Quenched moderate deviations principle for random walk in random environment*

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Abstract

We derive a quenched moderate deviations principle for a class of random walk in random environment, where the environment is assumed to be stationary and ergodic. The approach is based on hitting time decomposition. As a byproduct, we also get an explicit expression of the quenched variance of τ_1, the first passage time of 1 by the walk.

Key words: random walk in random environment, moderate deviation principle, large deviation principle.

AMS 2000 Subject Classification: Primary 60F15; secondary 60F10.

1. Introduction

Much attention has been focused on random walk in random environment (RWRE in short) in recent years. In one dimensional case, Solomon(see [6]) has derived the law of large numbers for random walk in i.i.d. environment, which has been generalized later to the stationary and ergodic setting , referring to Alili [1] and Zeitouni [7], where a central limit theorem has been got in different ways. And the large deviations principle (LDP in short) for RWRE has been proved by Grevenet et al [5] and Comets et al [2]. The main purpose of our present paper is to derive the moderate deviations principle (MDP in short) for RWRE in transient situation, whereas Comets and Popov (see [3]) considered the recurrent case (Sinai's walk).

Now we describe the model of interests to us. For any integer $i \in \mathbb{Z}$, denote the neighborhood of i by $N_i := \{i-1, i, i+1\}$. A probability measure ω_i on N_i is denoted by a triple $(\omega_i^-, \omega_i^0, \omega_i^+)$, where $\omega_i^-, \omega_i^0, \omega_i^+ \geq 0$ and $\omega_i^- + \omega_i^0 + \omega_i^+ = 1$. Let $M_1(N_i)$ be the collection of probability measures on N_i and equip $M_1(N_i)$ with the weak topology of probability measure, which makes it into a Polish space. Further this induces a Polish structure on $\Omega := \prod_{i\in\mathbb{Z}} M_1(N_i)$. Let $\mathcal{F}$ be the Borel σ-algebra on Ω. Given a probability measure P on $\mathcal{F}$, a *random environment* ω is an element of Ω distributed according to P.

We are now ready to define the class of random walks in random environment of interests to us. For each $\omega \in \Omega$, we define the *random walk in random environment* ω as the time-homogeneous Markov chain

*The project is partially supported by National Nature Science Foundation of China(Grant No.10721091) and NCET(No. 05-0143).

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$\{X_n\}$ taking values in $\mathbb{Z}^{\mathbb{N}}$ with transition probabilities

$$P_\omega\left(X_{n+1}=j \mid X_n=i\right)=\begin{cases}\omega_i^-, & if \quad j=i-1,\\ \omega_i^0, & if \quad j=i,\\ \omega_i^+, & if \quad j=i+1,\\ 0, & else.\end{cases}$$

For $v\in\mathbb{Z}$, we use P_ω^v to denote the law of $\{X_n\}$ on $(\mathbb{Z}^{\mathbb{N}},\mathcal{G})$, where $\mathcal{G}$ is the σ-algebra generated by the cylinder sets and $P_\omega^v(X_0=v)=1$. In this paper, we refer to $P_\omega^v(\cdot)$ as the *quenched* law of the random walk $\{X_n\}$. Note that for each $G\in\mathcal{G}$, the map $\omega\mapsto P_\omega^v(G)$ is $\mathcal{F}$-measurable. Therefore we may define the measure $\mathbb{P}^v:=P\otimes P_\omega^v$ on $(\Omega\times\mathbb{Z}^{\mathbb{N}},\mathcal{F}\times\mathcal{G})$ from the relation

$$\mathbb{P}^v(F\times G)=\int_F P_\omega^v(G)P(d\omega),\qquad F\in\mathcal{F},G\in\mathcal{G}.$$

The marginal of $\mathbb{P}^v$ on $\mathbb{Z}^{\mathbb{N}}$, denoted also by $\mathbb{P}^v$ whenever no confusion occurs, is called the *annealed* law of random walk $\{X_n\}$.

We introduce the hitting times of the walk $\{X_n\}$ which will serve us through out the paper. Let $T_0=0$, and

$$T_n=\min\{k:X_k=n\}$$

with the usual convention that the minimum over an empty set is ∞. Set $\tau_0=0$ and

$$\tau_n=T_n-T_{n-1},\ n\geq 1.$$

Similarly, set

$$T_{-n}=\min\{k:X_k=-n\}$$

and

$$\tau_{-n}=T_{-n}-T_{-n+1},\quad n\geq 1,$$

the convention being that $\tau_{\pm n}=\infty$ if $T_{\pm n}=\infty$. For fixed ω, whenever the walk is recurrent or transient to the right, it is easy to see, by the Markov property, that $\{\tau_n\}_{n=1}^\infty$ is an independent sequence under the quenched law $P_\omega^0(\cdot)$.

Let $\rho_i=\frac{\omega_i^-}{\omega_i^+}$. We introduce the following notations:

$\bar{S}=\sum_{i=1}^\infty\frac{1}{\omega_{(-i)}^+}\prod_{j=0}^{i-1}\rho_{(-j)}+\frac{1}{\omega_0^+}$;

$\bar{F}=\sum_{i=1}^\infty\frac{1}{\omega_i^-}\prod_{j=0}^{i-1}\rho_j^{-1}+\frac{1}{\omega_0^-}$;

$g(\omega)=E_\omega^0(\tau_1)$;

$\tilde{g}(\omega)=g(\omega)-\int_\Omega g(\omega)P(d\omega)$;

$\tilde{\tau}_k=\tau_k-E_\omega^0(\tau_k)=\tau_k-g(\theta^{k-1}\omega)$.

In the rest of the paper $\{b(n)\}$ is a sequence of numbers such that

$$\frac{b(n)}{\sqrt{n}}\to\infty,\ \text{and}\ \frac{b(n)}{n}\to 0$$

as $n\to\infty$. We will derive the quenched moderate deviations for T_n and X_n in the following Section 2 and Section 3 respectively.

2. Quenched MDP for T_n

Assumption (A)

(A1) *P is stationary, ergodic and uniformly elliptic, that is, $P(\omega_0^+ \geq \epsilon)P(\omega_0^- \geq \epsilon) = 1$, for some $\epsilon > 0$.*

(A2) $\rho_{max} := \sup[\rho : P(\rho_0 > \rho) > 0)] \in (0, 1)$. □

Remark 2.1 **1)** *It follows from (A1) that $E_P(\log \rho_0)$ is well defined. Then we have (see Zeitouni [6] lemma 2.1.12) $E_{\mathbb{P}^0}(\tau_1) = E_P\left(\bar{S}\right)$ and $E_{\mathbb{P}^0}(\tau_{-1}) = E_P\left(\bar{F}\right)$.*

2) *Under assumption (A) it follows from the definition of $\bar{S}$ that P-a.s. $\bar{S} \leq \epsilon^{-1}(1 + \sum_{i=1}^{\infty} \rho_{max}^i) = (\epsilon(1 - \rho_{max}))^{-1}$. Therefore $E_P\left(\bar{S}\right) < \infty$. Then we have (see Zeitouni [6], theorem 2.1.9) $\mathbb{P}^0$-a.s.,*

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{E_P\left(\bar{S}\right)} := v_P.$$

3) *If we use a new condition "(**B2**) $\rho_{min} := \inf[\rho : P(\rho_0 < \rho) > 0)] > 1$" instead of (A2), then an argument similar to above yields that $\mathbb{P}^0$-a.s.,*

$$\lim_{n\to\infty} \frac{X_n}{n} = -\frac{1}{E_P\left(\bar{F}\right)} := \tilde{v}_P.$$

□

We mention here that the rate functions of the MDP for both the hitting time and the walk are related to the second moment of τ_1. Hence we calculate the quenched second moment of τ_1 in the following proposition, but for the continuity consideration of our article, whose proof will be given as an appendix of the paper.

Proposition 2.1 *Denote*

$$A(\omega) =: 1 + \rho_{(-1)} + \rho_{(-1)}\rho_{(-2)} + \rho_{(-1)}\rho_{(-2)}\rho_{(-3)} + \cdots, \tag{1}$$

and

$$B(\omega) =: \frac{\omega_{(-1)}^0}{\omega_{(-1)}^+} + \rho_{(-1)}\frac{\omega_{(-2)}^0}{\omega_{(-2)}^+} + \rho_{(-1)}\rho_{(-2)}\frac{\omega_{(-3)}^0}{\omega_{(-3)}^+} + \cdots. \tag{2}$$

Then

$$\begin{aligned} E_\omega^0(\tilde{\tau}_1^2) &= V_\omega^0(W) = (\rho_0 + \rho_0^2)\left(2A(\omega) + B(\omega)\right)^2 + \frac{\omega_0^0}{\omega_0^+}\left(1 + \frac{\omega_0^0}{\omega_0^+}\right) + (2A(\omega) + B(\omega))\rho_0\frac{\omega_0^0}{\omega_0^+} \\ &+ \sum_{k=0}^{\infty}\left(\prod_{i=0}^{k}\rho_{(-i)}\left(\left(\rho_{(-k-1)} + \rho_{(-k-1)}^2\right)\left(2A(\theta^{-(k+1)}\omega) + B(\theta^{-(k+1)}\omega)\right)^2\right.\right. \\ &\left.\left. + \frac{\omega_{(-k-1)}^0}{\omega_{(-k-1)}^+}\left(1 + \frac{\omega_{(-k-1)}^0}{\omega_{(-k-1)}^+}\right) + \left(2A(\theta^{-(k+1)}\omega) + B(\theta^{-(k+1)}\omega)\right)\rho_{(-k-1)}\frac{\omega_{(-k-1)}^0}{\omega_{(-k-1)}^+}\right)\right). \end{aligned}$$

□

It follows from proposition 2.1 and assumption (A) that

$$E_\omega^0(\tilde{\tau}_1^2) \leq C + \frac{C\rho_{max}}{1 - \rho_{max}}, \text{ where } C = 2\left(\frac{2 + \epsilon^{-1}}{1 + \rho_{max}}\right)^2 + \frac{1}{\epsilon}\left(1 + \frac{1}{\epsilon}\right) + \left(\frac{2 + \epsilon^{-1}}{1 + \rho_{max}}\right)\frac{1}{\epsilon}.$$

Therefore $V_P(\tau_1) := E_P(E^0_\omega(\tilde{\tau}_1^2)) < \infty$. This enables us to define a finite function $G(\lambda, u) := \lambda u - \frac{\lambda^2}{2} V_P(\tau_1)$. Denote

$$I_P^{\tau,q}(x) := \sup_{\lambda \in R} G(\lambda, x) = \frac{x^2}{2V_P(\tau_1)}. \tag{3}$$

Similarly when $V_P(\tau_{-1}) := E_P(E^0_\omega(\tilde{\tau}_{-1}^2)) < \infty$, we define

$$I_P^{-\tau,q}(x) := \frac{x^2}{2V_P(\tau_{-1})}. \tag{4}$$

Theorem 2.1 *Suppose that assumption (A) holds. Then*

(a) for P-almost all environment ω, the random variables $\{\frac{1}{b(n)}(T_n - E^0_\omega(T_n))\}$ satisfy a LDP with speed $\frac{b(n)^2}{n}$ and a convex good rate function $I_P^{\tau,q}(x)$, where $I_P^{\tau,q}(x)$ is given in (3);

(b) furthermore, if the functional equation,

$$f - f \circ \theta = \tilde{g}, \tag{5}$$

has a bounded solution, then for P-almost all environment ω, the random variables $\{\frac{1}{b(n)}(T_n - nv_P^{-1})\}$ also satisfy a LDP with speed $\frac{b(n)^2}{n}$ and good rate function $I_P^{\tau,q}(x)$;

(c) if we use condition $(B2)$ instead of $(A2)$, then (a) and (b) hold with T_{-n} instead of T_n , $-\tilde{v}_P$ instead of v_P, and convex good rate function $I_P^{-\tau,q}$ in (4) instead of $I_P^{\tau,q}$.

Note that theorem 2.1 is actually a moderate deviations principle since the speed can vary without changing the rate function. The main tool we use to prove the theorem is Gärtner-Ellis theorem. Before proving the theorem, we give the following lemmas for preparation.

Lemma 2.1 *$I_P^{\tau,q}(x)$ is a convex good rate function and $I_P^{\tau,q}(0) = 0$.*

Proof. It is immediate from the expression of $I_P^{\tau,q}(x)$ in (3). □

Lemma 2.2 *Under assumption (A) there exist a $\lambda_{crit} > 0$ and a number $M_P < \infty$ such that P-a.s. $E^0_\omega(e^{\lambda\tau_1}) < M_P$ for all $\lambda < \lambda_{crit}$.*

Proof. For the proof of the lemma, see Comets et al [2], lemma 4. □

Lemma 2.3 *For $\lambda \in \mathbb{R}$ and $\omega \in \Omega$ define*

$$\Lambda_n(\lambda, \omega) = \frac{n}{b(n)^2} \log E^0_\omega\left(e^{\frac{\lambda b(n)}{n}(T_n - E^0_\omega(T_n))}\right).$$

Suppose that assumption (A) holds. Then P-a.s.,

$$\lim_{n\to\infty} \Lambda_n(\lambda, \omega) = \frac{\lambda^2}{2} V_P(\tau_1) =: \Lambda(\lambda).$$

Proof. Fix $\lambda \in \mathbb{R}$. Recall that $\tau_i, i \geq 1$ are independent under the quenched probability $P^0_\omega(\cdot)$. We have that P-a.s.,

$$\begin{aligned}\Lambda_n(\lambda,\omega) &= \frac{n}{b(n)^2}\log E^0_\omega\left(e^{\frac{\lambda b(n)}{n}(T_n - E^0_\omega(T_n))}\right) = \frac{n}{b(n)^2}\log E^0_\omega\left(e^{\frac{\lambda b(n)}{n}\sum_{i=1}^n \tilde{\tau}_i}\right)\\ &= \frac{n}{b(n)^2}\log\prod_{i=1}^n E^0_\omega\left(e^{\frac{\lambda b(n)}{n}\tilde{\tau}_i}\right) = \frac{n}{b(n)^2}\sum_{i=1}^n \log E^0_\omega\left(e^{\frac{\lambda b(n)}{n}\tilde{\tau}_i}\right)\\ &= \frac{n}{b(n)^2}\sum_{i=0}^{n-1}\log E^0_{\theta^i\omega}\left(e^{\frac{\lambda b(n)}{n}\left(\tau_1 - E^0_{\theta^i\omega}(\tau_1)\right)}\right).\end{aligned}$$

In the remainder of the proof of this lemma, we write $\tau_1 - E^0_{\theta^i\omega}(\tau_1)$ as $\bar{\tau}_1$ for simplicity. We also mention that $\bar{\tau}_1$ differs from $\tilde{\tau}_1$ which is $\tau_1 - E^0_\omega(\tau_1)$ by definition. Then we have

$$\Lambda_n(\lambda,\omega) = \frac{n}{b(n)^2}\sum_{i=0}^{n-1}\log E^0_{\theta^i\omega}\left(e^{\frac{\lambda b(n)}{n}\bar{\tau}_1}\right).$$

Since $\lim_{n\to 0}\frac{b(n)}{n} = 0$, lemma 2.2 implies that P-a.s. $E_\omega(e^{\frac{\lambda b(n)}{n}\tau_1}) < M_P$, for n large enough. Therefore for any $i \in \mathbb{Z}$, and n large enough there exists some $\xi_i \in (0,1)$ such that

$$\begin{aligned}E^0_{\theta^i\omega}\left(e^{\frac{\lambda b(n)}{n}\bar{\tau}_1}\right) &= E^0_{\theta^i\omega}\left(1 + \frac{\lambda b(n)}{n}\bar{\tau}_1 + \frac{\lambda^2 b(n)^2\bar{\tau}_1^2}{2n^2} + \frac{\lambda^3 b(n)^3}{6n^3}\bar{\tau}_1^3 e^{\frac{\xi_i\lambda b(n)}{n}\bar{\tau}_1}\right)\\ &= 1 + \frac{\lambda^2 b(n)^2}{2n^2}E^0_{\theta^i\omega}\left(\bar{\tau}_1^2\right) + \frac{\lambda^3 b(n)^3}{6n^3}E^0_{\theta^i\omega}\left(\bar{\tau}_1^3 e^{\frac{\xi_i\lambda b(n)}{n}\bar{\tau}_1}\right).\end{aligned}$$

Lemma 2.2 enables us to find a uniform bound $\bar{M}_P$ of $E^0_{\theta^i\omega}\left(|\bar{\tau}_1|^3 e^{\frac{\lambda_{crit}}{2}|\bar{\tau}_1|}\right)$. Define

$$\overline{\Lambda}_n(\lambda,\omega) := \frac{n}{b(n)^2}\sum_{i=0}^{n-1}\log\left(1 + \frac{\lambda^2 b(n)^2}{2n^2}E^0_{\theta^i\omega}\left(\bar{\tau}_1^2\right) + \frac{\lambda^3\bar{M}_P b(n)^3}{6n^3}\right)$$

and

$$\underline{\Lambda}_n(\lambda,\omega) := \frac{n}{b(n)^2}\sum_{i=0}^{n-1}\log\left(1 + \frac{\lambda^2 b(n)^2}{2n^2}E^0_{\theta^i\omega}\left(\bar{\tau}_1^2\right) - \frac{\lambda^3\bar{M}_P b(n)^3}{6n^3}\right).$$

Then for n large enough we have

$$\underline{\Lambda}_n(\lambda,\omega) \leq \Lambda_n(\lambda,\omega) \leq \overline{\Lambda}_n(\lambda,\omega). \tag{6}$$

Next we show that both $\overline{\Lambda}_n(\lambda,\omega)$ and $\underline{\Lambda}_n(\lambda,\omega)$ converge to $\frac{\lambda^2}{2}V_P(\tau_1)$ for P-a.s. ω as $n\to\infty$. In fact,

$$\left|\overline{\Lambda}_n(\lambda,\omega) - \frac{\lambda^2}{2}V_P(\tau_1)\right| \leq \left|\overline{\Lambda}_n(\lambda,\omega) - \frac{1}{n}\sum_{i=0}^{n-1}E^0_{\theta^i\omega}(\bar{\tau}_1^2)\frac{\lambda^2}{2}\right| + \left|\frac{1}{n}\sum_{i=0}^{n-1}E^0_{\theta^i\omega}(\bar{\tau}_1^2)\frac{\lambda^2}{2} - \frac{\lambda^2}{2}V_P(\tau_1)\right|.$$

Since Birkhoff's ergodic theorem implies that P-a.s., $\left|\frac{1}{n}\sum_{i=0}^{n-1}E^0_{\theta^i\omega}(\bar{\tau}_1^2)\frac{\lambda^2}{2} - \frac{\lambda^2}{2}V_P(\tau_1)\right| \to 0$ as $n\to\infty$, it is sufficient to show that P-a.s.,

$$\left|\overline{\Lambda}_n(\lambda,\omega) - \frac{1}{n}\sum_{i=0}^{n-1}E^0_{\theta^i\omega}(\bar{\tau}_1^2)\frac{\lambda^2}{2}\right| \to 0. \tag{7}$$

To see this,

$$
\begin{aligned}
&\Big|\overline{\Lambda}_n(\lambda,\omega) - \frac{1}{n}\sum_{i=0}^{n-1} E^0_{\theta^i\omega}(\bar\tau_1^2)\frac{\lambda^2}{2}\Big| \\
&\le \frac{1}{n}\sum_{i=0}^{n-1}\Big|\log\Big(1+\frac{\lambda^2 b(n)^2}{2n^2}E^0_{\theta^i\omega}(\bar\tau_1^2)+\frac{\lambda^3 b(n)^3}{6n^3}\bar M_P\Big)^{\frac{n^2}{b(n)^2}} - \frac{\lambda^2}{2}E^0_{\theta^i\omega}(\bar\tau_1^2)\Big| \\
&\qquad \text{writing } \frac{\lambda^2 b(n)^2}{2n^2}E^0_{\theta^i\omega}(\bar\tau_1^2)+\frac{\lambda^3 b(n)^3}{6n^3}\bar M_P \text{ as } S(\lambda,\theta^i\omega,n) \\
&\le \frac{1}{n}\sum_{i=0}^{n-1}\Big|\frac{\lambda^2}{2}E^0_{\theta^i\omega}(\bar\tau_1^2)\Big(\log(1+S(\lambda,\theta^i\omega,n))^{\frac{1}{S(\lambda,\theta^i\omega,n)}}-1\Big)\Big| \\
&+\frac{1}{n}\sum_{i=0}^{n-1}\frac{\lambda^3 b(n)}{6n}\bar M_P\log(1+S(\lambda,\theta^i\omega,n))^{\frac{1}{S(\lambda,\theta^i\omega,n)}}.
\end{aligned}
\tag{8}
$$

Since $(1+x)^{\frac{1}{x}} < e$ for all $x>0$, then the second term in the right-hand side of (8) is less than or equal to $\frac{\lambda^3 b(n)}{6n}\bar M_P \log e$ which is independent of ω and converges to 0 as $n\to\infty$.

To estimate the first term of the right-hand side of (8), note that

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=0}^{n-1}\Big|\frac{\lambda^2}{2}E^0_{\theta^i\omega}(\bar\tau_1^2)\Big(\log(1+S(\lambda,\theta^i\omega,n))^{\frac{1}{S(\lambda,\theta^i\omega,n)}}-1\Big)\Big| \\
&\quad= \frac{1}{n}\sum_{i=0}^{n-1}\frac{\lambda^2}{2}E^0_{\theta^i\omega}(\bar\tau_1^2)\Big(1-\log\big(1+S(\lambda,\theta^i\omega,n)\big)^{\frac{1}{S(\lambda,\theta^i\omega,n)}}\Big) \\
&\qquad\qquad \text{using the fact } (1+\tfrac{1}{x})^{\frac{1}{x}} < e < (1+\tfrac{1}{x})^{\frac{1}{x}+1} \text{ for all } x>0 \\
&\quad\le \frac{1}{n}\sum_{i=0}^{n-1}\frac{\lambda^2}{2}E^0_{\theta^i\omega}(\bar\tau_1^2)\Big(\log\big(1+S(\lambda,\theta^i\omega,n)\big)^{\frac{1}{S(\lambda,\theta^i\omega,n)}+1}-\log\big(1+S(\lambda,\theta^i\omega,n)\big)^{\frac{1}{S(\lambda,\theta^i\omega,n)}}\Big) \\
&\quad= \frac{1}{n}\sum_{i=0}^{n-1}\frac{\lambda^2}{2}E^0_{\theta^i\omega}(\bar\tau_1^2)\log\big(1+S(\lambda,\theta^i\omega,n)\big) \le \frac{1}{n}\sum_{i=0}^{n-1}\frac{\lambda^2}{2}E^0_{\theta^i\omega}(\bar\tau_1^2)S(\lambda,\theta^i\omega,n) \\
&\quad= \frac{1}{n}\sum_{i=0}^{n-1}\frac{\lambda^2}{2}E^0_{\theta^i\omega}(\bar\tau_1^2)\Big(\frac{\lambda^2 b(n)^2}{2n^2}E^0_{\theta^i\omega}(\bar\tau_1^2)+\frac{\lambda^3 b(n)^3}{6n^3}\bar M_P\Big),
\end{aligned}
$$

which converges to 0 P-a.s. by Birkhoff's ergodic theorem. Therefore (7) is proved. A similar proof yields that P-a.s.,

$$\Big|\underline{\Lambda}_n(\lambda,\omega) - \frac{1}{n}\sum_{i=0}^{n-1}E^0_{\theta^i\omega}(\bar\tau_1^2)\frac{\lambda^2}{2}\Big| \to 0$$

which together with (7) and (6) implies that P-a.s.,

$$\lim_{n\to\infty}\Lambda_n(\lambda,\omega) = \frac{\lambda^2}{2}V_P(\tau_1) = \Lambda(\lambda). \tag{9}$$

We emphasize that the P-null set in (9) may depend on λ. To deal with this, let Ω_1 be the set of ω such that (9) holds for all rational λ. Clearly $P(\Omega_1)=1$. For any $\lambda\in\mathbb{R}^+$, there exist rational sequences $\{s_i\}$ and $\{l_i\}$ such that s_i converges to λ increasingly and l_i converges to λ decreasingly.
Then for $\omega\in\Omega_1$,

$$\lim_{n\to\infty}\Lambda_n(s_i,\omega) \le \liminf_{n\to\infty}\Lambda_n(\lambda,\omega) \le \limsup_{n\to\infty}\Lambda_n(\lambda,\omega) \le \lim_{n\to\infty}\Lambda_n(l_i,\omega).$$

It follows that

$$\frac{s_i^2}{2}V_P(\tau_1) \le \liminf_{n\to\infty} \Lambda_n(\lambda,\omega) \le \limsup_{n\to\infty} \Lambda_n(\lambda,\omega) \le \frac{l_i^2}{2}V_P(\tau_1).$$

Let $i \to \infty$. We conclude that

$$\lim_{n\to\infty} \Lambda_n(\lambda,\omega) = \frac{\lambda^2}{2}V_P(\tau_1), \text{ for } \omega \in \Omega_1.$$

We can deal with $\lambda \in \mathbb{R}^-$ in the same way to complete the proof of the lemma. □

Proof of Theorem 2.1

We only give the proof of part (a) and part (b) of the theorem. Part (c) can be proved similarly by space reversal. Owning to lemma 2.1 and lemma 2.3, the proof of part (a) is just a simple application of Gärtner-Ellis theorem. We now prove part (b). Note that,

$$T_n - nv_P^{-1} = \sum_{i=1}^n (\tau_i - E_\omega^0(\tau_i)) + \sum_{i=1}^n (E_\omega^0(\tau_i) - v_p^{-1}).$$

Let $\gamma_n^\omega := \sum_{i=1}^n (E_\omega^0(\tau_i) - v_p^{-1}) = \sum_{i=0}^{n-1} \tilde{g}(\theta^k\omega)$. But by assumption, (5) has a bounded solution f. Hence γ_n^ω can be written as

$$\gamma_n^\omega = f(\omega) - f \circ \theta^n(\omega).$$

This implies in particular, that for all ω, $|\gamma_n^\omega| \le M$ for some constant $M > 0$. If we denote

$$\tilde{\Lambda}_n(\lambda,\omega) := \frac{n}{b(n)^2} \log E_\omega^0 \left(e^{\frac{\lambda b(n)}{n}(T_n - nv_P^{-1})} \right),$$

then

$$\begin{aligned}
\lim_{n\to\infty} \tilde{\Lambda}_n(\lambda,\omega) &= \lim_{n\to\infty} \frac{n}{b(n)^2} \log E_\omega^0 \left(e^{\frac{\lambda b(n)}{n}\left(\sum_{i=1}^n(\tau_i - E_\omega^0(\tau_i)) + \sum_{i=1}^n (E_\omega^0(\tau_i) - v_p^{-1})\right)} \right) \\
&= \lim_{n\to\infty} \frac{n}{b(n)^2} \log E_\omega^0 \left(e^{\frac{\lambda b(n)}{n}\sum_{i=1}^n(\tau_i - E_\omega^0(\tau_i))} e^{\frac{\lambda b(n)\gamma_n^\omega}{n}} \right) \\
&= \lim_{n\to\infty} \frac{n}{b(n)^2} \log E_\omega^0 \left(e^{\frac{\lambda b(n)}{n}\sum_{i=1}^n(\tau_i - E_\omega^0(\tau_i))} \right). \qquad (10)
\end{aligned}$$

Lemma 2.3 and (10) imply that

$$\lim_{n\to\infty} \tilde{\Lambda}_n(\lambda) = \Lambda(\lambda) \quad P\text{-}a.s.,$$

for any $\lambda \in \mathbb{R}$ where $\Lambda(\lambda) = \frac{\lambda^2}{2}V_P(\tau_1)$ by definition. Another application of Gärter-Ellis theorem yields part (b) of the theorem. Then the theorem is proved. □

3. Quenched MDP for X_n

Since a quenched MDP is proved for the sequence $\{T_n\}$, we can derive a quenched MDP for the walk $\{X_n\}$ itself. In the present section, let $b(n) = n^\beta$ particularly, where $\beta \in (\frac{1}{2}, 1)$.

Theorem 3.1 *Assume that assumption (A) holds and that (5) has a bounded solution. Define*

$$I_P^q(x) := v_P^{2\beta-1} I_P^{\tau,q}\left(-xv_P^{-(\beta+1)}\right) = \frac{x^2}{v_P^3 V_P(\tau_1)}, \text{ and } \tilde{I}_P^q(x) := \frac{x^2}{-\tilde{v}_P^3 V_P(\tau_{-1})},$$

where $\beta \in (\frac{1}{2}, 1)$. Then for any $A \in \mathscr{B}(\mathbb{R})$, P-a.s.,

$$\begin{aligned}
-I_P^q(A^\circ) &\le \liminf_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P_\omega^0\left(\frac{X_n - nv_P}{n^\beta} \in A\right) \\
&\le \limsup_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P_\omega^0\left(\frac{X_n - nv_P}{n^\beta} \in A\right) \le -I_P^q(\bar{A}),
\end{aligned}$$

where for $G \in \mathscr{B}(\mathbb{R})$, G^o denotes the interior of G, $\bar{G}$ denotes the closure of G and $I_P^q(G) = \inf_{x\in G} I_P^q(x)$; moreover if we use $(B2)$ instead of $(A2)$, the same result holds with $\tilde{v}_P$ instead of v_P, and $\tilde{I}_P^q(x)$ instead of $I_P^q(x)$.

Proof. We deal with only the first part. The second part can be proved by space reversal. Throughout, $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x. As a beginning, we prove the upper tail for the upper bound. Note that there is nothing to prove for $u \le 0$. Fix $u > 0$. For n large enough we have that,

$$\begin{aligned}
P_\omega^0\left(\frac{X_n - nv_P}{n^\beta} \ge u\right) &= P_\omega^0\left(X_n \ge n^\beta u + nv_P\right) \le P_\omega^0\left(T_{\lfloor n^\beta u + nv_P \rfloor} \le n\right) \\
&= P_\omega^0\left(\frac{T_{\lfloor n^\beta u + nv_P\rfloor} - \lfloor n^\beta u + nv_P \rfloor v_P^{-1}}{(\lfloor n^\beta u + nv_P \rfloor)^\beta} \le \frac{n - \lfloor n^\beta u + nv_P \rfloor v_P^{-1}}{(\lfloor n^\beta u + nv_P \rfloor)^\beta}\right) \\
&\le P_\omega^0\left(\frac{T_{\lfloor n^\beta u + nv_P\rfloor} - \lfloor n^\beta u + nv_P \rfloor v_P^{-1}}{(\lfloor n^\beta u + nv_P \rfloor)^\beta} \le \frac{-(n^\beta u - 1) v_P^{-1}}{(n^\beta u + nv_P + 1)^\beta}\right).
\end{aligned}$$

Taking logarithm, we have from theorem 2.1 that, P-a.s.,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P_\omega^0\left(\frac{X_n - nv_P}{n^\beta} \ge u\right) \\
&\le \limsup_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P_\omega^0\left(\frac{T_{\lfloor n^\beta u + nv_P\rfloor} - \lfloor n^\beta u + nv_P \rfloor v_P^{-1}}{(\lfloor n^\beta u + nv_P \rfloor)^\beta} \le \frac{-(n^\beta u - 1) v_P^{-1}}{(n^\beta u + nv_P + 1)^\beta}\right) \\
&\le \limsup_{n\to\infty} \frac{(\lfloor n^\beta u + nv_P \rfloor)^{2\beta-1}}{n^{2\beta-1}} \frac{1}{(\lfloor n^\beta u + nv_P \rfloor)^{2\beta-1}} \\
&\qquad \times \log P_\omega^0\left(\frac{T_{\lfloor n^\beta u + nv_P\rfloor} - \lfloor n^\beta u + nv_P \rfloor v_P^{-1}}{(\lfloor n^\beta u + nv_P \rfloor)^\beta} \le \frac{-(n^\beta u - 1) v_P^{-1}}{(n^\beta u + nv_P + 1)^\beta}\right) \\
&\le -v_P^{2\beta-1} \inf_{x \le -u v_P^{-(\beta+1)}} I_P^{\tau;q}(x) = -\inf_{x\ge u} I_P^q(x),
\end{aligned}$$

which completes the proof of the upper tail for the upper bound. We next derive the lower bound in a similar way. To do this, we claim first, for any $u \in \mathbb{R}$ and $0 < \eta < \delta$, that,

$$\{n - n^\beta \eta \le T_{\lfloor n^\beta u + nv_P \rfloor} \le n\} \subset \{nv_P + n^\beta(u - \delta) \le X_n \le n^\beta(u+\delta) + nv_P\}. \tag{11}$$

Indeed, note that,

$$\begin{aligned}
\{n - n^\beta \eta \le T_{\lfloor n^\beta u + nv_P \rfloor} \le n\} &\subset \{X_n \ge n^\beta u + nv_P - (n - T_{\lfloor n^\beta u + nv_P \rfloor})\} \\
&\subset \{X_n \ge n^\beta u + nv_P - (n - (n - n^\beta \eta))\} \\
&\subset \{X_n \ge nv_P + n^\beta(u - \eta)\} \subset \{X_n \ge nv_P + n^\beta(u-\delta)\},
\end{aligned}$$

and that,

$$\begin{aligned}
\{n - n^\beta \eta \le T_{\lfloor n^\beta u + nv_P \rfloor} \le n\} &\subset \{X_{\lfloor n - n^\beta \eta \rfloor} \le n^\beta u + nv_P\} \\
&\subset \{X_n \le n^\beta u + nv_P + n^\beta \eta\} \\
&\subset \{X_n \le n^\beta(u+\eta) + nv_P\} \subset \{X_n \le n^\beta(u+\delta) + nv_P\}.
\end{aligned}$$

Then we finish to prove (11). It follows from (11) and theorem 2.1 that, P-a.s.,

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n^{2\beta-1}}\log P_\omega^0\left(u-\delta<\frac{X_n-nv_P}{n^\beta}<u+\delta\right)\\
&\geq \liminf_{n\to\infty}\frac{1}{n^{2\beta-1}}\log P_\omega^0\left(n-n^\beta\eta\leq T_{\lfloor n^\beta u+nv_P\rfloor}\leq n\right)\\
&=\liminf_{n\to\infty}\frac{(\lfloor n^\beta u+nv_P\rfloor)^{2\beta-1}}{n^{2\beta-1}}\frac{1}{(\lfloor n^\beta u+nv_P\rfloor)^{2\beta-1}}\\
&\times\log P_\omega^0\left(\frac{n-n^\beta\eta-\lfloor n^\beta u+nv_P\rfloor v_P^{-1}}{(\lfloor n^\beta u+nv_P\rfloor)^\beta}\right.\\
&\qquad\left.\leq\frac{T_{\lfloor n^\beta u+nv_P\rfloor}-\lfloor n^\beta u+nv_P\rfloor v_P^{-1}}{(\lfloor n^\beta u+nv_P\rfloor)^\beta}\leq\frac{n-\lfloor n^\beta u+nv_P\rfloor v_P^{-1}}{(\lfloor n^\beta u+nv_P\rfloor)^\beta}\right)\\
&\geq -v_P^{2\beta-1}\inf_{x\in(-\eta v_P^{-\beta}-uv_P^{-(\beta+1)},-uv_P^{-(\beta+1)})}I_P^{\tau,q}(x),
\end{aligned}$$

where the last inequality follows from the lower bound of MDP for T_n. Moreover, since $\eta<\delta$ is arbitrary and since $I_P^{\tau,q}(\cdot)$ is continuous, letting $\eta\to 0$, we have that,

$$v_P^{2\beta-1}\inf_{x\in(-\eta v_P^{-\beta}-uv_P^{-(\beta+1)},-uv_P^{-(\beta+1)})}I_P^{\tau,q}(x)\to v_P^{2\beta-1}I_P^{\tau,q}(-uv_P^{-(\beta+1)})=I_P^q(u).$$

Therefore, P-a.s.,

$$\liminf_{n\to\infty}\frac{1}{n^{2\beta-1}}\log P_\omega^0\left(u-\delta<\frac{X_n-nv_P}{n^\beta}<u+\delta\right)\geq -I_P^q(u),$$

which completes the proof of the lower bound.

Next, we prove the lower tail for the upper bound to complete the proof of the theorem. For this purpose, we need some accurate estimation of $P_\omega^0\left(\inf_{l\geq 0}X_l\leq -i\right)$. Following Alili[1], lemma 5.3, we give this estimation in next lemma.

Lemma 3.1 *For all $\omega\in\Omega$, and all $i\in\mathbb{Z}$,*

$$P_\omega^0\left(\inf_{l\geq 0}X_l\leq -i\right)\leq\rho_{(-i+1)}\rho_{(-i+2)}\cdots\rho_0\sum_{k=0}^{\infty}\rho_1\ldots\rho_{k-1}\rho_k.$$

□

Now, we are ready to prove the lower tail for the upper bound. Note that there is nothing to prove for $u\geq 0$.

Fix $u<0$. Let $\alpha:=\frac{3\beta-1}{2}$. Note that for $\beta\in(\frac{1}{2},1)$ we always have $2\beta-1<\alpha<\beta$. Then, for n large enough,

$$\begin{aligned}
&P_\omega^0\left(\frac{X_n-nv_P}{n^\beta}\leq u\right)\leq P_\omega^0\left(\exists l\geq n: X_l\leq n^\beta u+nv_P\right)\\
&\leq P_\omega^0\left(T_{\lfloor n^\beta u+nv_P+n^\alpha\rfloor}\geq n\right)+P_\omega^0\left(\exists l\geq n: X_l\leq n^\beta u+nv_P, T_{\lfloor n^\beta u+nv_P+n^\alpha\rfloor}<n\right)\\
&\leq P_\omega^0\left(T_{\lfloor n^\beta u+nv_P+n^\alpha\rfloor}\geq n\right)\\
&\qquad+P_\omega^0\left(\inf_{l\geq T_{\lfloor n^\beta u+nv_P+n^\alpha\rfloor}}X_l\leq n^\beta u+nv_P, T_{\lfloor n^\beta u+nv_P+n^\alpha\rfloor}<n\right)\\
&\leq P_\omega^0\left(T_{\lfloor n^\beta u+nv_P+n^\alpha\rfloor}\geq n\right)\\
&\qquad+P_\omega^0\left(\inf_{l\geq T_{\lfloor n^\beta u+nv_P+n^\alpha\rfloor}}X_l-\lfloor n^\beta u+nv_P+n^\alpha\rfloor\leq -\lfloor n^\alpha\rfloor+1\right).
\end{aligned}\tag{12}$$

Since $\alpha < \beta$, the first probability in the right-hand side of (12) can be estimated by the MDP of T_n as

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P_\omega^0 \left(T_{\lfloor n^\beta u + nv_P + n^\alpha \rfloor} \geq n\right) \\
&\leq \limsup_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P_\omega^0 \Bigg(\frac{T_{\lfloor n^\beta u + nv_P + n^\alpha \rfloor} - \lfloor n^\beta u + nv_P + n^\alpha \rfloor v_P^{-1}}{(\lfloor n^\beta u + nv_P + n^\alpha \rfloor)^\beta} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \geq \frac{-(n^\beta u + n^\alpha + 1)v_P^{-1}}{(n^\beta u + nv_P + n^\alpha + 1)^\beta} \Bigg) \\
&\leq \limsup_{n\to\infty} \frac{(\lfloor n^\beta u + nv_P + n^\alpha \rfloor)^{2\beta-1}}{n^{2\beta-1}} \frac{1}{(\lfloor n^\beta u + nv_P + n^\alpha \rfloor)^{2\beta-1}} \\
&\qquad \times \log P_\omega^0 \left(\frac{T_{\lfloor n^\beta u + nv_P + n^\alpha \rfloor} - \lfloor n^\beta u + nv_P + n^\alpha \rfloor v_P^{-1}}{(\lfloor n^\beta u + nv_P + n^\alpha \rfloor)^\beta} \geq \frac{-(n^\beta u + n^\alpha + 1)v_P^{-1}}{(n^\beta u + nv_P + n^\alpha + 1)^\beta} \right) \\
&= -v_P^{2\beta-1} \inf_{x \geq -uv_P^{-(\beta+1)}} I_P^{\tau,q}(x) = -\inf_{x\leq u} I_P^q(x) \qquad P\text{-a.s.}.
\end{aligned} \tag{13}$$

We now turn to estimate the second probability in the right-hand side of (12).

$$\begin{aligned}
&P_\omega^0 \left(\inf_{l \geq T_{\lfloor n^\beta u + nv_P + n^\alpha \rfloor}} X_l - \lfloor n^\beta u + nv_P + n^\alpha \rfloor \leq -\lfloor n^\alpha \rfloor + 1 \right) \\
&= P^0_{\theta^{\lfloor n^\beta u + nv_P + n^\alpha \rfloor}\omega} \left(\inf_{l\geq 0} X_l \leq -\lfloor n^\alpha \rfloor + 1 \right).
\end{aligned}$$

Denote $c := c(n) = \lfloor n^\beta u + nv_P + n^\alpha \rfloor$. Then it follows from lemma 3.1 that, P-a.s.,

$$\begin{aligned}
&P^0_{\theta^{\lfloor n^\beta u + nv_P + n^\alpha \rfloor}\omega} \left(\inf_{l\geq 0} X_l \leq -\lfloor n^\alpha \rfloor + 1 \right) = P^0_{\theta^c\omega} \left(\inf_{l\geq 0} X_l \leq -\lfloor n^\alpha \rfloor + 1 \right) \\
&\leq \rho_{(-\lfloor n^\alpha \rfloor + 2 + c)} \rho_{(-\lfloor n^\alpha \rfloor + c + 1)} \cdots \rho_{(0+c)} \sum_{k=1}^\infty \rho_{(1+c)} \rho_{(2+c)} \cdots \rho_{(k+c)} \\
&\leq \rho_{max}^{\lfloor n^\alpha \rfloor - 1} \sum_{k=1}^\infty \rho_{max}^k = \frac{\rho_{max}^{\lfloor n^\alpha \rfloor}}{1 - \rho_{max}}.
\end{aligned}$$

Therefore P-a.s.,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P_\omega^0 \left(\inf_{l \geq T_{\lfloor n^\beta u + nv_P + n^\alpha \rfloor}} X_l - \lfloor n^\beta u + nv_P + n^\alpha \rfloor \leq -\lfloor n^\alpha \rfloor + 1 \right) \\
&\leq \limsup_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P^0_{\theta^{\lfloor n^\beta u + nv_P + n^\alpha \rfloor}\omega} \left(\inf_{l\geq 0} X_l \leq -\lfloor n^\alpha \rfloor + 1 \right) \\
&\leq \limsup_{n\to\infty} \frac{1}{n^{2\beta-1}} \log \frac{\rho_{max}^{\lfloor n^\alpha \rfloor}}{1-\rho_{max}} = \lim_{n\to\infty} \frac{n^\alpha}{n^{2\beta-1}} \log \rho_{max} \\
&= -\infty,
\end{aligned} \tag{14}$$

where the last equality holds because that $0 < 2\beta - 1 < \alpha$ and $0 < \rho_{max} < 1$. Then it follows(see Dembo and Zeitouni[4] lemma 1.2.15) from (12), (13) and (14) that P-a.s.,

$$\limsup_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P_\omega^0 \left(\frac{X_n - nv_P}{n^\beta} \leq u \right) \leq \max\{-\inf_{x\leq u} I_P^q(x), -\infty\} = -\inf_{x\leq u} I_P^q(x),$$

which completes the proof of the lower tail for the upper bound. Therefore, the theorem is proved. □

We can strengthen the result of theorem 3.1 due to the strict convexity and continuity of the rate function $I_P^q(x)$. We state it in the next corollary.

Corollary 3.1 *Assume that the conditions of theorem 3.1 hold. Then for any $u \in \mathbb{R}$, we have that P-a.s.,*

$$\lim_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P_\omega^0 \left(\frac{X_n - nv_P}{n^\beta} \le u \right) = - \inf_{x\le u} I_P^q(x),$$

and

$$\lim_{n\to\infty} \frac{1}{n^{2\beta-1}} \log P_\omega^0 \left(\frac{X_n - nv_P}{n^\beta} \ge u \right) = - \inf_{x\ge u} I_P^q(x).$$

□

Appendix A. On the second moment of T_1

Let us fix an environment ω. Our aim is to calculate $E_\omega^0\left(\tilde{\tau}_1^2\right)$, the variance of the first passage time T_1, given the environment ω.

For $i \le 0$, denote

$$N_i := \#\{k \in [0, T_1) : X_k = i\},$$
$$U_i := \#\{k \in [0, T_1) : X_k = i, X_{k+1} = i-1\}$$

and

$$Z_i := \#\{k \in [0, T_1) : X_k = i, X_{k+1} = i\}.$$

Moreover set $U_1 = 1$. Then it is easy to see that $N_i = U_i + U_{i+1} + Z_i$, and

$$T_1 = \sum_{i\le 0} N_i = \sum_{i\le 0} U_i + U_{i+1} + Z_i = 1 + \sum_{i\le 0} 2U_i + Z_i.$$

If we denote $W = \sum_{i\le 0} 2U_i + Z_i$, then $E_\omega^0\left(\tilde{\tau}_1^2\right) = E_\omega^0(\left(W - E_\omega^0(W \mid U_1 = 1)\right)^2) =: V_\omega^0(W)$. Therefore, we need only to calculate the variance of W.

Indeed, $\{(U_i, Z_i)\}_{i\le 1}$ with initial value $(U_1, Z_1) = (1, 0)$ forms a special two-type branching process in random environment with branching mechanisms

$$P_\omega^0\left((U_i, Z_i) = (m, k) | U_{i+1} = 1\right) = \binom{m+k}{m} (\omega_i^-)^m (\omega_i^0)^k \omega_i^+, \ i \le 0, \ m, k \ge 0.$$

Note that in this branching model, only type-1 particles can give birth to the next generation. Any particle of type-2 has no offspring. It is an easy task to find the marginal distributions of U_i and Z_i. In fact we have

$$P_\omega^0(U_i = m | U_{i+1} = 1) = \left(\frac{\omega_i^-}{\omega_i^- + \omega_i^+} \right)^m \frac{\omega_i^+}{\omega_i^- + \omega_i^+}$$

and

$$P_\omega^0(Z_i = k | U_{i+1} = 1) = \left(\frac{\omega_i^0}{\omega_i^0 + \omega_i^+} \right)^k \frac{\omega_i^+}{\omega_i^- + \omega_i^+}.$$

Therefore

$$\begin{aligned} E_\omega^0(U_i \mid U_{i+1}, \cdots, U_0) &= \rho_i U_{i+1}, \\ E_\omega^0(Z_i \mid U_{i+1}, \cdots, U_0) &= \frac{\omega_i^0}{\omega_i^+} U_{i+1}. \end{aligned} \tag{15}$$

Also by some easy calculations we have

$$E_\omega^0(U_i Z_i | U_{i+1} = 1) = 2\rho_i \frac{\omega_i^0}{\omega_i^+}, \tag{16}$$

and

$$V_\omega^0(U_i|U_{i+1}=1)=\rho_i(1+\rho_i),\ V_\omega^0(Z_i|U_{i+1}=1)=\frac{\omega_i^0}{\omega_i^+}\Big(1+\frac{\omega_i^0}{\omega_i^+}\Big). \tag{17}$$

To prove proposition, we give the following lemma.

Lemma 3.2 *For any fixed $\omega\in\Omega$, denote*

$$\tilde U_{-n}=U_{-n}-E_\omega^0(U_{-n}\mid U_{-n+1}),$$

$$\tilde Z_{-n}=Z_{-n}-E_\omega^0(Z_{-n}\mid U_{-n+1}).$$

Then we have that,

$$W-E_\omega^0(W\mid U_1)=\sum_{n=0}^\infty\Big(\tilde U_{-n}\left(2A(\theta^{-n}\omega)+B(\theta^{-n}\omega)\right)+\tilde Z_{-n}\Big).$$

Proof. If we denote $W_{-n}=\sum_{k=n}^\infty\left(2U_{-k}+Z_{-k}\right)$, we have for $n\geq 0$

$$2U_{-n}+Z_{-n}=W_{-n}-W_{-n-1}=E_\omega^0(W_{-n}-W_{-n-1}\mid U_{-n},Z_{-n}).$$

Then

$$W-E_\omega^0(W\mid U_1,Z_1)=\sum_{n=0}^\infty\left(E_\omega^0(W_{-n}\mid U_{-n},Z_{-n})-E_\omega^0(W_{-n}\mid U_{-n+1},Z_{-n+1})\right).$$

With $A(\omega)$ and $B(\omega)$ as in (1) and (2), it follows from (15) that

$$\begin{aligned}E_\omega^0(W_{-n}\mid U_{-n},Z_{-n})&=2U_{-n}(1+\rho_{(-n-1)}+\rho_{(-n-1)}\rho_{(-n-2)}+\cdots)+Z_{-n}\\&+U_{-n}\left(\frac{\omega^0_{(-n-1)}}{\omega^+_{(-n-1)}}+\rho_{(-n-1)}\frac{\omega^0_{(-n-2)}}{\omega^+_{(-n-2)}}+\rho_{(-n-1)}\rho_{(-n-2)}\frac{\omega^0_{(-n-3)}}{\omega^+_{(-n-3)}}+\cdots\right)\\&=2U_{-n}A(\theta^{-n}\omega)+U_{-n}B(\theta^{-n}\omega)+Z_{-n}\end{aligned}$$

and that

$$\begin{aligned}E_\omega^0(W_{-n}\mid U_{-n+1},Z_{-n+1})&=2U_{-n+1}\rho_{(-n)}(1+\rho_{(-n-1)}+\rho_{(-n-1)}\rho_{(-n-2)}+\cdots)+U_{-n+1}\frac{\omega^0_{(-n)}}{\omega^+_{(-n)}}\\&+U_{-n+1}\rho_{(-n)}\left(\frac{\omega^0_{(-n-1)}}{\omega^+_{(-n-1)}}+\rho_{(-n-1)}\frac{\omega^0_{(-n-2)}}{\omega^+_{(-n-2)}}+\rho_{(-n-1)}\rho_{(-n-2)}\frac{\omega^0_{(-n-3)}}{\omega^+_{(-n-3)}}+\cdots\right)\\&=2U_{-n+1}\rho_{(-n)}A(\theta^{-n}\omega)+U_{-n}\rho_{(-n)}B(\theta^{-n}\omega)+U_{-n+1}\frac{\omega^0_{(-n)}}{\omega^+_{(-n)}}.\end{aligned}$$

Then

$$\begin{aligned}W-E_\omega^0(W\mid U_1,Z_1)&=\sum_{n=0}^\infty\left(E_\omega^0(W_{-n}\mid U_{-n},Z_{-n})-E_\omega^0(W_{-n}\mid U_{-n+1},Z_{-n+1})\right)\\&=\sum_{n=0}^\infty\left(2A(\theta^{-n}\omega)+B(\theta^{-n}\omega)\right)\left(U_{-n}-E_\omega^0(U_{-n}\mid U_{-n+1})\right)\\&+\sum_{n=0}^\infty\left(Z_{-n}-E_\omega^0(Z_{-n}\mid U_{-n+1})\right).\end{aligned}$$

Therefore the lemma is proved. □

Now we come to the proof of proposition 2.1. Note that

$$\begin{aligned}
V_\omega^0(W) &= E_\omega^0((W - E_\omega^0(W \mid U_1))^2) = V_\omega^0\left(\tilde{Z}_0 + \tilde{U}_0\left(2A(\omega) + B(\omega)\right)\right) \\
&+E_\omega^0\left(V_\omega^0\left(\sum_{n=1}^{\infty}\left(\tilde{U}_{-n}\left(2A(\theta^{-n}\omega) + B(\theta^{-n}\omega)\right) + \tilde{Z}_{-n}\right) \mid U_0, Z_0\right)\right) \\
&+2E_\omega^0\left(E_\omega^0\left(\left(\tilde{Z}_0 + \tilde{U}_0\left(2A(\omega) + B(\omega)\right)\right)\right.\right. \\
&\quad\left.\left.\times\sum_{n=1}^{\infty}\left(\tilde{U}_{-n}\left(2A(\theta^{-n}\omega) + B(\theta^{-n}\omega)\right) + \tilde{Z}_{-n}\right) \mid U_0, Z_0\right)\right).
\end{aligned}$$

It is easy to see that the third term in the right-hand side of last equation is zero. Therefore

$$\begin{aligned}
&V_\omega^0(W) = V_\omega^0\left(\tilde{Z}_0 + \tilde{U}_0\left(2A(\omega) + B(\omega)\right)\right) \\
&+E_\omega^0\left(U_0 V_\omega^0\left(\sum_{n=1}^{\infty}\left(\tilde{U}_{-n}\left(2A(\theta^{-n}\omega) + B(\theta^{-n}\omega)\right) + \tilde{Z}_{-n}\right) \mid U_0 = 1\right)\right) \\
&= (\rho_0 + \rho_0^2)\left(2A(\omega) + B\left(\omega\right)\right)^2 + \frac{\omega_0^0}{\omega_0^+}\left(1 + \frac{\omega_0^0}{\omega_0^+}\right) + (2A(\omega) + B(\omega))\rho_0\frac{\omega_0^0}{\omega_0^+} + \rho_0 V_{\theta^{-1}\omega}^0(W), \quad (18)
\end{aligned}$$

where the last equality follows from (15),(16) and (17). Then it follows by induction from (18) that

$$\begin{aligned}
E_\omega^0(\tilde{\tau}_1^2) &= V_\omega^0(W) = (\rho_0 + \rho_0^2)\left(2A(\omega) + B\left(\omega\right)\right)^2 + \frac{\omega_0^0}{\omega_0^+}\left(1 + \frac{\omega_0^0}{\omega_0^+}\right) + (2A(\omega) + B(\omega))\rho_0\frac{\omega_0^0}{\omega_0^+} \\
&+ \sum_{k=0}^{\infty}\left(\prod_{i=0}^{k}\rho_{(-i)}\left(\left(\rho_{(-k-1)} + \rho_{(-k-1)}^2\right)\left(2A(\theta^{-(k+1)}\omega) + B(\theta^{-(k+1)}\omega)\right)^2\right.\right. \\
&\quad\left.\left.+\frac{\omega_{(-k-1)}^0}{\omega_{(-k-1)}^+}\left(1 + \frac{\omega_{(-k-1)}^0}{\omega_{(-k-1)}^+}\right) + \left(2A(\theta^{-(k+1)}\omega) + B(\theta^{-(k+1)}\omega)\right)\rho_{(-k-1)}\frac{\omega_{(-k-1)}^0}{\omega_{(-k-1)}^+}\right)\right)
\end{aligned}$$

which proves proposition 2.1. □

Acknowledgements

The authors would like to thank professor Ofer Zeitouni for his valuable suggestions and discussions.

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