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QUENCHED LARGE DEVIATION FOR SUPER-BROWNIAN MOTION WITH RANDOM IMMIGRATION

WENMING HONG

School of Mathematical Sciences and Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P. R. China
wmhong@bnu.edu.cn

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Quenched local large deviation is derived for the super-Brownian motion with super-Brownian immigration, in dimension $d \geq 4$. At the critical dimension $d = 4$, the quenched and annealed LDP are of the same speed but are different rate.

Keywords: Super-Brownian motion; large deviations; random immigration.

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1. Introduction and Statement of Results

As one kind of superprocess in random environment, super-Brownian motion with super-Brownian immigration (SBMSBI, for short), where the environment is determined by an immigration process which is controlled by the trajectory of another super-Brownian motion, has been studied recently. Many interesting limit properties for SBMSBI were described under the *annealed* probability (Refs. 5, 7 and 14, etc.). Recently, a CLT has been proved in Ref. 8 under the *quenched* probability, that is, conditioned upon a realization of the immigration process. In this paper, we study the *quenched* large deviation for $d \geq 4$.

We begin by recalling the SBMSBI model (we refer to Refs. 1 and 10 for a general introduction to the theory of superprocesses). Let $C(\mathbb{R}^d)$ denote the space of continuous bounded functions on $\mathbb{R}^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1 + |x|^2)^{-p/2}$ for $x \in \mathbb{R}^d$. Let $C_p(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : \sup |f(x)|/\phi_p(x) < \infty\}$. Let $M_p(\mathbb{R}^d)$ be the space of Radon measures μ on $\mathbb{R}^d$ such that $\langle \mu, f\rangle := \int f(x)\mu(dx) < \infty$ for all $f \in C_p(\mathbb{R}^d)$. We endow $M_p(\mathbb{R}^d)$ with the p-vague topology, that is, $\mu_k \to \mu$ if and only if $\langle \mu_k, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_p(\mathbb{R}^d)$. Then $M_p(\mathbb{R}^d)$ is metrizable.[9] We denote by λ the Lebesgue measure on $\mathbb{R}^d$, and note that $\lambda \in M_p(\mathbb{R}^d)$.

Let $S_{s,t}$ denote the heat semigroup in $\mathbb{R}^d$, that is, for $t > s$ and $f \in C(\mathbb{R}^d)$,

$$S_{s,t}f(x) = \frac{1}{(2\pi(t-s))^{d/2}} \int_{\mathbb{R}^d} e^{-|y-x|^2/2(t-s)} f(y)dy\,.$$

We write $S_t := S_{0,t}$ and G for the corresponding potential operator, that is $Gf = \int_0^\infty S_t f dt$, omitting the space variable x from the notation when no confusion may occur. Given $\mu \in M_p(\mathbb{R}^d)$, a *super-Brownian motion* $\varrho = (\varrho_t, P_\mu)$ is an $M_p(\mathbb{R}^d)$-valued Markov process with $\varrho_0 = \mu$ and Laplace transform given by

$$E_\mu \exp\{-\langle \varrho_t, f\rangle\} = \exp\{-\langle \mu, v(t,\cdot)\rangle\}\,, \qquad f \in C_p^+(\mathbb{R}^d)\,, \tag{1.1}$$

where $v(\cdot,\cdot)$ is the unique mild solution of the evolution equation

$$\begin{cases} \dot{v}(t) = \frac{1}{2}\Delta v(t) - v^2(t)\,, \\ v(0) = f\,, \end{cases} \tag{1.2}$$

and E_μ denotes expectation with respect to P_μ.

Given a super-Brownian motion $\varrho = (\varrho_t, P_\mu)$ as the "environment", we will consider another super-Brownian motion with the immigration rate controlled by the trajectory of ϱ, the (SBMSBI) $X^\varrho = (X_t^\varrho, P_\nu^\varrho)$ with $X_0^\varrho = \nu$, which is again an $M_p(\mathbb{R}^d)$-valued Markov process whose *quenched* probability law is determined by

$$E_\nu^\varrho \exp\{-\langle X_t^\varrho, f\rangle\} = \exp\left\{-\langle \nu, v(t,\cdot)\rangle - \int_0^t \langle \varrho_s, v(t-s,\cdot)\rangle ds\right\}\,. \tag{1.3}$$

Again, E_ν^ϱ denotes expectations with respect to P_ν^ϱ.

In the following we take $\mu = \nu = \lambda$, and write P^ϱ (resp. P) for P_λ^ϱ (resp. P_λ). We also use E^ϱ and E for the corresponding expectations. This model was considered in Refs. 7 and 5, see also Ref. 2, where some interesting and new phenomena were revealed under the *annealed* probability law:

$$\mathbb{P}(\cdot) := \int P^\varrho(\cdot)P(d\varrho)$$

with expectation denoted by $\mathbb{E}$.

Annealed LDP was obtained in Ref. 5, in this paper we will prove a quenched LDP for the SBMSBI, in dimension $d \geq 4$. As pointed out in Ref. 8, in the study of motion in random media, differences exist between quenched and annealed CLT behavior for the SBMSBI, see also Refs. 13 and 12 for several examples in RWRE. Our main result, Theorem 1.1 below, shows that this is also the case for the LDP.

The following estimation is useful in our proof, for any $f \in C_p^+(\mathbb{R}^d)$,

$$S_t f \leq c(1 \wedge t^{-d/2})\,. \tag{1.4}$$

where $c = \max\{(2\pi)^{-d/2}, \|f\|\}$ is a positive constant, and then $a := \int_0^\infty c(1 \wedge r^{-d/2})dr < \infty$ when $d \geq 3$.

We fix $f \in C_p^+(\mathbb{R}^d)$ such that $\langle \lambda, f\rangle = 1$ and define

$$\mathbf{W}(T) := \frac{1}{T}\langle X_T^\varrho, f\rangle \tag{1.5}$$

and

$$\Lambda_d^{\varrho}(T,\theta) := c_d(T)^{-1} \log E^{\varrho} \exp[\theta c_d(T) \mathbf{W}(T)]\,, \tag{1.6}$$

where the speed function is defined by $c_d(T) = T$, $d \geq 4$.

For $\theta \leq 0$, one can rewrite (1.3) and (1.2) as follows ($v \leftrightarrow -v$):

$$E^{\varrho} \exp\{\theta\langle X_t^{\varrho}, f\rangle\} = \exp\left\{\langle \lambda, v(t,\cdot;\theta)\rangle + \int_0^t \langle \varrho_s, v(t-s,\cdot;\theta)\rangle ds\right\}\,, \quad f \in C_p^+(\mathbb{R}^d)\,, \tag{1.7}$$

where $v(\cdot,\cdot;\theta)$ is the unique mild solution of the evolution equation

$$\begin{cases} \dot{v}(t) = \dfrac{1}{2}\Delta v(t) + v^2(t)\,, \\ v(0) = \theta f\,. \end{cases} \tag{1.8}$$

An important step is to extend (1.7) and (1.8) to some positive θ, which was proved[5] by use of Dynkin's moment method.

Lemma 1.1. (Hong, Ref. 5) *Let* $d \geq 3$, $|\theta| < \frac{1}{4a}$, *then Eq.* (1.8) *admits a unique mild solution* $v(t,x;\theta)$, *moreover it is analytic in* $|\theta| < \frac{1}{4a}$ *and*

$$|v(t,x;\theta)| \leq b(\theta) \cdot S_t f(x)\,, \tag{1.9}$$

where $b(\theta) = (2a)^{-1}[1 - (1 - 4a|\theta|)^{1/2}]$.

The main result of this paper is the following:

Theorem 1.1. (Quenched LDP) *Assume* $d \geq 4$, $|\theta| < \frac{1}{4a}$ *and* $f \in C_p^+(\mathbb{R}^d)$. *Then, for* P *a.e.* ϱ, *the law of* $\mathbf{W}_t$ *under* P^{ϱ} *admit the local LDP with speed function* t *and rate function* $I(\alpha)$, *i.e. there exists a neighborhood* O *of* 1 *such that if* $U \subset O$ *is open and* $C \subset O$ *is closed, then*

$$\liminf_{t\to\infty} \frac{1}{t} \log P^{\varrho}\{\mathbf{W}(t) \in U\} \geq -\inf_{\alpha\in U} I(\alpha)\,,$$

$$\limsup_{t\to\infty} \frac{1}{t} \log P^{\varrho}\{\mathbf{W}(t) \in C\} \leq -\inf_{\alpha\in C} I(\alpha)\,,$$

where $I(\alpha)$ *is the Legendre transform of* $\Lambda(\theta)$, *i.e.*

$$I(\alpha) := \sup_{|\theta|<\delta} [\alpha\theta - \Lambda(\theta)]$$

and

$$\Lambda(\theta) := \theta + \int_0^{\infty} \langle \lambda, [v(s,\cdot;\theta)]^2\rangle ds\,,$$

where $v(\cdot,\cdot)$ *is the unique mild solution of the evolution equation* (1.8). □

Remarks.

(1) For $d = 3$, an easy adaptation of our methods shows that the statement of Theorem 1.1 remains true with an almost sure statement being replaced by a statement in probability, with the same speed function being t^{-1}.

(2) In Ref. 6, a LDP for the *quenched mean* $E^{\varrho}\langle X_T^{\varrho}, f\rangle$ is proved for $d = 3$ with speed function $t^{-1/2}$, and it is easy to see that there is a degenerate LDP for the *quenched mean* for $d \geq 5$ and a LDP upper bound for $d = 4$ with speed function t^{-1}. Recall Ref. 5, where an *annealed* LDP is obtained for the SBMSBI with speed function being $t^{-1/2}$ when $d = 3$ and t^{-1} for $d \geq 4$. Comparing Theorem 1.1, one sees that two kinds of randomness reflected in the annealed LDP property: in $d = 3$ with speed function $t^{-1/2}$, the *quenched mean* (or the environment ϱ) dominates the behavior; whereas for $d \geq 5$ with speed function being t^{-1}, the *quenched* (i.e. for fixed ϱ, the randomness of the proces SBM) factor takes effect; and in dimension $d = 4$ with speed function being t^{-1}, both contribute to the rate of annealed LDP (i.e. the rate function of the *annealed* LDP is the combination of the rate of *quenched* LDP and the rate of *quenched mean* LDP).

(3) Only a local LDP is proved now, it is an interesting problem to prove the steepness of $\Lambda(\cdot)$ such that the full LDP being obtained.

2. Proof of Theorem 1.1

Set $d \geq 4$, $|\theta| < \frac{1}{4a}$ with $f \in C_p^+(\mathbb{R}^d)$. The mild solution of Eq. (1.8) is

$$v(r, x; \theta) = \theta S_r f(x) + \int_0^r S_{r-h} v(h, \cdot; \theta)^2(x) dh\,, \qquad 0 \leq r \leq T\,. \tag{2.1}$$

Based on Lemma 1.1, the Laplace expression (1.7) also holds for $0 < \theta < \frac{1}{4a}$ by properties of Laplace transform of probability measure on $[0, \infty)$ (cf. Ref. 11). Thus from (1.6) and (1.7), we have

$$\Lambda_d^{\varrho}(T, \theta) := T^{-1}\left[\langle \lambda, v(T, \cdot; \theta)\rangle + \int_0^T \langle \varrho_s, v(T - s, \cdot; \theta)\rangle ds\right]. \tag{2.2}$$

Note (1.9), the estimation of $v(t, x; \theta)$, it is easy to see that as $T \to \infty$,

$$T^{-1}\langle \lambda, v(T, \cdot; \theta)\rangle \longrightarrow 0\,. \tag{2.3}$$

In the sequel, we will take most of our effort to prove the following:

Proposition 2.1. *$d \geq 4$, $|\theta| < \frac{1}{4a}$, for P-a.e. ϱ,*

$$\Lambda_d^{\varrho}(T, \theta) \to \Lambda(\theta) := \theta + \int_0^\infty \langle \lambda, [v(s, \cdot; \theta)]^2\rangle ds\,. \tag{2.4}$$

Proof of Proposition 2.1. By (2.2) and (2.3), it is enough to prove the following two claims:

Claim 1. *For P-a.e. ϱ,*

$$n^{-1}\int_0^n \langle \varrho_s, v(n-s,\cdot;\theta)\rangle ds \to \theta + \int_0^\infty \langle \lambda, [v(s,\cdot;\theta)]^2\rangle ds\,. \tag{2.5}$$

Claim 2. *For P-a.e. ϱ,*

$$\max_{n\le t\le n+1}\left| t^{-1}\int_0^t \langle \varrho_s, v(t-s,\cdot;\theta)\rangle ds - n^{-1}\int_0^n \langle \varrho_s, v(n-s,\cdot;\theta)\rangle ds\right| \to 0\,. \tag{2.6}$$

We will prove the claims by means of Borel–Cantelli Lemma, to this end, some estimations for the moments should be investigated. We recall from Theorem 3.2 of Ref. 9 that for any $C_p(\mathbb{R}^d)^+$-valued continuous path $F(s)$, the Laplace transform $\int_0^t \langle \varrho_s, F(t-s)\rangle ds$ is given by

$$E\exp\left\{-\tau\int_0^t \langle \varrho_s, F(t-s)\rangle ds\right\} = \exp\{-\langle \lambda, u(t,\tau;\cdot)\rangle\}\,, \qquad \tau > 0\,, \tag{2.7}$$

where $u(s,\tau;x)$ is the non-negative solution of the following mild equation

$$u(s,\tau;x) = \tau\int_0^s S_{s-r}F(r)(x)dr - \int_0^s S_{s-r}u^2(r,\tau)(x)dr\,, \qquad 0\le s\le t\,. \tag{2.8}$$

Let

$$u^{(i)}(r,x) := \frac{\partial^i u(r,x,\theta)}{\partial\theta^i}|_{\theta=0}\,, \qquad i=1,2,3\,.$$

[In fact, (2.7) and (2.8) hold true for $|\tau| < c$ for c a small enough constant, see Ref. 5.] Differentiating with respect to τ in (2.7) and (2.8), and using that $u|_{\tau=0} = 0$, we obtain

$$E\left[\int_0^t \langle \varrho_s, F(t-s)\rangle ds\right] = \int_0^t \langle \lambda, F(s,\cdot)\rangle ds\,, \tag{2.9}$$

$$\mathrm{var}\left[\int_0^t \langle \varrho_s, F(t-s)\rangle ds\right] = 2\int_0^t \left\langle \lambda, \left(\int_0^s S_{s-r}F(r)dr\right)^2\right\rangle ds\,, \tag{2.10}$$

$$\begin{aligned} &E\left[\int_0^t \langle \varrho_s, F(t-s)\rangle ds - E\int_0^t \langle \varrho_s, F(t-s)\rangle ds\right]^4 \\ &\quad = 3\left(\int_0^t \langle \lambda, u^{(1)}(r)^2\rangle dr\right)^2 + 3\int_0^t \langle \lambda, u^{(2)}(r)^2\rangle dr + 4\int_0^t \langle \lambda, u^{(1)}(r)u^{(3)}(r)\rangle dr \\ &\quad := 3I^2 + 3J + 4K\,, \end{aligned} \tag{2.11}$$

where for $0\le r\le t$,

$$u^{(1)}(r) = \int_0^r S_{r-s}F(s)ds\,, \tag{2.12}$$

$$u^{(2)}(r) = -2\int_0^r S_{r-s}u^{(1)}(s)^2 ds\,, \tag{2.13}$$

$$u^{(3)}(r) = -6\int_0^r S_{r-s}u^{(1)}(s)u^{(2)}(s)ds\,. \tag{2.14}$$

In the sequel, we let A denote a constant whose value may change from line to line and which may depend on the dimension and on f, but not on s, t, x, etc.

Proof of Claim 1. First of all, from (2.9) and (2.1),

$$\begin{aligned} E\left[t^{-1}\int_0^t \langle \varrho_s, v(t-s,\cdot;\theta)\rangle ds\right] &= t^{-1}\int_0^t \langle \lambda, v(s,\cdot;\theta)\rangle ds \\ &= t^{-1}\int_0^t \langle \lambda, \theta S_s f + \int_0^s v(r,\cdot;\theta)^2 dr\rangle ds \\ &\to \theta\langle\lambda, f\rangle + \int_0^\infty \langle \lambda, v(r,\cdot;\theta)^2\rangle dr \end{aligned} \tag{2.15}$$

by l'Hospital's rule, which is finite for $|\theta| < \frac{1}{4a}$ when $d \geq 3$ by Lemma 1.1.

Let $F(s) := n^{-1}v(s,\cdot;\theta)$, from (2.12)–(2.14), combining (2.1) and (1.9), when $d \geq 4$ we have

$$\begin{aligned} u^{(1)}(r) &= \int_0^r S_{r-s}F(s)ds = n^{-1}\int_0^r S_{r-s}v(s,\cdot;\theta)ds\,, \\ &\leq n^{-1}b(\theta)\int_0^r S_{r-s}S_s f(x)ds = n^{-1}b(\theta)rS_r f\,, \end{aligned} \tag{2.16}$$

$$\begin{aligned} |u^{(2)}(r)| &= 2\int_0^r S_{r-s}u^{(1)}(s)^2 ds \leq 2n^{-2}b^2(\theta)\int_0^r S_{r-s}(sS_s f)^2 ds \\ &\leq An^{-2}b^2(\theta)\int_0^r s^2(1\wedge s^{-d/2})ds\cdot S_r f \leq An^{-2}b^2(\theta)(1\vee r)\cdot S_r f\,, \end{aligned} \tag{2.17}$$

$$\begin{aligned} u^{(3)}(r) &= -6\int_0^r S_{r-s}u^{(1)}(s)u^{(2)}(s)ds \\ &\leq An^{-3}b(\theta)^2\int_0^r S_{r-s}[(sS_s f)(1\vee s)S_s f]ds \\ &\leq An^{-3}b(\theta)^2(1\vee r)S_r f\,, \end{aligned} \tag{2.18}$$

where we used (1.4) many times. So from (2.11) when $d \geq 4$,

$$\begin{aligned} I^2 &= \left(\int_0^n \langle \lambda, u^{(1)}(r)^2\rangle dr\right)^2 \leq An^{-4}\left(\int_0^n \langle \lambda, (rS_r f)^2\rangle dr\right)^2 \\ &\leq An^{-4}\left(\int_0^n r^2(1\wedge r^{-d/2})dr\right)^2 \leq An^{-2}\,, \end{aligned} \tag{2.19}$$

$$J = \int_0^n \langle \lambda, u^{(2)}(r)^2 \rangle dr \le An^{-4} \int_0^n \langle \lambda, (1 \vee r)^2 (S_r f)^2 \rangle dr$$

$$\le An^{-4} \int_0^n r^2 (1 \wedge r^{-d/2})) dr \le An^{-3}, \tag{2.20}$$

$$K = \int_0^n \langle \lambda, u^{(1)}(r) u^{(3)}(r) \rangle dr \le An^{-4} \int_0^n \langle \lambda, r(1 \vee r)(S_r f)^2 \rangle dr$$

$$\le An^{-4} \int_0^n r(1 \vee r)(1 \wedge r^{-d/2}) dr \le An^{-3}. \tag{2.21}$$

Then,

$$E\left[n^{-1} \int_0^n \langle \varrho_s, v(n-s, \cdot; \theta) \rangle ds - n^{-1} E \int_0^n \langle \varrho_s, v(n-s, \cdot; \theta) \rangle ds \right]^4$$

$$= 3I^2 + 3J + 4K \le An^{-2}. \tag{2.22}$$

Thus by Borel–Cantelli lemma, P-a.s.,

$$\left| n^{-1} \int_0^n \langle \varrho_s, v(n-s, \cdot; \theta) \rangle ds - n^{-1} E \int_0^n \langle \varrho_s, v(n-s, \cdot; \theta) \rangle ds \right| \to 0, \tag{2.23}$$

combining (2.15) we get (2.5), and the proof is complete. □

Proof of Claim 2. Let

$$\Gamma(t) := t^{-1} \int_0^t \langle \varrho_s, v(t-s, \cdot; \theta) \rangle ds.$$

Let $n > 1$, $0 < \delta < 1$, $n \le t_1 < t_2 = t_1 + \delta \le n+1$, and

$$\begin{aligned} \Delta\Gamma(t_1, t_2) &:= \Gamma(t_2) - \Gamma(t_1) \\ &= t_1^{-1} \int_0^{t_1} \langle \varrho_s, v(t_2 - s) - v(t_1 - s) \rangle ds + t_2^{-1} \int_{t_1}^{t_2} \langle \varrho_s, v(t_2 - s) \rangle ds \\ &:= \Delta\Gamma_1(t_1, t_2) + \Delta\Gamma_2(t_1, t_2). \end{aligned} \tag{2.24}$$

For $\Delta\Gamma_2(t_1, t_2) = t_2^{-1} \int_{t_1}^{t_2} \langle \varrho_s, v(t_2 - s) \rangle ds$, let $F(s) := t_2^{-1} v(s) \le t_2^{-1} b(\theta) S_s f$ by (1.9), we need a counterpart of (2.7) and (2.8) as follows, for $\tau \ge 0$,

$$\begin{aligned} E \exp\left\{ -\tau \int_{t_1}^{t_2} \langle \varrho_r, F(t_2 - r) \rangle dr \right\} &= E \exp\{ -\langle \varrho_{t_1}, u(t_1, t_2, \tau; \cdot) \rangle \} \\ &= \exp\{ -\langle \lambda, w(0, t_1, \tau; \cdot) \rangle \}, \end{aligned}$$

where $u(s, t_2, \tau; \cdot)$ is the non-negative solution of the following mild equation:

$$u(s, t_2, \tau; x) = \tau \int_s^{t_2} S_{s,r} F(t_2 - r)(x) dr - \int_s^{t_2} S_{s,r} u^2(r, t_2, \tau)(x) dr, \quad t_1 \le s \le t_2,$$

and $w(s,t_1,\tau;x)$ is the non-negative solution of the following mild equation:

$$w(s,t_1,\tau;x) = S_{s,t_1}u(t_1,t_2,\tau;\cdot)(x) - \int_s^{t_1} S_{s,r}w^2(r,t_1,\tau)(x)dr\,, \qquad 0 \le s \le t_1\,.$$

Obviously,

$$\mathrm{var}[\Delta\Gamma_2(t_1,t_2)] = -\frac{\partial^2\langle\lambda, w(0,t_1,\tau;\cdot)\rangle}{\partial\tau^2}\bigg|_{\tau=0}\,.$$

Performing the differentiation and using that $u|_{\tau=0} = w|_{\tau=0} = 0$, we obtain

$$\begin{aligned}
&\mathrm{var}[\Delta\Gamma_2(t_1,t_2)]\\
&\quad = 2\left\langle\lambda, \int_{t_1}^{t_2} S_{0,s}\left[\int_s^{t_2} S_{s,r}F(t_2-r)dr\right]^2 ds\right\rangle\\
&\qquad + 2\left\langle\lambda, \int_0^{t_1} S_{0,s}\left[\int_{t_1}^{t_2} S_{s,r}F(t_2-r)dr\right]^2 ds\right\rangle\\
&\quad \le Ab(\theta)^2t_2^{-2}\left[\left\langle\lambda, \int_{t_1}^{t_2} S_{0,s}[(t_2-s)S_{t_2-s}f]^2ds\right\rangle + \left\langle\lambda, \int_0^{t_1} S_{0,s}[\delta S_{t_2-s}f]^2ds\right\rangle\right]\\
&\quad \le Ab(\theta)^2t_2^{-2}\left[\int_{t_1}^{t_2}(t_2-s)^2[1\wedge(t_2-s)^{-d/2}]ds + \delta^2\int_0^{t_1}[1\wedge(t_2-s)^{-d/2}]ds\right]\\
&\quad \le An^{-2}\delta^2\,,
\end{aligned} \tag{2.25}$$

in which (1.4) has been used several times. To consider $\Delta\Gamma_1(t_1,t_2)$, we need the following lemma which has been proved in Ref. 8.

Lemma 2.1. (Ref. 8) *There is a constant* A *such that for any* $t \ge \tau > 0$, *we have*

$$\sup_{0<s\le\tau\le t} s^{-1}|p(t+s,x,y)-p(t,x,y)| \le A\tau^{-1}[p(t+2\tau,x,y)+p(t,x,y)]\,. \tag{2.26}$$

□

Let $F_1(s) := t_1^{-1}[v(s+\delta)-v(s)]$, we rewrite

$$\Delta\Gamma_1(t_1,t_2) = t_1^{-1}\int_0^{t_1}\langle\varrho_{t_1-s}, v(s+\delta)-v(s)\rangle ds = \int_0^{t_1}\langle\varrho_{t_1-s}, F_1(s)\rangle ds\,.$$

Note that F_1 may be signed, but small in absolute value when n (so t_i) is large enough. By Lemma 2.5 of Ref. 5, (2.7)–(2.14) still hold while u is not necessarily non-negative. From (2.12),

$$u^{(1)}(r) = \int_0^r S_{r-s}F_1(s)ds = t_1^{-1}\int_0^r S_{r-s}[v(s+\delta)-v(s)]ds\,. \tag{2.27}$$

By (2.1),

$$
\begin{aligned}
&v(s+\delta)-v(s)\\
&\quad=\theta[S_{s+\delta}f-S_sf]+\int_0^s[S_{s+\delta-l}-S_{s-l}]v(l)^2dl+\int_s^{s+\delta}S_{s+\delta-l}v(l)^2dl\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|u^{(1)}(r)|&=t_1^{-1}\bigg|\theta r[S_{r+\delta}f-S_rf]+\int_0^r\int_0^s[S_{r+\delta-l}-S_{r-l}]v(l)^2dlds\\
&\qquad+\int_0^r\int_s^{s+\delta}S_{r+\delta-l}v(l)^2dlds\bigg|\\
&\le t_1^{-1}\bigg[|\theta r[S_{r+\delta}f-S_rf]|+\int_0^r|l[S_{l+\delta}-S_l]|v(r-l)^2dl\\
&\qquad+\int_0^r\int_s^{s+\delta}S_{r+\delta-l}v(l)^2dlds\bigg]\,, \qquad (2.28)
\end{aligned}
$$

but for $r>\delta$,

$$
\begin{aligned}
&\int_0^r|l[S_{l+\delta}-S_l]|v(r-l)^2dl\\
&\quad=\int_0^\delta|l[S_{l+\delta}-S_l]|v(r-l)^2dl+\int_\delta^r|l[S_{l+\delta}-S_l]|v(r-l)^2dl\\
&\quad\le\int_0^\delta[\delta[S_{l+\delta}+S_l]]v(r-l)^2dl+\int_\delta^r[\delta[S_{l+\delta}+S_l]]v(r-l)^2dl\\
&\quad\le Ab(\theta)^2\delta[S_{r+\delta}+S_r]f\,,
\end{aligned}
$$

where the second term in the second step is from Lemma 2.1 (with $\tau=t=l$, $s=\delta$ there), and (1.9), (1.4) have been used several times. Note that

$$\int_0^r\int_s^{s+\delta}S_{r+\delta-l}v(l)^2dlds\le Ab(\theta)^2\delta rS_{r+\delta}f\,,$$

we can continue (2.28) for $r>\delta$,

$$|u^{(1)}(r)|\le Ab(\theta)^2t_1^{-1}[|r(S_{r+\delta}f-S_rf)|+\delta(S_{r+\delta}+S_r)f+\delta rS_{r+\delta}f]\,, \qquad (2.29)$$

but for $r\le\delta$, from (2.27) and (1.9) it is easy to check that

$$|u^{(1)}(r)|\le Ab(\theta)t_1^{-1}\delta[(S_{r+\delta}+S_r)f]\,. \qquad (2.30)$$

Recall (2.11), we have

$$
\begin{aligned}
I&=\int_0^{t_1}\langle\lambda,u^{(1)}(r)^2\rangle dr=\int_0^\delta\langle\lambda,u^{(1)}(r)^2\rangle dr+\int_\delta^{t_1}\langle\lambda,u^{(1)}(r)^2\rangle dr\\
&\le At_1^{-1}\delta^2\,. \qquad (2.31)
\end{aligned}
$$

With (2.29) and (2.30) in hand, we can take the same calculations as in (2.17)–(2.22) and get to

$$E[\Delta\Gamma_1(t_1,t_2) - E\Delta\Gamma_1(t_1,t_2)]^4 \le At_1^{-2}\delta^2 \le An^{-2}\delta^2\,. \tag{2.32}$$

Now we proceed to the proof of claim 2. Let $\bar{\Gamma}(t) := \Gamma(t) - E\Gamma(t)$ denote the centered $\Gamma(t)$, and define $\Delta\overline{\Gamma_i}$ similarly. For any $\varepsilon > 0$,

$$\begin{aligned}
&P\left(\max_{n\le t\le(n+1)} |\bar{\Gamma}(t) - \bar{\Gamma}(n)| > \varepsilon\right)\\
&\le \sum_{k=1}^{\infty} P\left(\max_{0\le j\le 2^k-1} |\bar{\Gamma}(n+2^{-k}(j+1)) - \bar{\Gamma}(n+2^{-k}j)| > \varepsilon k^{-2}/2\right)\\
&= \sum_{k=1}^{\infty} P\left(\max_{0\le j\le 2^k-1} |\Delta\bar{\Gamma}(n+2^{-k}j, n+2^{-k}(j+1))| > \frac{\varepsilon}{2k^2}\right)\\
&\le \sum_{k=1}^{\infty}\sum_{i=1}^{2} 2^k \max_{0\le j\le 2^k} P\left(|\Delta\overline{\Gamma_i}(n+2^{-k}j, n+2^{-k}(j+1))| > \frac{\varepsilon}{4k^2}\right).
\end{aligned}$$

By Chebyshev's inequality and (2.32), for $i = 1$,

$$P\left(|\Delta\overline{\Gamma_1}(n+2^{-k}j, n+2^{-k}(j+1))| > \frac{\varepsilon}{4k^2}\right) \le 256A\varepsilon^{-4}k^8 2^{-2k} n^{-2}\,.$$

Similarly, using (2.25), we obtain for $i = 2$,

$$P\left(|\Delta\overline{\Gamma_2}(n+2^{-k}j, n+2^{-k}(j+1))| > \frac{\varepsilon}{4k^2}\right) \le 16A\varepsilon^{-2}k^4 2^{-2k} n^{-2}\,.$$

Thus,

$$P\left(\max_{n\le t\le(n+1)} |\bar{\Gamma}(t) - \bar{\Gamma}(n)| > \varepsilon\right) \le A\varepsilon^{-2}n^{-2}\sum_{k=1}^{\infty} k^8 2^{-k} \le A\varepsilon^{-2}n^{-2}\,.$$

By the Borel–Cantelli lemma, we get $\max_{n\le t\le(n+1)} |\bar{\Gamma}(t) - \bar{\Gamma}(n)| \to 0$, P-a.s. Combining (2.15) we obtain (2.6), which complete the proof of claim 2. □

Proof of Theorem 1.1. Recall $\Lambda(\theta) = \theta + \int_0^\infty \langle\lambda, [v(s,\cdot;\theta)]^2\rangle ds$. From (1.7)–(1.9), it is easy to get $\Lambda'(0) = 1$ and $\Lambda''(0) = 2\int_0^{+\infty}\langle\lambda, (P_s f)^2\rangle ds > 0$. Then there is $\delta > 0$ such that $\Lambda(\theta)$ is strictly convex, continuous differentiable in $|\theta| < \delta < \frac{1}{4a}$ with $\Lambda'(0) = 1$. By Proposition 2.1, the local large deviation principles is the consequence of *Gärtner–Ellis* theorem (cf. Theorem 2.3.6 of Ref. 3). The neighborhood O is that of $\{\Lambda'(\theta) : |\theta| < \delta < \frac{1}{4a}\}$, the set of exposed points of $I(\alpha)$ whose exposing hyperplane belong to the interior of $D_\Lambda := \{\theta : \Lambda(\theta) < \infty\}$. □

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