Generally speaking, random walks in random environment involves two kinds of randomness: the transition probability (we call it the "environment"), which is chosen from a specified distribution; and the random walk driven by the chosen "environment". Specifically, let $\Lambda = \{-1, 1, 2\}$ be the set of possible jump range of the random walks. Let $\mathcal{M}(\Lambda)$ be the collection of all probability measures on Λ. Then define an environment to be an element $\omega = \{(q(\omega)_x, p_1(\omega)_x, p_2(\omega)_x) : x \in \mathbb{Z}\} \in \mathcal{M}(\Lambda)^{\mathbb{Z}} =: \Omega$. Let P be a stationary and ergodic probability distribution on $(\Omega, \mathcal{F})$ and θ be the spatial shift, i.e., $(\theta\omega)_n = \omega_{n+1}$. Assume that θ is an invertible transformation on the probability space $(\Omega, \mathcal{F}, P)$, measurable as well as its inverse and preserving P. Moreover, assume that the environment is elliptic:

$$\exists\, \varepsilon > 0,\ \forall\, z \in \{1, 2\},\ (p_z/q) \geq \varepsilon,\ P\text{-a.s..}$$

Given an environment $\omega \in \Omega$, one can define a random walk $\{X_n\}$ in the environment ω to be a time-homogeneous Markov chain on $\mathbb{Z}$ with $X_0 = 0$ and the transition probabilities

$$P_\omega(X_{n+1} = x - 1 | X_n = x) := q(\theta^x\omega)_0 = q(\omega)_x,$$
$$P_\omega(X_{n+1} = x + z | X_n = x) := p_z(\theta^x\omega)_0 = p_z(\omega)_x,$$

for all $x \in \mathbb{Z}$ and $z \in \{1, 2\}$. For each ω, we use P_ω to denote the law induced on the space of paths $(\mathbb{Z}^{\mathbb{N}}, \mathcal{G})$. Then, define a probability measure $\mathbb{P} := P \otimes P_\omega$ on $(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathcal{F} \times \mathcal{G})$ by

$$\mathbb{P}(F \times G) := \int_F P_\omega(G) P(\mathrm{d}\omega), \quad F \in \mathcal{F},\ G \in \mathcal{G}.$$

Statements involving P_ω and $\mathbb{P}$ are called *quenched* and *annealed*, respectively. Generally, with a slight abuse of notation, $\mathbb{P}$ can also be used to denote the marginal on $\mathbb{Z}^{\mathbb{N}}$. Expectations under P_ω and $\mathbb{P}$ will be denoted by E_ω and $\mathbb{E}$, respectively.

We will need the following notations. $p_z(\theta^x\omega)_0$ will be simply denoted by $p_z(x)$. Any expression of the form $f(\theta^k\omega)$ will be simply denoted by $f(k)$. For the random walk $\{X_n\}$ and $k \in \mathbb{Z}\backslash\{0\}$, we write P_ω^k for the quenched probability starting at k, and E_ω^k for the corresponding expectation.

Assume that $X_0 = 0$ and the $(1, 2)$-random walk is transient to the right, i.e., $X_n \to \infty$, $\mathbb{P}$-a.s.. Let $T_0 = 0$, and

$$T_k = \inf\{n > T_{k-1} : X_n > X_{T_{k-1}}\}, \quad k \geq 1$$

be the sequence of *ladder times* of the random walks. Note that $T_k < \infty$, $\mathbb{P}$-a.s.. To calculate T_1 accurately, we will figure out a multitype branching processes by decomposing the path of the walk. Intuitively, if the walk from $i \leq 0$ take a step to $i-1$, it must crossing back to i or jumping over i (to $i+1$) because of $T_k < \infty$, $\mathbb{P}$-a.s., in which there are only three kind of backing ways: from $i-1$ to i, from $i-2$ to i and from $i-1$ to $i+1$. So we divide all the steps from i to $i-1$ into three kind of steps according the crossing back ways. Let $A(i), B(i)$, and $C(i)$ are the numbers of steps from i to $i-1$ before time T_1 with crossing-back from $i-1$ to i, $i-2$ to i and $i-1$ to $i+1$, respectively. And for the last step of T_1, we can consider it as a immigration for the multitype branching processes.

Set for $i \leq 0$,

$$U(i) = [A(i), B(i), C(i)],$$

Then we have the branching structure within the $(1, 2)$-RWRE as follows.

Theorem 1.1 *Assume $X_n \to \infty$, $\mathbb{P}$-a.s.. Then for P-a.s. ω, $\Big(U(i) = [A(i), B(i), C(i)]\Big)_{i\leq 0}$ is an inhomogeneous multitype branching process with immigration*

$$U(1) = [1, 0, 0], \quad \textit{with probability} \quad \frac{p_1(0)}{1-\alpha(0)-\beta(0)},$$
$$U(1) = [0, 1, 0], \quad \textit{with probability} \quad \frac{\gamma(0)}{1-\alpha(0)-\beta(0)},$$
$$U(1) = [0, 0, 1], \quad \textit{with probability} \quad \frac{p_2(0)}{1-\alpha(0)-\beta(0)}.$$

The offspring distribution is given by

$$P_\omega\Big(U(i) = [a, b, 0] \;\Big|\; U(i+1) = [1, 0, 0]\Big) = [1-\alpha(i)-\beta(i)] C_{a+b}^{a} \alpha(i)^a \beta(i)^b, \qquad (1.1)$$
$$P_\omega\Big(U(i) = [a, b, 1] \;\Big|\; U(i+1) = [0, 1, 0]\Big) = [1-\alpha(i)-\beta(i)] C_{a+b}^{a} \alpha(i)^a \beta(i)^b, \qquad (1.2)$$

$$P_\omega\Big(U(i)=[a,b,0]\;\Big|\;U(i+1)=[0,0,1]\Big)=[1-\alpha(i)-\beta(i)]C_{a+b}^a\alpha(i)^a\beta(i)^b, \qquad (1.3)$$

where

$$\gamma(i)=q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+1],$$
$$\alpha(i)=q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i]\cdot\frac{p_1(i-1)}{p_1(i-1)+\gamma(i-1)},$$
$$\beta(i)=q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i]\cdot\frac{\gamma(i-1)}{p_1(i-1)+\gamma(i-1)},$$

and $P_\omega^{i-1}[(-\infty,i-1),i-1+j]:=P_\omega^{i-1}\{$*reach* $[i,+\infty)$ *for the first time at the point* $i-1+j\}$ *for* $j=1,2$, *the exit probabilities which can be expressed in terms of the environment in Lemma 2.1.* □

As an immediate consequence of Theorem 1.1, We can get the offspring quenched mean matrix of the multitype branching process and the quenched mean of the T_1 .

Corollary 1.1 *Assume* $X_n\to\infty$, $\mathbb{P}$*-a.s.. Then for* P*-a.s.* ω, *and* $i\le 0$, *the offspring mean matrix of the* $(-i+1)$*-th generation of the multitype branching process is*

$$N(i)=\begin{pmatrix}\frac{\alpha(i)}{1-\alpha(i)-\beta(i)} & \frac{\beta(i)}{1-\alpha(i)-\beta(i)} & 0\\ \frac{\alpha(i)}{1-\alpha(i)-\beta(i)} & \frac{\beta(i)}{1-\alpha(i)-\beta(i)} & 1\\ \frac{\alpha(i)}{1-\alpha(i)-\beta(i)} & \frac{\beta(i)}{1-\alpha(i)-\beta(i)} & 0\end{pmatrix}.$$

Moreover,

$$E_\omega(T_1)=1+\Big\langle(2,2,1),\;\frac{1}{1-\alpha(0)-\beta(0)}\Big(p_1(0),\;\gamma(0),\;p_2(0)\Big)\cdot\sum_{i\le0}N(0)\cdots N(i)\Big\rangle.$$

□

As an application of the branching structure to the random walks in non-random environment, let

$$X_0=0,\quad X_n=\xi_1+\cdots+\xi_n,$$

where $\xi_1,\ \xi_2,\cdots$ is a series of i.i.d random variables with

$$P(\xi_1=-1)=q,\quad P(\xi_1=1)=p_1,\quad P(\xi_1=2)=p_2.$$

The computability of $E(T_1)$ by the branching structure as in Corollary 1.1 enable us to validate the Wald's equality.

Proposition 1.1 *Assume that* $E(\xi_1)=p_1+2p_2-q>0$. *Then the Wald's equality holds:*

$$E(X_{T_1})=E(T_1)\cdot E(X_1).$$

□

From the point of view "environment viewed from the particles", define $\overline{\omega}(n) = \theta^{X_n}\omega$. Then the process $\{\overline{\omega}(n)\}$ is a Markov process under either P_ω or $\mathbb{P}$, with state space Ω and transition kernel

$$K(\omega, \mathrm{d}\omega') = q\mathbf{1}_{\omega'=\theta^{-1}\omega} + p_1\mathbf{1}_{\omega'=\theta\omega} + p_2\mathbf{1}_{\omega'=\theta^2\omega}.$$

Set $\varphi^1_{\theta^k\omega} = P_{\theta^k\omega}(X_{T_1} = 1)$, and $\varphi^2_{\theta^k\omega} = P_{\theta^k\omega}(X_{T_1} = 2) = 1 - \varphi^1_{\theta^k\omega}$. Whenever $\mathbb{E}(T_1) < \infty$, define the measure

$$Q(B) = \mathbb{E}\Bigg(\frac{\mathbf{1}_{X_{T_1}=1}}{\varphi^1_\omega}\sum_{i=0}^{T_1-1}\mathbf{1}_{\{\overline{\omega}(i)\in B\}} + \frac{\mathbf{1}_{X_{T_1}=2}}{\varphi^2_\omega}\sum_{i=0}^{T_1-1}\mathbf{1}_{\{\overline{\omega}(i)\in B\}}\Bigg), \quad \overline{Q}(B) = \frac{Q(B)}{Q(\Omega)}.$$

The significance of the branching structure to the $(1,2)$-RWRE is that we can express the density of Q with respect to P *explicitly*.

Theorem 1.2 *Assume $\mathbb{E}(T_1) < \infty$. Then $Q(\cdot)$ is invariant under the Markov kernel K, that is,*

$$Q(B) = \iint \mathbf{1}_{\omega'\in B} K(\omega, d\omega')Q(d\omega).$$

Furthermore,

$$\frac{dQ}{dP} = \Pi(\omega),$$

where

$$\Pi(\omega) = \frac{2 + \Big\langle (1,1,1), \sum_{i\geq 1}\Big(\frac{p_1(i)}{p_1(i)+\gamma(i)}, \frac{\gamma(i)}{p_1(i)+\gamma(i)}, 1\Big)\cdot N(i)\cdots N(1)\Big\rangle}{1 - \alpha(0) - \beta(0)}.$$

□

Therefore, the LLN for the $(1,2)$-RWRE can be reproved with an *explicit* drift.

Theorem 1.3 *Assume $\mathbb{E}(T_1) < \infty$. Then*

$$\lim_{n\to\infty}\frac{X_n}{n} = v_P, \quad \mathbb{P}\text{-}a.s.,$$

where

$$v_P = \frac{E_P\Big[\frac{p_1(0)+2p_2(0)-p_{-1}(0)}{1-\alpha(0)-\beta(0)}\Big(2 + \Big\langle (1,1,1), \sum_{i\geq 1}\Big(\frac{p_1(i)}{p_1(i)+\gamma(i)}, \frac{\gamma(i)}{p_1(i)+\gamma(i)}, 1\Big)\cdot N(i)\cdots N(1)\Big\rangle\Big)\Big]}{E_P\Big[2 + \Big\langle (2,2,1), \sum_{i\geq 0}\Big(\frac{p_1(i)}{p_1(i)+\gamma(i)}, \frac{\gamma(i)}{p_1(i)+\gamma(i)}, 1\Big)\cdot N(i)\cdots N(0)\Big\rangle\Big]}.$$

□

We arrange the remainder of this paper as follows. In section 2, we formulate the intrinsic multitype branching structure within the $(1,2)$-RWRE under the assumption $X_n \to \infty$, i.e., Theorem 1.1, Corollary 1.1 and Proposition 1.1 are proved. In section 3, we specify the density of the absolutely continuous invariant measure for the environments seen from the particle *explicitly* based on the branching structure, i.e., Theorem 1.2 is proved. In section 4, as an application of the branching structure, the LLN will be reproved with an *explicit* drift, that is, Theorem 1.3 is proved. Finally in section 5, for general $R > 1$, we give the intrinsic multitype branching structure within the $(1,R)$-RWRE under the assumption $X_n \to \infty$.

2 Intrinsic multitype branching structure within the transient $(1,2)$-RWRE

We introduce the following exit probabilities of leaving a given interval from the right side. Consider integers a, b, k with $a \le k \le b$, define

$$P_\omega^k[(a,b), b+1] = P_\omega^k\{\text{reach } (b,+\infty) \text{ before } (-\infty,a) \text{ and at the point } b+1\},$$
$$P_\omega^k[(a,b), b+2] = P_\omega^k\{\text{reach } (b,+\infty) \text{ before } (-\infty,a) \text{ and at the point } b+2\}.$$

These exit probabilities play an important role in the offspring distribution of the branching structure, which can be expressed in terms of the environment (Brémont [5], page.1273-4, Lemma 2.1 and Proposition 2.2). Only a slight modification should be made: exiting from the left in [5] corresponds exiting from the right here. We still give the details of the proof for convenience.

Lemma 2.1 *For $n \ge 2$, we have*

$$P_\omega^i[(-n,i), i+1] = \frac{\langle e_1, [M(i) + \cdots + M(-n)\cdots M(i)]v\rangle}{1 + \langle e_1, [M(i) + \cdots + M(-n)\cdots M(i)]e_1\rangle},$$
$$P_\omega^i[(-n,i), i+2] = \frac{\langle e_1, [M(i) + \cdots + M(-n)\cdots M(i)]e_2\rangle}{1 + \langle e_1, [M(i) + \cdots + M(-n)\cdots M(i)]e_1\rangle},$$

where $e_1 = (1,0)'$, $e_2 = (0,1)'$, $v = e_1 - e_2$ and

$$M(i) := \begin{pmatrix} \frac{p_1(i)+p_2(i)}{q(i)} & \frac{p_2(i)}{q(i)} \\ 1 & 0 \end{pmatrix}.$$

Furthermore, if $X_n \to +\infty$, $\mathbb{P}$-a.s., then

$$P_\omega^i[(-\infty,i), i+1] + P_\omega^i[(-\infty,i), i+2] = 1. \tag{2.1}$$

Proof. Set $f(k) = P_\omega^k[(-n,i), i+1]$. For k such that $-n \le k \le i$, using Markov property, we get a harmonic-type recurrence equation:

$$f(k) = p_1(k)f(k+1) + p_2(k)f(k+2) + q(k)f(k-1),$$

with $f(i+1)=1,\ f(k)=0$ for $k\geq i+2$ or $k\leq -n-1$. Setting $g(k)=f(k)-f(k-1)$, we obtain

$$\begin{aligned} g(k) &= \frac{p_1(k)}{q(k)}g(k+1)+\frac{p_2(k)}{q(k)}\Big(g(k+2)+g(k+1)\Big) \\ &= \frac{p_1(k)+p_2(k)}{q(k)}g(k+1)+\frac{p_2(k)}{q(k)}g(k+2). \end{aligned}$$

Rewrite the equation as

$$\left(\begin{array}{c} g(k) \\ g(k+1) \end{array}\right)=\left(\begin{array}{cc} \frac{p_1(k)+p_2(k)}{q(k)} & \frac{p_2(k)}{q(k)} \\ 1 & 0 \end{array}\right)\left(\begin{array}{c} g(k+1) \\ g(k+2) \end{array}\right).$$

Set

$$U(k)=\left(\begin{array}{c} g(k) \\ g(k+1) \end{array}\right),\quad \text{and}\quad M(k)=\left(\begin{array}{cc} a_1(k) & a_2(k) \\ 1 & 0 \end{array}\right),$$

where $a_1(k)=\frac{p_1(k)+p_2(k)}{q(k)}$, $a_2(k)=\frac{p_2(k)}{q(k)}$. Thus, we obtain the relations

$$\begin{aligned} U(k) &= M(k)U(k+1) \\ &= M(k)\cdots M(i)U(i+1), \end{aligned}$$

with $U(i+1)=([1-f(i)],-1)'$ and $U(-n)=(f(-n),[f(-n+1)-f(-n)])'$. Summing from $-n$ to i, we deduce that

$$\begin{aligned} f(i) &= \Big\langle e_1, M(i)U(i+1)\Big\rangle+\cdots+\Big\langle e_1, M(-n)M(-n+1)\cdots M(i)U(i+1)\Big\rangle \\ &= \Big\langle e_1, \Big[M(i)+\cdots+M(-n)\cdots M(i)\Big]\cdot\left(\begin{array}{c} 1-f(i) \\ -1 \end{array}\right)\Big\rangle. \end{aligned}$$

The first formula then follows. By similar reasoning, one can get the second formula. The second part of the lemma is just the conclusion of Proposition 2.2 in [5]. □

Now we introduce the branching structure. Recall $X_0=0$ and the sequence of *ladder times* of the random walks: $T_0=0$ and

$$T_k=\inf\{n>T_{k-1}: X_n>X_{T_{k-1}}\},\quad k\geq 1.$$

Note that $T_k<\infty$, $\mathbb{P}$-a.s. if $X_n\to+\infty$, $\mathbb{P}$-a.s.. Define, for $i\leq 0$,

$$\begin{aligned} &\eta_{i,0}=\min\{k\leq T_1:\ X_k=i\}, \\ &\theta_{i,0}=\min\{\eta_{i,0}<k\leq T_1:\ X_{k-1}=i, X_k=i-1\}, \end{aligned}$$

and for $j\geq 1$,

$$\begin{aligned} &\alpha_{i,j}=\min\{\theta_{i,j-1}<k\leq T_1:\ X_{k-1}=i-1, X_k=i\}, \\ &\beta_{i,j}=\min\{\theta_{i,j-1}<k\leq T_1:\ X_{k-1}=i-2, X_k=i\}, \\ &\gamma_{i,j}=\min\{\theta_{i,j-1}<k\leq T_1:\ X_{k-1}=i-1, X_k=i+1\}, \end{aligned}$$

$$\eta_{i,j} = \min\{\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}\},$$
$$\theta_{i,j} = \min\{\eta_{i,j} < k \le T_1 : \ X_{k-1} = i, X_k = i-1\},$$

with the usual convention that the minimum over an empty set is $+\infty$. We refer to the time interval $[\theta_{i,j-1}, \eta_{i,j}]$ as the j-th excursion from $i-1$ to $\{i, i+1\}$. For any $j \ge 0$, any $i \le 0$, define

$$A_{i,j} = \#\{k \ge 0 : \theta_{i+1,j} < \theta_{i,k} < \eta_{i+1,j+1}, \text{ and } \eta_{i,k+1} = \alpha_{i,k+1}\},$$
$$B_{i,j} = \#\{l \ge 0 : \theta_{i+1,j} < \theta_{i,l} < \eta_{i+1,j+1}, \text{ and } \eta_{i,l+1} = \beta_{i,l+1}\},$$
$$C_{i,j} = \#\{m \ge 0 : \theta_{i+1,j} < \theta_{i,m} < \eta_{i+1,j+1}, \text{ and } \eta_{i,m+1} = \gamma_{i,m+1}\}.$$

Note that $A_{i,j}, B_{i,j}$, and $C_{i,j}$ are the numbers of steps from i to $i-1$ during the $(j+1)$-th excursion from i to $\{i+1, i+2\}$ with crossing-back from $i-1$ to i, $i-2$ to i and $i-1$ to $i+1$, respectively. Special attentions should be paid to the different ending ways of each excursion and their consequences, take the $(j+1)$-th excursion from i to $\{i+1, i+2\}$ for example:

- If $\eta_{i+1,j+1} = \alpha_{i+1,j+1}$, then the excursion $[\theta_{i+1,j+1}, \alpha_{i+1,j+1}]$ ends by jumping from i to $i+1$.
- If $\eta_{i+1,j+1} = \gamma_{i+1,j+1}$, then the excursion $[\theta_{i+1,j+1}, \gamma_{i+1,j+1}]$ ends by jumping from i to $i+2$.

Furthermore, in the above two cases, excursions from $i-1$ to $\{i, i+1\}$ included in the $(j+1)$-th excursion from i to $\{i+1, i+2\}$ must end at i and consequently, $C_{i,j} = 0$.

- If $\eta_{i+1,j+1} = \beta_{i+1,j+1}$, then the excursion $[\theta_{i+1,j+1}, \beta_{i+1,j+1}]$ ends by jumping from $i-1$ to $i+1$.

Moreover, in this case, the last excursion from $i-1$ to $\{i, i+1\}$ included in the $j+1$-th excursion from i to $\{i+1, i+2\}$ also ends by jumping from $i-1$ to $i+1$, that is, if θ_{i,m_0} is the beginning of it, then $\eta_{i,m_0+1} = \gamma_{i,m_0+1} = \beta_{i+1,j+1} = \eta_{i+1,j+1}$. Therefore, $C_{i,j} = 1$.

Define for $i \le 0$,

$$A(i) = \sum_{j\ge 0} A_{i,j}, \ B(i) = \sum_{j\ge 0} B_{i,j}, \ C(i) = \sum_{j\ge 0} C_{i,j}$$

to be the numbers of steps from i to $i-1$ before time T_1 with crossing-back from $i-1$ to i, $i-2$ to i and $i-1$ to $i+1$, respectively. Define for $i \le 0$,

$$U(i) = [A(i), B(i), C(i)].$$

Then $A(i) + B(i) + C(i)$ is the total number of steps the walk jumping from i to $i-1$ before T_1. The number of steps crossing back from $i-1$ to i is $A(i) + B(i)$, because $X_n \to \infty$, $\mathbb{P}$-a.s. and $C(i) = B(i+1)$ is counted as steps crossing back from i to $i+1$. Thus we have

$$\begin{aligned} T_1 &= 1 + \sum_{i\le 0} \Big(2A(i) + 2B(i) + C(i)\Big) \\ &= 1 + \Big\langle (2,2,1), \sum_{i\le 0} U(i) \Big\rangle. \end{aligned} \tag{2.2}$$

In order to study T_1, we consider $\{U(i)\}_{i\le 0}$ instead. We need the following probabilities. For $i \le 0$,

$$\begin{aligned} \alpha(i) &= P_\omega^i[\text{ the walk jumping from } i \text{ to } i-1, \text{ and finally jumping back} \\ &\quad \text{from } i-1 \text{ to } i \text{ before } T_1], \end{aligned} \tag{2.3}$$

$$\begin{aligned} \beta(i) &= P_\omega^i[\text{ the walk jumping from } i \text{ to } i-1, \text{ and finally jumping back} \\ &\quad \text{from } i-2 \text{ to } i \text{ before } T_1], \end{aligned} \tag{2.4}$$

$$\begin{aligned} \gamma(i) &= P_\omega^i[\text{ the walk jumping from } i \text{ to } i-1, \text{ and finally jumping back} \\ &\quad \text{from } i-1 \text{ to } i+1 \text{ before } T_1]. \end{aligned} \tag{2.5}$$

Expressions of these probabilities will be given later in terms of the exit probabilities. By Markov property, and the different ending cases explained above, we obtain, for $i \le 0$,

$$P_\omega\Big(U(i) = [a,b,0], U(i+1) = [1,0,0] \;\Big|\; |U(i+1)| = 1\Big) = C_{a+b}^a \alpha(i)^a \beta(i)^b p_1(i), \tag{2.6}$$

$$P_\omega\Big(U(i) = [a,b,1], U(i+1) = [0,1,0] \;\Big|\; |U(i+1)| = 1\Big) = C_{a+b}^a \alpha(i)^a \beta(i)^b \gamma(i), \tag{2.7}$$

$$P_\omega\Big(U(i) = [a,b,0], U(i+1) = [0,0,1] \;\Big|\; |U(i+1)| = 1\Big) = C_{a+b}^a \alpha(i)^a \beta(i)^b p_2(i). \tag{2.8}$$

Additionally, for $i = 0$, we define $U(1) = [A(1), B(1), C(1)]$ for consistency. Noting that the walk starts at 0, and the excursion from 0 to $\{1,2\}$ ends at time T_1, there is only one jump crossing up from 0 to 1 in the time interval $[0, T_1]$, which can be regarded as $|U(1)| = 1$ if $T_1 < \infty$, $\mathbb{P}$-a.s.. If the excursion from 0 to $\{1,2\}$ ends by jumping from 0 to 1, then set $A(1) = 1$, in other words, $U(1) = [1,0,0]$. By similar reasoning, $U(1) = [0,1,0]$ if the ending jump is from -1 to 1 and $U(1) = [0,0,1]$ if the ending jump is from 0 to 2. Summing over $a, b \ge 0$ in (2.6), we have

$$\begin{aligned} & P_\omega\Big(U(i+1) = [1,0,0] \;\Big|\; |U(i+1)| = 1\Big) \\ = & \sum_{a,b\ge 0} P_\omega\Big(U(i) = [a,b,0], U(i+1) = [1,0,0] \;\Big|\; |U(i+1)| = 1\Big) \\ = & \sum_{a,b\ge 0} C_{a+b}^a \alpha(i)^a \beta(i)^b p_1(i) = \frac{p_1(i)}{1-\alpha(i)-\beta(i)}. \end{aligned} \tag{2.9}$$

Similarly,

$$P_\omega\Big(U(i+1) = [0,1,0] \;\Big|\; |U(i+1)| = 1\Big) = \frac{\gamma(i)}{1-\alpha(i)-\beta(i)}, \tag{2.10}$$

$$P_\omega\Big(U(i+1) = [0,0,1] \;\Big|\; |U(i+1)| = 1\Big) = \frac{p_2(i)}{1-\alpha(i)-\beta(i)}. \tag{2.11}$$

Now we are ready to calculate $\alpha(i)$, $\beta(i)$, and $\gamma(i)$. Firstly, by the definitions (2.3)–(2.5), we can see

$$\begin{aligned} \alpha(i) &= P_\omega\Big(U(i) = [1,0,0] \;\Big|\; |U(i)| = 1\Big), \\ \beta(i) &= P_\omega\Big(U(i) = [0,1,0] \;\Big|\; |U(i)| = 1\Big), \\ \gamma(i) &= P_\omega\Big(U(i) = [0,0,1] \;\Big|\; |U(i)| = 1\Big). \end{aligned}$$

Thus from (2.9) and (2.10), we have the ratio

$$\alpha(i) : \beta(i) = p_1(i-1) : \gamma(i-1). \tag{2.12}$$

On the other hand, by the definitions (2.3)–(2.5) again, we know

$$\begin{aligned}\alpha(i)+\beta(i) &= P^i_\omega[\text{ the walk jumping from } i \text{ to } i-1, \\ &\qquad\qquad \text{and finally jumping back to } i \text{ before } T_1], \\ \gamma(i) &= P^i_\omega[\text{ the walk jumping from } i \text{ to } i-1, \\ &\qquad\qquad \text{and finally jumping back to } i+1 \text{ before } T_1].\end{aligned}$$

Recall the exit probabilities, we obtain

$$\alpha(i)+\beta(i) = q(i)\cdot P^{i-1}_\omega[(-\infty, i-1), i], \tag{2.13}$$

$$\gamma(i) = q(i)\cdot P^{i-1}_\omega[(-\infty, i-1), i+1]. \tag{2.14}$$

Finally, combining (2.12) and (2.13) we get

$$\alpha(i) = q(i)\cdot P^{i-1}_\omega[(-\infty, i-1), i]\cdot \frac{p_1(i-1)}{p_1(i-1)+\gamma(i-1)}, \tag{2.15}$$

$$\beta(i) = q(i)\cdot P^{i-1}_\omega[(-\infty, i-1), i]\cdot \frac{\gamma(i-1)}{p_1(i-1)+\gamma(i-1)}. \tag{2.16}$$

Proof of Theorem 1.1. By (2.6)–(2.8) and (2.9)–(2.11), we obtain

$$P_\omega\Big(U(i)=[a,b,0] \;\Big|\; U(i+1)=[1,0,0]\Big) = [1-\alpha(i)-\beta(i)]C^a_{a+b}\alpha(i)^a\beta(i)^b, \tag{2.17}$$

$$P_\omega\Big(U(i)=[a,b,1] \;\Big|\; U(i+1)=[0,1,0]\Big) = [1-\alpha(i)-\beta(i)]C^a_{a+b}\alpha(i)^a\beta(i)^b, \tag{2.18}$$

$$P_\omega\Big(U(i)=[a,b,0] \;\Big|\; U(i+1)=[0,0,1]\Big) = [1-\alpha(i)-\beta(i)]C^a_{a+b}\alpha(i)^a\beta(i)^b. \tag{2.19}$$

(2.17)–(2.19) give the offspring distribution for the $(-i+1)$-th generation ($i \le 0$). Indeed, it is enough to show

$$\begin{aligned}\sum_{a,b\ge 0}\Big[&P_\omega\Big(U(i)=[a,b,0] \;\Big|\; U(i+1)=[1,0,0]\Big)\cdot P_\omega\Big(U(i+1)=[1,0,0] \;\Big|\; |U(i+1)|=1\Big) \\ &+P_\omega\Big(U(i)=[a,b,1] \;\Big|\; U(i+1)=[0,1,0]\Big)\cdot P_\omega\Big(U(i+1)=[0,1,0] \;\Big|\; |U(i+1)|=1\Big) \\ &+P_\omega\Big(U(i)=[a,b,0] \;\Big|\; U(i+1)=[0,0,1]\Big)\cdot P_\omega\Big(U(i+1)=[0,0,1] \;\Big|\; |U(i+1)|=1\Big)\Big] = 1,\end{aligned}$$

which is equivalent to check

$$\frac{p_1(i)+\gamma(i)+p_2(i)}{1-\alpha(i)-\beta(i)} = 1.$$

By (2.1), (2.13), and (2.14), we know $\alpha(i)+\beta(i)+\gamma(i) = q(i)$. The conclusion therefore follows from $p_1(i)+p_2(i)+q(i) = 1$.

Since $X_0 = 0$ and $X_n \to \infty$, $\mathbb{P}$-a.s., we have $T_1 < +\infty$, $\mathbb{P}$-a.s., so we imagine that there are different types of particles immigrating into the system $U(1) = [A(1), B(1), C(1)]$ and hence $|U(1)| = 1$ $\mathbb{P}$-a.s.. By (2.9), we know

$$P_\omega\Big(U(1) = [1,0,0]\Big) = P_\omega\Big(U(1) = [1,0,0] \;\Big|\; |U(1)| = 1\Big) = \frac{p_1(0)}{1-\alpha(0)-\beta(0)}.$$

By similar argument,

$$P_\omega\Big(U(1) = [0,1,0]\Big) = \frac{\gamma(0)}{1-\alpha(0)-\beta(0)},$$
$$P_\omega\Big(U(1) = [0,0,1]\Big) = \frac{p_2(0)}{1-\alpha(0)-\beta(0)}.$$

Thus the immigration of the multi-type branching process follows. This completes the proof. □

Proof of Corollary 1.1. Summing over $b \geq 0$ in (2.17), we obtain

$$P_\omega\Big(A(i) = a \;\Big|\; U(i+1) = [1,0,0]\Big) = \frac{1-\alpha(i)-\beta(i)}{1-\beta(i)}\Big(\frac{\alpha(i)}{1-\beta(i)}\Big)^a.$$

So the expected number of type-A offspring of a single type-A particle in one generation is

$$N_{11}(i) = \sum_{a\geq 0} a \cdot P_\omega\Big(A(i) = a \;\Big|\; U(i+1) = [1,0,0]\Big) = \frac{\alpha(i)}{1-\alpha(i)-\beta(i)}.$$

By the same argument, we can get all the elements of the offspring mean matrix $N(i)$. For the second result, noting that by the multitype branching process, we have

$$E_\omega[U(i)] = E_\omega[U(i+1)] \cdot N(i) = E_\omega[U(1)] \cdot N(0) \cdots N(i),$$

and

$$\begin{aligned}
&E_\omega[U(1)] \\
&= [1,0,0] \cdot P_\omega\Big(U(1) = [1,0,0]\Big) + [0,1,0] \cdot P_\omega\Big(U(1) = [0,1,0]\Big) \\
&\qquad\qquad\qquad\qquad\qquad + [0,0,1] \cdot P_\omega\Big(U(1) = [0,0,1]\Big) \\
&= \Big(\frac{p_1(0)}{1-\alpha(0)-\beta(0)}, \frac{\gamma(0)}{1-\alpha(0)-\beta(0)}, \frac{p_2}{1-\alpha(0)-\beta(0)}\Big) \\
&= \frac{1}{1-\alpha(0)-\beta(0)}\Big(p_1(0),\ \gamma(0),\ p_2(0)\Big).
\end{aligned}$$

The desired conclusion follows by taking the quenched expectation in (2.2). □

As an immediate application of the branching structure, consider random walks in non-random environment. Let $\xi_1, \xi_2, \cdots$ be independent variables with common distribution

$$P(\xi_1 = -1) = q, \quad P(\xi_1 = 1) = p_1, \quad P(\xi_1 = 2) = p_2.$$

The induced random walk is the sequence of random variables

$$X_0 = 0, \quad X_n = \xi_1 + \cdots + \xi_n.$$

Define $T_1 = \inf\{n : X_n > 0\}$ as before. The countability of T_1 in Corollary 1.1 enable us to check the Wald's equality ([8], pg. 397) directly.

Proof of Proposition 1.1. First, we have

$$E(X_{T_1}) = 1 \cdot P(X_{T_1} = 1) + 2 \cdot P(X_{T_1} = 2).$$

By Lemma 2.1, we obtain

$$P(X_{T_1} = 1) = P^0[(-\infty, 0), 1] = \lim_{n\to\infty} \frac{\langle e_1, [M + M^2 + \cdots + M^n]v\rangle}{1 + \langle e_1, [M + M^2 + \cdots + M^n]e_1\rangle}, \tag{2.20}$$

$$P(X_{T_1} = 2) = P^0[(-\infty, 0), 2] = \lim_{n\to\infty} \frac{\langle e_1, [M + M^2 + \cdots + M^n]e_2\rangle}{1 + \langle e_1, [M + M^2 + \cdots + M^n]e_1\rangle}, \tag{2.21}$$

where

$$M := \begin{pmatrix} \frac{p_1+p_2}{q} & \frac{p_2}{q} \\ 1 & 0 \end{pmatrix}.$$

Note that

$$M^n = A\Lambda^n A^{-1} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \cdot \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix},$$

where $\lambda_{1,2} = \frac{p_1+p_2 \pm \sqrt{(p_1+p_2)^2 + 4qp_2}}{2q}$ are the eigenvalues of M. Then

$$\langle e_1, M^n v\rangle = (1 + \lambda_2)\lambda_1^{n+1} - (1 + \lambda_1)\lambda_2^{n+1},$$
$$\langle e_1, M^n e_2\rangle = (-\lambda_2)\lambda_1^{n+1} + \lambda_1\lambda_2^{n+1},$$
$$\langle e_1, M^n e_1\rangle = \lambda_1^{n+1} - \lambda_2^{n+1}.$$

Since $EX_1 > 0$, we have $\lambda_1 > 1$, and $\lambda_2 \in (-1, 0)$. Hence, by (2.20) and (2.21), we obtain

$$P(X_{T_1} = 1) = 1 + \lambda_2,$$
$$P(X_{T_1} = 2) = -\lambda_2.$$

Therefore, $E(X_{T_1}) = 1 - \lambda_2$.

The next step is to calculate $E(T_1)$, which is done by the branching process (Corollary 1.1). Since the environment is not random, there is no site index or integration with respect to P any more. Applying Corollary 1.1 to this random walk, we have

$$E(T_1) = 1 + \left\langle (2, 2, 1), \frac{1}{1 - \alpha - \beta}\Big(p_1,\ \gamma,\ p_2\Big) \cdot \sum_{n\geq 1} N^n \right\rangle.$$

where

$$N := \begin{pmatrix} \frac{\alpha}{1-\alpha-\beta} & \frac{\beta}{1-\alpha-\beta} & 0 \\ \frac{\alpha}{1-\alpha-\beta} & \frac{\beta}{1-\alpha-\beta} & 1 \\ \frac{\alpha}{1-\alpha-\beta} & \frac{\beta}{1-\alpha-\beta} & 0 \end{pmatrix},$$

Using the eigenvalues and eigenvectors of N, and noting that the norm of the greatest eigenvalue of N is less then 1, we have

$$\sum_{n\geq 1} N^n = \frac{1}{1-2\alpha-3\beta} \begin{pmatrix} \alpha & \beta & \beta \\ 2\alpha & 2\beta & 1-2\alpha-\beta \\ \alpha & \beta & \beta \end{pmatrix}. \tag{2.22}$$

Thus,

$$E(T_1) = \frac{\gamma+1-\alpha-\beta}{(1-\alpha-\beta)(1-2\alpha-3\beta)}.$$

Substituting $\gamma = -q\lambda_2$, $\alpha = q(1+\lambda_2)\frac{p_1}{p_1-q\lambda_2}$, and $\beta = q(1+\lambda_2)\frac{-q\lambda_2}{p_1-q\lambda_2}$, we have

$$\begin{aligned} \gamma+1-\alpha-\beta &= \Delta, \\ 1-\alpha-\beta &= \frac{1}{2}(1-q+\Delta), \\ 1-2\alpha-3\beta &= 1-\frac{(1+q-\Delta)(p_1-3p_2+3\Delta)}{2(p_1-p_2+\Delta)}, \end{aligned}$$

where $\Delta = \sqrt{(p_1+p_2)^2+4p_2q}$. Then

$$\begin{aligned} E(T_1) &= \frac{2\Delta}{(1-q+\Delta)-\frac{(1-q+\Delta)(1+q-\Delta)(p_1-3p_2+3\Delta)}{2(p_1-p_2+\Delta)}} \\ &= \frac{2\Delta}{(1-q+\Delta)-\frac{2q(p_1-p_2+\Delta)(p_1-3p_2+3\Delta)}{2(p_1-p_2+\Delta)}} \\ &= \frac{2\Delta}{1-q-qp_1+3qp_2+(1-3q)\Delta} =: \frac{a(p_1,p_2,q)}{b(p_1,p_2,q)}, \end{aligned}$$

and

$$\frac{E(X_{T_1})}{E(X_1)} = \frac{1-\lambda_2}{p_1+2p_2-q} = \frac{3q-1+\Delta}{2q(p_1+2p_2-q)} =: \frac{c(p_1,p_2,q)}{d(p_1,p_2,q)}.$$

In order to prove $E(X_{T_1}) = E(T_1)\cdot E(X_1)$, it suffices to show

$$a(p_1,p_2,q)d(p_1,p_2,q) = b(p_1,p_2,q)c(p_1,p_2,q).$$

In fact,

$$\begin{aligned} a(p_1,p_2,q)d(p_1,p_2,q) &= 4q(p_1+2p_2-q)\Delta \\ &= 4q(1-p_2-q+2p_2-q)\Delta \\ &= 4q(1+p_2-2q)\Delta. \end{aligned}$$

And,

$$\begin{aligned}
& b(p_1,p_2,q)c(p_1,p_2,q) \\
= & [(1-q-qp_1+3qp_2)+(1-3q)\Delta](3q-1+\Delta) \\
= & (1-q-qp_1+3qp_2)(3q-1)+(1-3q)\Delta^2+[(1-q-qp_1+3qp_2)+(1-3q)(3q-1)]\Delta \\
= & (3q-1)(1-q-qp_1+3qp_2-(1-q)^2-4qp_2) \\
& \qquad +[1-q-q(1-q-p_2)+3qp_2-(1-6q+9q^2)]\Delta \\
= & (3q-1)(q(1-p_1-p_2)-q^2)+[4q+4qp_2-8q^2]\Delta \\
= & 4q(1+p_2-2q)\Delta.
\end{aligned}$$

Thus, the desired conclusion follows. □

Remark Proposition 1.1 is a strong evidence to validate the branching structure.

3 Density of the absolutely continuous invariant measure

Now we introduce the machinery of the "environment viewed from the particle". The first step consists of introducing an auxiliary Markov chain. Starting from the RWRE $\{X_n\}$, define $\overline{\omega}(n) = \theta^{X_n}\omega$. The sequence $\{\overline{\omega}(n)\}$ is a process with paths in $\Omega^{\mathbb{N}}$. This process is in fact a Markov process. The proof is the same as Zeitouni ([19], page 204, Lemma 2.1.18), we omit the details.

Lemma 3.1 *The process $\{\overline{\omega}(n)\}$ is a Markov process under either P_ω or $\mathbb{P}$, with state space Ω and transition kernel*

$$K(\omega, d\omega') = q\mathbf{1}_{\omega'=\theta^{-1}\omega} + p_1\mathbf{1}_{\omega'=\theta\omega} + p_2\mathbf{1}_{\omega'=\theta^2\omega}.$$

□

The next step is to construct an invariant measure for the transition kernel K. Assume that $X_n \to \infty$, $\mathbb{P}$-a.s., implying $T_n < \infty$, $\mathbb{P}$-a.s.. Set $\varphi^1_{\theta^k\omega} = P_{\theta^k\omega}(X_{T_1} = 1)$, and $\varphi^2_{\theta^k\omega} = P_{\theta^k\omega}(X_{T_1} = 2) = 1 - \varphi^1_{\theta^k\omega}$. Whenever $\mathbb{E}(T_1) < \infty$, define the measure

$$Q(B) = \mathbb{E}\bigg(\frac{\mathbf{1}_{X_{T_1}=1}}{\varphi^1_\omega}\sum_{i=0}^{T_1-1}\mathbf{1}_{\{\overline{\omega}(i)\in B\}} + \frac{\mathbf{1}_{X_{T_1}=2}}{\varphi^2_\omega}\sum_{i=0}^{T_1-1}\mathbf{1}_{\{\overline{\omega}(i)\in B\}}\bigg), \quad \overline{Q}(B) = \frac{Q(B)}{Q(\Omega)}.$$

Note that $\overline{Q}$ is a probability measure.

Proof of Theorem 1.2—invariant measure. We will show

$$Q(B) = \iint \mathbf{1}_{\omega'\in B}K(\omega, \mathrm{d}\omega')Q(\mathrm{d}\omega).$$

On one hand,

$$Q(B) = \sum_{i=0}^{\infty}\mathbb{E}\bigg(\frac{\mathbf{1}_{X_{T_1}=1}}{\varphi^1_\omega}; T_1 > i; \overline{\omega}(i)\in B\bigg) + \sum_{i=0}^{\infty}\mathbb{E}\bigg(\frac{\mathbf{1}_{X_{T_1}=2}}{\varphi^2_\omega}; T_1 > i; \overline{\omega}(i)\in B\bigg).$$

On the other hand,

$$
\begin{aligned}
&\iint \mathbf{1}_{\omega'\in B}K(\omega,\mathrm{d}\omega')Q(\mathrm{d}\omega)\\
=\ & \sum_{i=0}^{\infty}\mathbb{E}\Big(\frac{\mathbf{1}_{X_{T_1}=1}}{\varphi^1_\omega};\ T_1>i;\ \overline{\omega}(i+1)\in B\Big)+\sum_{i=0}^{\infty}\mathbb{E}\Big(\frac{\mathbf{1}_{X_{T_1}=2}}{\varphi^2_\omega};\ T_1>i;\ \overline{\omega}(i+1)\in B\Big)\\
=\ & \sum_{j=1}^{\infty}\mathbb{E}\Big(\frac{\mathbf{1}_{X_{T_1}=1}}{\varphi^1_\omega};\ T_1>j;\ \overline{\omega}(j)\in B\Big)+\sum_{j=1}^{\infty}\mathbb{E}\Big(\frac{\mathbf{1}_{X_{T_1}=2}}{\varphi^2_\omega};\ T_1>j;\ \overline{\omega}(j)\in B\Big)\\
&\quad +\mathbb{E}\Big(\frac{\mathbf{1}_{X_{T_1}=1}}{\varphi^1_\omega};\ T_1<\infty;\ \overline{\omega}(T_1)\in B\Big)+\mathbb{E}\Big(\frac{\mathbf{1}_{X_{T_1}=2}}{\varphi^2_\omega};\ T_1<\infty;\ \overline{\omega}(T_1)\in B\Big).
\end{aligned}
$$

It only needs to show

$$\mathbb{E}\Big(\frac{\mathbf{1}_{X_{T_1}=i}}{\varphi^i_\omega};\ T_1>0;\ \overline{\omega}(0)\in B\Big)=\mathbb{E}\Big(\frac{\mathbf{1}_{X_{T_1}=i}}{\varphi^i_\omega};\ T_1<\infty;\ \overline{\omega}(T_1)\in B\Big),\ i=1,2.$$

Indeed,

$$
\begin{aligned}
\text{R.S.} &= E_P\Big[\frac{1}{\varphi^i_\omega}E_\omega\Big(\mathbf{1}_{X_{T_1}=i},\ \overline{\omega}(T_1)\in B, T_1<\infty\Big)\Big]\\
&= E_P\Big[P_\omega\Big(\overline{\omega}(T_1)\in B,\ T_1<\infty\Big|X_{T_1}=i\Big)\Big]\\
&=\ P\ \Big[\theta^i\omega\in B\cdot P_\omega\Big(T_1<\infty\Big|X_{T_1}=i\Big)\Big]\\
&=\ P\ \Big(\theta^i\omega\in B\Big)\ (\text{since } P_\omega(T_1<\infty)=1).
\end{aligned}
$$

Similarly, L.S.$=P(\omega\in B)$. Then, by the invariance of the environment, we get L.S.=R.S.. This completes the proof. □

Proof of Theorem 1.2—density. Let $f:\Omega\to\mathbb{R}$ be measurable. Then,

$$
\begin{aligned}
\int f\mathrm{d}Q &= \mathbb{E}\Big(\sum_{i=0}^{T_1-1}f(\overline{\omega}(i))\frac{\mathbf{1}_{X_{T_1}=1}}{\varphi^1_\omega}+\sum_{i=0}^{T_1-1}f(\overline{\omega}(i))\frac{\mathbf{1}_{X_{T_1}=2}}{\varphi^2_\omega}\Big)\\
&= \mathbb{E}\Big(\sum_{i\le 0}f(\theta^i\omega)V_i\frac{\mathbf{1}_{X_{T_1}=1}}{\varphi^1_\omega}+\sum_{i\le 0}f(\theta^i\omega)V_i\frac{\mathbf{1}_{X_{T_1}=2}}{\varphi^2_\omega}\Big),
\end{aligned}
$$

where $V_i=\#\{k\in[0,T_1):X_k=i\}$. Using the shift invariance of P, we get

$$
\begin{aligned}
\int f\mathrm{d}Q &= \sum_{i\le 0}E_P\Big(f(\theta^i\omega)\Big[\frac{1}{\varphi^1_\omega}E_\omega(V_i\cdot\mathbf{1}_{X_{T_1}=1})+\frac{1}{\varphi^2_\omega}E_\omega(V_i\cdot\mathbf{1}_{X_{T_1}=2})\Big]\Big)\\
&= \sum_{i\le 0}E_P\Big(f(\omega)\Big[\frac{1}{\varphi^1_{\theta^{-i}\omega}}E_{\theta^{-i}\omega}(V_i\cdot\mathbf{1}_{X_{T_1}=1})+\frac{1}{\varphi^2_{\theta^{-i}\omega}}E_{\theta^{-i}\omega}(V_i\cdot\mathbf{1}_{X_{T_1}=2})\Big]\Big)\\
&= E_P\Big(f(\omega)\sum_{i\le 0}\Big[\frac{1}{\varphi^1_{\theta^{-i}\omega}}E_{\theta^{-i}\omega}(V_i\cdot\mathbf{1}_{X_{T_1}=1})+\frac{1}{\varphi^2_{\theta^{-i}\omega}}E_{\theta^{-i}\omega}(V_i\cdot\mathbf{1}_{X_{T_1}=2})\Big]\Big)
\end{aligned}
$$

$$= \quad E_P\Big(f(\omega)\sum_{i\le 0}\Big[E_{\theta^{-i}\omega}(V_i|X_{T_1}=1)+E_{\theta^{-i}\omega}(V_i|X_{T_1}=2)\Big]\Big).$$

Hence,

$$\frac{\mathrm{d}Q}{\mathrm{d}P}=\sum_{i\le 0}\Big[E_{\theta^{-i}\omega}(V_i|X_{T_1}=1)+E_{\theta^{-i}\omega}(V_i|X_{T_1}=2)\Big].$$

The calculation of $E_{\theta^{-i}\omega}(V_i|X_{T_1}=k),\ (k=1,2)$ is based on the branching process. Note that $V_i=A(i+1)+B(i+1)+C(i+1)+A(i)+B(i)=|U(i+1)|+A(i)+B(i)$. Then

$$\begin{aligned}
E_\omega\Big(V_0\Big|X_{T_1}=1\Big) &= E_\omega\Big[E_\omega\Big(|U(1)|+A(0)+B(0)\ \Big|\ U(1)\Big)\Big|X_{T_1}=1\Big]\\
&= E_\omega\Big[|U(1)|+\frac{\alpha(0)}{1-\alpha(0)-\beta(0)}|U(1)|+\frac{\beta(0)}{1-\alpha(0)-\beta(0)}|U(1)|\Big|X_{T_1}=1\Big]\\
&= \frac{1}{1-\alpha(0)-\beta(0)}\Big|E_\omega\Big(U(1)\Big|X_{T_1}=1\Big)\Big|\\
&= \frac{1}{1-\alpha(0)-\beta(0)}\Big\langle(1,1,1),E_\omega\Big(U(1)\Big|X_{T_1}=1\Big)\Big\rangle.
\end{aligned}$$

Similarly, for $i\le -1$,

$$\begin{aligned}
E_\omega\Big(V_i\Big|X_{T_1}=1\Big) &= E_\omega\Big[E_\omega\Big(V_i\Big|U(i+1)\Big)\Big|X_{T_1}=1\Big]\\
&= \frac{1}{1-\alpha(i)-\beta(i)}\Big|E_\omega\Big(U(i+1)\Big|X_{T_1}=1\Big)\Big|\\
&= \frac{1}{1-\alpha(i)-\beta(i)}\Big|E_\omega\Big(U(1)\Big|X_{T_1}=1\Big)\cdot N(0)\cdots N(i+1)\Big|\\
&= \frac{1}{1-\alpha(i)-\beta(i)}\Big\langle(1,1,1),E_\omega\Big(U(1)\Big|X_{T_1}=1\Big)\cdot N(0)\cdots N(i+1)\Big\rangle.
\end{aligned}$$

Hence, for $i\le -1$,

$$E_{\theta^{-i}\omega}\Big(V_i\Big|X_{T_1}=1\Big)=\frac{1}{1-\alpha(0)-\beta(0)}\Big\langle(1,1,1),E_{\theta^{-i}\omega}\Big(U(1)\Big|X_{T_1}=1\Big)\cdot N(-i)\cdots N(1)\Big\rangle.$$

By the same argument,

$$E_\omega\Big(V_0\Big|X_{T_1}=2\Big)=\frac{1}{1-\alpha(0)-\beta(0)}\Big\langle(1,1,1),E_\omega\Big(U(1)\Big|X_{T_1}=2\Big)\Big\rangle.$$

And for $i\le -1$,

$$E_{\theta^{-i}\omega}\Big(V_i\Big|X_{T_1}=2\Big)=\frac{1}{1-\alpha(0)-\beta(0)}\Big\langle(1,1,1),E_{\theta^{-i}\omega}\Big(U(1)\Big|X_{T_1}=2\Big)\cdot N(-i)\cdots N(1)\Big\rangle.$$

By branching process, we have

$$\begin{aligned}
&E_\omega\Big(U(1)\Big|X_{T_1}=1\Big)\\
=\ &(1,0,0)\cdot P_\omega(U(1)=(1,0,0)|X_{T_1}=1)\\
&\quad+(0,1,0)\cdot P_\omega(U(1)=(0,1,0)|X_{T_1}=1)+(0,0,1)\cdot P_\omega(U(1)=(0,0,1)|X_{T_1}=1)
\end{aligned}$$

$$
\begin{aligned}
&= (1,0,0)\cdot \frac{p_1(0)}{1-\alpha(0)-\beta(0)}\Big/\Big(\frac{p_1(0)+\gamma(0)}{1-\alpha(0)-\beta(0)}\Big) \\
&\qquad +(0,1,0)\cdot \frac{\gamma(0)}{1-\alpha(0)-\beta(0)}\Big/\Big(\frac{p_1(0)+\gamma(0)}{1-\alpha(0)-\beta(0)}\Big)+(0,0,1)\cdot 0 \\
&= \Big(\frac{p_1(0)}{p_1(0)+\gamma(0)},\frac{\gamma(0)}{p_1(0)+\gamma(0)},0\Big).
\end{aligned}
$$

And,

$$
E_\omega\Big(U(1)\Big|X_{T_1}=2\Big)=(0,0,1).
$$

Consequently, we get that

$$
\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}P} &= \sum_{i\le 0}\Big(\sum_{k=1,2}E_{\theta^{-i}\omega}\Big(V_i\Big|X_{T_1}=k\Big)\Big) \\
&= \frac{2+\Big\langle(1,1,1),\sum_{i\ge1}\Big[\sum_{k=1,2}E_{\theta^i\omega}\Big(U(1)\Big|X_{T_1}=k\Big)\Big]\cdot N(i)\cdots N(1)\Big\rangle}{1-\alpha(0)-\beta(0)} \\
&= \frac{2+\Big\langle(1,1,1),\sum_{i\ge1}\Big(\frac{p_1(i)}{p_1(i)+\gamma(i)},\frac{\gamma(i)}{p_1(i)+\gamma(i)},1\Big)\cdot N(i)\cdots N(1)\Big\rangle}{1-\alpha(0)-\beta(0)}.
\end{aligned}
$$

□

4 The law of large numbers

We are now ready to prove the law of large number with an *explicit* drift based on the branching structure by the method of "environment viewed from the particle".

Lemma 4.1 *Under the law induced by $\overline{Q}\otimes P_\omega$, the sequence $\{\overline{\omega}(n)\}$ is stationary and ergodic.*

Proof . Since $\overline{Q}$ is an invariant measure under the transition kernel K by Theorem 1.2, we obtain that the process $\{\overline{\omega}(n)\}$ is stationary under $\overline{Q}\otimes P_\omega$. For the ergodicity, the proof is similar as Zeitouni ([19], page 207, Corollary 2.1.25), we omit the details. □

Proof of Theorem 1.3. The idea of the proof for the LLN is the same as Zeitouni ([19], page 208, Theorem 2.1.9), however, we pay attention to the drift here.

Define the local drift at site x in the environment ω, as $d(x,\omega)=E_\omega^x(X_1-x)$. The ergodicity of $\{\overline{\omega}(i)\}$ under $\overline{Q}\otimes P_\omega$ implies that:

$$
\frac{1}{n}\sum_{k=0}^{n-1}d(X_k,\omega)=\frac{1}{n}\sum_{k=0}^{n-1}d(0,\overline{\omega}(k))\longrightarrow\int d(0,\omega)\ \mathrm{d}\overline{Q},\ \text{ as } n\to\infty,\ \overline{Q}\otimes P_\omega\text{-a.s.}
$$

On the other hand,

$$
X_n=\sum_{i=1}^n(X_i-X_{i-1}) \quad = \quad \sum_{i=1}^n\Big(X_i-X_{i-1}-d(X_{i-1},\omega)\Big)+\sum_{i=1}^n d(X_{i-1},\omega)
$$

$$:= \quad M_n + \sum_{i=1}^{n} d(X_{i-1}, \omega).$$

Under P_ω, M_n is a martingale, with $|M_n - M_{n-1}| \le 3$ for $\omega \in \Omega$. Hence, with $\mathcal{G}_n = \sigma(M_1, \cdots, M_n)$,

$$\begin{aligned} E_\omega(\mathrm{e}^{\lambda M_n}) &= E_\omega\Big(\mathrm{e}^{\lambda M_{n-1}} E_\omega(\mathrm{e}^{\lambda(M_n - M_{n-1})} | \mathcal{G}_n)\Big) \\ &\le E_\omega\Big(\mathrm{e}^{\lambda M_{n-1}} \mathrm{e}^{3\lambda^2}\Big), \end{aligned}$$

and hence, iterating, $E_\omega(\mathrm{e}^{\lambda M_n}) \le \mathrm{e}^{3n\lambda^2}$ (this is a version of Azuma's inequality, see [**?**], Corollary 2.4.7). Then Chebyshev's inequality implies that

$$\frac{M_n}{n} \to 0, \ \mathbb{P}\text{-a.s.}$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n} X_n = \int d(0,\omega) \ \mathrm{d}\overline{Q} = v_P,$$

Observe that

$$\begin{aligned} v_P &= \int d(0,\omega) \mathrm{d}\overline{Q} = E_{\overline{Q}}(X_1) \\ &= \frac{E_P\Big[\Pi(\omega)\Big(p_1(0) + 2p_2(0) - p_{-1}(0)\Big)\Big]}{Q(\Omega)} \\ &= \frac{E_P\Big[\frac{p_1(0)+2p_2(0)-p_{-1}(0)}{1-\alpha(0)-\beta(0)}\Big(2 + \Big\langle(1,1,1), \sum_{i\ge 1}\Big(\frac{p_1(i)}{p_1(i)+\gamma(i)}, \frac{\gamma(i)}{p_1(i)+\gamma(i)}, 1\Big)\cdot N(i)\cdots N(1)\Big)\Big\rangle\Big]}{\mathbb{E}(T_1|X_{T_1}=1) + \mathbb{E}(T_1|X_{T_1}=2)}. \end{aligned}$$

Moreover, by (2.2) and the proof of the density of Theorem 1.2, we have

$$\begin{aligned} \mathbb{E}(T_1|X_{T_1}=1) &= E_P\Big[E_\omega\Big(1 + \Big\langle(2,2,1), \sum_{i\le 0} U(i)\Big\rangle\Big| X_{T_1} = 1\Big)\Big] \\ &= E_P\Big[1 + \Big\langle(2,2,1), \sum_{i\le 0} E_\omega\Big(U(i)\Big| X_{T_1}=1\Big)\Big\rangle\Big] \\ &= E_P\Big[1 + \Big\langle(2,2,1), \sum_{i\le 0} E_\omega\Big(U(1)\Big| X_{T_1}=1\Big)\cdot N(0)\cdots N(i)\Big\rangle\Big] \\ &= E_P\Big[1 + \Big\langle(2,2,1), \sum_{i\le 0}\Big(\frac{p_1(0)}{p_1(0)+\gamma(0)}, \frac{\gamma(0)}{p_1(0)+\gamma(0)}, 0\Big)\cdot N(0)\cdots N(i)\Big\rangle\Big], \end{aligned}$$

and by the same argument,

$$\mathbb{E}(T_1|X_{T_1}=2) = E_P\Big[1 + \Big\langle(2,2,1), \sum_{i\le 0}\Big(0,0,1\Big)\cdot N(0)\cdots N(i)\Big\rangle\Big].$$

Therefore,

$$\begin{aligned}
& \mathbb{E}(T_1|X_{T_1}=1)+\mathbb{E}(T_1|X_{T_1}=2) \\
= \;& E_P\Big[2+\Big\langle(2,2,1),\sum_{i\le 0}\Big(\frac{p_1(0)}{p_1(0)+\gamma(0)},\frac{\gamma(0)}{p_1(0)+\gamma(0)},1\Big)\cdot N(0)\cdots N(i)\Big\rangle\Big] \\
= \;& E_P\Big[2+\Big\langle(2,2,1),\sum_{i\ge 0}\Big(\frac{p_1(i)}{p_1(i)+\gamma(i)},\frac{\gamma(i)}{p_1(i)+\gamma(i)},1\Big)\cdot N(i)\cdots N(0)\Big\rangle\Big],
\end{aligned}$$

where the last equation holds by the invariance of P. Consequently, the drift in this case can be written as

$$v_P=\frac{E_P\Big[\frac{p_1(0)+2p_2(0)-p_{-1}(0)}{1-\alpha(0)-\beta(0)}\Big(2+\sum_{i\ge1}\Big\langle(1,1,1),\Big(\frac{p_1(i)}{p_1(i)+\gamma(i)},\frac{\gamma(i)}{p_1(i)+\gamma(i)},1\Big)\cdot N(i)\cdots N(1)\Big\rangle\Big)\Big]}{E_P\Big[2+\Big\langle(2,2,1),\sum_{i\ge0}\Big(\frac{p_1(i)}{p_1(i)+\gamma(i)},\frac{\gamma(i)}{p_1(i)+\gamma(i)},1\Big)\cdot N(i)\cdots N(0)\Big\rangle\Big]}.$$

□

Remark Consider the random walk in non-random environment defined in Proposition 1.1. Then the drift v_P reduces to $E(X_1)=p_1+2p_2-q$. In fact, when the environment is not random, the drift in Theorem 1.3 can be written as

$$v_P=(p_1+2p_2-q)\frac{\frac{1}{1-\alpha-\beta}\Big(2+\Big\langle(1,1,1),\Big(\frac{p_1}{p_1+\gamma},\frac{\gamma}{p_1+\gamma},1\Big)\cdot\sum_{n\ge1}N^n\Big\rangle\Big)}{2+\Big\langle(2,2,1),\Big(\frac{p_1}{p_1+\gamma},\frac{\gamma}{p_1+\gamma},1\Big)\cdot\sum_{n\ge1}N^n\Big\rangle}.$$

By (2.22), we obtain

$$\begin{aligned}
v_P \;=\;& (p_1+2p_2-q) \\
& \times\frac{\frac{1}{1-\alpha-\beta}\Big[2+\frac{1}{(p_1+\gamma)(1-2\alpha-3\beta)}\Big\langle(1,1,1),\Big(\alpha(2p_1+3\gamma),\ \beta(2p_1+3\gamma),\ 2\beta p_1+(1-2\alpha)\gamma\Big)\Big\rangle\Big]}{2+\frac{1}{(p_1+\gamma)(1-2\alpha-3\beta)}\Big\langle(2,2,1),\Big(\alpha(2p_1+3\gamma),\ \beta(2p_1+3\gamma),\ 2\beta p_1+(1-2\alpha)\gamma\Big)\Big\rangle} \\
=\;& (p_1+2p_2-q)\times\frac{\frac{2p_1+3\gamma}{(p_1+\gamma)(1-2\alpha-3\beta)}}{\frac{2p_1+3\gamma}{(p_1+\gamma)(1-2\alpha-3\beta)}} \\
=\;& p_1+2p_2-q,
\end{aligned}$$

as it should be. □

5 The general bounded jump case: $(1,R)$-RWRE

In this section, we consider $(1,R)$-RWRE, in which the possible jumps to the right are $1,2,\cdots,R$, where R is some fixed positive integer. In this case, the environment is an element

$$\omega=\{(q(\omega)_z,p_1(\omega)_z,\cdots,p_R(\omega)_z):z\in\mathbb{Z}\}\in\mathcal{M}(\widetilde{\Lambda})^{\mathbb{Z}}=:\widetilde{\Omega},$$

where $\widetilde{\Lambda} = \{-1, 1, \cdots, R\}$ is the set of all possible jumps. Following the idea in section 2, we can define a branching process with $(1+2+\cdots+R)$-type species. In fact, for a jump from i to $i-1$, there are $(1+2+\cdots+R)$ crossing-back ways, i.e., jumping from $i-k_1$ to i for $k_1 \in \{1, 2, \cdots, R\}$, jumping from $i-k_2$ to $i+1$ for $k_2 \in \{1, 2, \cdots, R-1\}$, $\cdots$, jumping from $i-k_R$ to $i+R-1$ for $k_R = 1$.

First, we need the following lemma dealing with exit probabilities for this general case.

Lemma 5.1 *For $n \geq 2$ and $1 \leq j \leq R$, we have*

$$P_\omega^i[(-n, i), i+j] = \frac{\langle e_1, [\widetilde{M}(i) + \cdots + \widetilde{M}(-n) \cdots \widetilde{M}(i)](e_j - e_{j+1})\rangle}{1 + \langle e_1, [\widetilde{M}(i) + \cdots + \widetilde{M}(-n) \cdots \widetilde{M}(i)]e_1\rangle},$$

with $e_{R+1} = 0$ and

$$\widetilde{M}(i) := \begin{pmatrix} \frac{p_1(i)+\cdots+p_R(i)}{q(i)} & \cdots & \frac{p_{R-1}(i)+p_R(i)}{q(i)} & \frac{p_R(i)}{q(i)} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The proof is the same as Lemma 2.1. □

Define for $i \leq 0$,

$$U(i) = \Big(U_1(i), \cdots, U_R(i), U_{R+1}(i), \cdots, U_{R+R-1}(i), \cdots, U_{1+2+\cdots+R}(i)\Big),$$

where, for $k = 1, 2, \cdots, (1+2+\cdots+R)$, $U_k(i)$ is the number of steps from i to $i-1$ before time T_1 with crossing-back from $i-1$ to i, $\cdots$, $i-R$ to i; $i-1$ to $i+1$, $\cdots$, $i-R+1$ to $i+1$; $\cdots$; $i-1$ to $i+R-1$; respectively. For probabilities, let

$$\begin{aligned}
p_{(1)}(i) &= P_\omega^i[\text{the walk jumping from } i \text{ to } i-1, \text{ and finally jumping back} \\
&\qquad \text{from } i-1 \text{ to } i \text{ before } T_1], \\
&\vdots \\
p_{(R)}(i) &= P_\omega^i[\text{the walk jumping from } i \text{ to } i-1, \text{ and finally jumping back} \\
&\qquad \text{from } i-R \text{ to } i \text{ before } T_1], \\
p_{(R+1)}(i) &= P_\omega^i[\text{the walk jumping from } i \text{ to } i-1, \text{ and finally jumping back} \\
&\qquad \text{from } i-1 \text{ to } i+1 \text{ before } T_1], \\
&\vdots \\
p_{(R+R-1)}(i) &= P_\omega^i[\text{the walk jumping from } i \text{ to } i-1, \text{ and finally jumping back} \\
&\qquad \text{from } i-R+1 \text{ to } i+1 \text{ before } T_1], \\
&\vdots \\
p_{(1+2+\cdots+R)}(i) &= P_\omega^i[\text{the walk jumping from } i \text{ to } i-1, \text{ and finally jumping back} \\
&\qquad \text{from } i-1 \text{ to } i+R-1 \text{ before } T_1].
\end{aligned}$$

Expressions of these probabilities can be calculated by using the exit probabilities as this is done in section 2. Firstly, we have

$$\begin{aligned}
p_{(1)}+\cdots+p_{(R)} &= P_\omega^i[\text{the walk jumping from } i \text{ to } i-1, \text{ and finally} \\ &\qquad \text{jumping back to } i \text{ before } T_1], \\
p_{(R+1)}+\cdots+p_{(R+R-1)} &= P_\omega^i[\text{the walk jumping from } i \text{ to } i-1, \text{ and finally} \\ &\qquad \text{jumping back to } i+1 \text{ before } T_1], \\
&\vdots \\
p_{((1+2+\cdots+R)-2)}(i)+p_{((1+2+\cdots+R)-1)}(i) &= P_\omega^i[\text{the walk jumping from } i \text{ to } i-1, \text{ and finally} \\ &\qquad \text{jumping back to } i+R-2 \text{ before } T_1], \\
p_{(1+2+\cdots+R)}(i) &= P_\omega^i[\text{the walk jumping from } i \text{ to } i-1, \text{ and finally} \\ &\qquad \text{jumping back to } i+R-1 \text{ before } T_1].
\end{aligned}$$

Recall the exit probabilities, we obtain

$$\begin{aligned}
p_{(1)}+\cdots+p_{(R)} &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i], \\
p_{(R+1)}+\cdots+p_{(R+R-1)} &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+1], \\
&\vdots \\
p_{((1+2+\cdots+R)-2)}(i)+p_{((1+2+\cdots+R)-1)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+R-2], \qquad (5.1) \\
p_{(1+2+\cdots+R)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+R-1].
\end{aligned}$$

Observe that

$$p_{((1+2+\cdots+R)-2)}(i) : p_{((1+2+\cdots+R)-1)}(i) = p_{R-1}(i-1) : p_{(1+2+\cdots+R)}(i-1).$$

Thus by (5.1), we get

$$\begin{aligned}
p_{((1+2+\cdots+R)-2)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+R-2]\cdot\frac{p_{R-1}(i-1)}{p_{R-1}(i-1)+p_{(1+2+\cdots+R)}(i-1)}, \\
p_{((1+2+\cdots+R)-1)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+R-2]\cdot\frac{p_{(1+2+\cdots+R)}(i-1)}{p_{R-1}(i-1)+p_{(1+2+\cdots+R)}(i-1)}.
\end{aligned}$$

Using the same argument, we can get the expression of all the probabilities:

$$\begin{aligned}
p_{(1+2+\cdots+R)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+R-1], \\
p_{((1+2+\cdots+R)-2)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+R-2]\cdot\frac{p_{R-1}(i-1)}{p_{R-1}(i-1)+p_{(1+2+\cdots+R)}(i-1)}, \\
p_{((1+2+\cdots+R)-1)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+R-2]\cdot\frac{p_{(1+2+\cdots+R)}(i-1)}{p_{R-1}(i-1)+p_{(1+2+\cdots+R)}(i-1)}, \\
&\vdots \\
p_{(R+1)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+1] \\
&\qquad \times\frac{p_2(i-1)}{p_2(i-1)+p_{(R+R-1+1)}(i-1)+\cdots+p_{(R+R-1+R-2)}(i-1)},
\end{aligned}$$

$$
\begin{aligned}
&\vdots\\
p_{(R+R-1)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i+1]\\
&\qquad\times\frac{p_{(R+R-1+R-2)}(i-1)}{p_2(i-1)+p_{(R+R-1+1)}(i-1)+\cdots+p_{(R+R-1+R-2)}(i-1)},\\
p_{(1)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i]\cdot\frac{p_1(i-1)}{p_1(i-1)+p_{(R+1)}(i-1)+\cdots+p_{(R+R-1)}(i-1)},\\
p_{(2)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i]\cdot\frac{p_{(R+1)}(i-1)}{p_1(i-1)+p_{(R+1)}(i-1)+\cdots+p_{(R+R-1)}(i-1)},\\
&\vdots\\
p_{(R)}(i) &= q(i)\cdot P_\omega^{i-1}[(-\infty,i-1),i]\cdot\frac{p_{(R+R-1)}(i-1)}{p_1(i-1)+p_{(R+1)}(i-1)+\cdots+p_{(R+R-1)}(i-1)}.
\end{aligned}
$$

Now we are ready to give the theorem about $U(i)$.

Theorem 5.1 *Assume* $X_n \to \infty$, $\mathbb{P}$-*a.s.. Then* $\Big(U(i)\Big)_{i\le 0}$ *is an inhomogeneous multitype branching process in* $\mathbb{R}^{1+2+\cdots+R}$ *with immigration*

$$
\begin{aligned}
&U(1)=e_1, \quad \text{with probability} \quad \frac{p_1(0)}{1-p_{(1)}(0)-\cdots-p_{(R)}(0)},\\
&U(1)=e_2, \quad \text{with probability} \quad \frac{p_{(R+1)}(0)}{1-p_{(1)}(0)-\cdots-p_{(R)}(0)},\\
&\quad\vdots\\
&U(1)=e_R, \quad \text{with probability} \quad \frac{p_{(R+R-1)}(0)}{1-p_{(1)}(0)-\cdots-p_{(R)}(0)},\\
&U(1)=e_{R+1}, \quad \text{with probability} \quad \frac{p_2(0)}{1-p_{(1)}(0)-\cdots-p_{(R)}(0)},\\
&\quad\vdots\\
&U(1)=e_{2+3+\cdots+R}, \quad \text{with probability} \quad \frac{p_{(1+2+\cdots+R)}(0)}{1-p_{(1)}(0)-\cdots-p_{(R)}(0)},\\
&U(1)=e_{1+2+\cdots+R}, \quad \text{with probability} \quad \frac{p_R(0)}{1-p_{(1)}(0)-\cdots-p_{(R)}(0)},
\end{aligned}
$$

and the following offspring distribution:

$$
\begin{aligned}
&P_\omega\Big(U(0)=(u_1,\cdots,u_R,0,\cdots,0),U(1)=e_1\Big||U(1)|=1\Big)\\
&\qquad\qquad=\frac{(u_1+\cdots+u_R)!}{u_1!\cdots u_R!}p_{(1)}^{u_1}(0)\cdots p_{(R)}^{u_R}(0)p_1(0),\\
&P_\omega\Big(U(0)=(u_1,\cdots,u_R,1,0,\cdots,0),U(1)=e_2\Big||U(1)|=1\Big)\\
&\qquad\qquad=\frac{(u_1+\cdots+u_R)!}{u_1!\cdots u_R!}p_{(1)}^{u_1}(0)\cdots p_{(R)}^{u_R}(0)p_{(R+1)}(0),
\end{aligned}
$$

$$\vdots$$

$$P_\omega\Big(U(0)=(u_1,\cdots,u_R,0\cdots,0,\overset{(R+R\text{-}1)th}{1},0,\cdots,0),U(1)=e_R\Big||U(1)|=1\Big)$$
$$=\frac{(u_1+\cdots+u_R)!}{u_1!\cdots u_R!}p_{(1)}^{u_1}(0)\cdots p_{(R)}^{u_R}(0)p_{(R+R-1)}(0),$$
$$P_\omega\Big(U(0)=(u_1,\cdots,u_R,0,\cdots,0),U(1)=e_{R+1}\Big||U(1)|=1\Big)$$
$$=\frac{(u_1+\cdots+u_R)!}{u_1!\cdots u_R!}p_{(1)}^{u_1}(0)\cdots p_{(R)}^{u_R}(0)p_2(0),$$
$$\vdots$$
$$P_\omega\Big(U(0)=(u_1,\cdots,u_R,0,\cdots,0,1),U(1)=e_{2+3+\cdots+R}\Big||U(1)|=1\Big)$$
$$=\frac{(u_1+\cdots+u_R)!}{u_1!\cdots u_R!}p_{(1)}^{u_1}(0)\cdots p_{(R)}^{u_R}(0)p_{(1+2+\cdots+R)}(0),$$
$$P_\omega\Big(U(0)=(u_1,\cdots,u_R,0,\cdots,0),U(1)=e_{1+2+\cdots+R}\Big||U(1)|=1\Big)$$
$$=\frac{(u_1+\cdots+u_R)!}{u_1!\cdots u_R!}p_{(1)}^{u_1}(0)\cdots p_{(R)}^{u_R}(0)p_R(0).$$

Furthermore, for $i\le 0$, the offspring mean matrix of the $(-i+1)$-th generation is:

$$\widetilde{N}(i)=(\widetilde{N}_1(i)\ \widetilde{N}_2),$$

where

$$\widetilde{N}_1(i)=\begin{pmatrix}\frac{p_{(1)}(i)}{1-p_{(1)}(i)-\cdots-p_{(R)}(i)} & \cdots & \frac{p_{(R)}(i)}{1-p_{(1)}(i)-\cdots-p_{(R)}(i)}\\ \vdots & \vdots & \vdots\\ \frac{p_{(1)}(i)}{1-p_{(1)}(i)-\cdots-p_{(R)}(i)} & \cdots & \frac{p_{(R)}(i)}{1-p_{(1)}(i)-\cdots-p_{(R)}(i)}\end{pmatrix}_{(1+2+\cdots+R)\times R},$$

and

$$\widetilde{N}_2=\begin{pmatrix}Z_{1,R-1} & Z_{1,R-2} & \cdots & Z_{1,1}\\ I_{R-1} & Z_{R-1,R-2} & \cdots & Z_{R-1,1}\\ Z_{1,R-1} & Z_{1,R-2} & \cdots & Z_{1,1}\\ Z_{R-2,R-1} & I_{R-2} & \cdots & Z_{R-2,1}\\ Z_{1,R-1} & Z_{1,R-2} & \cdots & Z_{1,1}\\ \vdots & \vdots & \vdots & \vdots\\ Z_{1,R-1} & Z_{1,R-2} & \cdots & I_1\\ Z_{1,R-1} & Z_{1,R-2} & \cdots & Z_{1,1}\end{pmatrix}_{(1+2+\cdots+R)\times(1+2+\cdots+R-1)},$$

in which $Z_{m,n}$ is the zero matrix of dimension $m\times n$, and I_m is the identity matrix of dimension $m\times m$.

Remark Following the ideas in section 2, Theorem 5.1 can be proved analogously. We omit the details.

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