Branching structure for an (L-1) random walk in random environment and its applications

Wenming Hong, Huaming Wang

School of Mathematical Sciences & Key Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R.China;

Email: wmhong@bnu.edu.cn, huamingking@mail.bnu.edu.cn

Abstract

By decomposing the random walk path, we construct a multitype branching process with immigration in random environment for corresponding random walk with bounded jumps in random environment. Then we give two applications of the branching structure. Firstly, we specify the explicit invariant density by a method different with the one used in Brémont [3] and reprove the law of large numbers of the random walk by a method known as " the environment viewed from particles". Secondly, the branching structure enables us to prove a stable limit law, generalizing the result of Kesten-Kozlov-Spitzer [11] for the nearest random walk in random environment. As a byproduct, we also prove that the total population of a multitype branching process in random environment with immigration before the first regeneration belongs to the domain of attraction of some κ-stable law.

Keywords: random walk, branching process, random environment, stable law.
AMS Subject Classification: Primary 60K37; secondary 60F05.

1. Introduction

Random Walk in Random Environment (RWRE for short) has been extensively studied (see e.g. Sznitman [20] or Zeitouni [21] for a comprehensive survey), and has wide range of applications both in probability theory and physics, for example, in metal physics and crystallography (see Hughes [9] for an introduction). Two kinds of randomness are involved in RWRE: first the transition probability chosen randomly at each state position (called random environment); and second the random walk, a time homogeneous Markov chain driven by the chosen transition probability.

Random walk in random environment with bounded jumps ((L-R) RWRE, that is, the walk, in every unit of time, jumping no more than L to the left and no more than R to the right, where R and L are positive integers) was first introduced in Key [13]. Further developments can be found in Brémont [3, 4] and Lëtchikov [16, 17]. We mention here that (L-R) RWRE is a special case of random walk in random environment on a strip. RWRE on a strip was first introduced in Bolthausen-Goldsheid [2], where the authors provided a criteria for the recurrence and transience of the walk. For further development of RWRE on a strip, one can refer to Goldsheid [7, 8] and Roitershtein [19].

Branching structure played an important role in proving the limit properties for the nearest neighborhood RWRE. When the walk is transient to the right, a branching structure was found in Kesten-Kozlov-Spitzer [11] and an elegant stable limit law was obtained; and the renewal theorem was proved also relying on the branching structure (Kesten [12]). Those fine results were proved because the steps of the walk could be calculated accurately under the branching structure. By using the branching structure, Alili [1] (see also Zeitouni [21]) got the invariant density and consequently proved the Law of Large Numbers (LLN for short) by " the environment viewed from particles", a method which goes back to Kozlov [15].

Brémont has also systematically studied (L-1) RWRE in [3], where the recurrence and transience, LLN and some central limit theorem have been proved. One of the main purpose of this paper is to prove a stable limit theorem to generalize Kesten-Kozlov-Spitzer [11] which dealt the nearest setting.

To get a stable limit theorem for the (L-1) RWRE, one of the crucial steps is to derive the branching structure. In this paper, we will formulate a multitype branching process for the walk transient to the right. However, it is much more complicated than the nearest setting, for (1) There are overlaps between different steps, that is, there may be jumps down from i to $i-1, i-2, ..., i-L$. Consequently (2) one could not use jumps down from i directly as the number of $(n-i)$-th generation of the branching process any more, because one cannot figure out the exact parents of particles in $(n-i)$-th generation.

The idea to deal with this difficulty is to imagine that a jump of size l down from i by the walk can be remembered by location $i-1, i-2, ..., i-l$. In this way we can construct a Multitype Branching Process in Random Environment with one type-1 Immigrant in each generation (MBPREI for short) to analyze T_n, the first hitting time of state $n > 0$.

After specifying the corresponding MBPREI, we give two applications. Firstly we can figure out the invariant density directly from the branching structure and avoid introducing the **(IM)** condition in Brémont [3]. Consequently we can reprove directly the LLN for the (L-1) RWRE by a method known as " the environment viewed from particles"; secondly we prove a stable limit Theorem for the (L-1) RWRE, generalizing Kesten-Kozlov-Spitzer [11] for the nearest one.

We now define precisely the model of interests to us.

1.1 Description of the model

Let $\Lambda = \{-L, ..., 1\}/\{0\}$, $\Sigma = \{(x_l)_{l\in\Lambda} \in \mathbb{R}^L : \sum_{l\in\Lambda} x_l = 1, x_l \geq 0, l \in \Lambda\}$ the simplex in $\mathbb{R}^{L+1}$, and $\Omega = \Sigma^{\mathbb{Z}}$. Let μ be a probability measure on Σ and $\omega_0 = (\omega_0(z))_{z\in\Lambda}$ be a Σ-valued random vector with distribution μ, satisfying $\sum_{z\in\Lambda} \omega_0(z) = 1$. Let $\mathbb{P} = \otimes^{\mathbb{Z}}\mu$ on Ω making $\omega_x, x \in \mathbb{Z}$ i.i.d. and satisfying

$$\mathbb{P}(\omega_0(z)/\omega_0(1) \geq \varepsilon, \forall z \in \Lambda) = 1 \text{ for some } \varepsilon > 0. \tag{1}$$

Define the shift operator θ on Ω by the relation

$$(\theta\omega)_i = \omega_{i+1}. \tag{2}$$

The pair $(\Omega, \mathbb{P})$ will serve as the space of environment for both the random walk with bounded jumps and the multitype branching process which we will give next. The random walk in random environment ω with bounded jumps is the Markov chain defined by $X_0 = x$ and transition probabilities

$$P_{x,\omega}(X_{n+1} = y + z | X_n = y) = \omega_y(z), \forall y \in \mathbb{Z}, z \in \Lambda.$$

In the sequel we refer to $P_{x,\omega}(\cdot)$ as the "quenched" law. One also defines the "annealed" laws

$$P_x := \mathbb{P} \times P_{x,\omega} \quad \text{for } x \in \mathbb{Z}.$$

In the rest of the paper, we use respectively $\mathbb{E}$ corresponding to $\mathbb{P}$, $E_{x,\omega}$ corresponding to $P_{x,\omega}$ and E_x corresponding to P_x to denote the expectations. And for simplicity, we may use P, P_ω, E, and E_ω instead of P_0, $P_{0,\omega}$, E_0 and $E_{0,\omega}$.

1.2 Notations, basic conditions and known results

All the vectors, both row vectors and column vectors, involved in this paper are in $\mathbb{R}^L$ except otherwise stated. All the matrices involved are in $\mathbb{R}^{L\times L}$. Let x be a vector and M be a matrix. We put

$$|x| = \sum_{i=1}^{L} |x_i|, \ \|M\|_c = \max_{|y|=1} |My|, \text{ and } \|M\| = \max_{|x|=1} |xM|.$$

One should note that the matrix norms $\|\cdot\|$ and $\|\cdot\|_c$ are different.

Let $S_{L-1} := \{x \in \mathbb{R}^L : |x| = 1\}$ being the unit ball in $\mathbb{R}^L$, and $S_+ = \{x \in S_{L-1} : x_i \geq 0\}$.

For $i \in \mathbb{Z}$ let

$$M_i = \begin{pmatrix} b_i(1) & \cdots & b_i(L-1) & b_i(L) \\ 1+b_i(1) & \cdots & b_i(L-1) & b_i(L) \\ \vdots & \ddots & \vdots & \vdots \\ b_i(1) & \cdots & 1+b_i(L-1) & b_i(L) \end{pmatrix}, \ \overline{M}_i = \begin{pmatrix} a_i(1) & \cdots & a_i(L-1) & a_i(L) \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad (3)$$

with $a_i(l) = \frac{\omega_i(-l)+\cdots+\omega_i(-L)}{\omega_i(1)}, b_i(l) = \frac{\omega_i(-l)}{\omega_i(1)}, 1 \leq l \leq L.$

We also introduce the following special vectors. e_i is the unit row vector with i-th component being 1. The black $\mathbf{1} = (1, ..., 1)$, $e_0 = (\frac{1}{L}, ..., \frac{1}{L})$, $x_0 = (2, 1, ..., 1)^T$, and $\overline{x}_0 = (2, -1, 0, ..., 0)^T$.

For $n \geq 0$, define

$$T_n = \inf[k \geq 0 : X_k = n].$$

Note that T_n is the first time the walk reaching n.

Let $\rho = \frac{1-\omega_0(1)}{\omega_0(1)}$. We use the following conditions in the paper.

Condition C
(C1) $\mathbb{E}\left(\log^+ \rho\right) < \infty$, with $\log^+ x := 0 \vee \log x$.
(C2) $\mathbb{P}\left(\rho > 1\right) > 0$.
Note that under **(C2)** it is an easy task to show that there exists some $\kappa_0 > 2$ such that

$$\mathbb{E}\Big[\Big(\min_{1\leq i\leq L}\{\sum_{j=1}^{L} M_0(i,j)\}\Big)^{\kappa_0}\Big] = \mathbb{E}(\rho^{\kappa_0}) > 1. \quad (4)$$

We mention that (4) corresponds to (1.13) of Kesten [10]. But they have different forms since the norm used here differs from the one used in Kesten [10]. Now fixing such κ_0 we give a new condition
(C3) $\mathbb{E}(\rho^{\kappa_0} \log^+ \rho) < \infty$.
Let ϱ be the greatest eigenvalue of M_0. One follows from (1) that $\varrho > 0$.
(C4) The group generated by supp$[\log \varrho]$ is dense in $\mathbb{R}$. □

In the remainder of the paper, except otherwise stated, we always assume that Condition C holds.

Remark 1.1 *The above conditions look more or less like the conditions of the theorem in Kesten-Kozlov-Spitzer [11]. (C1) implies that* $\mathbb{E}(\log^+ \|M_0\|) < \infty$*, enabling us to calculate the Lyapunov exponents. Condition (C2) excludes the biased trivial case and it also ensures the existing of a number* $\kappa \in (0, \kappa_0]$ *such that*

$$\lim_{n\to\infty}\{\mathbb{E}\left(\| M_0 M_{-1} \cdots M_{-n+1} \|^{\kappa}\right)\}^{\frac{1}{n}} = 1$$

(see (12) below). One follows from (C3) that

$$\mathbb{E}\left(\|M_0\|^{\kappa_0} \log^+ \|M_0\|\right) < \infty, \textit{and} \max_{1\leq l\leq L} \mathbb{E}\left((\omega_0(-l)/\omega_0(1))^{\kappa_0}\right) < \infty,$$

which will be used many times in this paper. Also, (C3) implies (C1). Condition (C4) is the request of Kesten [10] for the proof of the renewal theory of the products of random matrices.

The (L-1) RWRE has been studied intensively in Brémont [3], where the recurrence and transience criteria, the LLN and some central limits theorem have been derived. We state only the recurrence and transience of the model here. Under condition (C1) one can calculate the greatest Ljapounov exponents of $\{\overline{M}_i\}$ and $\{M_i\}$ under both the norms $\|\cdot\|_c$ and $\|\cdot\|$. Indeed in Proposition 3.1 below, we show that $\{M_i\}$ and $\{\overline{M}_i\}$ share the same greatest Lyapunov exponent. Also as $\{M_i\}$ and $\{\overline{M}_i\}$ to be considered, it causes no difference to calculate the Lyapunov exponents under different norms $\|\cdot\|_c$ and $\|\cdot\|$. So in the remainder of the paper, *we use γ_L to denote the greatest Lyapunov exponent of both $\{M_i\}$ and $\{\overline{M}_i\}$.* The number γ_L provides the criteria for the transience and recurrence of (L-1) RWRE. We have

Theorem A (Brémont) *The (L-1) RWRE $\{X_n\}_{n\geq 0}$ is P-a.s. recurrent, transient to the right or transient to the left according as $\gamma_L = 0$, $\gamma_L < 0$ or $\gamma_L > 0$.*

1.3 Main results

When the walk $\{X_n\}$ is transient to the right (P-a.s.), we can formulate a related MBPREI (with *negative time*) to calculate the steps of the walk. We first define an MBPREI $\{Z_{-n}\}_{n\geq 0}$, with negative time. For each integer k we define $Z(k,m)$ to be the L-type branching process in random environment which begins at time k. That is to say, conditioned on ω,

$$\begin{aligned} &P_\omega(Z(k,m)=\mathbf{0})=1,\ \text{if } m>k,\\ &P_\omega(Z(k,k)=e_1)=1, \end{aligned} \tag{5}$$

and for $m<k$

$$P_\omega(Z(k,m)=(u_1,u_2,...,u_L)\big|Z(k,m+1)=e_1)$$
$$=\frac{(u_1+u_2+\cdots+u_L)!}{u_1!u_2!\cdots u_L!}\omega_{m+1}(-1)^{u_1}\omega_{m+1}(-2)^{u_2}\cdots\omega_{m+1}(-L)^{u_L}\omega_{m+1}(1), \tag{6}$$

$$P_\omega\left(Z(k,m)=(u_1,...,u_{l-2},u_{l-1}+1,u_l,...,u_L)\big|Z(k,m+1)=e_l\right)$$
$$=\frac{(u_1+u_2+\cdots+u_L)!}{u_1!u_2!\cdots u_L!}\omega_{m+1}(-1)^{u_1}\omega_{m+1}(-2)^{u_2}\cdots\omega_{m+1}(-L)^{u_L}\omega_{m+1}(1),$$
$$l=2,3,...,L. \tag{7}$$

In addition to assume that conditioned on ω, each of the process $Z(k,*)$ has independent lines of descent, we also assume that conditioned on ω, the processes $Z(k,*)$ are independent.

Let

$$Z_{-n}=\sum_{k=0}^{n-1}Z(-k,-n),\quad n>0. \tag{8}$$

Z_{-n} is the total number of offspring, born at time $-n$ to the immigrants who arrived between 0 and $-n+1$, of an MBPREI beginning at time zero.

Next we consider the path of the (L-1) RWRE $\{X_n\}$ with initial value $X_0=0$.

Fix $n>0$. For $-\infty<i<n$, $1\leq l\leq L$, let

$$U_{i,l}^n=\#\{0<k<T_n: X_{k-1}>i, X_k=i-l+1\}$$

recording all steps by the walk between time interval $(0,T_n)$ from above i to $i-l+1$. Set

$$U_i^n:=(U_{i,1}^n,U_{i,2}^n,\cdots,U_{i,L}^n).$$

One sees that $|U_i^n|$ is the total number of steps by the walk reaching or crossing i downward from above i.

Let $I_k=:\{X_m: T_k\leq m<T_{k+1}\}, k=0,1,...,n-1$, decomposing the the random walk path before time T_n into n independent and non-intersecting pieces. Define for $1\leq l\leq L$, $i<k$,

$$U_l^n(k,i)=\#\{T_k\leq m<T_{k+1}: X_{m-1}>i, X_m=i-l+1\},$$

counting the steps in I_k from above i to $i-l+1$. Let

$$U^n(k,i) = (U_1^n(k,i), U_2^n(k,i), ..., U_L^n(k,i))$$

recording all steps in I_k reaching or crossing i downward from above i. Set $U^n(k,i) = \mathbf{0}$ for $i > k$ and set $U^n(k,k) = e_1$. One can see from the definitions of U_i^n and $U^n(k,i)$ that

$$U_i = \sum_{k=(i+1)\vee 0}^{n-1} U^n(k,i).$$

The relationship between the (L-1) RWRE and the MBPREI $\{Z_{-n}\}_{n\geq 0}$ is summarized in the following theorem which will be proved in Section 2.

Theorem 1.1 *Suppose that $\gamma_L < 0$ (implying that $X_n \to \infty$ P-a.s. by Theorem A). Then one has that*

(a)

$$T_n = n + \sum_{i=-\infty}^{n-1} |U_i^n| + \sum_{i=-\infty}^{n-1} U_{i,1}^n = n + \sum_{i=-\infty}^{n-1} U_i^n x_0; \tag{9}$$

(b) Each of the processes $U^n(k,*)$, $0 \leq k \leq n-1$, is an inhomogeneous multitype branching process beginning at time k with branching mechanism

$$\begin{aligned} &P_\omega(U^n(k,i-1) = (u_1,...,u_L) \big| U^n(k,i) = e_1) \\ &\qquad = \frac{(u_1 + \cdots + u_L)!}{u_1! \cdots u_L!} \omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L} \omega_i(0), \end{aligned} \tag{10}$$

and for $2 \leq l \leq L$,

$$\begin{aligned} &P_\omega\left(U^n(k,i-1) = (u_1,...,1+u_{l-1},...,u_L) \big| U^n(k,i) = e_l\right) \\ &\qquad = \frac{(u_1 + \cdots + u_L)!}{u_1! \cdots u_L!} \omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L} \omega_i(0). \end{aligned} \tag{11}$$

Moreover conditioned on ω, $U^n(k,*), k = 0,1,...,n-1$ are mutually independent and each of the branching processes $U^n(k,*)$ has independent line of descent. Consequently $U_{n-1}^n = 0, U_{n-2}^n, \cdots, U_1^n, U_0^n$ are the first n generations of an inhomogeneous multitype branching process with a type-1 immigration in each generation in random environment.

(c) $U_{n-1}^n = 0, U_{n-2}^n, \cdots, U_1^n, U_0^n$ has the same distribution with the first n generations of the inhomogeneous MBPREI $\{Z_{-n}\}_{n\leq 0}$ defined in (8).

We have immediately the following corollary about the offspring matrices of the multitype branching process $\{U_i^n\}_{0\leq i\leq n-1}$.

Corollary 1.1 *For the process $\{U_i^n\}_{i=0}^{n-1}$, let M_i be the $L \times L$ matrix whose l-th row is the expected number of offspring born to a type-l parent of the $(n-i)$-th generation conditioned on ω, that is, $E_\omega\left(U^n(i,i-1)|U^n(i,i) = e_l\right)$. Then one has that*

$$M_i = \begin{pmatrix} b_i(1) & \cdots & b_i(L-1) & b_i(L) \\ 1 + b_i(1) & \cdots & b_i(L-1) & b_i(L) \\ \vdots & \ddots & \vdots & \vdots \\ b_i(1) & \cdots & 1 + b_i(L-1) & b_i(L) \end{pmatrix}$$

which coincides with the definition of M_i in (3). Similarly for the process $\{Z_{-n}\}_{n\geq 0}$ let M_{-i} be the $L\times L$ matrix whose l-th row is the expected number of offspring born to a type-l parent at time $-i$, conditioned on ω : $E_\omega\left(Z(-i,-i-1)|Z(-i,-i)=e_l\right)$. Then one has that

$$M_{-i}=\begin{pmatrix} b_{-i}(1) & \cdots & b_{-i}(L-1) & b_{-i}(L)\\ 1+b_{-i}(1) & \cdots & b_{-i}(L-1) & b_{-i}(L)\\ \vdots & \ddots & \vdots & \vdots\\ b_{-i}(1) & \cdots & 1+b_{-i}(L-1) & b_{-i}(L) \end{pmatrix}.$$

Part (c) of Theorem 1.1 says that $U_{n-1}^n=0, U_{n-2}^n,\cdots,U_1^n,U_0^n$ has the same distribution with the first n generations of the inhomogeneous MBPREI $\{Z_{-n}\}_{n\leq 0}$ defined in (8). Instead of studying the limiting behaviors of the hitting time T_n directly, one turns to study that of the L-type branching process $\{Z_{-n}\}_{n\geq 0}$ first.

Let $\nu_0\equiv 0$, and define recursively

$$\nu_n=\min\{m>\nu_{n-1}:Z_{-m}=\mathbf{0}\}\text{ for }n>0,$$

being the successive regeneration times of MBPREI $\{Z_{-n}\}_{n\geq 0}$. For simplicity we write ν_1 as ν.

One sees that the regeneration time ν of the MBPREI $\{Z_{-n}\}_{n\geq 0}$ corresponds to the regeneration position (some position where the walk will never go back after passing it) for $\{X_n\}_{n\geq 0}$.

Define also

$$W=\sum_{k=0}^{\nu-1}Z_{-k},$$

the total number of offspring born before the regeneration time ν.

Kesten[10] (see Theorem 5.2 below) has proved that if Condition C holds and the greatest Lyapunov exponent γ_L of $\{M_{-n}\}_{n\geq 0}$ is strictly negative, there exists a unique $\kappa\in(0,\kappa_0]$, such that

$$\log\rho(\kappa)=\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left(\|\ M_0M_{-1}\cdots M_{-n+1}\ \|^\kappa\right)=0. \tag{12}$$

Then we have the following limiting theorem of MBPREI $\{Z_{-n}\}_{\geq 0}$.

Theorem 1.2 *Let κ be the number in (12). Suppose that Condition C holds and $\gamma_L<0$. If $\kappa>2$, then $E((Wx_0)^2)<\infty$; if $\kappa\leq 2$, then there exists some $0<K_3<\infty$ such that*

$$\lim_{t\to\infty}t^\kappa P(Wx_0\geq t)=K_3. \tag{13}$$

For $n\geq 0$ define $\overline{\omega}(n)=\theta^{X_n}\omega$. Then $\{\overline{\omega}(n)\}$ is a Markov chain with transition kernel

$$\overline{P}(\omega,d\omega')=\omega_0(1)\delta_{\theta\omega=\omega'}+\sum_{l=1}^{L}\omega_0(-l)\delta_{\theta^{-l}\omega=\omega'}.$$

In [3] an **(IM)** condition is said to be satisfied if there is $\pi(\omega)$ such that

$$\int\tilde{\pi}(\omega)\mathbb{P}(d\omega)=1\text{ and }\tilde{\pi}(\omega)=\overline{P}*\tilde{\pi}(\omega),$$

where $\tilde{\pi}(\omega)=\pi(\omega)[\mathbb{E}(\pi(\omega))]^{-1}$. Under **(IM)** condition Brémont showed an LLN of $\{X_n\}$ in [3]. But the **(IM)** condition was not given directly in the words of environment ω. So one has to check the existence of the invariant density $\pi(\omega)$. In [3], Brémont showed the existence of $\pi(\omega)$ by analyzing its definition and the transition probability of the walk.

What makes difference in this article is that, with the help of the branching structure, we specify the invariant density $\pi(\omega)$ directly by analyzing a multitype branching process. Therefore we avoid introducing the **(IM)** condition and show directly that $\{X_n\}$ satisfies an LLN with a positive speed under the assumption "$\mathbb{E}(\pi(\omega)) < \infty$". Also the speed has a simple explicit form $[\mathbb{E}(\pi(\omega))]^{-1}$.

Define $\pi(\omega) := \frac{1}{\omega_0(1)}\left(1 + \sum_{i=1}^{\infty} e_1 \overline{M}_i \cdots \overline{M}_1 e_1^T\right)$. Let $\tilde{\pi}(\omega) = \frac{\pi(\omega)}{\mathbb{E}(\pi(\omega))}$. Then we have

Theorem 1.3 *Suppose that $\mathbb{E}(\pi(\omega)) < \infty$. Then we have that*

(i) $\gamma_L < 0$;

(ii) $\tilde{\pi}(\omega)\mathbb{P}(d\omega)$ *is invariant under the kernel* $\overline{P}(\omega, d\omega')$, *that is*

$$\int 1_B \tilde{\pi}(\omega)\mathbb{P}(d\omega) = \iint 1_{\omega' \in B} \overline{P}(\omega, d\omega')\tilde{\pi}(\omega)\mathbb{P}(d\omega); \tag{14}$$

(iii) and $\mathbb{P}$*-a.s.,* $\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\mathbb{E}(\pi(\omega))}$.

Remark 1.2 *The independence assumption of the environment is unnecessary. It is enough if $(\Omega, \mathbb{P}, \theta)$ is an ergodic system.*

Also for the (L-1) RWRE we have the following stable limit theorem, generalizing Kesten-Kozlov-Spitzer [11] which dealt with the nearest setting.

Theorem 1.4 *Suppose that Condition C holds and that $\gamma_L < 0$. Let κ be the number in (12). Let $L_\kappa(x)$ be a κ-stable law (L_κ is concentrated on $[0, \infty)$ if $\kappa < 1$ and has mean zero if $\kappa > 1$). Then with $0 < A_\kappa, B_i < \infty$ suitable constants,* $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$,

(i) if $0 < \kappa < 1$,

$$\lim_{n\to\infty} P(n^{-\frac{1}{\kappa}} T_n \le x) = L_\kappa(x),$$
$$\lim_{n\to\infty} P(n^{-\kappa} X_n \le x) = 1 - L_\kappa(x^{-\frac{1}{\kappa}});$$

(ii) if $\kappa = 1$, then for suitable $D(n) \sim \log n$ and $\delta(n) \sim (A_1 \log n)^{-1} n$,

$$\lim_{n\to\infty} P(n^{-1}(T_n - A_1 n D(n\mu^{-1})) \le x) = L_1(x),$$
$$\lim_{n\to\infty} P(n^{-1}(\log n)^2 (X_n - \delta(n)) \le x) = 1 - L_1(-A_1^2 x);$$

(iii) if $1 < \kappa < 2$,

$$\lim_{n\to\infty} P\left(n^{-\frac{1}{\kappa}}(T_n - A_\kappa n) \le x\right) = L_\kappa(x),$$
$$\lim_{n\to\infty} P\left(n^{-\frac{1}{\kappa}}\left(X_n - \frac{n}{A_\kappa}\right) \le x\right) = 1 - L_\kappa(-x A_\kappa^{1+\kappa^{-1}});$$

(iv) if $\kappa = 2$,

$$\lim_{n\to\infty} P\Big(\frac{T_n - A_2 n}{B_1 \sqrt{n \log n}} \le x\Big) = \Phi(x),$$

$$\lim_{n\to\infty} P\left(A_2^{\frac{3}{2}} B_1^{-1}(n\log n)^{-\frac{1}{2}}\left(X_n - \frac{n}{A_2}\right) \le x\right) = \Phi(x);$$

(v) if $\kappa > 2$,

$$\lim_{n\to\infty} P\Big(\frac{T_n - B_3 n}{B_2\sqrt{n}} \le x\Big) = \Phi(x),$$
$$\lim_{n\to\infty} P\left(B_3^{\frac{3}{2}} B_2^{-1} n^{-\frac{1}{2}}\left(X_n - \frac{n}{B_3}\right) \le x\right) = \Phi(x).$$

Notes: The stable limit law for the nearest neighborhood RWRE ((1-1) RWRE) was shown in Kesten-Kozlov-Spitzer [11]. But to prove Theorem 1.4 is far more than a trivial work for the following reasons:

(1) The branching structure (MBPREI $\{Z_{-n}\}_{n\ge 0}$) for (L-1) RWRE was never seen in literatures we are aware of. But it is crucial to construct such branching structure to prove Theorem 1.4.

(2) After constructing the branching structure, to prove Theorem 1.4, a key step is to show Theorem 1.2, that is, to show that W, the total population of $\{Z_{-n}\}_{n\ge 0}$ before the first regeneration, belongs to the domain of attraction of a κ-stable law:

$$\lim_{t\to\infty} t^\kappa P(W x_0 \ge t) = K_3. \tag{15}$$

For this purpose one should use the tail of the series of the products of random matrices of Kesten [10], that is,

$$\lim_{t\to\infty} t^\kappa \mathbb{P}\left(\sum_{n=0}^{\infty} x M_0 M_{-1}\cdots M_{-n+1} x_0\right) = K(x, x_0), \tag{16}$$

where K is a constant depending on positive $x \in \mathbb{R}^L$.

But to prove (15) one needs to find out how the constant K depending on x explicitly. For general random matrix, this is still open. But for $\{M_i\}$, by the similarity between M_i and $\overline{M}_i$, we prove a finer result based on (16). We show that

$$\lim_{t\to\infty} t^\kappa \mathbb{P}\left(\sum_{n=0}^{\infty} x M_0 M_{-1}\cdots M_{-n+1} x_0\right) = K_2 |xB|^\kappa,$$

where K_2 is independent of x.

For the reason we list above, although Key, E.S. indicated that "the general argument for finding limiting distributions for $\{X_n\}$ seems to go through except now $\{Z_t\}$ (some branching process) is multitype; the only part that seems not to be line by line rewriting of Kesten, Kozlov and Spitzer's argument is the proof of

$$P(\nu > t) < K_4 \exp(-K_5 t)\text{''} \tag{17}$$

ν being the regeneration time of some multitype branching process $\{Z_t\}$(see [14] page 350), we think it makes sense to prove Theorem 1.4. □

Since the proof of Theorem 1.4 will be a long march. We describe the skeleton of its proof.

In order to determine the limit law of X_n, we consider first the limit law of the hitting time T_n. One gets from Theorem 1.1 that

$$T_n = n + \sum_{i=-\infty}^{n-1} |U_i^n| + \sum_{i=-\infty}^{n-1} U_{i,1}^n = n + \sum_{i=-\infty}^{n-1} U_i^n x_0.$$

Note that when the walk transient to the right, there are only finite steps in $(-\infty,0)$, that is, P-a.s., $\frac{1}{n}\sum_{i<0}U_i^n x_0 \to 0$. Consequently, it suffices to show that $\sum_{i=0}^{n-1}U_i^n x_0$ converges to L_κ in distribution after suitable normalization. Also in Theorem 1.1 we observe that

$$U_{n-1}^n=0, U_{n-2}^n, \cdots, U_1^n, U_0^n$$

has the same distribution with the first n generations of an inhomogeneous MBPREI $\{Z_{-n}\}_{n\geq 0}$ such that conditioned on ω, $E_\omega(Z_{-t}|Z_{-k}, 0\leq k<t)=(Z_{-t+1}+e_1)M_{-t+1}$.

Let $W_k:=\sum_{\nu_k\leq t<\nu_{k+1}}Z_{-t}$ be the total offspring born between time interval $[\nu_k,\nu_{k+1})$. Then due to the independence of the environment, $(\nu_{k+1}-\nu_k, W_k),\ k=0,1,2,...$ are independent and identically distributed.

The key step is to show that

$$W_0x_0 \text{ is in the domain of attraction of a } \kappa\text{-stable law,} \tag{18}$$

which is proved in Theorem 1.2. Then Theorem 1.4 follows by a standard argument. To show (18), we could approximate Wx_0 by some random variable of the form $\Gamma(R+I)x_0$, where Γ is a random row vector with positive components and independent of R, and Rx_0 has the same distribution with

$$\eta_0x_0:=\sum_{m=1}^{\infty}M_0M_{-1}\cdots M_{-m+1}x_0.$$

It remains to show that for all row vector x with positive components, $x\eta_0x_0$ belongs the domain of attraction of κ-stable law, that is

$$P(x\eta_0x_0>t)\sim K_2|xB|^\kappa t^{-\kappa} \text{ as } n\to\infty \tag{19}$$

with the constant K_2 independent of x.

To this end we first analyze the connections between the matrices M_i and $\overline{M}_i$ and find that the projection of η_0x_0 on different directions $e_l, 1\leq l\leq L$, that is $e_l\eta_0x_0$, have the same distributions up to certain linear transformations (see Proposition 3.2 below).

This fine property of the random variable η_0 together with Kesten's results of the products of random matrices enables us to show that the constant K_2 of (19) is independent of x (see Theorem 5.1 below).

We arrange the remainder of this paper as follows. In Section 2, we formulate the related branching structure MBPREI and express the hitting time T_n by the MBPREI, i.e., Theorem 1.1 is proved. In Section 3 we study the connections between the matrices M_i and $\overline{M}_i$ which will be important to show (19). In Section 4, we give the invariant density from the point of branching structure and provide an alternative proof of LLN (Theorem 1.3). Finally the long Section 5 is devoted to studying the tail of Wx_0 (Theorem 1.2) and to proving the stable limit law (Theorem 1.4).

2 Branching structure and hitting times

In this section we assume the walk transient to the right, i.e., $X_n\to\infty$ P-a.s., and always use notation $J_{s,h}$ to denote a jump (a piece of random walk path) taken by $\{X_n\}$ from s to h. That is

$$J_{s,h}=\{(X_n,X_{n+1}): X_n=s, X_{n+1}=h\}, n\in\mathbb{N}.$$

We are going to find a multitype branching process (i.e., MBPREI) to analyze T_n. To see this, for some $1\leq l\leq L$, suppose we have a jump, say $J_{i,i-l}$, by the walk from i downward to $i-l$ before time T_n. Though throughout this jump, the walk will not stop at $i-1, i-2, ..., i-l+1$, we can imagine that it will cross $i-1, i-2, \cdots, i-l+1$, and reach $i-l$ at last. For $-\infty < i < n$, we want to record how much times the walk will cross or reach i downward from above i. So for $-\infty < i < n$, $1 \le l \le L$, we define

$$U_{i,l}^n = \#\{0 < k < T_n : X_{k-1} > i, X_k = i-l+1\},$$

being the records of steps by the walk from above i to $i-l+1$, and let

$$U_i^n = (U_{i,1}^n, U_{i,2}^n, \cdots, U_{i,L}^n).$$

Then $|U_i^n|$ is the total number of times the walk crossing or reaching i downward from above i before T_n. In particular $U_{i,1}^n$ is the total number of jumps taken by the walk downward from above i which reach i. But every jump taken by the walk downward must reach some i. From this point of view, the total number of steps taken by the walk downward before time T_n is $\sum_{i=-\infty}^{n-1} U_{i,1}^n$.

On the other hand, for the walk transient to the right, since we are considering the (L-1) RWRE, every record of the walk reaching or crossing i downward from above i, an individual of U_i^n, corresponds to a jump taken by the walk upward from i to $i+1$. Therefore the total number of the jumps taken by the walk upward before time T_n is $\sum_{i=-\infty}^{n-1} |U_i^n|$. Then we conclude from the above discussion that

$$T_n = n + \sum_{i=-\infty}^{n-1} |U_i^n| + \sum_{i=-\infty}^{n-1} U_{i,1}^n = n + \sum_{i=-\infty}^{n-1} U_i^n x_0,$$

where $x_0 = (2, 1, ..., 1)^T \in \mathbb{R}^L$. So instead of studying T_n directly we consider $\{U_i^n\}_{i<n}$.

We first divide the path between 0 and T_n into n pieces which do not have intersection. For $k = 0, 1, .., n-1$, define

$$I_k =: \{X_m : T_k \le m < T_{k+1}\}, \tau_k = \{t : T_k \le t < T_{k+1}\}$$

Then one follows from the strong Markov property that

$$(I_0, \tau_0), (I_1, \tau_1), ..., (I_{n-1}, \tau_{n-1})$$

are mutually independent under the quenched probability P_ω^0. We will see next that each of the pieces (I_k, τ_k), $0 \le k \le n-1$, corresponds to an immigration structure.

Now we fix $0 \le k \le n-1$. We want to construct an L-type branching process from the random walk path I_k.

Define for $1 \le l \le L$, $i < k$,

$$U_l^n(k, i) = \#\{T_k \le m < T_{k+1} : X_{m-1} > i, X_m = i-l+1\},$$

counting the steps in I_k from above i to $i-l+1$. Let

$$U^n(k, i) = (U_1^n(k, i), U_2^n(k, i), ..., U_L^n(k, i))$$

recording all steps in I_k reaching or crossing i downward from above i.

Note that if $i \ge k$, there is no step between time interval τ_k by the walk reaching or crossing i downward from above i. So we set very naturally $U^n(k, i) = 0$ for $i > k$.

But note also that there may be some steps between time interval τ_k by the walk reaching or crossing $k-1$ down ward from above. If we want to consider these jumps as the particles of a branching process at time $k-1$, we must figure out their parents. So we can assume that there is an immigrant entering the system at time k. Therefore we set $U^n(k, k) := e_1$, representing the immigrant entering at time k.

We show next that $\{U^n(k, i)\}_{i \le k}$ is an inhomogeneous L-type branching process beginning at time k.

Fix $i < k$. Let $\eta_{i,0}^k = T_k$. Define recursively for $j \ge 1$

$$\eta_{i,j}^k = \min\{\eta_{i,j-1}^k < m < T_{k+1} : X_{m-1} > i, X_m \le i\}.$$

Then by definition $\eta_{i,j}^k, j \geq 1$ are the successive time of steps in I_k taken by the walk crossing or reaching i downward from above i. For $i \leq k-1$, $1 \leq l \leq L$, we have

$$U_l^n(k,i) = \sum_{j=1}^{\infty} 1_{[T_k < \eta_{i,j}^k < T_{k+1}, X_{\eta_{i,j}^k} = i-l+1]}. \tag{20}$$

Also define

$$\xi_{i,l}^{k,j} = \#\{\eta_{i+1,j}^k \leq m < \eta_{i+1,j+1}^k < T_{k+1} : X_{m-1} > i, X_m = i-l+1\},$$

recording the steps by the walk from above i to $i-l+1$ between the j-th and the $j+1$-th excursions reaching or crossing $i+1$ in the time interval τ_k, and define

$$\xi_i^{k,j} = (\xi_{i,1}^{k,j}, ..., \xi_{i,L}^{k,j}).$$

Then it follows from the path decomposition of I_k and the strong Markov property that $\xi_i^{k,j}$, $j = 1, 2, ...$ are i.i.d. for fixed k and i.

In the definition of $\xi_{i,l}^{k,j}$, where things get delicate is the first step in the time interval $[\eta_{i+1,j}^k, \eta_{i+1,j+1}^k)$. Note that on the event $\{X_{\eta_{i+1,j}^k} = i+1\}$ there is no jump of the form $J_{s,h}$ with $s > i+1$ and $h \leq i$ during the time interval $[\eta_{i+1,j}^k, \eta_{i+1,j+1}^k)$. Hence for $1 \leq l \leq L$

$$\begin{aligned} \xi_{i,l}^{k,j} &= \#\{\eta_{i+1,j}^k \leq m < \eta_{i+1,j+1}^k < T_n : X_{m-1} > i, X_m = i-l+1\} \\ &= \#\{\eta_{i+1,j}^k < m < \eta_{i+1,j+1}^k < T_n : X_{m-1} = i+1, X_m = i-l+1\} \\ &= \#\{\text{jumps of the kind } J_{i+1,i+1-l} \text{ by the walk during time interval } (\eta_{i+1,j}^k, \eta_{i+1,j+1}^k)\}. \end{aligned}$$

For $2 \leq m \leq L$, on the event $\{X_{\eta_{i+1,j}^k} = i-(m-1)+1\}$, the first step in the time interval $[\eta_{i+1,j}^k, \eta_{i+1,j+1}^k)$ taken by the walk from the above of $i+1$ reaches or crosses i downward and hence contributes unconditionally one particle to $\xi_{i,m-1}^{k,j}$. There is no other step of the form $J_{s,h}$ with $s > i+1$ and $h \leq i$ in such interval. Hence

$$\xi_{i,m-1}^{k,j} = 1 + \#\{\text{jumps of the kind } J_{i+1,i-(m-1)+1} \text{ by the walk during time interval } (\eta_{i+1,j}^k, \eta_{i+1,j+1}^k)\},$$

and for $2 \leq l \leq L, l \neq m-1$

$$\xi_{i,l}^{n,j} = \#\{\text{jumps of the kind } J_{i+1,i+1-l} \text{ by the walk during time interval } (\eta_{i+1,j}^k, \eta_{i+1,j+1}^k)\}.$$

Then it follows from the above discussion and the definition of $\{U^n(k,i)\}_{i\leq k}$ that

$$\begin{aligned} &P_\omega(U^n(k,i-1) = (u_1, ..., u_L) \big| U^n(k,i) = e_1) \\ &\quad = P_\omega(\xi_{i-1,m}^{k,j} = u_m, 1 \leq m \leq L \big| X_{\eta_{i,j}^k} = i) \\ &\quad = \frac{(u_1 + \cdots + u_L)!}{u_1! \cdots u_L!} \omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L} \omega_i(0), \end{aligned} \tag{21}$$

and for $2 \leq l \leq L$,

$$\begin{aligned} &P_\omega(U^n(k,i-1) = (u_1, ..., 1 + u_{l-1}, ..., u_L) \big| U^n(k,i) = e_l) \\ &\quad = P_\omega(\xi_{i-1,l-1}^{k,j} = u_{l-1} + 1, \xi_{i-1,m}^{k,j} = u_m, m \neq l-1, 1 \leq m \leq L \big| X_{\eta_{i,j}^k} = i+1-l) \\ &\quad = \frac{(u_1 + \cdots + u_L)!}{u_1! \cdots u_L!} \omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L} \omega_i(0). \end{aligned} \tag{22}$$

Then one can conclude that $\{U^n(k,i)\}_{i\leq k}$ is an inhomogeneous L-type branching process beginning at time k.

It follows from the independence of the $\{(I_k,\tau_k)\}_{k=0}^{n-1}$ that $U^n(k,*), k=n-1,...,1,0$ is mutually independent. Also by the above discussion of path decomposition inner the random walk piece I_k, we found that particles of $U^n(k,i)$ generate offspring independently.

The above two kinds of independence correspond to the independence we imposed on the branching processes $Z(k,*)$.

Note that by the definition of $U_{i,l}^n$, for $1\le l\le L$, one has that

$$U_{i,l}^n=\sum_{k=(i+1)\vee 0}^{n-1}\sum_{j=1}^{\infty}1_{[\eta_{i,j}^k<T_n,X_{\eta_{i,j}^k}=i-l+1]}. \tag{23}$$

Then it follows from (20) and (23) that

$$U_{i,l}^n=\sum_{k=(i+1)\vee 0}^{n-1}U_l^n(k,i),$$

which implies that

$$U_i^n=\sum_{k=(i+1)\vee 0}^{n-1}U^n(k,i) \tag{24}$$

Comparing the above (21), (22) with (6), (7) in the definition of $\{Z_{-n}\}_{n\ge 0}$, since $\omega_{n-1},\omega_{n-2}...,\omega_1$ have the same joint distribution as $\omega_0,\omega_{-1},...,\omega_{-n+1}$, it follows that

$$U_{n-1}^n=0,U_{n-2}^n,...,U_1^n,U_0^n$$

has the same distribution with the first n generations of the inhomogeneous MBPREI $\{Z_{-n}\}_{n\ge 0}$ defined in Section 1.

Summarizing the discussion above, we obtain the proof of Theorem 1.1.

3 Connections between matrices M_i and $\overline{M}_i$

Recall that in [3], the author used the greatest Lyapunov exponent γ_L of $\{\overline{M}_i\}$ to characterize the recurrence and transience of (L-1) RWRE $\{X_n\}$ (see also Theorem A above). One may be curious that in Corollary 1.1 the expected offspring matrices of the branching processes $\{U_i^n\}_{0\le i\le n-1}$ and $\{Z_{-n}\}_{n\ge 0}$ are matrices in the sequence $\{M_i\}_{i\in Z}$. From this point of view, one may guess that there must be some intrinsic connections between the matrices M_i and $\overline{M}_i$. So the main task of this short section is to find some specific relations between $\{M_i\}$ and $\{\overline{M}_i\}$.

Firstly, one easily sees that M_i and $\overline{M}_i$ are similar to each other. Indeed, introduce the deterministic matrix

$$B=\begin{pmatrix}1&&&\\1&1&&\\\vdots&\vdots&\ddots&\\1&1&\cdots&1\end{pmatrix},\ \text{with inverse } B^{-1}=\begin{pmatrix}1&&&\\-1&1&&\\&\ddots&\ddots&\\&&-1&1\end{pmatrix} \tag{25}$$

the entries in the blank being all zero. Then one has that

$$\overline{M}_i=B^{-1}M_iB. \tag{26}$$

Then the similarity between M_i and $\overline{M}_i$ follows.

One notes that since $\mathbb{P}$ makes $\{\omega_i\}_{i\in\mathbb{Z}}$ an i.i.d. sequence, it also makes $\{\overline{M}_i\}_{i\in\mathbb{Z}}$ and $\{M_i\}_{i\in\mathbb{Z}}$ two i.i.d. sequences as well. These two random sequences of matrices are of great importance to us. Under condition **(C1)** one can apply Oseledec's multiplicative ergodic theorem (see [18]) to both $\{\overline{M}_i\}$ and $\{M_i\}$, with the shift operator defined in (2). Write $\gamma_L(\overline{M},\theta) \geq \gamma_{L-1}(\overline{M},\theta) \geq \cdots \geq \gamma_1(\overline{M},\theta)$ for the Lyapunov exponents of $\{\overline{M}_i\}$ and $\gamma_L(M,\theta) \geq \gamma_{L-1}(M,\theta) \geq \cdots \geq \gamma_1(M,\theta)$ for the Lyapunov exponents of $\{M_i\}$ under the matrix norm $\|\cdot\|_c$. For simplicity we write $\gamma_L(\overline{M},\theta)$ as $\gamma_L(\overline{M})$ and $\gamma_L(M,\theta)$ as $\gamma_L(M)$ respectively. Due to the positivity of both M_0 and $\overline{M}_0$, we have for all $x \in S_+$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\gamma_L(\overline{M}) &= \lim_{n\to\infty}\frac{1}{n}\log\|\overline{M}_{n-1}\cdots\overline{M}_1\overline{M}_0\|_c = \lim_{n\to\infty}\frac{1}{n}\log|\overline{M}_{n-1}\cdots\overline{M}_1\overline{M}_0x| \\ &= \lim_{n\to\infty}\frac{1}{n}\mathbb{E}(\log\|\overline{M}_{n-1}\cdots\overline{M}_1\overline{M}_0\|_c)\end{aligned} \tag{27}$$

and

$$\begin{aligned}\gamma_L(M) &= \lim_{n\to\infty}\frac{1}{n}\log\|M_{n-1}\cdots M_1M_0\|_c = \lim_{n\to\infty}\frac{1}{n}\log|M_{n-1}\cdots M_1M_0x| \\ &= \lim_{n\to\infty}\frac{1}{n}\mathbb{E}(\log\|M_{n-1}\cdots M_1M_0\|_c).\end{aligned}$$

Similarly we can calculate the greatest Lyapunov exponent $\overline{\gamma}_L(M)$ of $\{M_i\}_{i\leq 0}$ and $\overline{\gamma}_L(\overline{M})$ of $\{\overline{M}_i\}_{i\leq 0}$ under matrix norm $\|\cdot\|$ as

$$\begin{aligned}\overline{\gamma}_L(\overline{M}) &= \lim_{n\to\infty}\frac{1}{n}\log\|\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}\| = \lim_{n\to\infty}\frac{1}{n}\log|x\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}| \\ &= \lim_{n\to\infty}\frac{1}{n}\mathbb{E}(\log\|\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}\|)\end{aligned}$$

and

$$\begin{aligned}\overline{\gamma}_L(M) &= \lim_{n\to\infty}\frac{1}{n}\log\|M_0M_{-1}\cdots M_{-n+1}\| = \lim_{n\to\infty}\frac{1}{n}\log|xM_0M_{-1}\cdots M_{-n+1}| \\ &= \lim_{n\to\infty}\frac{1}{n}\mathbb{E}(\log\|M_0M_{-1}\cdots M_{-n+1}\|).\end{aligned}$$

In fact we have the following simple but interesting result about the above the greatest Lyapunov exponents.

Proposition 3.1 *Suppose that condition **(C1)** holds. Then we have*

$$\gamma_L(\overline{M}) = \gamma_L(M) = \overline{\gamma}_L(\overline{M}) = \overline{\gamma}_L(M).$$

Proof. The equality $\gamma_L(M) = \gamma_L(\overline{M})$ follows directly from the similarity of $\overline{M}_n$ and M_n. Indeed, for any n we see from the definition of $\overline{M}_n$ and M_n that $\overline{M}_n = B^{-1}M_nB$. Since for any matrices A and B we always have $\|AB\|_c \leq \|A\|_c\|B\|_c$, then

$$\begin{aligned}\gamma_L(\overline{M}) &= \lim_{n\to\infty}\frac{1}{n}\log\|\overline{M}_{n-1}\cdots\overline{M}_0\|_c = \lim_{n\to\infty}\frac{1}{n}\log\|B^{-1}M_{n-1}\cdots M_0B\|_c \\ &\leq \lim_{n\to\infty}\frac{1}{n}\log(\|B^{-1}\|_c\|M_{n-1}\cdots M_0\|_c\|B\|_c) = \gamma_L(M).\end{aligned}$$

In the same way we have the inverse inequality $\gamma_L(M) \leq \gamma_L(\overline{M})$ to finish the proof of the first equality $\gamma_L(\overline{M}) = \gamma_L(M)$. The third equality follows from the same reason as the first equality.

Next we show that $\gamma_L(\overline{M}) = \overline{\gamma}_L(\overline{M})$. Note that for $1 \leq l \leq L-1$

$$\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}e_l^T = \overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+2}((a_{-n+1}(l)e_1^T + e_{l+1}^T).$$

Then one has that

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}(\log e_1\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}e_l^T)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log e_1\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+2}((a_{-n+1}(l)e_1^T + e_{l+1}^T) \\
&= \max\Big\{ \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log a_{-n+1}(l)e_1\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+2}e_1^T), \\
&\qquad\qquad \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log e_1\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+2}e_{l+1}^T)\Big\} \\
&\geq \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(\log e_1\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+2}e_{l+1}^T\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(\log e_1\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}e_{l+1}^T\right).
\end{aligned}
$$

Therefore it follows that

$$
\begin{aligned}
\overline{\gamma}_L(\overline{M}) &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log\|\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}\|) \\
&= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log e_1\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}\mathbf{1}^T) \\
&= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big(\log\sum_{l=1}^{L} e_1\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}e_l^T\Big) \\
&= \max_{1\leq l\leq L}\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big(\log\sum_{l=1}^{L} e_1\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}e_l^T\Big) \\
&= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big(\log e_1\overline{M}_0\overline{M}_{-1}\cdots\overline{M}_{-n+1}e_1^T\Big).
\end{aligned}
$$

Then one follows from stationarity that

$$
\overline{\gamma}_L(\overline{M}) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big(\log e_1\overline{M}_{n-1}\cdots\overline{M}_1\overline{M}_0e_1^T\Big). \tag{28}
$$

But on the other hand

$$
\begin{aligned}
\gamma_L(\overline{M}) &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log\mathbf{1}\overline{M}_{n-1}\cdots\overline{M}_1\overline{M}_0e_1^T) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(\log\Big(\sum_{l=1}^{L} e_1\overline{M}_{n-l}\cdots\overline{M}_1\overline{M}_0e_1^T\Big)\right) \\
&= \max_{1\leq l\leq L}\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(\log(e_1\overline{M}_{n-l}\cdots\overline{M}_1\overline{M}_0e_1^T)\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(\log(e_1\overline{M}_{n-1}\cdots\overline{M}_1\overline{M}_0e_1^T)\right). \qquad (29)
\end{aligned}
$$

Then (28) and (29) imply that $\gamma_L(\overline{M}) = \overline{\gamma}_L(\overline{M})$. □

Since $\gamma_L(\overline{M}) = \gamma_L(M) = \overline{\gamma}_L(\overline{M}) = \overline{\gamma}_L(M)$, we write all of them as γ_L in the remainder of our article. We recall that γ_L characterizes the transience and recurrence of RWRE X_n (see Theorem A of Section 1).

Recall that $Z(-k,-m)$ is the $m-k$-th generation of the branching process beginning at time $-k$. Let

$$
Y_{-k} = \sum_{m=k+1}^{\infty} Z(-k,-m) \tag{30}
$$

being the total number of progeny of the immigrant at times $-k$. Sinece for $m > k$, $E_\omega(Z(-k,-m)) = M_{-k}M_{-k-1}\cdots M_{-m+1}$, one has that

$$
\eta_{-k} := \sum_{m=k+1}^{\infty} M_{-k}...M_{-m+1}, \tag{31}
$$

is the expectation matrix of Y_{-k}. In next proposition we find that the projection of $\eta_0 x_0$ on different directions $e_l, 1 \le l \le L$, that is $e_l \eta_0 x_0$, have the same distribution up to certain linear transformations.

Proposition 3.2 *Let $x_0 = (2, 1, ..., 1)^T \in \mathbb{R}^L$ and $\overline{x}_0 = (2, -1, 0, ..., 0)^T \in \mathbb{R}^L$. Then we have*
i) for all $2 \le l \le L$,

$$\sum_{n=1}^{\infty} e_l M_0 M_{-1} \cdots M_{-n+1} x_0 \stackrel{\mathscr{D}}{=} l + l \sum_{n=1}^{\infty} e_1 M_0 M_{-1} \cdots M_{-n+1} x_0;$$

ii) for all $2 < l \le L$

$$\sum_{n=1}^{\infty} e_l \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0 \stackrel{\mathscr{D}}{=} 1 + \sum_{n=1}^{\infty} e_1 \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0,$$

and

$$\sum_{n=1}^{\infty} e_2 \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0 \stackrel{\mathscr{D}}{=} 2 + \sum_{n=1}^{\infty} e_1 \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0.$$

Proof. We mention that all equalities in distribution in our proof follow from the stationarity of the environment. For fixed $1 < l \le L$,

$$\begin{aligned}\sum_{n=1}^{\infty} e_l M_0 M_{-1} \cdots M_{-n+1} x_0 &= \sum_{n=1}^{\infty} e_l B \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} B^{-1} x_0 \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{l} e_k \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0.\end{aligned}$$

For $l = 2$, note that

$$\begin{aligned}\sum_{n=1}^{\infty} e_2 \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0 &= 2 + \sum_{n=1}^{\infty} e_1 \overline{M}_{-1} \overline{M}_{-2} \cdots \overline{M}_{-n} \overline{x}_0 \\ &\stackrel{\mathscr{D}}{=} 2 + \sum_{n=1}^{\infty} e_1 \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0 \qquad (32)\end{aligned}$$

Then we have

$$\begin{aligned}\sum_{n=1}^{\infty} e_2 M_0 M_{-1} \cdots M_{-n+1} x_0 &= \sum_{n=1}^{\infty} (e_1 + e_2) \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0 \\ &\stackrel{\mathscr{D}}{=} 2 + 2 \sum_{n=1}^{\infty} e_1 \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0 = 2 + 2 \sum_{n=1}^{\infty} e_1 B^{-1} M_0 M_{-1} \cdots M_{-n+1} B \overline{x}_0 \\ &= 2 + 2 \sum_{n=1}^{\infty} e_1 M_0 M_{-1} \cdots M_{-n+1} x_0.\end{aligned}$$

For $2 < l \le L$ note that

$$\begin{aligned}\sum_{n=1}^{\infty} e_l \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0 &= 1 + \sum_{n=1}^{\infty} e_1 \overline{M}_{-l+1} \overline{M}_{-l} \cdots \overline{M}_{-l-n+2} \\ &\stackrel{\mathscr{D}}{=} 1 + \sum_{n=1}^{\infty} e_1 \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0. \qquad (33)\end{aligned}$$

Then we have

$$\sum_{n=1}^{\infty} e_l M_0 M_{-1} \cdots M_{-n+1} x_0 \overset{\mathscr{D}}{=} l + l \sum_{n=1}^{\infty} e_1 \overline{M}_0 \overline{M}_{-1} \cdots \overline{M}_{-n+1} \overline{x}_0$$
$$= l + l \sum_{n=1}^{\infty} e_1 B^{-1} M_0 M_{-1} \cdots M_{-n+1} B \overline{x}_0 = l + l \sum_{n=1}^{\infty} e_1 M_0 M_{-1} \cdots M_{-n+1} x_0.$$

Then the first assertion follows. The second assertion was also proved (see the above equations (32) and (33)). □

4 An alternative proof of the LLN of X_n

For $n \geq 0$ define $\overline{\omega}(n) = \theta^{X_n}\omega$. Then $\{\overline{\omega}(n)\}$ is a Markov chain with transition kernel

$$\overline{P}(\omega, d\omega') = \omega_0(1)\delta_{\theta\omega=\omega'} + \sum_{l=1}^{L} \omega_0(-l)\delta_{\theta^{-l}\omega=\omega'}.$$

In [3] an **(IM)** condition is said to be satisfied if there is $\pi(\omega)$ such that

$$\int \tilde{\pi}(\omega)\mathbb{P}(d\omega) = 1 \text{ and } \tilde{\pi}(\omega) = \overline{P} * \tilde{\pi}(\omega),$$

where $\tilde{\pi}(\omega) = \pi(\omega)[\mathbb{E}(\pi(\omega))]^{-1}$. Under **(IM)** condition Brémont showed an LLN of $\{X_n\}$ in [3]. But the **(IM)** condition was not given directly in the words of environment ω. So one has to check the existence of the invariant density $\pi(\omega)$. In [3], Brémont showed the existence of $\pi(\omega)$ by analyzing its definition and the transition probability of the walk.

What makes difference in our article is that, with the help of the branching structure, we specify the invariant density $\pi(\omega)$ directly by analyzing a multitype branching process. Therefore we can avoid introducing the **(IM)** condition and show directly that $\{X_n\}$ satisfies an LLN with a positive speed under the assumption "$\mathbb{E}(\pi(\omega)) < \infty$". Also the speed has a simple explicit form $[\mathbb{E}(\pi(\omega))]^{-1}$.

The result of this section was stated in Theorem 1.3 in the introduction section. Before giving the proof, we explain how we get the explicit expression of the invariant density $\pi(\omega)$ from the MBPREI constructed in Section 2.

For $i \leq 0$ define $N_i = \#\{0 \leq k \leq T_1, X_k = i\}$. Note that conditioned on the event $\{X_n \to \infty\}$, $N_i = U_{i,1}^1 + |U_{i-1}^1|$. Then omitting the superscript "1", for $i < 0$, we have

$$E_\omega(N_i | U_i, U_{i+1}, ..., U_0) = U_{i,1} + |U_i M_i|.$$

Hence

$$\begin{aligned} E_\omega(N_i) &= e_1 M_0 \cdots M_{i+1} e_1^T + |e_1 M_0 \cdots M_{i+1} M_i| \\ &= e_1 M_0 \cdots M_{i+1} e_1^T + e_1 M_0 \cdots M_{i+1} M_i \mathbf{1}^T \\ &= (1 + a_i(1)) e_1 M_0 \cdots M_{i+1} \mathbf{1}^T = \frac{1}{\omega_i(1)} e_1 B \overline{M}_0 \cdots \overline{M}_{i+1} B^{-1} \mathbf{1}^T \\ &= \frac{1}{\omega_i(1)} e_1 \overline{M}_0 \cdots \overline{M}_{i+1} e_1^T. \end{aligned} \tag{34}$$

Note also that $E_\omega(N_0) = 1 + E_\omega(|U_{-1}|) = 1 + \sum_{l=1}^{l} b_0(l) = \frac{1}{\omega_0(1)}$. Then

$$E_\omega(T_1) = E_\omega\left(1 + \sum_{i=1}^{\infty} U_{-i,1} + |U_{-i}|\right) = \sum_{i=0}^{\infty} E_\omega(N_{-i})$$

$$= \frac{1}{\omega_0(1)} + \sum_{i=1}^{\infty} \frac{1}{\omega_i(1)} e_1 \overline{M}_0 \cdots \overline{M}_{-i+1} e_1^T.$$

Then we can define

$$\pi(\omega) := \frac{1}{\omega_0(1)} \left(1 + \sum_{i=1}^{\infty} e_1 \overline{M}_i \cdots \overline{M}_1 e_1^T \right). \tag{35}$$

Remark 4.1 *Indeed, one can simply follows from (9) that $T_1 = 1 + \sum_{i<0} U_i x_0$ implying that*

$$E_\omega(T_1) = 1 + \sum_{i<0} E_\omega(U_i(2,1,...,1)^T) = 1 + \sum_{i=1}^{\infty} e_1 M_0 \cdots M_{-i+1}(2,1,...,1)^T.$$

Therefore one can also define

$$\pi(\omega) = 1 + \sum_{i=1}^{\infty} e_1 M_i \cdots M_1 (2,1,...,1)^T. \tag{36}$$

One sees that the right-hand sides of (35) and (36) have different forms. But it follows from the second line of (34) that they are the same indeed.

Proof of Theorem 1.3: Since the MBPREI $\{U_i\}$ makes sense only on the event $\{X_n \to \infty\}$ we first show that $\gamma_L < 0$, implying that P-a.s., $X_n \to \infty$. Indeed for any $i > 0$ by Jensen's inequality we have

$$\mathbb{E}(e_1 \overline{M}_i \cdots \overline{M}_1 e_1^T) = \mathbb{E}\left(e^{\log e_1 \overline{M}_i \cdots \overline{M}_1 e_1^T}\right) \geq e^{\mathbb{E}\left(\log e_1 \overline{M}_i \cdots \overline{M}_1 e_1^T\right)}.$$

But it follows from (27) that

$$\begin{aligned}
\gamma_L &= \lim_{n\to\infty} \frac{1}{n} \mathbb{E}(\log \mathbf{1} \overline{M}_{n-1} \cdots \overline{M}_0 e_1^T) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left(\log\left(\sum_{l=1}^{L} e_1 \overline{M}_{n-l} \cdots \overline{M}_0 e_1^T\right)\right) \\
&= \max_{1\leq l\leq L} \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left(\log(e_1 \overline{M}_{n-l} \cdots \overline{M}_0 e_1^T)\right) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left(\log(e_1 \overline{M}_{n-1} \cdots \overline{M}_0 e_1^T)\right).
\end{aligned}$$

Therefore as $i \to \infty$,

$$e^{\gamma_L i} \sim e^{\mathbb{E}\left(\log e_1 \overline{M}_i \cdots \overline{M}_1 e_1^T\right)} \leq \mathbb{E}(e_1 \overline{M}_i \cdots \overline{M}_1 e_1^T) \to 0,$$

since $\mathbb{E}(\pi(\omega)) < \infty$. Then we have $\gamma_L < 0$. (i) is proved. We just give the idea of the proof (ii) and (iii) since it is similar to Zeitouni [21] (see the second version of the proof of Theorem 2.1.9). It follows from the stationarity that $E(T_1) = \mathbb{E}(\pi(\omega)) < \infty$. Define

$$Q(B) = E\left(\sum_{i=0}^{T_1 - 1} 1_{\{\overline{\omega}(i) \in B\}}\right), \quad \overline{Q}(B) = \frac{Q(B)}{Q(\Omega)} = \frac{Q(B)}{E(T_1)}.$$

Then $Q(\cdot)$ is invariant under kernel $\overline{P}$, that is

$$Q(B) = \iint 1_{\omega' \in B} \overline{P}(\omega, d\omega') Q(d\omega),$$

and $\frac{dQ}{d\mathbb{P}} = \sum_{i\leq 0} N_i = \pi(\omega)$. Then (14) is proved. Also under $\overline{Q} \otimes P_\omega$ the sequence $\{\overline{\omega}(n)\}$ is stationary and ergodic. Define the local drift $d(x,\omega) = E_{x,\omega}(X_1 - x)$. Then

$$X_n = \sum_{i=1}^{n} (X_i - X_{i-1} - d(X_{i-1}, \omega)) + \sum_{i=1}^{n} d(X_{i-1}, \omega)$$

$$:= \overline{R}_n + \sum_{i=1}^{n} d(X_{i-1}, \omega)$$

where $\{\overline{R}_n\}$ is a P_ω-martingale and P-a.s., $\frac{\overline{R}_n}{n} \to 0$. We have from the ergodicity under $\overline{Q} \otimes P_\omega$ that P-a.s.,

$$\lim_{n\to\infty} \frac{X_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d(0, \overline{\omega}(i-1)) = E_{\overline{Q}}(d(0, \overline{\omega}(0))).$$

But

$$\begin{aligned}
E_{\overline{Q}}(d(0,\overline{\omega}(0))) &= \frac{1}{\mathbb{E}(\pi(\omega))}\mathbb{E}\left(\pi(\omega)\Big(\omega_0(1) - \sum_{l=1}^{L} l\omega_0(-l)\Big)\right) \\
&= \frac{1}{\mathbb{E}(\pi(\omega))}\mathbb{E}\left(\frac{1}{\omega_0(1)}\Big(1 + \sum_{i=1}^{\infty} e_1\overline{M}_i\cdots\overline{M}_1 e_1^T\Big)\Big(\omega_0(1) - \sum_{l=1}^{L} l\omega_0(-l)\Big)\right) \\
&= \frac{1}{\mathbb{E}(\pi(\omega))}\mathbb{E}\left(\Big(1 + \sum_{i=1}^{\infty} e_1\overline{M}_i\cdots\overline{M}_1 e_1^T\Big)\Big(1 - \sum_{l=1}^{L} a_0(l)\Big)\right) \\
&= \frac{1}{\mathbb{E}(\pi(\omega))}\mathbb{E}\left(1 + \sum_{i=1}^{\infty} e_1\overline{M}_i\cdots\overline{M}_1 e_1^T - \sum_{l=1}^{L} a_0(l)(1 + \sum_{i=1}^{\infty} e_1\overline{M}_i\cdots\overline{M}_1 e_1^T)\right). \qquad (37)
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{i=1}^{\infty} e_1\overline{M}_i\cdots\overline{M}_1 e_1^T &= (a_1(1),...,a_1(L))e_1^T + \sum_{i=1}^{\infty}(a_{i+1}(1),...,a_{i+1}(L))\overline{M}_i\cdots\overline{M}_1 e_1^T \\
&\overset{\mathscr{D}}{=} (a_0(1),...,a_0(L))e_1^T + \sum_{i=1}^{\infty}(a_0(1),...,a_0(L))\overline{M}_i\cdots\overline{M}_1 e_1^T \\
&= a_0(1) + \sum_{i=1}^{\infty}\sum_{l=1}^{L} a_0(l)e_l\overline{M}_i\cdots\overline{M}_1 e_1^T \\
&\overset{\mathscr{D}}{=} \sum_{l=1}^{L}\left(a_0(l) + a_0(l)\sum_{i=1}^{\infty} e_1\overline{M}_i\cdots\overline{M}_1 e_1^T\right), \qquad (38)
\end{aligned}$$

where the last step follows similarly as (32) and (33). Substituting (38) to (37), using stationarity we have $E_{\overline{Q}}(d(0,\overline{\omega}(0))) = \frac{1}{\mathbb{E}(\pi(\omega))}$. □

5 The stable limit law of X_n

To begin with, we introduce some random variables relate to the process $\{Z_{-n}\}_{n\geq 0}$. First recall that $Y_{-k} = \sum_{m=k+1}^{\infty} Z(-k,-m)$ is the total number of progeny of the immigrant at times $-k$, and $\eta_k := \sum_{m=k+1}^{\infty} M_{-k}...M_{-m+1}$ is the corresponding expectation random matrix.

Next, let $\nu_0 \equiv 0$, and define recursively

$$\nu_n = \min\{m > \nu_{n-1} : Z_{-m} = \mathbf{0}\} \text{ for } n > 0,$$

being the successive regeneration times of MBPREI $\{Z_{-n}\}_{n\geq 0}$. For simplicity we write ν_1 as ν.

Define also

$$W = \sum_{k=0}^{\nu-1} Z_{-k},$$

the total number of offspring born before regenerating time ν.

Finally, for $A > 0$, we introduce the stopping time

$$\sigma = \sigma(A) = \inf\{m : |Z_{-m}| > A\}$$

which is the time the number of particles of the process $\{Z_n\}$ exceeding A.

5.1 The tail of the expectation of the total number of $\{Z(0, -k)\}_{k\geq 0}$

To study the limit law of RWRE with bounded jumps $\{X_n\}$, a key step is to prove that random variable Wx belongs to the domain of attraction of some κ-stable law. For this purpose it is crucial to show first that $x\eta_0 x_0$ belongs to the domain of attraction of a κ-stable law for any positive $x \in \mathbb{R}^L$. Indeed, we have

Theorem 5.1 *Suppose that $\gamma_L < 0$. Then under Condition C, for κ of (12) (see also (40) below) and for some $K_2 = K_2(x_0) \in (0, \infty)$, we have*

$$\lim_{t\to\infty} t^\kappa \mathbb{P}(x\eta_0 x_0 \geq t) = K_2 |xB|^\kappa$$

for all $x \in \mathbb{R}^L$ with positive components such that $|x| > 0$.

To proof Theorem 5.1, we need some classical results of random matrices in Kesten's paper [10]. We rewrite them in terms of $\{M_{-n}\}_{n\geq 0}$. Recall that

$$\gamma_L = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}(\log \|M_0 M_{-1} \cdots M_{-n+1}\|)$$

is the greatest Lyapunov exponent of $\{M_i\}_{i\leq 0}$.

Theorem 5.2 (Kesten [10]) *Suppose that Condition C holds and $\gamma_L < 0$. Then*

1) *for every $\alpha \in [0, \kappa_0]$ the limits*

$$\log \rho(\alpha) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left(\| M_0 M_{-1} \cdots M_{-n+1} \|^\alpha\right) \tag{39}$$

exist and $\log \rho(\alpha)$ is a strictly convex function of α. Hence

2) *there exists a unique $\kappa \in (0, \kappa_0]$, such that*

$$\log \rho(\kappa) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left(\| M_0 M_{-1} \cdots M_{-n+1} \|^\kappa\right) = 0. \tag{40}$$

3) *Let $\{Q_{-n}\}_{n\geq 0}$, with the law of Q_0 being $\mathbb{Q}$, be a random sequence of L-(column) vectors such that $\{M_{-n}, Q_{-n}\}_{n\geq 0}$ are i.i.d.. Assume also $\mathbb{Q}(Q_0 = \mathbf{0}) < 1$, $\mathbb{Q}(Q_0 \geq \mathbf{0}) = 1$, $E_{\mathbb{Q}}|Q_0|^\kappa < \infty$ for κ of (40), where $Q_0 \geq \mathbf{0}$ means that all components of Q_0 are nonnegative. Then for each $x \in S_{L-1}$, with an abuse use of notation $\mathbb{P}$, the limit*

$$\lim_{t\to\infty} t^\kappa \mathbb{P}\Big(\sum_{n=1}^{\infty} x M_0 M_{-1} \cdots M_{-n+1} Q_{-n} > t\Big)$$

exists and is finite. In particular there exist constants $K_1 = K_1(\mathbb{P}, M, \mathbb{Q}) \in (0, \infty)$ and $r = r(x, M) \in (0, \infty)$ such that

$$\lim_{t\to\infty} t^\kappa \mathbb{P}\Big(\sum_{n=1}^{\infty} x M_0 M_{-1} \cdots M_{-n+1} Q_{-n} \geq t\Big) = K_1(\mathbb{P}, M, \mathbb{Q}) r(x, M) \tag{41}$$

for $x \in S_+$.

Remark 5.1 *(i) The first two parts of the theorem can be concluded from the proof of Theorem 3 (see step 4) of Kesten [10]. The third part corresponds to Theorem 4 of Kesten [10].*
(ii) We mention that $|x|$ *denotes* $\left(\sum_{i=1}^L x_i^2\right)^{\frac{1}{2}}$ *in [10]. But all proofs go through under* l_1*-norm* $|x| := \sum_{i=1}^L |x_i|$.

Now we are ready to present the proof of Theorem 5.1.
Proof of Theorem 5.1: Fix $x = (x_1, ..., x_L) \in \mathbb{R}^L$ such that $x_i \geq 0,\ i = 1, ..., L$ and $|x| > 0$. We have

$$\mathbb{P}(x\eta_0 x_0 \geq t) = \mathbb{P}\Big(\sum_{n=1}^{\infty} xM_0M_{-1}\cdots M_{-n+1}x_0 \geq t\Big) = \mathbb{P}\Big(\sum_{n=1}^{\infty}\sum_{l=1}^{L} x_l e_l M_0 M_{-1}\cdots M_{-n+1}x_0 \geq t\Big).$$

It follows from Proposition 3.2 that the rightmost-hand side of above expression equals to

$$\mathbb{P}\Big(\sum_{l=2}^{L} lx_l + \sum_{l=1}^{L} lx_l \sum_{n=1}^{\infty} e_1 M_0 M_{-1}\cdots M_{-n+1}x_0 \geq t\Big)$$
$$= \mathbb{P}\Big(\sum_{l=2}^{L} lx_l + |xB| \sum_{n=1}^{\infty} e_1 M_0 M_{-1}\cdots M_{-n+1}x_0 \geq t\Big).$$

Then one gets from the third part of Theorem 5.2 that

$$\lim_{t\to\infty} t^\kappa \mathbb{P}(x\eta_0 x_0 > t) = \lim_{t\to\infty} t^\kappa \mathbb{P}\Big(|xB| \sum_{n=1}^{\infty} e_1 M_0 M_{-1}\cdots M_{-n+1}x_0 \geq t\Big)$$
$$= |xB|^\kappa K_1(\mathbb{P}, M, \delta_{x_0}) r(e_1, M) =: K_2 |xB|^\kappa,$$

which finishes the proof of Theorem 5.1. □

5.2 The tail of the population size of MBPREI before regeneration

Recall that ν is the regeneration time of the MBPREI and $W = \sum_{0<n\leq \nu-1} Z_{-n}$ is the total number of particles born to the immigrants entering before time ν. The main purpose of this section is to find how large the population size will be before regeneration.

Theorem 5.3 *Suppose that Condition C holds and* $\gamma_L < 0$*. If* $\kappa > 2$*, then* $E((Wx_0)^2) < \infty$*; if* $\kappa \leq 2$*, then there exists some* $0 < K_3 < \infty$ *such that*

$$\lim_{t\to\infty} t^\kappa P(Wx_0 \geq t) = K_3. \tag{42}$$

To prove the theorem, we need some preparations. To begin with we show that the tail probability of ν vanishes with exponential rate. This can follow from Theorem 4.2 of Key [14]. But for our MBPREI condition (iii): $\mathbb{P}(P_\omega(Z(0,-1) = \mathbf{0}|Z(0,0) = e_l) > 0 \ for\ l = 1, 2, ..., L) > 0$, in that theorem does not hold. However, we can prove the results directly, since we have

Lemma 5.1 *Suppose that condition **(C1)** holds and* $\gamma_L < 0$*. Then* $\lim_{m\to\infty} P(Z_{-m} = v) = \pi(v)$*, a probability distribution on* $\mathbb{Z}^L$ *with* $\pi(\boldsymbol{0}) > 0$.

Proof. The first assertion follows as Theorem 3.3 in [14]. To show that $\pi(\mathbf{0}) > 0$ we proceed by contradiction. If $\pi(\mathbf{0}) = 0$, then $\lim_{m\to\infty} P(Z_{-m} = \mathbf{0}) = 0$. Hence we have $\mathbb{P}$-a.s.,

$$\lim_{m\to\infty} P_\omega(Z_{-m} = \mathbf{0}) = 0. \tag{43}$$

For $v \in \mathbb{Z}^L$, $m > 0$, let

$$q(m, v) := P_\omega(Z_{-m-L} = \mathbf{0} | Z_{-m} = v).$$

Then for each $v \in \mathbb{Z}^L$

$$P_\omega(Z_{-m-L} = \mathbf{0}) \geq P_\omega(Z_{-m} = v) q(m, v). \tag{44}$$

Taken together, (43) and (44) imply that for all $v \in \mathbb{Z}^L$, $\mathbb{P}$-a.s.,

$$\lim_{m\to\infty} P_\omega(Z_{-m} = v) q(m, v) = 0.$$

It follows by stationarity that $P_\omega(Z_{-m} = v)q(m, v)$ and $P_\omega(Z'_{-m} = v)q(0, v)$ have the same distribution, where $Z'_{-m} = \sum_{k=0}^{m-1} Z(k, 0)$. Therefore for all $v \in \mathbb{Z}^L$, $\mathbb{P}$-a.s.,

$$\lim_{m\to\infty} P_\omega(Z'_{-m} = v) q(0, v) = 0.$$

Then on the event $\{q(0, v) > 0\}$,

$$\lim_{m\to\infty} P_\omega(Z'_{-m} = v) = 0.$$

If we can show that

$$\mathbb{P}(q(0, v) > 0) = 1 \tag{45}$$

then it follows that $\mathbb{P}$-a.s.,

$$\sum_{v\in\mathbb{Z}^L} \lim_{m\to\infty} P_\omega(Z'_{-m} = v) = 0$$

which will contradict that π is a probability distribution (Here we mention that Z'_{-m} and Z_{-m} have the same limit distributions, see Lemma 2.1 and Lemma 3.2 of Key [14]). It remains to show (45). Since

$$\max_{1\leq l\leq L} \mathbb{E}\Big(\log^+ \frac{\omega_0(-l)}{\omega_0(1)}\Big) < \infty,$$

$\mathbb{P}(\omega_0(1) > 0) = 1$. Then we have $\mathbb{P}$-a.s.,

$$P_\omega(Z_{-L} = \mathbf{0} | Z_0 = (v_1, v_2, ..., v_L)) \geq \prod_{k=0}^{L-1} \omega_{-k}(1)^{1+\sum_{l=k+1}^{L} v_l} > 0$$

which proves (45). $\square$

Remark 5.2 *We adopted the idea of the proof of Theorem 3.3 of Key [14] to prove this lemma. The only difference here is that we replace condition*

$$(iii) : \mathbb{P}(P_\omega(Z(0, -1) = \mathbf{0} \big| Z(0, 0) = e_l) > 0 \ for\ l = 1, 2, ..., L) > 0,$$

in that theorem with $\mathbb{P}(\omega_0(1) > 0) > 0$, *which is implied in condition* ***(C1)***.

With Lemma 5.1 in hands, the next theorem follows verbatim as Theorem 4.2 in [14].

Theorem 5.4 (Key[14]) *Suppose that* $\gamma_L < 0$ *and that condition* ***(C3)*** *holds. Then there exist positive constants* K_4 *and* K_5 *such that*

$$P(\nu > t) < K_4 \exp(-K_5 t).$$

In the following three lemmas, i.e., Lemma 5.2, Lemma 5.3 and Lemma 5.4, some estimations for the related probabilities and moments are given. The proofs are technical and follow almost verbatim as Kesten-Kozlov-Spitzer [11]. But the proofs will be long journeys. Therefore, for continuity consideration, we delay the proofs of these lemmas to the Appendix section.

In Lemma 5.2, Lemma 5.3 and Lemma 5.4, we always make the assumption that all conditions of Theorem 5.3 hold.

Lemma 5.2 *If $\kappa \le 2$, then there exists for all $\epsilon > 0$ an $A_0 = A_0(\epsilon) < \infty$ such that*

$$P\Big(\sum_{\sigma \le k < \nu} |Y_{-k}| \ge \epsilon x\Big) \le \epsilon x^{-\kappa} \text{ for } A \ge A_0(\epsilon).$$

Lemma 5.3 *If $\kappa \le 2$, then for fixed A*

$$E\left(|Z_{-\sigma}|^{\kappa}; \sigma < \nu\right) < \infty. \tag{46}$$

If $\kappa > 2$ then

$$E(|W|^2) < \infty.$$

Next we introduce

$$S_{-\sigma,-m} = \text{number of progeny alive at time } -m \text{ of the } Z_{-\sigma} \text{ particles present at } -\sigma > -m.$$

Let $S_{-\sigma,-\sigma} = Z_{-\sigma}$, and

$$S_{-\sigma} = \sum_{m=\sigma}^{\infty} S_{-\sigma,-m} = Z_{-\sigma} + \text{total progeny of the } Z_{-\sigma} \text{ particles at } -\sigma.$$

Lemma 5.4 *If $\kappa \le 2$, then there exists for all $\epsilon > 0$ an $A_1 = A_1(\epsilon)$ such that for $A > A_1$*

$$P\Big(\Big|\sum_{m=\sigma}^{\infty}\Big(S_{-\sigma,-m} - Z_{-\sigma}\prod_{i=\sigma}^{m-1} M_{-i}\Big)\Big| \ge \epsilon x, \sigma < \nu\Big) \le \epsilon x^{-\kappa} E(|Z_{-\sigma}|^{\kappa}; \sigma < \nu).$$

Proof of Theorem 5.3: Since in Lemma 5.3 we have shown that $E(|W|^2) < \infty$ when $\kappa > 2$, it follows immediately that $E((Wx_0)^2) < 4E(|W|^2) < \infty$. The first part of the theorem follows. To prove the second part, recall that W is the number of particles born before $-\nu$. Then on the event $\{\sigma < \nu\}$ we have

$$W = \sum_{s=0}^{\sigma-1} Z_{-s} + S_{-\sigma} + \sum_{\sigma \le s < \nu} Y_{-s}.$$

As an immediate corollary of Theorem 5.4 we have for all $\epsilon > 0,\ A > 0$

$$P(Wx_0 \ge \epsilon x, \sigma(A) \ge \nu) \le P(2A\nu \ge \epsilon x) = o(x^{-\kappa}), \qquad x \to \infty, \tag{47}$$

since $Wx_0 < |2W| = |2\sum_{t=0}^{\nu-1} Z_{-t}| \le 2A\nu$ on the event $\{\sigma \ge \nu\}$. Similarly we have

$$P\Big(\Big|\sum_{t=0}^{\sigma-1} Z_{-t}x_0\Big| \ge \epsilon x, \sigma(A) < \nu\Big) \le P(2A\nu \ge \epsilon x) = o(x^{-\kappa}), \qquad x \to \infty. \tag{48}$$

Taken together (47), (48) and Lemma 5.2 imply that for sufficiently large A and x

$$P(\sigma < \nu, S_{-\sigma}x_0 \ge x)$$

$$
\begin{aligned}
&\leq P(Wx_0 \geq x) = P(Wx_0 \geq x, \sigma < \nu) + P(Wx_0 \geq x, \sigma \geq \nu) \\
&\leq P(\sigma < \nu, S_{-\sigma}x_0 \geq x - \sum_{s=0}^{\sigma-1} Z_{-s}x_0 - \sum_{\sigma \leq s < \nu} Y_{-s}x_0) + \epsilon x^{-\kappa} \\
&\leq P(\sigma < \nu, S_{-\sigma}x_0 \geq x(1-2\epsilon)) + 3\epsilon x^{-\kappa}. \qquad (49)
\end{aligned}
$$

Since

$$
\begin{aligned}
&P\Big(\sigma < \nu, Z_{-\sigma} \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} M_{-i}x_0 \geq (1+\epsilon)x\Big) \\
&\quad \leq P\Big(\sigma < \nu, \sum_{t=\sigma}^{\infty} S_{-\sigma,-t}x_0 \geq x\Big) + P\Big(\sigma < \nu, \Big|\Big(Z_{-\sigma} \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} M_{-i} - \sum_{t=\sigma}^{\infty} S_{-\sigma,-t}\Big)x_0\Big| \geq \epsilon x\Big) \\
&\quad \leq P\Big(\sigma < \nu, \sum_{t=\sigma}^{\infty} S_{-\sigma,-t}x_0 \geq x\Big) + P\Big(\sigma < \nu, 2\Big|Z_{-\sigma} \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} M_{-i} - \sum_{t=\sigma}^{\infty} S_{-\sigma,-t}\Big| \geq \epsilon x\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
P(\sigma < \nu, S_{-\sigma}x_0 > x(1-2\epsilon)) &\leq P\Big(\sigma < \nu, Z_{-\sigma} \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} M_{-i}x_0 \geq (1-3\epsilon)x\Big) \\
&\quad + P\Big(\sigma < \nu, \Big|\Big(Z_{-\sigma} \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} M_{-i} - \sum_{t=\sigma}^{\infty} S_{-\sigma,-t}\Big)x_0\Big| \geq \epsilon x\Big) \\
&\leq P\Big(\sigma < \nu, Z_{-\sigma} \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} M_{-i}x_0 \geq (1-3\epsilon)x\Big) \\
&\quad + P\Big(\sigma < \nu, 2\Big|Z_{-\sigma} \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} M_{-i} - \sum_{t=\sigma}^{\infty} S_{-\sigma,-t}\Big| \geq \epsilon x\Big),
\end{aligned}
$$

then it follows from Lemma 5.4 and (49) that for sufficiently large A

$$
\begin{aligned}
&P\Big(\sigma < \nu, Z_{-\sigma} \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} M_{-i}x_0 \geq (1+\epsilon)x\Big) - \epsilon x^{-\kappa} E(|Z_{-\sigma}|^{\kappa}; \sigma < \nu) \\
&\quad \leq P(Wx_0 \geq x) \\
&\quad \leq P\Big(\sigma < \nu, Z_{-\sigma} \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} M_{-i}x_0 \geq (1-3\epsilon)x\Big) + \epsilon x^{-\kappa}(3 + E(|Z_{-\sigma}|^{\kappa}; \sigma < \nu)). \qquad (50)
\end{aligned}
$$

Since

$$
\sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} M_{-i} = I + \eta_{-\sigma},
$$

we can write (50) as

$$
\begin{aligned}
&P(\sigma < \nu, Z_{-\sigma}(I + \eta_{-\sigma})x_0 \geq (1+\epsilon)x) - \epsilon x^{-\kappa} E(|Z_{-\sigma}|^{\kappa}; \sigma < \nu) \\
&\quad \leq P(Wx_0 \geq x) \\
&\quad \leq P(\sigma < \nu, Z_{-\sigma}(I + \eta_{-\sigma})x_0 \geq (1-3\epsilon)x) + \epsilon x^{-\kappa}(3 + E(|Z_{-\sigma}|^{\kappa}; \sigma < \nu)).
\end{aligned}
$$

Then, to prove (42), it suffices to prove that for each fixed A

$$
0 < \lim_{x\to\infty} x^{\kappa} P(\sigma = \sigma(A) < \nu, Z_{-\sigma}(I + \eta_{-\sigma})x_0 \geq x) = K_2 E(|Z_{-\sigma}B|^{\kappa}; \sigma < \nu) < \infty. \qquad (51)
$$

Indeed, this follows immediately from Lemma 5.3 and Theorem 5.1 since

$$\begin{aligned}
&\lim_{x\to\infty} x^\kappa P(\sigma < \nu, Z_{-\sigma}(I+\eta_{-\sigma})x_0 \ge x) \\
&= \lim_{x\to\infty} x^\kappa \int_{|s|\ge A}^{\infty} P(\sigma<\nu, Z_{-\sigma}\in ds)P(s(I+\eta_{-\sigma})x_0 \ge x) \\
&= K_2 \int_{|s|\ge A}^{\infty} P(\sigma<\nu, Z_{-\sigma}\in ds)|sB|^\kappa \\
&= K_2 E(|Z_{-\sigma}B|^\kappa; \sigma<\nu) < L^\kappa K_2 E(|Z_{-\sigma}|^\kappa;\sigma<\nu) < \infty.
\end{aligned}$$

Also, we have

$$E(|Z_{-\sigma}B|^\kappa;\sigma<\nu) \ge E(|Z_{-\sigma}|^\kappa;\sigma<\nu) \ge A^\kappa \mathbb{E}(P_\omega(|Z_{-1}|>A)) > A^\kappa \mathbb{E}(\omega_0(-1)^{A+1}\omega_0(1)) > 0.$$

Thus Theorem 5.3 follows. □.

Proof of Theorem 1.4: From here on the proof of Theorem 1.4 is standard. Recall that

$$T_n = n + \sum_{i=-\infty}^{n-1} |U_i^n| + \sum_{i=-\infty}^{n-1} U_{i,1}^n = n + \sum_{i=-\infty}^{n-1} U_i^n x_0.$$

When the walk is transient to the right, it only takes finite steps in $(-\infty, 0)$, i.e., P-a.s.,

$$\sum_{i<0} U_i^n x_0 < \infty.$$

Therefore to determine the limit distribution of T_n, we need only to consider

$$n + \sum_{i=0}^{n-1} U_i^n x_0,$$

which, by Theorem 1.1, has the same distribution with

$$n + \sum_{t=0}^{n-1} Z_{-t} x_0.$$

In Theorem 5.3 we have proved that if $\kappa > 2$, $E((Wx_0)^2) < \infty$ while for $\kappa \le 2$, $P(Wx_0 > x) \sim K_3 x^\kappa$, as $x \to \infty$. Then it follows that (see, Feller [5], Chap. XVII, Sec. 5 Theorem 2 or Gendenko-Kolmogorov [6] Chap.7, Sec. 35, Theorem 2) Wx_0 belongs to the domain of attraction of a κ-stable law. Recall that $\nu_0 = 0, \nu_1, \nu_2, \ldots$ are the successive regeneration times of the MBPREI. Now put

$$W_k = \sum_{\nu_k \le t < \nu_{k+1}} Z_{-t},$$

then the pairs $\{(\nu_{k+1}-\nu_k), W_k\}_{k\ge 0}$ are independent, all with distribution of the (ν, W) because (ν, W) coincides with $(\nu_1 - \nu_0, W_0)$. Then the proof of the theorem is standard. It follows exactly with the proof of the theorem in Kesten-Kozlov-Spitzer [10]. We will not repeat it here. □

Appendix: Proofs of Lemma 5.2, Lemma 5.3 and Lemma 5.4

Proof of Lemma 5.2: Since that $\sum_{k=1}^\infty k^{-2} = \frac{\pi^2}{6}$, we have

$$P\Big(\sum_{\sigma\le k<\nu} |Y_{-k}| \ge \epsilon x\Big) = P\left(\sum_{k=1}^\infty 1_{[\sigma\le k<\nu]}|Y_{-k}| \ge 6\pi^{-2}\epsilon x \sum_{k=1}^\infty k^{-2}\right)$$

$$\leq \sum_{k=1}^{\infty} P\left(\sigma \leq k < \nu, |Y_{-k}| \geq \frac{1}{2}\epsilon x k^{-2}\right). \tag{52}$$

Note that Y_{-k} is $\sigma\{\omega_{-k}, \omega_{-k-1}, \cdots\}$-measurable, the event $\{\sigma \leq k < \nu\}$ being defined in terms of $Z_0, Z_{-1} \cdots, Z_{-k}$ is $\sigma\{\omega_0, \omega_{-1} \cdots, \omega_{-k+1}\}$-measurable and Y_{-k} has the same distribution as Y_0. Then

$$\begin{aligned} P\Big(\sum_{\sigma \leq k < \nu} |Y_{-k}| \geq \epsilon x\Big) &\leq \sum_{k=1}^{\infty} P(\sigma \leq k < \nu) P\Big(|Y_{-k}| \geq \frac{1}{2}\epsilon x k^{-2}\Big) \\ &= \sum_{k=1}^{\infty} P(\sigma \leq k < \nu) P\Big(|Y_0| \geq \frac{1}{2}\epsilon x k^{-2}\Big). \end{aligned}$$

Thus if we can prove that

$$P(|Y_0| \geq x) \leq K_6 x^{-\kappa} \tag{53}$$

for some $K_6 < \infty$, then it follows that

$$\begin{aligned} P\Big(\sum_{\sigma \leq k < \nu} |Y_{-k}| \geq \epsilon x\Big) &\leq x^{-\kappa} 2^{\kappa} \epsilon^{-\kappa} K_6 \sum_{k=1}^{\infty} k^{2\kappa} P(\sigma \leq k < \nu) \\ &\leq x^{-\kappa} 2^{\kappa} \epsilon^{-\kappa} K_6 E\left(\nu^{2\kappa+1}; \sigma < \nu\right) \leq \epsilon x^{-\kappa} \end{aligned}$$

for A large enough since $E\left(\nu^{2\kappa+1}\right) < \infty$ by Lemma 5.2 and $\sigma(A) \to \infty$ in probability as $A \to \infty$.

To prove (53), observe that $\eta_{-m} = M_{-m}(I + \eta_{-m-1})$ and consequently with $Z(0,0) = \mathbf{0}$

$$Y_0 = \sum_{m=1}^{\infty} Z(0,-m) = \sum_{m=1}^{\infty} (Z(0,-m) - Z(0,-m+1)M_{-m+1})(I + \eta_{-m}).$$

Let $e_0 = (\frac{1}{L}, ..., \frac{1}{L})$. Using the independence of $(I + \eta_{-m})$ and $M_{-m+1}, Z(0,-m+1), Z(0,-m)$, similarly as (52) we have

$$\begin{aligned} P(|Y_0| \geq x) &\leq \sum_{m=1}^{\infty} P\left(|(Z(0,-m) - Z(0,-m+1)M_{-m+1})(I + \eta_{-m})| \geq \frac{1}{2} m^{-2} x\right) \\ &\leq \sum_{m=1}^{\infty} \int P(|Z(0,-m) - Z(0,-m+1)M_{-m+1}| \in ds) \\ &\times \ P\left(e_0(I + \eta_0)e_0^T \geq (2sL^2m^2)^{-1} x\right). \end{aligned}$$

From Theorem 5.2 there exists a $0 < K_7 < \infty$ for which

$$P\left(e_0(I + \eta_0)e_0^T \geq (2sL^2m^2)^{-1} x\right) \leq K_7 (2sL^2m^2)^{\kappa} x^{-\kappa}. \tag{54}$$

Then it follows that

$$\begin{aligned} P(|Y_0| \geq x) &\leq x^{-\kappa} 2^{\kappa} L^{2\kappa} K_7 \sum_{m=1}^{\infty} m^{2\kappa} E\left(|Z(0,-m) - Z(0,-m+1)M_{-m+1}|^{\kappa}\right) \\ &\leq x^{-\kappa} 2^{\kappa} L^{2\kappa} K_7 \sum_{m=1}^{\infty} m^{2\kappa} \mathbb{E}\left(E_\omega\left(|Z(0,-m) - Z(0,-m+1)M_{-m+1}|^2\right)^{\frac{\kappa}{2}}\right) \end{aligned} \tag{55}$$

since by assumption of the lemma $\kappa \leq 2$. We prove next the convergence of the last series in above expression to complete the proof of the lemma. For this purpose note that

$$E_\omega\left(|Z(0,-m) - Z(0,-m+1)M_{-m+1}|^2 | Z(0,-m+1)\right)$$

$$= |Z(0,-m+1)|\Big(\sum_{l=1}^{L}(b_{-m+1}(l)+b^2_{-m+1}(l))+2\sum_{1\le l<k\le L}b_{-m+1}(l)b_{-m+1}(k)\Big)$$
$$=: |Z(0,-m+1)|R(M_{-m+1}). \tag{56}$$

Then it follows that

$$\begin{aligned}&\mathbb{E}\left(E_\omega\left(|Z(0,-m)-Z(0,-m+1)M_{-m+1}|^2\right)^{\frac{\kappa}{2}}\right)\\ &= \mathbb{E}(\{E_\omega(|Z(0,-m+1)|R(M_{-m+1})\}^{\frac{\kappa}{2}}))\\ &= \mathbb{E}\Big(|e_1M_0M_{-1}...M_{-m+2}|^{\frac{\kappa}{2}}R(M_{-m+1})^{\frac{\kappa}{2}}\Big)\\ &= \mathbb{E}\left(|e_1M_0M_{-1}...M_{-m+2}|^{\frac{\kappa}{2}}\right)\mathbb{E}(R(M_0)^{\frac{\kappa}{2}}),\end{aligned}$$

using independence and stationarity in the last step. Since $\kappa<2$, then condition **(C3)** implies that $\mathbb{E}(R(M_0)^{\frac{\kappa}{2}})<\infty$, and 1), 2) of Theorem 5.2 imply that

$$\mathbb{E}\Big(|e_1M_0M_{-1}\cdots M_{-m+2}|^{\frac{\kappa}{2}}\Big)\le L^{\frac{\kappa}{2}}\mathbb{E}\Big(\|M_0M_{-1}\cdots M_{-m+2}\|^{\frac{\kappa}{2}}\Big)\sim L^{\frac{\kappa}{2}}e^{-c(m-1)}$$

as m tends to ∞ for some constant $c>0$. Thus the convergence of the last series in (55) follows. □

Proof of Lemma 5.3: We have on $\{\sigma<\nu\}$

$$|Z_{-\sigma}|=(|Z_{-\sigma+1}|+1)\frac{|Z_{-\sigma}|}{|Z_{-\sigma+1}|+1}\le(A+1)\frac{|Z_{-\sigma}|}{|Z_{-\sigma+1}|+1}\le(A+1)\sum_{1\le m\le\nu}\frac{|Z_{-m}|}{|Z_{-m+1}|+1}. \tag{57}$$

As a matter of fact, if $\kappa\ge1$

$$\begin{aligned}(E\left(|Z_{-\sigma}|^\kappa;\sigma<\nu\right))^{\frac{1}{\kappa}}&\le(A+1)\Big(E\Big[\Big(\sum_{m\ge1}\frac{|Z_{-m}|}{|Z_{-m+1}|+1}1_{[m\le\nu]}\Big)^\kappa\Big]\Big)^{\frac{1}{\kappa}}\\ &\le(A+1)\sum_{m\ge1}\left(E\Big(\Big(\frac{|Z_{-m}|}{|Z_{-m+1}|+1}\Big)^\kappa 1_{[m\le\nu]}\Big)\right)^{\frac{1}{\kappa}}.\end{aligned} \tag{58}$$

Conditioned on ω and Z_{-m+1} we have

$$|Z_{-m}|\le\sum_{j=1}^{|Z_{-m+1}|+1}(|V_j|+1),$$

where

$$P_\omega(V_j=(u_1,\cdots,u_L))=\frac{(u_1+\cdots+u_L)!}{u_1!\cdots u_L!}\omega_{-m+1}(-1)^{u_1}\cdots\omega_{-m+1}(-L)^{u_L}\omega_{-m+1}(1)$$

and $V_j,\ j=1,2,...$ are i.i.d.. Then for $1\le\kappa\le2$ we have

$$\begin{aligned}(E_\omega(|Z_{-m}|^\kappa\big|Z_0,Z_{-1},\cdots,Z_{-m+1}))^{\frac{1}{\kappa}}&\le\sum_{j=1}^{|Z_{-m+1}|+1}(E_\omega((|V_j|+1)^2\,\big|Z_0,Z_{-1},\cdots,Z_{-m+1}))^{\frac{1}{2}}\\ &=(|Z_{-m+1}|+1)\Big(1+2\sum_{l=1}^{L}b_{-m+1}(l)\\ &\qquad+\sum_{l=1}^{L}(b_{-m+1}(l)+2b^2_{-m+1}(l))+4\sum_{1\le k<l\le L}b_{-m+1}(k)b_{-m+1}(l)\Big)^{\frac{1}{2}}\end{aligned}$$

$$=: (|Z_{-m+1}|+1)\tilde{R}(M_{-m+1})^{\frac{1}{2}}. \tag{59}$$

It follows that

$$E\Big(\Big(\frac{|Z_{-m}|}{|Z_{-m+1}|+1}\Big)^{\kappa}1_{[m\le\nu]}\Big) = E\Big(E_\omega\Big(\Big(\frac{|Z_{-m}|}{|Z_{-m+1}|+1}\Big)^{\kappa}\Big|Z_0, Z_{-1}\cdots, Z_{-m+1}\Big); m\le\nu\Big)$$

by (59) and independence and stationarity

$$\le E(R(M_{-m+1})^{\frac{\kappa}{2}}; m\le\nu) = \mathbb{E}(\tilde{R}(M_0)^{\frac{\kappa}{2}})P(\nu > m-1).$$

For $1\le\kappa\le 2$, since condition **(C3)** implies that $\mathbb{E}(\tilde{R}(M_0)^{\frac{\kappa}{2}})<\infty$, then the inequality (46) follows from (58) and Theorem 5.4. For $\kappa<1$ we have from (57) that

$$\begin{aligned}
E(|Z_{-\sigma}|^{\kappa};\sigma<\nu) &\le (A+1)^{\kappa}\sum_{m\ge1}E\Big(\Big(\frac{|Z_{-m}|}{|Z_{-m+1}|+1}\Big)^{\kappa}1_{[m\le\nu]}\Big)\\
&\le (A+1)^{\kappa}\sum_{m\ge1}E\left(1_{[m\le\nu]}(|Z_{-m+1}|+1)^{-\kappa}(E_\omega(|Z_{-m}|\big|Z_0,Z_{-1},\cdots,Z_{-m+1}))^{\kappa}\right)\\
&\le (A+1)^{\kappa}\sum_{m\ge1}E\Big(1_{[m\le\nu]}\Big(1+\sum_{l=1}^{L}b_{-m+1}(l)\Big)^{\kappa}\Big)\\
&= (A+1)^{\kappa}\sum_{m\ge1}P(\nu>m-1)\mathbb{E}\Big(\Big(1+\sum_{l=1}^{L}b_0(l)\Big)^{\kappa}\Big)<\infty.
\end{aligned}$$

For $\kappa>2$, note that

$$W=\sum_{0\le m<\nu}Y_{-m}=\sum_{m=0}^{\infty}Y_{-m}1_{[m<\nu]}.$$

Then by the independence of Y_{-m} and $1_{[m<\nu]}$ we have

$$(E|W|^2)^{\frac{1}{2}}\le\sum_{m=0}^{\infty}(E\left(|Y_{-m}|^2 1_{[m<\nu]}\right))^{\frac{1}{2}}=\sum_{m=0}^{\infty}(E|Y_0|^2)^{\frac{1}{2}}(P(\nu>m))^{\frac{1}{2}}.$$

Theorem 5.4 implies that $E|W|^2$ is finite if we can show that

$$E|Y_0|^2<\infty.$$

In fact,

$$\begin{aligned}
(E\left(|Y_0|^2\right))^{\frac{1}{2}} &= \Big(E\Big(\Big(\sum_{t=1}^{\infty}|Z(0,-t)|\Big)^2\Big)\Big)^{\frac{1}{2}}\\
&\le \sum_{t=1}^{\infty}\left(E|Z(0,-t)|^2\right)^{\frac{1}{2}}=\sum_{t=1}^{\infty}\left(\mathbb{E}\left[E_\omega|Z(0,-t)|^2\right]\right)^{\frac{1}{2}}\\
&= \sum_{t=1}^{\infty}\Big(\mathbb{E}\left([E_\omega|Z(0,-t)|]^2\right)+\mathbb{E}\left[V_\omega(|Z(0,-t)|)\right]\Big)^{\frac{1}{2}}\\
&\le \sum_{t=1}^{\infty}\Big(\mathbb{E}\left([E_\omega|Z(0,-t)|]^2\right)\Big)^{\frac{1}{2}}+\sum_{t=1}^{\infty}\Big(\mathbb{E}\left[V_\omega(|Z(0,-t)|)\right]\Big)^{\frac{1}{2}}. \qquad (60)
\end{aligned}$$

To estimate the first term in the rightmost-hand side of (60), note that

$$\mathbb{E}[(E_\omega|Z(0,-t)|)^2]=\mathbb{E}(|e_1M_0M_{-1}\cdots M_{-t+1}|^2)\le L^2\mathbb{E}(\parallel M_0M_{-1}\cdots M_{-t+1}\parallel^2).$$

Since $\kappa > 2$, (39) and (40) of Theorem 5.2 imply that for some $\beta < 0$

$$\mathbb{E}(\| M_0 M_{-1} \cdots M_{-t+1} \|^2) \sim e^{\beta t} \text{ as } t \to \infty. \tag{61}$$

Now we estimate the second term in the rightmost-hand side of (60). Note that

$$\begin{aligned}
V_\omega(|Z(0,-t)|) &= E_\omega([|Z(0,-t)| - |E_\omega(Z(0,-t))|]^2) \leq E_\omega(|Z(0,-t) - E_\omega(Z(0,-t))|^2) \\
&= E_\omega\left(E_\omega\left[|Z(0,-t) - Z(0,-t+1)M_{-t+1}|^2 \middle| Z(0,-t+1)\right]\right) \\
&= E_\omega\left(|Z(0,-t+1)| E_\omega\left[|Z(0,-t) - Z(0,-t+1)M_{-t+1}|^2 \middle| Z(0,-t+1) = e_1\right]\right) \\
&= |e_1 M_0 M_{-1} \cdots M_{-t+2}| \Big(\sum_{l=1}^{L} (b_{-t+1}(l) + b_{-t+1}(l)^2) + 2 \sum_{1 \leq k < l \leq L} b_{-t+1}(l) b_{-t+1}(k) \Big)
\end{aligned}$$

Then we have that

$$\begin{aligned}
&\mathbb{E}(V_\omega(|Z(0,-t)|)) \\
&\quad \leq L\mathbb{E}(\| M_0 M_{-1} \cdots M_{-t+2} \|) \mathbb{E}\Big(\sum_{l=1}^{L} (b_0(l) + b_0(l)^2) + 2 \sum_{1 \leq k < l \leq L} b_0(l) b_0(k) \Big) \\
&\quad \leq C\mathbb{E}(\| M_0 M_{-1} \cdots M_{-t+2} \|) \sim Ce^{\gamma t}
\end{aligned} \tag{62}$$

for some $C > 0$ and $\gamma < 0$ for the same reason as (61). Then (61) together with (62) implies the convergence of the series in the rightmost-hand side of (60). Therefore $E|Y_0|^2 < \infty$. $\square$

Recall

$$S_{-\sigma,-m} = \text{number of progeny alive at time } -m \text{ of the } Z_{-\sigma} \text{ particles present at } -\sigma > -m.$$

Let $S_{-\sigma,-\sigma} = Z_{-\sigma}$, and

$$S_{-\sigma} = \sum_{m=\sigma}^{\infty} S_{-\sigma,-m} = Z_{-\sigma} + \text{total progeny of the } Z_{-\sigma} \text{ particles at } -\sigma.$$

Proof of Lemma 5.4: Observe that

$$S_{-\sigma,-m} - Z_{-\sigma} \prod_{i=\sigma}^{m-1} M_{-i} = \sum_{\sigma+1 \leq l \leq m} \Big(S_{-\sigma,-l} \prod_{i=l}^{m-1} M_{-i} - S_{-\sigma,-l+1} \prod_{i=l-1}^{m-1} M_{-i} \Big),$$

the convention being that empty product is I, and therefore

$$\begin{aligned}
\sum_{m=\sigma}^{\infty} \Big(S_{-\sigma,-m} - Z_{-\sigma} \prod_{i=\sigma}^{m-1} M_{-i} \Big) &= \sum_{l=\sigma+1}^{\infty} \sum_{m=l}^{\infty} \Big(S_{-\sigma,-l} \prod_{i=l}^{m-1} M_{-i} - S_{-\sigma,-l+1} \prod_{i=l-1}^{m-1} M_{-i} \Big) \\
&= \sum_{l=\sigma+1}^{\infty} (S_{-\sigma,-l} - S_{-\sigma,-l+1} M_{-l+1}) \sum_{m=l}^{\infty} \prod_{i=l}^{m-1} M_{-i} \\
&= \sum_{l=\sigma+1}^{\infty} (S_{-\sigma,-l} - S_{-\sigma,-l+1} M_{-l+1})(I + \eta_{-l}).
\end{aligned}$$

Then we have

$$P\Big(\Big| \sum_{m=\sigma}^{\infty} \Big(S_{-\sigma,-m} - Z_{-\sigma} \prod_{i=\sigma}^{m-1} M_{-i} \Big) \Big| \geq \epsilon x, \sigma < \nu \Big)$$

$$= \sum_{j=1}^{\infty} P\Big(\Big| \sum_{m=\sigma}^{\infty} \Big(S_{-\sigma,-m} - Z_{-\sigma} \prod_{i=\sigma}^{m-1} M_{-i} \Big) \Big| \geq \epsilon x, \sigma < \nu, \sigma = j \Big)$$
$$= \sum_{j=1}^{\infty} P\Big(\Big| \sum_{l=\sigma+1}^{\infty} (S_{-\sigma,-l} - S_{-\sigma,-l+1} M_{-l+1})(I + \eta_{-l}) \Big| \geq \epsilon x, \sigma < \nu, \sigma = j \Big)$$
$$\leq \sum_{j=1}^{\infty} P\Big(\sum_{l=\sigma+1}^{\infty} \Big| (S_{-\sigma,-l} - S_{-\sigma,-l+1} M_{-l+1}) \Big| |\mathbf{1}(I + \eta_{-l})| \geq \epsilon x, \sigma < \nu, \sigma = j \Big).$$

By a similar argument as (52), the rightmost-hand side of the last expression is less than or equal to

$$\sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} P(|S_{-\sigma,-l} - S_{-\sigma,-l+1} M_{-l+1}| e_0 (I + \eta_{-l}) e_0^T \geq \frac{1}{2} (l - \sigma)^{-2} L^{-2} \epsilon x, \sigma < \nu, \sigma = j)$$
$$= \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} \int P(|S_{-j,-l} - S_{-j,-l+1} M_{-l+1}| \in ds, j < \nu, \sigma = j,$$
$$e_0 (I + \eta_{-l}) e_0^T \geq (2s)^{-1} (l - j)^{-2} L^{-2} \epsilon x).$$

Note that $\{|S_{-j,-l} - S_{-j,-l+1} M_{-l+1}| \in ds, j < \nu, \sigma = j\}$, defined in term of $\{Z_0, Z_{-1}, ..., Z_{-l}\}$, depends only on $\sigma(\omega_{-i}; i < l)$ and that $\{e_0 (I + \eta_{-l}) e_0^T \geq (2s)^{-1} (l - j)^{-2} L^{-2} \epsilon x\}$ depends only on $\sigma(\omega_{-i}; i \geq l)$. Then by the independence and stationarity of the environment, the right-hand side of the above equality equals to

$$\sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} \int P(|S_{-j,-l} - S_{-j,-l+1} M_{-l+1}| \in ds, j < \nu, \sigma = j)$$
$$\times P(e_0 (I + \eta_0) e_0^T \geq (2s)^{-1} (l - j)^{-2} L^{-2} \epsilon x)$$

which, by Theorem 5.2 with K_7 as in (54), is less than or equal to

$$x^{-\kappa} (\frac{2}{\epsilon})^{\kappa} K_7 L^{2\kappa} \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} (l - j)^{2\kappa} \int s^{\kappa} P(|S_{-j,-l} - S_{-j,-l+1} M_{-l+1}| \in ds, j < \nu, \sigma = j)$$
$$= x^{-\kappa} (\frac{2}{\epsilon})^{\kappa} K_7 L^{2\kappa} \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} (l - j)^{2\kappa} E(|S_{-j,-l} - S_{-j,-l+1} M_{-l+1}|^{\kappa}; j < \nu, \sigma = j)$$
$$= x^{-\kappa} (\frac{2}{\epsilon})^{\kappa} K_7 L^{2\kappa} E\Big(\sum_{l=\sigma+1}^{\infty} (l - \sigma)^{2\kappa} |S_{-\sigma,-l} - S_{-\sigma,-l+1} M_{-l+1}|^{\kappa}; \sigma < \nu \Big)$$
$$= x^{-\kappa} (\frac{2}{\epsilon})^{\kappa} K_7 L^{2\kappa} E\Big(\sum_{l=\sigma+1}^{\infty} (l - \sigma)^{2\kappa}$$
$$\times E\Big(|S_{-\sigma,-l} - S_{-\sigma,-l+1} M_{-l+1}|^{\kappa}; \sigma < \nu \Big| \omega, \sigma, Z_0, Z_{-1}, \cdots, Z_{-\sigma} \Big) \Big).$$

Recalling that $\kappa \leq 2$, Jensen's inequality implies that the rightmost-hand side of above expression is less than or equal to

$$x^{-\kappa} (\frac{2}{\epsilon})^{\kappa} K_7 L^{2\kappa} E\Big(\sum_{l=\sigma+1}^{\infty} (l - \sigma)^{2\kappa} \times$$
$$\Big\{ E\Big(|S_{-\sigma,-l} - S_{-\sigma,-l+1} M_{-l+1}|^2; \sigma < \nu \Big| \omega, \sigma, Z_0, Z_{-1}, \cdots, Z_{-\sigma} \Big) \Big\}^{\frac{\kappa}{2}} \Big). \tag{63}$$

Again as in (56) we have

$$\begin{aligned}
&E\left(|S_{-\sigma,-l}-S_{-\sigma,-l+1}M_{-l+1}|^2\big|\omega,\sigma,Z_0,Z_{-1},\cdots,Z_{-\sigma},S_{-\sigma,-l+1}\right)\\
&=|S_{-\sigma,-l+1}|\Big(\sum_{j=1}^{L}(b_{-l+1}(j)+b^2_{-l+1}(j))+2\sum_{1\le i<j\le L}b_{-l+1}(i)b_{-l+1}(j)\Big)\\
&=:|S_{-\sigma,-l+1}|R(M_{-l+1})
\end{aligned}$$

and

$$\begin{aligned}
&(E\left(|S_{-\sigma,-l}-S_{-\sigma,-l+1}M_{-l+1}|^2\big|\omega,\sigma,Z_0,Z_{-1},\cdots,Z_{-\sigma}\right))^{\frac{\kappa}{2}}\\
&=(E\left(|S_{-\sigma,-l+1}|\big|\omega,\sigma,Z_0,Z_{-1},\cdots,Z_{-\sigma}\right))^{\frac{\kappa}{2}}R(M_{-l+1})^{\frac{\kappa}{2}}\\
&=\Big|Z_{-\sigma}\prod_{i=\sigma}^{l-2}M_{-i}\Big|^{\frac{\kappa}{2}}R(M_{-l+1})^{\frac{\kappa}{2}}.
\end{aligned}$$

Substituting to (63), we get that

$$\begin{aligned}
&P\Big(\Big|\sum_{m=\sigma}^{\infty}\Big(S_{-\sigma,-m}-Z_{-\sigma}\prod_{i=\sigma}^{m-1}M_{-i}\Big)\Big|\ge\epsilon x;\sigma<\nu\Big)\\
&\le x^{-\kappa}(\frac{2}{\epsilon})^{\kappa}K_7L^{2\kappa}E\Big(\sum_{l=\sigma+1}^{\infty}(l-\sigma)^{2\kappa}\Big|Z_{-\sigma}\prod_{i=\sigma}^{l-2}M_{-i}\Big|^{\frac{\kappa}{2}}R(M_{-l+1})^{\frac{\kappa}{2}};\sigma<\nu\Big)\\
&\le x^{-\kappa}(\frac{2}{\epsilon})^{\kappa}K_7L^{2\kappa}\sum_{m=1}^{\infty}\sum_{l=m+1}^{\infty}(l-m)^{2\kappa}E\Big(|Z_{-m}|^{\frac{\kappa}{2}}\Big\|\prod_{i=m}^{l-2}M_{-i}\Big\|^{\frac{\kappa}{2}}R(M_{-l+1})^{\frac{\kappa}{2}};m=\sigma<\nu\Big).
\end{aligned}$$

Again, using independence and stationarity, the rightmost-hand side of the above expression

$$\begin{aligned}
&=x^{-\kappa}(\frac{2}{\epsilon})^{\kappa}K_7L^{2\kappa}\\
&\quad\times\sum_{m=1}^{\infty}\sum_{l=m+1}^{\infty}(l-m)^{2\kappa}\mathbb{E}\Big(\Big\|\prod_{i=0}^{l-m-2}M_{-i}\Big\|^{\frac{\kappa}{2}}\Big)\mathbb{E}\left(R(M_0)^{\frac{\kappa}{2}}\right)E\Big(|Z_{-m}|^{\frac{\kappa}{2}};m=\sigma<\nu\Big)\\
&=x^{-\kappa}(\frac{2}{\epsilon})^{\kappa}K_7L^{2\kappa}\\
&\quad\times\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}l^{2\kappa}\mathbb{E}\Big(\Big\|\prod_{i=0}^{l-2}M_{-i}\Big\|^{\frac{\kappa}{2}}\Big)\mathbb{E}\left(R(M_0)^{\frac{\kappa}{2}}\right)E\Big(|Z_{-m}|^{\frac{\kappa}{2}};m=\sigma<\nu\Big)\\
&\qquad\text{using 1) and 2) of Theorem 5.2 and condition } \textbf{(C3)}\\
&\le K_8(\epsilon x)^{-\kappa}E(|Z_{-\sigma}|^{\frac{\kappa}{2}};\sigma<\nu)\le K_8(\epsilon x)^{-\kappa}A^{-\frac{\kappa}{2}}E(|Z_{-\sigma}|^{\kappa};\sigma<\nu)\\
&\le\epsilon x^{-\kappa}E(|Z_{-\sigma}|^{\kappa};\sigma<\nu)
\end{aligned}$$

for $A\ge A_1(\epsilon)$, and some $K_8>0$. □

Acknowledgements: The authors would like to thank Professor Zenghu Li for his useful discussions on the multitype branching processes, we also thank Ms. Hongyan Sun and Lin Zhang for the stimulating discussions. The project is partially supported by National Nature Science Foundation of China (Grant No.10721091) and NCET(No. 05-0143).

References

1. Alili, S.: Asymptotic behavior for random walks in random environments. *J. Appl. Prob. Vol. 36, pp. 334-349(1999)*

2. Bolthausen, E. and Goldsheid, I.: Recurrence and transience of random walks in random environments on a strip. *Comm. Math. Physics, 214, pp. 429-447(2000)*

3. Brémont, J.: On some random walks on $\mathbb{Z}$ in random medium. *Ann. prob. Vol. 30, No. 3, pp. 1266-1312(2002)*

4. Brémont, J.: Random walks on $\mathbb{Z}$ in random medium and Lyapunov spectrum. *Annales de l'I.H.P. Prob/Stat., Vol 40, No.3, pp. 309-336(2004)*

5. Feller, F.: An introduction to probability theory and its applications, *vol. II, 2nd ed. 1971*

6. Gendenko, B.V., Kolmogorov, A.N.: Limit distributions for sums of independent random variables. *Addison-Wesley Publ.Co. 1954*

7. Goldsheid, I.: Simple transient random walks in one-dimensional random environment. *Prob. Theory Relat. Fields 139, pp. 41-64(2007)*

8. Goldsheid, I.: Linear and sub-linear growth and the CLT for hitting times of a random walk in random environment on a strip. *Prob. Theory Related Fields 141, pp. 471-511(2008)*

9. Hughes, B.D.: Random walks and random environments. Volume 2: Random environments. *Clarendon Press, Oxford 1996*

10. Kesten, H.: Random difference equations and renewal theory of products of random matrices. *Acta Math.131, pp. 208-248(1973)*

11. Kesten, H., Kozlov, M.V., Spitzer, F.: A limit law for random walk in a random environment. *Compositio Math. 30, pp. 145-168(1975)*

12. Kesten, H.: A renewal theorem for random walk in a random environment. *Proceedings of Symposia in Pure Mathematics, Volume 31, pp. 67-77(1977)*

13. Key, E.S.: Recurrence and transience criteria for random walk in a random environment. *Ann. Prob. 12, pp. 529-560(1984)*

14. Key, E.S.: Limiting distributions and regeneration times for multitype branching processes with immigration in a random environment. *Ann. prob. Vol. 15, No. 1, pp. 344-353(1987)*

15. Kozlov, S.M.: The method of averaging and walks in inhomogeneous environments. *Russian Math. Surveys 40, pp. 73-145(1985)*

16. Lëtchikov, A. V.: A limit theorem for a recurrent random walk in a random environment. *Dokl. Akad. Nauk SSSR 304, pp. 25-28(1989); English transl. in Soviet Math. Dokl. 38.*

17. Lëtchikov, A. V.: A criterion for linear drift, and the central limit theorem for one-dimensional random walks in a random environment. *Russian Acad. Sci. Sb. Math. Vol. 79, No. 1, pp. 73-92(1994)*

18. Oseledec, V.L.: A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trudy Moskov. Mat. Obshch. 19, pp. 197-231(1968)*

19. Roitershtein, A.: Transient random walks on a strip in a random environment *Ann. Probab. Vol. 36, pp. 2354-2387(2008)*

20. Sznitman, A.S.: Lectures on random motions in random media. *In DMV seminar 32, Birkhauser, Basel 2002*

21. Zeitouni, O.: Random walks in random environment. *LNM 1837, J. Picard (Ed.), pp. 189-312, Springer-Verlag Berlin Heidelberg 2004*