Alternative Proof for the Recurrence and Transience of Random Walks in Random Environment with Bounded Jumps*

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Abstract

We derive a new proof of a recurrence and transience criteria for a class of random walks in random environment with bounded jumps, where the environment is assumed to be stationary and ergodic. Martingale convergence method is used in this paper, comparing the original one (Brémont (2002)) by the method of computing the exit probability.

Key words: random walks in random environment, martingale convergence theorem, recurrence, transience.

1. Introduction

Random walks in random environment (RWRE, for short) have attracted much intention in recent years, we refer for example Zeitouni ([9], 2004) or Sznitman ([8], 2004) as a general introduction. In one dimension, many results have been obtained, especially for the recurrence and transience criteria. Solomon ([7], 1975) has derived the recurrence and transience criterion for random walks in i.i.d environment with nearest jump, and has been generalized later to the case that the environment is stationary and ergodic, referring, for example, to Alili ([1],1999) and Zeitouni [9]. Key ([6], 1975) proved a recurrence criterion for random walks in i.i.d environment with bounded jump, involving the Rth and $(R+1)$th Lyapunov exponents with respect to a random matrix M of dimension $(R+L)\times(R+L)$ built with the environment. In ([3]), Brémont (2002) proved a recurrence criterion for RWRE with bounded jump (one nearest step to the right) only involving the largest Lyapunov exponent with respect to a random matrix M of dimension $L\times L$ by the method of computing the exit probability. In this note, we will give a new proof for the Brémont's recurrence criterion by martingale convergence method, which is stimulated by Sznitman ([8], 2002) in the case of nearest jump. The key step is to construct a martingale, we get it intuitively from the point of view "resistor" of the electric networks.

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We introduce the model briefly at first, and we adapt the notation as in Brémont ([3]). Let $(\Omega, \mathcal{F}, \mu, T)$ be an invertible dynamical system, that is, a probability space $(\Omega, \mathcal{F}, \mu)$ with an invertible transformation T, measurable as well as its inverse and preserving μ. We suppose that T is ergodic with respect to μ. The space Ω is interpreted as the space of environments.

Let $L \geq 1$ and $R \geq 1$ be two fixed integers and introduce the set of consecutive integers $\Lambda = \{-L, \cdots, R\}$ which will be the set of possible jumps of the random walks. Let $(p_z)_{z\in\Lambda}$ be a collection of positive random variables on $(\Omega, \mathcal{F})$ satisfying $\sum_{z\in\Lambda} p_z(\omega) = 1$, μ-a.e., and an ellipticity condition:

$$\exists \varepsilon > 0, \forall z \in \Lambda \text{ and } z \neq 0, \qquad (p_z/p_R) \geq \varepsilon,\ \ \mu - a.e. \tag{1}$$

For a fixed medium ω, let $\{X_n\}_{n\geq 0}$ be the Markov chain on $\mathbb{Z}$ defined by $X_0 = 0$ and the transition probabilities

$$\forall x \in \mathbb{Z}, \qquad P_\omega(X_{n+1} = x + z | X_n = x) := p_z(T^x\omega).$$

Let P_ω^x be the measure induced by $X_n(\omega)_{n\geq 0}$ with $X_0 = x$ on the space of jumps $\Lambda^{\mathbb{N}}$, called the "quenched" probability, in contrast to the "annealed" probability $\int_\Omega P_\omega d\mu(\omega)$. E_ω^x denote the expectation under the probability P_ω^x, and write P_ω (respectively E_ω) when $x = 0$.

We present a few conventions. In the rest of the paper the dependence on ω will always be implicit. Any expression of the form $f(T^k\omega)$ will simply be denoted by $T^k f$ or $f(k)$.

In what follows, we will restrict $R = 1$. For $1 \leq i \leq L$, introduce the quantities

$$a_i = (p_{-i} + \cdots + p_{-L})/p_1. \tag{2}$$

We define the following invertible nonnegative random matrices of size $L \times L$ that will play a central role in this paper:

$$M := \begin{pmatrix} a_1 & \cdots & a_{L-1} & a_L \\ 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

For $k \geq l \in \mathbb{Z}$, denote $M(k,l) = M(k) \cdots M(l)$, recall that $M(k) = T^k M$. We write $(e_i)_{1\leq i\leq L}$ for the canonical basis of $\mathbb{R}^L$. And introduce the notation:

$$\delta(0,k) := \begin{cases} \langle e_1,\ M(k,0)e_1\rangle, & if \quad k \geq\ 0, \\ \quad 1, & if \quad k = -1 \end{cases} \tag{3}$$

Let $\gamma(M,T)$ be the largest Liapounov exponents of M with respect to $(\Omega, \mathcal{F}, \mu, T)$. Consider $\mathbb{R}^L$ with its canonical basis and 1-norm, that is, $\|x\| = \sum_{i=1}^L |x_i|$, for $x = (x_i)_{1\leq i\leq n}$. Introduce the cone $C = \{x \in \mathbb{R}^L, x_i > 0\}$ and its intersection with the sphere: $B = C \cap \{x \in \mathbb{R}^L, \|x\| = 1\}$.

We note that the product of L matrices of the same form as M has strictly positive entries. Many properties of nonnegative matrices were stated in Brémont ([3]), we summarize it as following

Proposition 1.1 *(Brémont ([3]))*

(i) $\forall x \in B$, *μ-a.e.*

$$\gamma(M,T) = \lim_{n\to\infty} \frac{1}{n} \log ||M(n-1,0)x||.$$

(ii)There is a unique random vector V in B and a unique scalar map λ such that $MV = \lambda TV$. Moreover $M(n-1,0)V = (T^{n-1}\lambda \cdots \lambda)T^n V$, and

$$\gamma(M,T) = \int \log(\lambda) d\mu.$$

(iii) There exists a constant $C > 0$ such that ("$\preceq$" for the partial order on R^L)

$$\frac{1}{C} M(L-1,0)e_1 \preceq M(L-1,0)V \preceq CM(L-1,0)e_1$$

(iv) The vector V is strictly in the interior of the positive cone of $\mathbb{R}^L$,uniformly on ω; that is,there exists a constant $\delta > 0$ such that $\langle e_i, V\rangle \geq \delta$ for $1 \leq i \leq L$. Consequently, there exists a constant $C > 0$ such that, for all $k \geq 0$,

$$\frac{1}{C}(T^k\lambda \cdots T^0\lambda) \leq \delta(0,k) \leq C(T^k\lambda \cdots T^0\lambda)$$

□

A recurrence and transience criterion for the RWRE with bounded jumps has been proved in Brémont ([3]) by computing the exit probability.

Theorem 1.1 *(Brémont ([3])) The asymptotic behavior of random walks is the following:*

(i) If $E\log\lambda < 0$, then $\lim_{n\to\infty} X_n(\omega) = +\infty$, P_ω-a.e., μ-a.e.

(ii) If $E\log\lambda > 0$, then $\lim_{n\to\infty} X_n(\omega) = -\infty$, P_ω-a.e., μ-a.e.

(iii) If $E\log\lambda = 0$, then $-\infty = \liminf_{n\to\infty} X_n(\omega) < \limsup_{n\to\infty} X_n(\omega) = +\infty$, P_ω-a.e., μ-a.e.

2. Alternative proof of Theorem 1.1

Brémont ([3]) introduced $\delta(0,k)$ in (3) for $k \geq -1$. To formulate the martingale, we will first extend the definition of $\delta(0,k)$ in (3) from $k \geq -1$ to the whole line, this is the key step in our proof. Intuitively, $\delta(0,k)$ is the "resistor" between k and $k+1$ in the electric networks for $k \geq -1$, from this point of view it is natural to extend $\delta(0,k)$ to the whole line as following,

$$\delta(0,k) := \begin{cases} \langle e_1,\ M(k,0)e_1\rangle, & if \quad k \geq\ 0, \\ \quad 1, & if \quad k = -1 \\ \langle e_1,\ M(-1,k+1)^{-1}e_1\rangle, & if \quad k \leq -2, \end{cases} \tag{4}$$

We note that the product of L matrices of the same form as M^{-1} has strictly positive entries. Thus we can prove some properties of $\delta(0,k)$ for $k \leq -2$.

Proposition 2.1 *(i) For $-L \leq k \leq -2$,*

$$\delta(0,k) = 0$$

(ii) For $k < -L$,

$$\frac{1}{C}(T^k\lambda^{-1} \cdots T^{-1}\lambda^{-1}) \leq \delta(0,k) \leq C(T^k\lambda^{-1} \cdots T^{-1}\lambda^{-1})$$

PROOF. (i) By simple calculation we can get the form of M^{-1} :

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ * & * & * & \cdots & * \end{pmatrix} \quad \text{where } * \text{ is positive.}$$

So the product of two matrices of the same form as M^{-1} has the following form:

$$\begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \end{pmatrix} \quad \text{where } * \text{ is positive.}$$

Then we can get that the top left corner of the product of $|k|$ matrices of the same form as M^{-1} should always be 0 when $-L \leq k \leq -2$, that is $\delta(0,k) = 0$.

(ii) At first note that for $k < -L$, $\delta(0,k) > 0$. Since $MV = \lambda TV$, let $V' = TV$, so we have $M^{-1}V' = \lambda^{-1}T^{-1}V'$. Replace V' by V, that is, $M^{-1}V = \lambda^{-1}T^{-1}V$. We can then obtain some properties for (M^{-1}, T^{-1}) as in Proposition 1.1 for (M,T) . Then we can get

$$\frac{1}{C}(T^k\lambda^{-1}\cdots T^{-1}\lambda^{-1}) \leq \delta(0,k) \leq C(T^k\lambda^{-1}\cdots T^{-1}\lambda^{-1}). \quad \square$$

Next lemma was proved in Atkinson [2], which will be used in proving the recurrent situation.

Lemma 2.1 *Let $(\Omega, \mathcal{F}, \mu, T)$ be an ergodic dynamical system and $\phi \in L^1(\mu)$. If $\int \phi \, d\mu = 0$, then μ-a.e.*

$$\exists \; n_i(\omega) \to +\infty, \qquad \sum_{k=0}^{n_i(\omega)-1} \phi(T^k\omega) \to 0 \text{ as } i \to +\infty.$$

2.1 Construction of a proper martingale

Stimulated by Sznitman ([8]) in the case of nearest jump. For $\forall$ environment $\omega \in \Omega$, we define

$$f(x,\omega) := \begin{cases} -\sum\limits_{z=-1}^{x-1} \delta(0,z), & if \quad x \geq -1, \\ \sum\limits_{z=x}^{-2} \delta(0,z), & if \quad x \leq -2, \end{cases} .$$

By convention $\sum_r^s = 0$ when $s < r$. Then we have the following conclusion which will play an important role when we prove the recurrence and transience property of RWRE.

Proposition 2.2 *For any fixed environment $\omega \in \Omega$, $f(X_n, \omega)$ is a martingale under P_ω, where X_n is the random walk determined by ω.*

PROOF. To prove $f(X_n,\omega)$ is a martingale, i.e., $E_\omega[f(X_{n+1},\omega)|\mathscr{F}_n] = f(X_n,\omega)$. It suffices to prove $E_\omega^{f(x,\omega)}[f(X_1,\omega)] = f(x,\omega)$ because of the Markov property and homogeneous. That is,

$$p_1(x)f(x+1,\omega)+\sum_{j=1}^{L}p_{-j}(x)f(x-j,\omega)=f(x,\omega), \tag{5}$$

which is

$$p_1(x)[f(x+1,\omega)-f(x,\omega)]+\sum_{j=1}^{L}p_{-j}(x)[f(x-j,\omega)-f(x,\omega)]=0, \tag{6}$$

by the definition of f and a simple calculation, (6) is equivalent to

$$p_1(x)\delta(0,x)=(p_{-1}(x)+\cdots+p_{-L}(x))\delta(0,x-1)+\cdots+(p_{-L+1}(n)+p_{-L}(x))\delta(0,x-L+1)+p_{-L}(x)\delta(0,x-L)$$

By the definition in (2), it suffice to prove

$$\delta(0,x)=a_1(x)\delta(0,x-1)+\cdots+a_L(x)\delta(0,x-L). \tag{7}$$

proof of (7): a) For $x>-1$,

(a1) we can first consider $x=n\geq L$,

$$\begin{aligned}
\delta(0,n) &= \langle e_1, M(n,0)e_1\rangle \\
&= \langle e_1, M(n)M(n-1,0)e_1\rangle \\
&= \langle e_1, \left[\begin{pmatrix} a_1(n) & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_2(n) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & \cdots & a_L(n) \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right] \\
&\cdot \quad M(n-1,0)e_1\rangle \\
&= a_1(n)\delta(0,n-1)+\langle e_1, \begin{pmatrix} 0 & a_2(n) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} M(n-1)\cdot M(n-2,0)e_1\rangle \qquad (8) \\
&+ \quad \cdots + \langle e_1, \begin{pmatrix} 0 & 0 & \cdots & a_L(n) \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cdot M(n-1,0)e_1\rangle \\
&= a_1(n)\delta(0,n-1)+\langle e_1, \begin{pmatrix} a_2(n) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} M(n-2,0)e_1\rangle \qquad (9) \\
&+ \quad \cdots + \langle e_1, \begin{pmatrix} 0 & 0 & \cdots & a_L(n) \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cdot M(n-1,0)e_1\rangle \qquad (10) \\
&= \cdots\cdots \\
&= a_1(n)\delta(0,n-1)+a_2(n)\delta(0,n-2)+\cdots+a_L(n)\delta(0,n-L). \qquad (11)
\end{aligned}$$

Conclusion (11) is due to the multiplying operations of matrices as the form of M, which can drag a_i to the left top corner position.

(a2) If $-1 < x = n < L$, then we can also prove (7) similarly as the above because of (i) in Proposition 2.1.

b) For $x \leq -2, \quad x \in \mathbb{Z}$.

We can rewrite $\delta(0,x)$ as the following,

$$\begin{aligned}
\delta(0,x) &= \langle e_1, (M_{-1}\cdots M_{x+1})^{-1}e_1\rangle \\
&= \langle e_1, M_x M_{x-1}\cdots M_{x-L}(M_{-1}\cdots M_{x-L})^{-1}e_1\rangle \qquad (12)\\
\delta(0,x-1) &= \langle e_1, (M_{-1}\cdots M_{x})^{-1}e_1\rangle \\
&= \langle e_1, M_{x-1}\cdots M_{x-L}(M_{-1}\cdots M_{x-L})^{-1}e_1\rangle \qquad (13)\\
&\cdots\cdots\cdots \\
\delta(0,x-L) &= \langle e_1, (M_{-1}\cdots M_{x-L+1})^{-1}e_1\rangle \\
&= \langle e_1, M_{x-L}(M_{-1}\cdots M_{x-L})^{-1}e_1\rangle. \qquad (14)
\end{aligned}$$

Then we can also decompose $\delta(0,x)$ by the same procedures as in a) to get (7), for example,

$$\begin{aligned}
\delta(0,x) &= \langle e_1, M_x\cdot M_{x-1}\cdots M_{x-L}(M_{-1}\cdots M_{x-L})^{-1}e_1\rangle \\
&= \langle e_1, \left[\begin{pmatrix} a_1(x) & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_2(x) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & \cdots & a_L(x) \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right] \\
&\cdot \quad M_{x-1}\cdots M_{x-L}(M_{-1}\cdots M_{x-L})^{-1}e_1\rangle \\
&\cdots\cdots\cdots \\
&= a_1(x)\delta(0,x-1) + a_2(x)\delta(0,x-2) + \cdots + a_L(x)\delta(0,x-L).
\end{aligned}$$

c) For $x = -1$, we need only to note that $\delta(0,-1-L) = \langle e_1,\ M(-1,-L)^{-1}e_1\rangle = \frac{1}{a_L(-1)}$. Thus (7) is proved. $\square$

2.2 Proof of Theorem 1.1

PROOF. (i)When $E\log\lambda < 0$. For large $x > 0$, $f(x,\omega)$ is nonincreasing in x,

$$\begin{aligned}
f(x,\omega) &= -\sum_{z=-1}^{x-1}\delta(0,z) \\
&\geq -C(\sum_{z=0}^{x-1} T^z\lambda\cdots T^0\lambda + 1) \\
&= -C(\sum_{z=0}^{x-1} e^{\sum_{i=0}^{z}\log T^i\lambda} + 1) \\
&= -C(\sum_{z=0}^{x-1} e^{z(E\log\lambda + o(1))} + 1), \qquad (15)
\end{aligned}$$

$\mu - a.s.\omega$, where the inequality is due to Proposition 1.1 and (15) is due to Birkhoff's ergodic theorem. The series converge as $x \to +\infty$ when $E\log\lambda < 0$, then f is lower-bounded. Thus, $\exists$ finite $K(\omega) > 0$, such that $\mu - a.s.\omega$,

$$\lim_{x\to+\infty} f(x,\omega) = -K(\omega).$$

Similarly, $\mu - a.s.\omega$,

$$\begin{aligned}
\lim_{x\to-\infty} f(x,\omega) &= \lim_{x\to-\infty} \sum_{z=x}^{-2} \delta(0,z) \\
&\geq \lim_{x\to-\infty} \frac{1}{C} (\sum_{z=x}^{-2} T^z\lambda^{-1} \cdots T^{-1}\lambda^{-1}) \\
&= \lim_{x\to-\infty} \frac{1}{C} (\sum_{z=x}^{-2} e^{\sum_{i=1}^{z} \log T^{-i}\lambda^{-1}}) \\
&= \lim_{x\to-\infty} \frac{1}{C} \sum_{z=x}^{-2} e^{-|z|(E\log\lambda + o(1))} && (16) \\
&= +\infty. && (17)
\end{aligned}$$

With Proposition 2.2, we know $f(X_n,\omega)$ is a martingale under P_ω. By martingale convergence theorem, we have $f(X_n,\omega)$ converges to a finite limit, P_ω-a.e., thus X_n must has a limit (maybe ∞), as $n\to+\infty$, P_ω-a.e., μ-a.e, because of the explicit expression of $f(x,\omega)$. We can conclude that $X_n \to +\infty$, P_ω-a.e., μ-a.e. (If $X_n \to -\infty$, P_ω-a.e., μ-a.e., it contradicts (17). If $X_n \to$ a finite limit, P_ω-a.e., μ-a.e., it contradicts (1)).

(ii) Analogously we obtain the conclusion as the discussion in (i) when $E\log\lambda > 0$.

(iii) When $E\log\lambda = 0$,

$$\begin{aligned}
\lim_{x\to+\infty} f(x,\omega) &= -\lim_{x\to+\infty} \sum_{z=-1}^{x-1} \delta(0,z) \\
&\leq -\frac{1}{C} \lim_{x\to+\infty} \sum_{z=0}^{x-1} T^z\lambda \cdots T^\lambda T^0\lambda \\
&= -\frac{1}{C} \lim_{x\to+\infty} \sum_{z=0}^{x-1} e^{\sum_{i=0}^{z} \log T^i\lambda},
\end{aligned}$$

where the inequality is due to Proposition 1.1. Since $E\log\lambda = 0$, by Lemma 2.1, μ a.e. $\exists\, z(i) \to +\infty$, s.t.

$$\sum_{i=1}^{z(i)} \log T^i\lambda \to 0, \text{ as } i \to +\infty$$

Hence $\lim_{x\to+\infty} f(x,\omega) = -\infty$, μ-a.e.. Similarly, $\lim_{x\to-\infty} f(x,\omega) = \infty$, μ-a.e..
For $\forall A \geq 0$, Let

$$T = \inf\{k \geq 0, X_k \geq A\} \text{ and } S = \inf\{k \geq 0, X_k \leq -A\}.$$

Since $f(X_{n\wedge T},\omega)$ is a martingale, by martingale convergence theorem, P_ω-a.e.,

$$f(x_{n\wedge T},\omega) \to \text{ a finite limit, as } n \to \infty.$$

As discussed in (i), we get $X_{n\wedge T} \to$ a finite limit, P_ω-a.e. as $n \to \infty$. If $T = +\infty$, that is, $x_n \to$ a finite limit. So $\exists$ large n, s.t. $p_j(n)|_{j\in\Lambda} = o$, which contradicts (1). Hence $T < +\infty$. Similarly, $S < +\infty$. That is, $P_\omega(T < +\infty$ and $S < +\infty) = 1$, μ-a.e. Letting $A \to +\infty$, one gets

$$-\infty = \liminf_{n\to\infty} X_n(\omega) < \limsup_{n\to\infty} X_n(\omega) = +\infty,\ P_\omega - a.e.,\ \mu - a.e. \quad \square$$

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