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Moderate deviations for the quenched mean of the super-Brownian motion with random immigration[1]

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Abstract. Moderate deviations for the quenched mean of the super-Brownian motion with random immigration are proved for $3 \le d \le 6$, which fills in the gap between the CLT and LDP.

Key words: Super-Brownian motion, Large deviation, Moderate deviation, Random immigration.

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1 Introduction and statement of results

Super-Brownian motion with super-Brownian immigration (SBMSBI, for short) was considered by Hong and Li ([8]), see also [3]. As one kind of superprocess in random environment, some interesting phenomena were revealed under the *annealed* probability law ([6], [8]) and a *quenched* CLT was obtained ([9]). For the quenched mean, a CLT ($d \ge 3$) and a LDP ($d = 3$) were proved ([7]). In this paper, we will consider the moderate deviation for the quenched mean, which fills in the gap between the CLT and LDP.

To state our results, we begin by recalling the SBMSBI model (we refer to [2] and [12] for a general introduction to the theory of superprocesses). Let $C(I\!R^d)$ denote the space of continuous bounded functions on $I\!R^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1 + |x|^2)^{-p/2}$ for $x \in I\!R^d$. Let $C_p(I\!R^d) := \{f \in C(I\!R^d) : \sup |f(x)|/\phi_p(x) < \infty\}$. Let $M_p(I\!R^d)$ be the space of Radon measures μ on $I\!R^d$ such that $\langle \mu, f\rangle := \int f(x)\mu(dx) < \infty$ for all $f \in C_p(I\!R^d)$. We endow $M_p(I\!R^d)$ with the p-vague topology, that is, $\mu_k \to \mu$ if and only if $\langle \mu_k, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_p(I\!R^d)$. Then $M_p(I\!R^d)$ is metrizable ([10]). We denote by λ the Lebesgue measure on $I\!R^d$, and note that $\lambda \in M_p(I\!R^d)$.

Let $S_{s,t}$ denote the heat semigroup in $I\!R^d$, that is, for $t > s$ and $f \in C(I\!R^d)$,

$$S_{s,t}f(x) = \frac{1}{(2\pi(t-s))^{d/2}} \int_{I\!R^d} e^{-|y-x|^2/2(t-s)} f(y)dy\,.$$

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We write $S_t := S_{0,t}$. Given $\mu \in M_p(I\!R^d)$, a *super-Brownian motion* $\varrho = (\varrho_t, P_\mu)$ is an $M_p(I\!R^d)$-valued Markov process with $\varrho_0 = \mu$ and Laplace transform given by

$$E_\mu \exp\{-\langle \varrho_t, f\rangle\} = \exp\{-\langle \mu, v(t,\cdot)\rangle\}, \quad f \in C_p^+(I\!R^d), \tag{1.1}$$

where $v(\cdot,\cdot)$ is the unique mild solution of the evolution equation

$$\begin{cases} \dot{v}(t) = \frac{1}{2}\Delta v(t) - v^2(t) \\ v(0) = f\,, \end{cases} \tag{1.2}$$

and E_μ denotes expectation with respect to P_μ.

Given a super-Brownian motion $\varrho = (\varrho_t, P_\mu)$ as the "environment", we will consider another super-Brownian motion with the immigration rate controlled by the trajectory of ϱ, the (SBMSBI) $X^\varrho = (X_t^\varrho, P_\nu^\varrho)$ with $X_0^\varrho = \nu$, which is again an $M_p(I\!R^d)$-valued Markov process whose *quenched* probability law is determined by

$$E_\nu^\varrho \exp\{-\langle X_t^\varrho, f\rangle\} = \exp\{-\langle \nu, v(t,\cdot)\rangle - \int_0^t \langle \varrho_s, v(t-s,\cdot)\rangle ds\}. \tag{1.3}$$

Again, E_ν^ϱ denotes expectations with respect to P_ν^ϱ.

In the following we take $\mu = \nu = \lambda$, and write P^ϱ (resp. P) for P_λ^ϱ (resp. P_λ). We also use E^ϱ and E for the corresponding expectations. This model was considered in [8] and [6]–[9], see also [3], where some interesting and new phenomena were revealed under the *annealed* probability law:

$$\mathbb{P}(\cdot) := \int P^\varrho(\cdot) P(d\varrho)$$

with expectation denoted by $\mathbb{E}$.

Recall that a CLT for the quenched mean $E^\varrho\langle X_t^\varrho, f\rangle$ has been proved in [7]. i.e., for the centered functional $F(\varrho.,T;f)$

$$F(\varrho.,T;f) := a_d(T)^{-1}\{E^\varrho\langle X_T^\varrho, f\rangle - \mathbb{E}\langle X_T^\varrho, f\rangle\}, \quad f \in C_p^+(I\!R^d), \tag{1.4}$$

where

$$a_d(T) = \begin{cases} T^{(6-d)/4}, & 3 \le d \le 5 \\ (\log T)^{1/2}, & d = 6 \\ 1, & d \ge 7. \end{cases} \tag{1.5}$$

Theorem 1.1 (Hong, 2005) *For $d \ge 3$, $f \in C_p^+(I\!R^d)$, then as $T \to \infty$, $F(\varrho.,T;f) \Rightarrow \xi(f)$ in distribution under the law P, $\xi(f)$ is determined by*

$$E\exp\{-\theta\xi(f)\} = \exp\{F(\theta, f)\}, \quad \theta \ge 0$$

where

$$F(\theta,f) = \begin{cases} C_d\theta^2\langle\lambda, f\rangle^2, & 3 \le d \le 6 \\ \int_0^\infty \langle \lambda, u_\theta^2(s,\cdot)\rangle ds, & d \ge 7. \end{cases}$$

$C_d = (4\pi)^{-d/2}\int_0^1 s^{2-d/2}ds$ for $3 \le d \le 5$, $C_6 = (4\pi)^{-3}$; and $u_\theta(t,x)$ is the mild solution of equation

$$u_\theta(t,x) = \theta t S_t f(x) - \int_0^t S_{t-s}u_\theta^2(s,\cdot)(x)ds.$$

□

Remark 1.1(a) Note that when T large enough, $E^{\varrho}\langle X_T^{\varrho}, f\rangle \sim T\langle\lambda, f\rangle \ +\xi$ by (1.4) and (1.5).

(b) A LDP for the quenched mean $E^{\varrho}\langle X_t^{\varrho}, f\rangle$ has been obtained for dimension $d=3$ as well ([7]), with norming $a_3(T)=T$ and speed $b_3(T)=T^{-1/2}$. By a further observation, we found that an upper bound LDP for dimension $d=4$ and a degenerate form (rate function be 0) for dimensions $d\geq 5$ can be obtained with norming $a_d(T)=T$ and speed $b_d(T)=T^{-1}$.

In this paper, we will prove a moderate deviation for the quenched mean $E^{\varrho}\langle X_t^{\varrho}, f\rangle$ in dimensions $3\leq d\leq 6$, thus we give a complete picture about the limit behavior of the quenched mean regarding to (a) and (b) in the remark 1.1.

For $3\leq d\leq 6$, fix $f\in C_p^+(I\!R^d)$ satisfying $\langle\lambda, f\rangle=1$ and let

$$\mathbf{W}(T):=a_d(T)^{-1}[E^{\varrho}\langle X_T^{\varrho}, f\rangle-\mathbb{E}\langle X_T^{\varrho}, f\rangle], \tag{1.6}$$

and

$$\Lambda(T,\theta):=b_d(T)^{-1}\log E\exp[\theta b_d(T)\mathbf{W}(T)], \tag{1.7}$$

where the norming $a_d(T)$ and the speed $b_d(T)$ satisfy the following conditions,

$$\begin{cases} (1)\ b_d(T)\cdot a_d(T)^{-1}:=l_d(T)\to 0,\\ (2)\ b_d(T)=a_d(T)\cdot l_d(T)\to\infty,\\ (3)\ a_d(T)^{-2}\cdot b_d(T)=b_d(T)^{-1}\cdot l_d(T)^2=\begin{cases} T^{d/2-3}, & 3\leq d\leq 5.\\ (\log T)^{-1}, & d=6.\end{cases}\\ \text{in addition,}\\ (4)\ \text{ for } d=3, a_d(T)\cdot T^{-1}=l_d(T)\cdot T^{1/2}\to 0.\end{cases} \tag{1.8}$$

Theorem 1.2 *For* $3\leq d\leq 6$, *define* $C_d=(4\pi)^{-d/2}\int_0^1 s^{2-d/2}ds$ *for* $3\leq d\leq 5$, $C_6=(4\pi)^{-3}$; *and* $I(x)=\frac{x^2}{4C_d}$. *The law of* $\mathbf{W}_T$ *under* P *admit the LDP with speed function* $b_d(T)$ *and rate function* $I(x)$, *i.e., for any* $U\subset O$ *is open and* C *is closed, then*

$$\liminf_{T\to\infty} b_d(T)^{-1}\log P\{\mathbf{W}(T)\in U\}\geq -\inf_{x\in U} I(x),$$

$$\limsup_{T\to\infty} b_d(T)^{-1}\log P\{\mathbf{W}(T)\in C\}\leq -\inf_{x\in C} I(x).$$

□

Remark 1.2 As an example, one can choose the norming $a_d(T)$ and the speed $b_d(T)$ satisfying the condition (1.8) as following

$$\begin{aligned} &(1) d=3,\ a_3(T)=T^{1-\alpha}, b_3(T)=T^{1/2-2\alpha}, \alpha\in(0,1/4),\\ &(2) d=4,\ a_4(T)=T^{1-\alpha}, b_4(T)=T^{1-2\alpha}, \alpha\in(0,1/2),\\ &(3) d=5,\ a_5(T)=T^{1/2-\alpha}, b_5(T)=T^{1/2-2\alpha}, \alpha\in(0,1/4),\\ &(4) d=6,\ a_6(T)=(\log T)^{1-\alpha}, b_6(T)=(\log T)^{1-2\alpha}, \alpha\in(0,1/2).\end{aligned} \tag{1.9}$$

Note that when α takes the right endpoint value, the correspond norming $a_d(T)$ is the same as (1.5) for the central limit theorem. On the other hand, when α being 0, the left endpoint value, the corresponding norming $a_d(T)$ and speed $b_d(T)$ are just those for the LDP in dimension $d=3,4$. The limit property in Theorem 1.2 reveals the behavior between CLT and LDP, which is the so called *moderate deviation.* Interesting phenomena happened for $d=5,6$: the LDP is degenerate, but we still get the moderate deviation as stated above.

2 Proof of Theorem 1.2

Let $f_T := l_d(T)f$, where $l_d(T) = b_d(T) \cdot a_d(T)^{-1}$ is as in (1.8). The mild solution of equation (1.2) is

$$v_T(t,x) = S_t f_T(x) - \int_0^t S_{t-s} v_T(s,\cdot)^2(x)ds, \quad t \geq 0 \quad f \in C_p^+(I\!R^d). \tag{2.1}$$

From equation (1.3) it is easy to get the quenched mean of $\langle X_T^\varrho, f_T\rangle$,

$$E^\varrho \langle X_T^\varrho, f_T\rangle = \langle \lambda, f_T\rangle + \int_0^T \langle \varrho_s, S_{T-s} f_T\rangle ds, \tag{2.2}$$

which is the functional of the process $\varrho = (\varrho_t, P)$ and is determined by the Laplace functional (see Iscoe ([10])), for $\theta \leq 0$,

$$E \exp\{\theta E^\varrho \langle X_t^\varrho, f_T\rangle\} = \exp\{\theta\langle \lambda, f_T\rangle + \langle \lambda, u_T(t,\cdot)\rangle\}, \tag{2.3}$$

where $u_T(t,x;\theta)$ is the mild solution of the following evolution equation,

$$\begin{cases} \dot{u}_T(t) = \Delta u_T(t) + u_T^2(t) + \theta S_t f_T & 0 < t \leq T \\ u_T(0) = 0, \end{cases} \tag{2.4}$$

i.e.,

$$u_T(t,x) = \theta t S_t f_T(x) + \int_0^t S_{t-s} u_T^2(s,\cdot)(x)ds, \quad 0 < t \leq T. \tag{2.5}$$

Actually, (2.3)–(2.5) are just the Laplace transformation for $\theta \leq 0$ (in which $-\theta \leftrightarrow \theta$, $-u \leftrightarrow u$), the key step here is to extend them to some $\theta > 0$, such that we can get the limit of $\Lambda_d(T,\theta)$ for θ in a neighborhood of zero as $T \to \infty$. In what follows, we will firstly to prove the existence and smoothness of the solution of equation (2.4) for θ in a neighborhood of zero by means of series expansion which was used in Hong [6]. Although the proof below can be got by modifying those in [6] almost word by word, we prefer to give the details here because the difference of the constant $c_d(T)$ makes us to obtain a full LDP for $\mathbf{W}(T)$ (whereas only a local LDP being proved in [6]). Secondly we extend the Laplace expression (2.3) to θ in a neighborhood of zero.

For any functions $g(t,\cdot), h(t,\cdot) \in C_p(R^d)$, $\forall t \geq 0$, $p > 1$, we define the convolution

$$g(t,x) * h(t,x) := \int_0^t S_s[g(t-s,\cdot) \cdot h(t-s,\cdot)](x)ds. \tag{2.6}$$

Let

$$\begin{cases} g^{*1}(t,x) := g(t,x) \\ g(t,x)^{*n} := \sum_{k=1}^{n-1} g(t,x)^{*k} * g(t,x)^{*(n-k)}, \end{cases} \tag{2.7}$$

and $\{B_n, n \geq 1\}$ is a sequence of positive numbers determined by

$$\begin{cases} B_1 = B_2 = 1 \\ B_n = \sum_{k=1}^{n-1} B_k B_{n-k}, \end{cases} \tag{2.8}$$

see Dynkin [5] or Wang [13]. Let us recall an estimation which is useful in our proof, for any $f \in C_p^+(I\!R^d)$,

$$S_t f \leq c(1 \wedge t^{-d/2}). \tag{2.9}$$

where $c = \max\{(2\pi)^{-d/2}, ||f||\}$ is a positive constant.

Lemma 2.1. *Let $3 \leq d \leq 6$ and $F_T(t,x) = tS_t f_T(x)$, then*

$$F_T(t,x)^{*n} \leq B_n c_d(T)^{n-1} \cdot tS_t f_T(x) \tag{2.10}$$

where $c_d(T) = c \cdot l_d(T)$, for $4 \leq d \leq 6$; $c_3(T) = 2c \cdot l_3(T) T^{1/2}$.

Proof. We will prove (2.10) by induction in n. It is trivial for $n = 1$. When $n = 2$, from the definition we have

$$\begin{aligned} F_T(t,x)^{*2} &= \int_0^t S_s[(t-s)S_{t-s}f_T]^2(x)ds \\ &\leq c \cdot l_d(T)^2 \int_0^t (t-s)^2[1 \wedge (t-s)^{d/2}]ds \cdot S_t f \\ &= c_d(T) \cdot tS_t f_T(x) \end{aligned}$$

as desired, where we use (2.9) at the second step. If (2.10) is true for all $k < n$, by (2.7) and (2.8) we get

$$\begin{aligned} F_T(t,x)^{*n} &\leq \sum_1^{n-1} B_k c_d(T)^{k-1} \cdot tS_t f_T(x) * B_{n-k} c_d(T)^{n-k-1} \cdot tS_t f_T(x) \\ &= B_n c_d(T)^{n-2} \cdot F_T(t,x)^{*2} \\ &\leq B_n c_d(T)^{n-1} \cdot tS_t f_T(x), \end{aligned}$$

and then the proof is complete by induction. $\square$.

Lemma 2.2. *Let $3 \leq d \leq 6$, $|\theta| < \frac{1}{4c_d(T)}$, then the equation (2.5) admits an unique mild solution $u_T(t,x;\theta)$, moreover it is analytic in $|\theta| < \frac{1}{4c_d(T)}$ and*

$$|u_T(t,x;\theta)| \leq C_d(T,\theta) \cdot tS_t f_T(x), \tag{2.11}$$

where $C_d(T,\theta) = (2c_d(T))^{-1}[1 - (1 - 4c_d(T)|\theta|)^{1/2}]$.

Proof. We can rewrite equation (2.5) by convolution as

$$u_T(t,x;\theta) = \theta F_T(t,x) + u_T(t,x;\theta) * u_T(t,x;\theta). \tag{2.12}$$

Then one gets the solution of (2.12)

$$u_T(t,x;\theta) = \sum_{n=1}^{\infty} F_T(t,x)^{*n} \theta^n \tag{2.13}$$

by Dynkin [5] (see also Wang [13]) once the convergence of the series on the right hand is proved, where $F_T(t,x)$ is given in Lemma 2.1. By Lemma 2.1, the series is dominated by

$$|u_T(t,x;\theta)| \leq \sum_{n=1}^{\infty} B_n c_d(T)^{n-1} |\theta|^n \cdot tS_t f_T(x). \tag{2.14}$$

On the other hand, we know (see Dawson [1], also Dynkin [5] and Wang [13]) that the function $g(z) = \frac{1}{2}[1 - (1-4z)^{1/2}]$ can be expanded as a power series

$$g(z) = \frac{1}{2}[1 - (1-4z)^{1/2}] = \sum_{n=1}^{\infty} B_n z^n,$$

when $|z| < 1/4$, where B_n is given in (2.8). So the series (2.13) is absolutely convergence for $|\theta| < \frac{1}{4c_d(T)}$, and from (2.14) we get

$$|u_T(t,x;\theta)| \leq (2c_d(T))^{-1}[1 - (1 - 4c_d(T)|\theta|)^{1/2}] \cdot tS_t f_T(x) := C_d(T,\theta) \cdot tS_t f_T(x),$$

as desired. $\square$.

As a consequence of Lemma 2.2, we can extend (2.3) to $|\theta| < \frac{1}{4c_d(T)}$ by properties of Laplace transform of probability measure on $[0,\infty)$ (cf. [14]). Note that $c_d(T) \to 0$ by the condition (1.8), thus for any θ, we can choose T_θ large enough such that $|\theta| < \frac{1}{4c_d(T_\theta)}$. So (2.3) is valid for any θ as T goes to infinity.

From (1.7), (2.3) and (2.5), and note that $\mathbb{E}\langle X_T^{\varrho}, f\rangle = \langle \lambda, f_T\rangle + T\langle \lambda, f_T\rangle$, $\langle \lambda, u_T(T,\cdot)\rangle = \theta T\langle \lambda, f_T(x)\rangle + \int_0^T \langle \lambda, u_T^2(s,\cdot)\rangle ds$. For $|\theta| < \frac{1}{4c_d(T)}$ we have

$$\begin{aligned}\Lambda(T,\theta) \quad &:= b_d(T)^{-1} \log E \exp[\theta b_d(T)\mathbf{W}(T)] \\ &= b_d(T)^{-1} \int_0^T \langle \lambda, u_T^2(s,\cdot)\rangle ds. \end{aligned} \tag{2.15}$$

Lemma 2.3. *Let $3 \leq d \leq 6$, for any θ fixed,*

$$A_d(T,f) := b_d(T)^{-1} \int_0^T \langle \lambda, (\theta t S_t f_T)^2\rangle dt. \tag{2.16}$$

We have

$$\lim_{T\to\infty} A_d(T,f) = C_d \theta^2 \langle \lambda, f\rangle^2, \tag{2.17}$$

where $C_d = (4\pi)^{-d/2} \int_0^1 s^{2-d/2} ds$ for $3 \leq d \leq 5$, $C_6 = (4\pi)^{-3}$.

Proof. From (2.16)

$$\begin{aligned} A_d(T,f) &= b_d(T)^{-1} l_d(T)^2 \theta^2 \int_0^T \langle \lambda, (tS_t f)^2\rangle dt \\ &= b_d(T)^{-1} l_d(T)^2 \theta^2 \int_0^T t^2 dt \int\int p(2t,y,z) f(y) f(z) dy dz. \end{aligned}$$

When $3 \leq d \leq 5$,

$$\begin{aligned} \lim_{T\to\infty} A_d(T,f) &= \lim_{T\to\infty} b_d(T)^{-1} l_d(T)^2 T^{3-d/2} \theta^2 \int_0^1 t^2 dt \int\int (4\pi t)^{-d/2} e^{-\frac{|y-z|^2}{2Tt}} f(y) f(z) dy dz \\ &= C_d \theta^2 \langle \lambda, f\rangle^2 \end{aligned}$$

by dominated convergence theorem and condition (1.8). When $d=6$,

$$\begin{aligned}\lim_{T\to\infty} A_d(T,f) &= \lim_{T\to\infty} b_d(T)^{-1}l_d(T)^2\theta^2\int_1^T t^2dt\int\int p(2t,y,z)f(y)f(z)dydz \\ &= \lim_{T\to\infty} b_d(T)^{-1}l_d(T)^2\theta^2\log T\int_0^1 T^{3s}ds\int\int(4\pi T^s)^{-3}e^{-\frac{|y-z|^2}{2T^s}}f(y)f(z)dydz \\ &= C_d\theta^2\langle\lambda,f\rangle^2\end{aligned}$$

by dominated convergence theorem, where we have taken the transformation $t=T^s$ at the second step. Completes the proof. □

Lemma 2.4. *Let $3\le d\le 6$, for any θ fixed,*

$$\varepsilon_d(T,f):=b_d(T)^{-1}\left|\int_0^T\langle\lambda,(\theta tS_tf_T)^2\rangle dt-\int_0^T\langle\lambda,u_T^2(t,\cdot)\rangle dt\right|, \tag{2.18}$$

we have

$$\lim_{T\to\infty}\varepsilon_d(T,f)=0. \tag{2.19}$$

Proof. From equation (2.5),

$$\begin{aligned}\left|\theta tS_tf_T(x)\right]^2-u_T^2(t,x)\right| &\le 2[|\theta|tS_tf_T(x)\int_0^t S_{t-s}u_T^2(s,\cdot)(x)ds+\left[\int_0^t S_{t-s}u_T^2(s,\cdot)(x)ds\right]^2 \\ &:= I+II.\end{aligned}$$

Let C denotes a positive constant and it may be different values at different line. Recall the useful inequality $S_tf(x)\le C(1\wedge t^{-d/2})$, by (2.11),

$$\begin{aligned}b_d(T)^{-1}\int_0^T\langle\lambda,I\rangle dt &\le 2|\theta|C_d(T,\theta)b_d(T)^{-1}\int_0^T\langle\lambda,[tS_tf_T]\int_0^t S_{t-s}(sS_sf_T)^2ds\rangle dt \\ &\le C|\theta|C_d(T,\theta)b_d(T)^{-1}l_d(T)^3\int_0^T t\langle\lambda,(S_tf)^2\int_0^t s^2(1\wedge s^{-d/2})ds\rangle dt \\ &\le C|\theta|C_d(T,\theta)b_d(T)^{-1}l_d(T)^3\int_0^T t(1\wedge t^{-d/2})dt\cdot\int_0^T s^2(1\wedge s^{-d/2})dt \\ &\longrightarrow 0,\end{aligned}$$

by condition (1.8) and note that $C_d(T,\theta)\to\theta$ as $T\to\infty$. Similar calculation can be done for the second part II, we omit the details. Completes the proof. □

Proof of Theorem 1.2. By (2.15), Lemma 2.3 and Lemma 2.4, when $3\le d\le 6$ for any θ,

$$\begin{aligned}\Lambda(T,\theta) &= b_d(T)^{-1}\int_0^T\langle\lambda,u_T^2(s,\cdot)\rangle ds \\ &\longrightarrow C_d\theta^2\langle\lambda,f\rangle^2:=\Lambda_d(\theta),\end{aligned} \tag{2.20}$$

as $T\to\infty$, where C_d is given in Lemma 2.3. It is easy to get the Legendre transform of $\Lambda_d(\theta)$, i.e.,

$$I(x):=\sup_\theta[\theta x-\Lambda_d(\theta)]=\frac{x^2}{4C_d}.$$

Thus the results followed from the general large deviation result *Gärtner – Ellis* Theorem [cf. Dembo & Zeitouni [4]]. □

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