Light-tailed behavior of stationary distribution for state-dependent random walks on a strip*

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Abstract: In this paper, we consider the state-dependent reflecting random walk on a half-strip.We provide explicit criteria for (positive) recurrence, and an explicit expression for the stationary distribution. As a consequence, the light-tailed behavior of the stationary distribution is proved under appropriate conditions. The key idea of the method employed here is the decomposition of the trajectory of the random walk and the main tool is the intrinsic branching structure buried in the random walk on a strip, which is different from the matrix-analytic method.

Keywords: random walk on a strip, stationary distribution, light-tailed behavior, branching process, recurrence, state-dependent.

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1 Introduction and Main results

Let $d \geq 1$ be any integer and denote $\mathscr{D} = \{1, 2, \cdots, d\}$. We consider the reflecting space-inhomogeneous and state-dependent reflecting random walk on a half-strip $S = \{0, 1, 2, \ldots\} \times \mathscr{D}$. This model is often referred to as the state-dependent quasi-birth-and-death (QBD) process in queueing theory. Studies on the state-dependent QBD process has been centered at its stationary distribution such as properties of the rate matrices, efficient algorithms for computations, often through the matrix-analytic approach or the censoring techniques (e.g. [12], [2] and [16]). In this paper, we propose a different method to decompose the trajectory of the random walk on the strip using the intrinsic branching structure ([8]), through which, we provide criteria for (positive) recurrence, obtain an expression for the stationary distribution of the walk, and characterize the exponential tail asymptotic behavior of the stationary distribution for the walk.

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We consider the random walk $X_n = (\xi_n, Y_n)$, $n = 0, 1, \ldots$, on a half-strip, which is a Markov chain with $\xi_n \in \{0, 1, 2, \ldots\}$, referred to as the layer (or level), and $Y_n \in \mathscr{D}$. Let $L_i = \{(i, r);\ r = 1, 2, \cdots, d\}$. Then, the half-strip S can be expressed as $S = \cup_{i=0}^{+\infty} L_i$. The transition probability of the walk is given by

$$\breve{P} = \begin{pmatrix} \mathbf{0} & P_0 & & & & \\ Q_1 & R_1 & P_1 & & & \\ & Q_2 & R_2 & P_2 & & \\ & & Q_3 & R_3 & P_3 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

where P_0 is a $d \times d$ stochastic matric and $\{(P_n, Q_n, R_n),\ n \in \mathbb{Z}^+ = \{1, 2, \ldots\}\}$, satisfies $(P_n + Q_n + R_n)\mathbf{1} = \mathbf{1}$ with $\mathbf{1}$ being a column vector of ones. For a matrix A, its (i, j)th component is denoted by $A(i, j)$, $1 \le i, j \le d$. Following [1], we assume the following conditions, under which the process is irreducible:

C_1**.** $\log(1 - \|R_n + P_n\|)^{-1} < \infty$ and $\log(1 - \|R_n + Q_n\|)^{-1} < \infty$;

C_2**.** For any n and any j, $\sum_{i=1}^{d} P_n(i, j) > 0$ and $\sum_{i=1}^{d} Q_n(i, j) > 0$;

C_3**.** The layer 0 is in one communication class.

To state the main results, we introduce the following notation or definitions: $\mathbf{e}_i = (\underbrace{0, \ldots, 0, 1, 0, \ldots, 0}_{i\text{th component}})$; I_A is the indicator function of the set A; B' and $\mathbf{b}'$ are the transposes of matrix B and vector $\mathbf{b}$, respectively; $\mathbf{1} = (1, 1, \cdots, 1)'$; $P_{\mu_n}(\cdot)$ represents the probability given that the random walk starts from layer n with the distribution μ_n; $P_{(n,i)}(\cdot)$ means the probability given that the walk starts from the site (state) (n, i); $E_{\mu_n}(\cdot)$ and $E_{(n,i)}(\cdot)$ are similarly defined.

Define the hitting times T_n and T_n^+ by $T_n = \inf\{t \ge 0 :\ X_t \in L_n\}$ and $T_n^+ = \inf\{t \ge 1 :\ X_t \in L_n\}$, respectively. Define $f_{x,y}^{(n)}$ and $f_{x,y}$ respectively as $f_{x,y}^{(n)} = P_x(T_y^+ = n) = P_x(X_n \in L_y,\ X_m \notin L_y,\ 1 \le m < n)$ and $f_{x,y} = \sum_{n=1}^{+\infty} f_{x,y}^{(n)}$. Also, define

$$E_x(T_y^+) = \begin{cases} \sum_{n=1}^{+\infty} n f_{x,y}^{(n)} & \text{if } f_{x,y} = 1, \\ +\infty & \text{if } f_{x,y} < 1. \end{cases}$$

Definition 1. The layer y is (layer) recurrent if $f_{y,y} = 1$, otherwise it is (layer) transient. If $E_y(T_y^+) < +\infty$, the layer y is called (layer) positive recurrent.

Define recursively for $n \in \{0, 1, 2, \ldots\}$,

$$\zeta_0^+ = P_0 \quad \text{and} \quad \zeta_n^+ = (I - Q_n\zeta_{n-1}^+ - R_n)^{-1} P_n, \quad n = 1, 2, \ldots. \tag{1.2}$$

The existence of $(I - Q_n\zeta_{n-1}^+ - R_n)^{-1}$ is a consequence of assumption (C_2). Note that P_0 is stochastic, so is ζ_n^+ (see [1]). Define also $\mathbf{u}_0^+ = \mathbf{1}$, and for $n \ge 1$,

$$A_n^+ = (I - Q_n\zeta_{n-1}^+ - R_n)^{-1} Q_n \quad \text{and} \quad \mathbf{u}_n^+ = (I - Q_n\zeta_{n-1}^+ - R_n)^{-1}\mathbf{1}. \tag{1.3}$$

Remark 1.1 *$\zeta_n^+(i,j)$ has an interpretation as the probability of the random walk starting from (n,i) and with the reflection at layer 0 reaches layer $n+1$ at point $(n+1,j)$, often referred to as the exit probability and denoted as η_n as well in our model. Also, $A_n^+(i,j)$ can be interpreted as the expected number of steps from (n,i) to $(n-1,j)$ caused by a step from layer $n+1$ to (n,i) (see (2.3), i.e., the mean offspring of the "father" step from layer $n+1$ to (n,i)). It is worthwhile to mention that the rate matrix and the fundamental period matrix are key probabilistic quantities in studying the level-independent QBD process, They are generalized into two matrix sequences R_n^+ and G_n^-, respectively, when the method is used to study the level-dependent QBD process (e.g., [12] and [16]). Their dual versions R_n^- and G_n^+ also play an important role in the study using the matrix-analytic method (e.g. [13]). The matrix ζ_n^+ is the same as G_n^+, while A_n^+ is a unique quantity from the branching process method, which is not a usual measure used in the matrix-analytic method.*

The first group of results are conditions for recurrence and positive recurrence of the walk.

Theorem 1.1 *For the random walk starts from layer 0 with an initial distribution μ, define*

$$\beta^+ = \sum_{k=0}^{+\infty} \mu_k A_k^+ A_{k-1}^+ \cdots A_1^+ \mathbf{1},$$

where $\mu_k = \mu\zeta_0^+\zeta_1^+ \cdots \zeta_{k-1}^+$. Then the random walk is recurrent if and only if $\beta^+ = \infty$.

Remark 1.2 *Actually, β^+ in the Theorem is the expectation number of the visiting times by the random walk at layer 0 , which can be calculated by the means of the intrinsic branching structure within the walk.*

To state the criteria for the positive recurrence, we need the "exit probability" from the other direction. Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $a \in \mathbb{Z}$. For $n \leq a$, define recursively

$$\zeta_{a,a}^- = \rho \quad \text{and} \quad \zeta_{n,a}^- = (I - P_n\zeta_{n+1,\ a}^- - R_n)^{-1}Q_n, \quad n < a, \tag{1.4}$$

where ρ is stochastic, i.e., $\rho\mathbf{1} = \mathbf{1}$. Then under condition C, the limit $\zeta_n^- = \lim_{a\to\infty}\zeta_{n,a}^-$ exist and satisfy the following equation (Theorem 1, [1]),

$$\zeta_n^- = (I - P_n\zeta_{n+1}^- - R_n)^{-1}Q_n, \quad n \in \mathbb{Z}. \tag{1.5}$$

Define for $n \geq 1$,

$$A_n^- = (I - P_n\zeta_{n+1}^- - R_n)^{-1}P_n \quad \text{and} \quad \mathbf{u}_n^- = (I - P_n\zeta_{n+1}^- - R_n)^{-1}\mathbf{1}. \tag{1.6}$$

Remark 1.3 *$\zeta_n^-(i,j)$ has an interpretation as the probability of the random walk starting from (n,i) reaches layer $n-1$ at point $(n-1,j)$, which is the same as G_n^-. Also, $A_n^-(i,j)$ is the same as $A_n^+(i,j)$ but from the other direction, can be interpreted as the expected number of steps from (n,i) to $(n+1,j)$ caused by a step from layer $n-1$ to (n,i), which is unique to the branching process method.*

For convenience, let $\mathbf{u}_0^- = 1$.

Theorem 1.2 *For the random walk starting from layer* 0 *with an initial distribution* μ, *define*

$$\varrho_1^+ = \mathbf{1}' P_0 \left(\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^- \right) + d.$$

Then the random walk is positive recurrent if and only if $\varrho_1^+ < \infty$.

Remark 1.4 *Actually,* $\varrho^+ = \mu P_0 \left(\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^-\right) + \mu \mathbf{1}$ *is the expectation of the first return time of the random walk start at layer* 0 *, which can be calculated by means of the intrinsic branching structure within the walk. Note that* $\varrho^+ \leq \varrho_1^+$*, the criteria is independent of the initial distribution of the walk start at layer* 0.

When the walk is state-independent, i.e., $(P_n, Q_n, R_n) = (P, Q, R)$ for $n > 0$, we denote the walk as $\{\overline{X}_n,\ n \geq 0\}$, and have correspondingly

$$\zeta^- = (I - P\zeta^- - R)^{-1} Q,$$

and

$$A^- = (I - P\zeta^- - R)^{-1} P,\ \mathbf{u}^- = (I - P\zeta^- - R)^{-1} \mathbf{1}. \tag{1.7}$$

Corollary 1.3 *Suppose that the random walk* $\{\overline{X}_n,\ n \geq 0\}$ *starts from layer* 0 *with an initial distribution* $\overline{\mu}$. *Then*

(1) The random walk is positive recurrent if and only if

$$\bar{\varrho}_1^+ = \mathbf{1}' P \ (\sum_{k\geq 1} \ (A^-)^{k-1} \ \mathbf{u}^-) + d < \infty,$$

(2) Denote the maximum eigenvalues of A^- *as* λ_{A^-}. *Then* $\lambda_{A^-} < 1$ *whenever* $\bar{\varrho}_1^+ < \infty$.

We now state the main result for the stationary distribution. We assume that the walk is positive recurrent and start from layer 0 with a "proper" distribution. The so called "proper" distribution is the "censored measure", a terminology borrowed from queueing theory (e.g. [16]). Define $\breve{P}_1$ by

$$\breve{P}_1 = \begin{pmatrix} R_1 & P_1 & & & \\ Q_2 & R_2 & P_2 & & \\ & Q_3 & R_3 & P_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let $S_0 = L_0$ and let $S_1 = S/S_0$ be a partition of the state space S. Then $\breve{P}$ can be partitioned according to S_0 and S_1 as

$$\breve{P} = \begin{pmatrix} \breve{P}^0 & U \\ D & \breve{P}_1 \end{pmatrix},$$

where $\breve{P}^0 = R_0$, $U = (P_0, O, O, \ldots)$ and $D = (Q_1, O, O, \ldots)^T$.

The censored matrix $\breve{P}^{(S_0)}$ of $\breve{P}$ with the censoring set S_0 is defined by

$$\breve{P}^{(S_0)} = \breve{P}^0 + U\widehat{\breve{P}^1}D,$$

where $\widehat{\breve{P}^1} = \sum_{k=0}^{+\infty}(\breve{P}^1)^k$ is called the fundamental matrix of $\breve{P}^1$. $\breve{P}^{(S_0)}$ is a $d \times d$ matric, and the censored matrix $\breve{P}^{(S_0)}$ has a probabilistic interpretation: it is the probability that the next state visited in S_0 is j, given that the process starts in state $i \in S_0$.

A measure $\breve{\mu}_0$ which satisfies

$$\breve{\mu}_0\breve{P}^{(S_0)} = \breve{\mu}_0 \tag{1.8}$$

is called as censored measure with censoring set S_0.

Theorem 1.4 *If $\varrho_1^+ < \infty$ and the walk starts from layer 0 with the censored measure $\breve{\mu}_0$, then the stationary distribution $\{\nu_n,\ n = 0, 1, 2, \ldots\}$ exists and unique, which can be expressed explicitly as*

$$\nu_n = \frac{\breve{\mu}_0 P_0 A_1^- A_2^- \cdots A_{n-1}^- \widetilde{u}_n^-}{\breve{\mu}_0 P_0 (\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^-) + \breve{\mu}_0 \mathbf{1}}, \quad n > 0; \tag{1.9}$$

and

$$\nu_0 = \frac{\breve{\mu}_0}{\breve{\mu}_0 P_0 (\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^-) + \breve{\mu}_0 \mathbf{1}}, \tag{1.10}$$

where $\breve{\mu}_0 P_0 (\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^-) + \breve{\mu}_0 \mathbf{1} < \varrho_1^+ < \infty$ and $\widetilde{u}_n^- = (I - P_n\zeta_{n+1}^- - R_n)^{-1}$.

Remark 1.5 *We can show that the expression in Theorem 1.4 is consistent with the matrix-product form solution given by the matrix-analytic method:*

$$\nu_n = \nu_0 R_1^+ R_2^+ \cdots R_n^+.$$

To see it, we notice that $\tilde{u}_n^- = (I - P_n\zeta_{n+1}^- - R_n)^{-1}$ is the entry $\widehat{P}_{n,n}^{(n)}$ of the fundamental matrix. Then, according to

$$R_n^+ = P_{n-1}\widehat{P}_{n,n}^{(n)},$$

we can have the equivalence. For details, readers may refer to [12], [15] and [16].

The expression of the stationary distribution for the state-dependent walk in Theorem 1.4 enable us to obtain the following asymptotic behavior. Let $D = \{(P, Q, R) :\ (P + Q + R)\mathbf{1} = \mathbf{1},\ \varrho_1^+ < \infty\}$.

Theorem 1.5 *For the random walk on a strip,*

(1) *If $(P, Q, R) \in D$, we have $\lambda_{A^-} < 1$.*

(2) *Suppose that the random walk starts from layer 0 with the censored measure $\breve{\mu}_0$, and the transition probabilities satisfy $(P_n, Q_n, R_n) \to (P, Q, R)$ as $n \to \infty$ with $(P, Q, R) \in D$. Then the random walk is positive recurrent and the stationary distribution $\{\nu_n,\ n \geq 0\}$ defined in (1.9) is light-tailed, with the decay rate $0 < \lambda_{A^-} \leq 1$ along the layer direction, that is, for each fixed $1 \leq j \leq d$,*

$$\lim_{n\to\infty} \frac{\log \nu_n(j)}{n} = \log \lambda_{A^-}, \tag{1.11}$$

where λ_{A^-} is the maximum eigenvalues of A^- (given in(1.7)).

Example 1.6 (A retrial queue with a state-dependent retrial rate) This model is a modification of the standard $M/M/c$ retrial queue (for example, see Falin and Templeton [5]). In the modified model, instead of the retrial rate $n\theta$, we assume the total retrial rate is θ_n, where n is the number of customers in the retrial orbit. For this model, let $N(t)$ and $C(t)$ be the number of retrial customers in the orbit and the number of busy servers at time t, respectively. Then, it is easy to see that $(N(t), C(t))$ is a continuous-time Markov chain. We show how to apply Theorem 1.5 to obtain the exponential decay rate. For this purpose, assume that $\theta_n \to \theta < \infty$ as $n \to \infty$. Then, the generator of the limiting chain is given by

$$Q = \begin{pmatrix} B_0 & A & & & \\ C & B & A & & \\ & C & B & A & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.12}$$

where

$$B = \begin{pmatrix} -(\lambda+\theta) & \lambda & & & \\ \mu & -(\lambda+\mu+\theta) & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & (c-1)\mu & -[\lambda+(c-1)\mu+\theta] & \lambda \\ & & & c\mu & -(\lambda+c\mu) \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & \lambda \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & \theta & & & \\ & 0 & \theta & & \\ & & \ddots & \ddots & \\ & & & 0 & \theta \\ & & & & 0 \end{pmatrix}.$$

Without loss of generality, we assume $\lambda + c\mu + \theta = 1$. Upon uniformization, we can convert the generator to a transition matrix $\breve{P} = I - Q$ to have (P, Q, R). To determine the condition for positive recurrence and λ_{A^-}, we use Theorem 1.4 and (1.7), respectively. First, (1.7) is equivalent to the equation $R^+ = P + R^+R + {R^+}^2Q$. To solve this equation, we notice that

$$R^+ = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ r_1 & r_2 & \cdots & r_c \end{bmatrix},$$

which greatly simplify the calculations. Also, we can find that the chain is positive recurrent if and only if $r_c < 1$. For example, when $c = 1$, $\lambda_{A^-} = r_c = \lambda(\lambda+\theta)/\mu\theta$ and when $c = 2$,

$$\lambda_{A^-} = r_c = \frac{\lambda}{\theta\mu}\frac{(\lambda+\theta)^2+\theta\mu}{3\lambda+2\mu+2\theta}.$$

As c gets larger, the formula becomes cumbersome and is less interesting.

We arrange the remainder of this paper as follows. As the main tool of this paper, the intrinsic branching structure within random walk on a strip is briefly reviewed in Section 2; and then the proofs for the Theorems are followed in Section 3.

2 A brief review for the intrinsic branching structure within random walk on a strip

The intrinsic branching structure within a random walk is a very powerful tool in the research on the limit property about random walk. For the neighborhood nearest random walk on the line, Dwass ([4], 1975) and Kesten et al. ([10], 1975) observed a Galton-Watson process with a geometric offspring distribution hidden in it. Kesten *et al.* ([10]) proved a stable law by using the branching structure for the random walk in a random environment. For other random walks, e.g., the random walk with bounded jumps, the branching structure were revealed by Hong & Wang ([6], 2009) for the $(L,1)$-case and Hong & Zhang ([7], 2010) for the $(1,R)$-case.

The intrinsic branching structure within random walk on a strip has been revealed by Hong & Zhang ([8], 2012), which enables us to provide explicit criteria for (positive) recurrence and to obtain an explicit expression for the stationary distribution. As a consequence, it allows us to consider the tail asymptotic of the stationary distribution. The key point is the trajectory decomposition for the random walk. If the walk starts from layer $n > 0$, the trajectory has the "upper" and "lower" parts, which are introduced in the following subsections.

2.1 The "lower" branching structure

Assume that $X_0 \in L_k$, the random walk starts from layer k with initial distribution μ_k or $\mu_k(i) = P(\xi_0 = k,\ Y_0 = i)$. For $0 < n \le k$ and $i \in \{1,2,\cdots,d\}$, define U_n^i as the number of steps from layer n to $(n-1,i)$ before the hitting time T_{k+1}, and Z_n^i as the number of steps from layer n to (n,i) before T_{k+1}. Define

$$\mathbf{U}_n = (U_n^1, U_n^2, \cdots, U_n^d), \quad \mathbf{Z}_n = (Z_n^1, Z_n^2, \cdots, Z_n^d),$$

and $|\mathbf{U}_n| = \mathbf{U}_n\mathbf{1}$, $|\mathbf{Z}_n| = \mathbf{Z}_n\mathbf{1}$.

Theorem 2.1 *(Hong & Zhang, 2012) Suppose that Condition C is satisfied, and the random walk starts from layer k with initial distribution μ_k. Then* $\{|\mathbf{U}_n|,\ 1 < n \le k\}$

and $\{|\mathbf{Z}_n|,\ 1 < n \le k\}$ *are inhomogeneous branching processes with immigration. The offspring distribution* $(1 < n \le k)$ *is given as:*

$$\begin{aligned} P\big(|\mathbf{U}_n| = m\big|\mathbf{U}_{n+1} = \mathbf{e}_i\big) &= \mathbf{e}_i[(I-R_n)^{-1}Q_n\zeta_{n-1}^+]^m(I-R_n)^{-1}P_n\mathbf{1}, \\ P\big(|\mathbf{Z}_n| = K\big|\mathbf{U}_{n+1} = \mathbf{e}_i\big) &= \mathbf{e}_i[(I-Q_n\zeta_{n-1}^+)^{-1}R_n]^K(I-Q_n\zeta_{n-1}^+)^{-1}P_n\mathbf{1}, \end{aligned}$$

with immigration

$$P\big(\mathbf{U}_{k+1} = \mathbf{e}_i\big) = \mu_k(i), \quad i \in \mathscr{D}.$$

□

The key idea in the construction of the branching mechanism is that the position of the walk corresponds to the time of the branching process. $|\mathbf{U}_n|$ as the number of steps from layer n to $n-1$ layer is indeed the $(k-n)$-th generation of the branching process. The condition $T_{k+1} < \infty$ is obviously satisfied in our reflecting model.

Proposition 2.2 *Denote* N_n^i *as the number of steps visited at* (n, i) *before time* T_{k+1}, *and* $\mathbf{N}_n = (N_n^1, N_n^2, \cdots, N_n^d)$ *with* $|\mathbf{N}_n| = \mathbf{N}_n\mathbf{1}$. *Suppose that Condition C is satisfied, and the random walk starts from layer* k *with initial distribution* μ_k. *Then for any* $0 < n \le k$,

$$\begin{aligned} E_{\mu_k}(\mathbf{N}_n) &= \mu_k A_k^+ A_{k-1}^+ \cdots A_{n+2}^+ A_{n+1}^+ (I - Q_n\zeta_{n-1}^+ - R_n)^{-1}, \\ E_{\mu_k}(|\mathbf{N}_n|) &= \mu_k A_k^+ A_{k-1}^+ \cdots A_{n+2}^+ A_{n+1}^+ \mathbf{u}_n^+, \end{aligned} \tag{2.1}$$

and for $n = 0$,

$$E_{\mu_k}(\mathbf{N}_0) = \mu_k A_k^+ A_{k-1}^+ \cdots A_1^+, \quad E_{\mu_k}(|\mathbf{N}_0|) = \mu_k A_k^+ A_{k-1}^+ \cdots A_1^+\mathbf{1}. \tag{2.2}$$

Proof. We provide key steps here and readers may refer to [8] for details. Note that

$$E(\mathbf{U}_n|\mathbf{U}_{n+1}) = \mathbf{U}_{n+1}\sum_{m=1}^{+\infty}[(I-R_n)^{-1}Q_n\zeta_{n-1}^+]^{m-1}(I-R_n)^{-1}Q_n = \mathbf{U}_{n+1}A_n^+, \tag{2.3}$$

$$\begin{aligned} E(\mathbf{Z}_n|\mathbf{U}_{n+1}) &= \mathbf{U}_{n+1}\sum_{K=1}^{+\infty}[(I-Q_n\zeta_{n-1}^+)^{-1}R_n]^{m-1}(I-Q_n\zeta_{n-1}^+)^{-1}R_n \\ &= \mathbf{U}_{n+1}(I-Q_n\zeta_{n-1}^+ - R_n)^{-1}R_n. \end{aligned}$$

With the help of the branching structure in Theorem 2.1, we have for any $0 < n \le k$,

$$\begin{aligned} E_{\mu_k}(\mathbf{N}_n) &= E_{\mu_k}(\mathbf{U}_n\zeta_{n-1}^+ + \mathbf{Z}_n + \mathbf{U}_{n+1}) \\ &= E_{\mu_k}\big[E_{\mu_k}(\mathbf{U}_n\big|\mathbf{U}_{n+1})\zeta_{n-1}^+ + E_{\mu_k}(\mathbf{Z}_n\big|\mathbf{U}_{n+1}) + E_{\mu_k}(\mathbf{U}_{n+1}\big|\mathbf{U}_{n+1})\big] \\ &= E_{\mu_k}(\mathbf{U}_{n+1})(I-Q_n\zeta_{n-1}^+ - R_n)^{-1}, \end{aligned} \tag{2.4}$$

and then $E_{\mu_k}(|\mathbf{N}_n|) = E_{\mu_k}(\mathbf{U}_{n+1})\mathbf{u}_n$.

For $n = 0$, the expected number of steps visiting the reflecting layer 0 before time T_{k+1} is $E_{\mu_k}(\mathbf{N}_0) = E_{\mu_k}(\mathbf{Z}_0 + \mathbf{U}_1) = E_{\mu_k}(\mathbf{U}_1)$, and then $E_{\mu_k}(|\mathbf{N}_0|) = E_{\mu_k}(\mathbf{U}_1)$. Together with

$$E_{\mu_k}(\mathbf{U}_{n+1}) = E_{\mu_k}[E_{\mu_k}(\mathbf{U}_{n+1}|\mathbf{U}_{n+2})] = E_{\mu_k}(\mathbf{U}_{n+2})A_{n+1}^+ = \cdots = \mu_k A_k^+ A_{k-1}^+ \cdots A_{n+2}^+ A_{n+1}^+,$$

the proof is finished. We usually denote $\mathbf{u}_0^+ = \mathbf{1}$ to make the expression uniform. □

2.2 The "upper" branching structure

Assume that $X_0 \in L_k$, the random walk starts from layer k with initial distribution μ_k or $\mu_k(i) = P(\xi_0 = k,\ Y_0 = i)$. Similarly, For $n \geq k+1$ and $i \in \{1, 2, \cdots, d\}$, define

$$\mathbf{W}_n = (W_n^1, W_n^2, \cdots, W_n^d), \quad \text{and} \quad \mathbf{Z}_n^- = (Z_n^{-,1}, Z_n^{-,2}, \cdots, Z_{-,n}^d),$$

where W_n^i is the number of steps from layer n to $(n+1, i)$ before the hitting time T_{k-1}; and $Z_n^{-,i}$ is the number of steps from layer n to (n, i) before the hitting time T_{k-1}.

Define $|\mathbf{W}_n| = \mathbf{W}_n \mathbf{1}$ and $|\mathbf{Z}_n^-| = \mathbf{Z}_n^- \mathbf{1}$.

Theorem 2.3 *(Hong & Zhang, 2012) Suppose that Condition C is satisfied, and the random walk starts from layer k with initial distribution μ_k and $T_{k-1} < +\infty$. Then $\{|\mathbf{W}_n|,\ n \geq k+1\}$ and $\{|\mathbf{Z}_n^-|,\ n \geq k+1\}$ are inhomogeneous branching processes with immigration. The offspring distribution ($n \geq k+1$) is given by*

$$\begin{aligned} P\big(|\mathbf{W}_n| = m \big| \mathbf{W}_{n-1} = \mathbf{e}_i\big) &= \mathbf{e}_i[(I - R_n)^{-1} P_n \zeta_{n+1}^-]^m (I - R_n)^{-1} Q_n \mathbf{1}, \\ P\big(|\mathbf{Z}_n^-| = K \big| \mathbf{W}_{n-1} = \mathbf{e}_i\big) &= \mathbf{e}_i[(I - P_n \zeta_{n+1}^-)^{-1} R_n]^K (I - P_n \zeta_{n+1}^-)^{-1} Q_n \mathbf{1}, \end{aligned}$$

with immigration

$$P\big(\mathbf{U}_{k-1} = \mathbf{e}_i\big) = \mu_k(i), \quad i \in \mathscr{D}.$$

□

In parallel, for $n \geq k+1$, denote

$$N_n^{-,i} = \sharp\{k \in [0, T_{-1}) :\ X_k = (n, i)\},$$

where $N_n^{-,i}$ is the number of steps visited at (n, i) before the hitting time T_{k-1}. Define $\mathbf{N}_n^- = (N_n^{-,1}, N_n^{-,2}, \cdots, N_n^{-,d})$ and $|\mathbf{N}_n^-| = \mathbf{N}_n^- \mathbf{1}$.

Proposition 2.4 *Suppose that Condition C is satisfied, and the random walk starts from layer k with initial distribution μ_k. Then for any $n \geq k+1$,*

$$\begin{aligned} E_{\mu_k}(\mathbf{N}_n^-) &= E_{\mu_k}(\mathbf{W}_{n-1})(I - P_n \zeta_{n+1}^- - R_n)^{-1} \\ &= \mu_k A_k^- A_{k+1}^- \cdots A_{n-2}^- A_{n-1}^- (I - P_n \zeta_{n+1}^- - R_n)^{-1}, \end{aligned}$$

$$E_{\mu_k}(|\mathbf{N}_n^-|) = E_{\mu_k}(\mathbf{W}_{n-1})(I - P_n \zeta_{n+1}^- - R_n)^{-1} \mathbf{1} = \mu_k A_k^- A_{k+1}^- \cdots A_{n-2}^- A_{n-1}^- \mathbf{u}_n^-. \tag{2.5}$$

3 Proofs

3.1 Criteria for recurrence—*Proof of Theorem 1.1*

Let $T_y^0 = 0$, and let $T_y^k = \inf\{n > T_y^{k-1} : X_n \in L_y\}$ for $k \geq 1$, or T_y^k is the time of the k-th return to layer y. Note that $T_y^1 > 0$. Hence, a possible visit at time 0 does not count, and T_y^1 equals to T_y^+ defined above.

We firstly extend a basic property about Markov chains to the random walk on a strip, which is stated in the following lemma.

Lemma 3.1 *Layer y is recurrent if and only if $E_y(|\mathbf{N}_y|) = +\infty$.*

Proof. Denote $f_{x,y} = P_x(T_y^+ < \infty)$. Then,

$$P_x(T_y^k < \infty) = f_{x,y} f_{y,y}^{k-1}.$$

This is clear, since in order to visit layer y for exactly the k-th time, the walk has to go from layer x to layer y first, and then return to layer y $k-1$ times. A detailed formal proof is similar to that for Theorem 3.1 in [3] for the random walk on a line.

Recall that $|\mathbf{N}_y| = \mathbf{N}_y \mathbf{1} = \sum_{m=1}^{+\infty} I_{\{X_m \in L_y\}}$ is the number of visits to layer y at positive times. By the definition, layer y is transient if and only if $f_{y,y} < 1$. Suppose that layer y is transient, then

$$\begin{aligned} E_x(|\mathbf{N}_y|) &= \sum_{k=1}^{+\infty} P_x(\mathbf{N}_y \mathbf{1} \geq k) = \sum_{k=1}^{+\infty} P_x(T_y^k < \infty) \\ &= \sum_{k=1}^{+\infty} f_{x,y} f_{y,y}^{k-1} = \frac{f_{x,y}}{1 - f_{y,y}} < +\infty. \end{aligned}$$

Thus, layer y is recurrent if and only if $E_y(|\mathbf{N}_y|) = +\infty$. □

Proof of Theorem 1.1 Because the random walk is irreducible, we only need to calculate the $E_\mu(|\mathbf{N}_0|)$, where the walk starts at layer 0 with distribution μ, and $|\mathbf{N}_0| = \sum_{i=0}^{\infty} 1_{(X_i \in L_0)}$ is the occupation time of the walk at layer 0. We can decompose the trajectory of the walk as the summation of infinite "pieces", each "piece" is an immigration ("lower") branching structure as considered in Theorem 2.1. In fact, by recalling the definition of the hitting times $T_k = \inf\{i : X_i \in L_k\}$ for layer k and denoting $X^{(\tau_k)} = \{X_i, T_k < i \leq T_{k+1}\}$, we can write

$$\{X_i, i > 0\} = \bigcup_{k=0}^{+\infty} \{X_i, T_k < i \leq T_{k+1}\} = \bigcup_{k=0}^{+\infty} X^{(\tau_k)}, \tag{3.1}$$

and as a consequence,

$$|\mathbf{N}_0| = \sum_{i=0}^{+\infty} 1_{(X_i \in L_0)} = \sum_{k=0}^{+\infty} \sum_{i=T_k}^{T_{k+1}} 1_{(X_i \in L_0)} = \sum_{k=0}^{+\infty} |\mathbf{N}_0|^{(\tau_k)}, \tag{3.2}$$

where the superscript is used to emphasize that the process starts at T_k. For $k = 0, 1, \ldots$, each trajectory "piece" $X^{(\tau_k)} = \{X_i, T_k < i \le T_{k+1}\}$ formulates a branching structure with immigration $P\big(\mathbf{U}_{k+1} = \mathbf{e}_i\big) = \mu_k(i)$, where $\mu_k = \mu\zeta_0^-\zeta_1^+ \cdots \zeta_{k-1}^+$. By (2.2) of Proposition 2.2 we have,

$$E_{\mu_k}(|\mathbf{N}_0|^{(\tau_k)}) = \mu_k A_k^+ A_{k-1}^+ \cdots A_1^+ \mathbf{1}.$$

Combining with (3.2),

$$E_\mu|\mathbf{N}_0| = \sum_{k=0}^{+\infty} E_{\mu_k}|\mathbf{N}_0|^{(\tau_k)} = \sum_{k=0}^{+\infty} \mu_k A_k^+ A_{k-1}^+ \cdots A_1^+ \mathbf{1} = \beta^+.$$

The proof is complete. □

3.2 Criteria for positive recurrence—*Proof of Theorem 1.2*

Define $\overline{T}_n^n$ as the return time of layer n when the random walk starting from layer n, $\overline{T}_n^{(n-1,j)}$ the hitting time of layer n when the random walk starting from $(n-1, j)$, and $\overline{T}_n^{(n+1,j)}$ the hitting time of layer n when the random walk starting from $(n+1, j)$. Then by the path decomposition,

$$\overline{T}_n^n = \sum_j I_{X_1=(n-1,j)}\overline{T}_n^{(n-1,j)} + \sum_j I_{X_1=(n+1,j)}\overline{T}_n^{(n+1,j)} + \sum_j I_{X_1=(n,j)},$$

and therefore,

$$\begin{aligned}
E_{\mu_n}(T_n^+) &= E_{\mu_n}(\overline{T}_n^n) \\
&= E_{\mu_n}\big(\sum_j I_{X_1=(n-1,j)}(\overline{T}_n^{(n-1,j)} + 1) + \sum_j I_{X_1=(n+1,j)}(\overline{T}_n^{(n+1,j)} + 1) + \sum_j I_{X_1=(n,j)}\big) \\
&= \sum_j P_{\mu_n}(X_1 = (n-1, j))(E_{(n-1,j)}T_n^+ + 1) \\
&\quad + \sum_j P_{\mu_n}(X_1 = (n+1, j))(E_{(n+1,j)}T_n^+ + 1) + \sum_j P_{\mu_n}(X_1 = (n, j)).
\end{aligned}$$

Note that

$$E_{(n-1,j)}(T_n^+) = \sum_{k=0}^{n-1} E_{(n-1,j)}(|\mathbf{N}_k|) \quad \text{and} \quad E_{(n+1,j)}(T_n^+) = \sum_{k=n+1}^{+\infty} E_{(n+1,j)}(|\mathbf{N}_n^-|).$$

It follows from (2.1) and (2.5) that

$$\begin{aligned}
E_{(n-1,j)}(T_n^+) &= \mathbf{e}_j \sum_{k=0}^{n-1} A_{n-1}^+ A_{n-2}^+ \cdots A_{k+2}^+ A_{k+1}^+ \mathbf{u}_k^+, \\
E_{(n+1,j)}(T_n^+) &= \mathbf{e}_j \sum_{k=n+1}^{+\infty} A_{n+1}^- A_{n+2}^- \cdots A_{k-1}^- \mathbf{u}_k^-.
\end{aligned}$$

Hence we have

$$\begin{aligned}E_{\mu_n}(T_n^+) &= \mu_n Q_n(\sum_{k=0}^{n-1} A_{n-1}^+ A_{n-2}^+ \cdots A_{k+1}^+ \mathbf{u}_k^+) + \sum_j P_{\mu_n}\big(X_1 = (n-1,j)\big) \\ &\quad + \mu_n P_n(\sum_{k=n+1}^{+\infty} A_{n+1}^- A_{n+2}^- \cdots A_{k-1}^- \mathbf{u}_k^-) + \sum_j P_{\mu_n}\big(X_1 = (n+1,j)\big) \\ &\quad + \sum_j P_{\mu_n}\big(X_1 = (n,j)\big) \\ &= \mu_n Q_n(\sum_{k=0}^{n-1} A_{n-1}^+ A_{n-2}^+ \cdots A_{k+1}^+ \mathbf{u}_k^+) + \mu_n P_n(\sum_{k=n+1}^{+\infty} A_{n+1}^- A_{n+2}^- \cdots A_{k-1}^- \mathbf{u}_k^-) + \mu_n \mathbf{1}.\end{aligned}$$

Particularly, if the random walk starts from layer 0 with an initial distribution μ, we have

$$E_\mu(T_0^+) = \mu P_0(\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^-) + \mu \mathbf{1}. \tag{3.3}$$

Thus the reflecting random walk on a strip is positive recurrent (independent of the initial distribution μ) if and only if $\varrho_1^+ < \infty$. □

3.3 Stationary distribution—*Proof of Theorem 1.4*

Suppose that the random walk starts from layer 0 with a censored measure $\breve{\mu}_0$, which satisfies

$$\breve{\mu}_0 \breve{P}^{(S_0)} = \breve{\mu}_0.$$

The following lemma modifies Thm 5.4.3 in [3] about a stationary measure of a general Markov chain on $\mathbb{Z}^d$ to our model, and defines a stationary measure for the random walk on a strip.

Lemma 3.2 *Suppose that the random walk starts from layer* 0 *with a censored measure* $\breve{\mu}_0$*, and layer* 0 *is a recurrent layer. Then* $\{\overline{\nu}_n,\ n \in \mathbb{N}\}$ *defines a stationary measure, where*

$$\overline{\nu}_n(i) = E_{\breve{\mu}_0}\left(\sum_{m=0}^{T_0^+-1} I_{\{X_m=(n,i)\}}\right) = \sum_{m=0}^{+\infty} P_{\breve{\mu}_0}(X_m = (n,i),\ m < T_0^+). \tag{3.4}$$

Proof. The key idea of the proof is to use the "cycle trick". $\overline{\nu}_n(i)$ is the expected number of visits to (n,i) at times $0,1,\ldots,T_0^+-1$. And $\sum_{y,j} \overline{\nu}_y(j) p\big[(y,j),(n,i)\big]$ is the expected number of visits to (n,i) at times $1,2,\ldots,T_0^+$, which equals to $\overline{\nu}_n(i)$ since $X_{T_0^+} \sim \breve{\mu}_0$ if $X_0 \sim \breve{\mu}_0$ based on the property of $\breve{\mu}_0 \breve{P}^{(S_0)} = \breve{\mu}_0$.

The goal is to prove that $\overline{\nu}_n$ defined in (3.4) is a stationary measure, that is,

$$\sum_{y,j} \overline{\nu}_y(j) p\big[(y,j),(n,i)\big] = \overline{\nu}_n(i). \tag{3.5}$$

By Fibini's theorem, we get

$$\sum_{y,j}\overline{\nu}_y(j)p\big[(y,j),(n,i)\big]=\sum_{m=0}^{+\infty}\sum_{y,j}P_{\breve{\mu}_0}(X_m=(y,j),\ m<T_0^+)p\big[(y,j),(n,i)\big].$$

Case 1: $n\neq 0$. In this case, we have

$$\begin{aligned}&\sum_{y,j}P_{\breve{\mu}_0}(X_m=(y,j),\ m<T_0^+)p\big[(y,j),(n,i)\big]\\=&\sum_{y,j}P_{\breve{\mu}_0}(X_m=(y,j),\ m<T_0^+,\ X_{m+1}=(n,i))\\=&\ P_{\breve{\mu}_0}(T_0^+>m+1,\ X_{m+1}=(n,i)),\end{aligned}$$

and then

$$\begin{aligned}\sum_{y,j}\overline{\nu}_y(j)p\big[(y,j),(n,i)\big]&=\sum_{m=0}^{+\infty}\sum_{y,j}P_{\breve{\mu}_0}(X_m=(y,j),\ m<T_0^+)p\big[(y,j),(n,i)\big]\\&=\sum_{m=0}^{+\infty}P_{\breve{\mu}_0}\big(T_0^+>m+1,\ X_{m+1}=(n,i)\big).\\&=\sum_{m=0}^{+\infty}P_{\breve{\mu}_0}(X_m=(n,i),\ m<T_0^+)=\overline{\nu}_n(i),\end{aligned}\tag{3.6}$$

because $P_{\breve{\mu}_0}(T_0^+>0,\ X_0=(n,i))=0$.

Case 2: $n=0$. At first, note that the process starts from layer 0 with the initial distribution $\breve{\mu}_0$, i.e., the right hand side of (3.5) is $\breve{\mu}_0$. For the left hand side of (3.5), we calculate

$$\begin{aligned}&\sum_{y,j}P_{\breve{\mu}_0}\big(X_m=(y,j),\ m<T_0^+\big)p\big[(y,j),(n,i)\big]\\=&\sum_{y,j}P_{\breve{\mu}_0}\big(X_m=(y,j),\ m<T_0^+,\ X_{m+1}=(n,i)\big)\\=&\ P_{\breve{\mu}_0}\big(T_0^+=m+1,\ X_{m+1}=(0,i)\big),\end{aligned}$$

and then

$$\begin{aligned}\sum_{y,j}\overline{\nu}_y(j)p\big[(y,j),(n,i)\big]&=\sum_{m=0}^{+\infty}\sum_{y,j}P_{\breve{\mu}_0}(X_m=(y,j),\ m<T_0^+)p\big[(y,j),(n,i)\big]\\&=\sum_{m=0}^{+\infty}P_{\breve{\mu}_0}(T_0^+=m+1,\ X_{m+1}=(0,i)).\end{aligned}$$

Note that $T_0^+\geq 1$. Therefore, $P_{\breve{\mu}_0}(T_0^+=0,\ X_0=(0,i))=0$, and we have

$$\sum_{m=0}^{+\infty}P_{\breve{\mu}_0}\big(T_0^+=m+1,\ X_{m+1}=(0,i)\big)=\sum_{m=0}^{+\infty}P_{\breve{\mu}_0}\big(T_0^+=m,\ X_m=(0,i)\big)=\breve{\mu}_0,$$

because $\breve{\mu}_0$ is the censored measure. The proof is complete now. $\square$

Proof of Theorem 1.4 First, we can calculate the stationary measure in (3.4) by using the branching structure. The stationary measure is given by $\overline{\nu}_n(i) = E_{\breve{\mu}_0}\left(\sum_{m=0}^{T_0^+-1} I_{\{X_m=(n,i)\}}\right)$, which is the expected number of visits to (n,i) before time T_0^+ (but not contains the time T_0^+). So, $\overline{\nu}_n(i)$ $(n>0)$ equals to $E_1\mathbf{N}_n^-$ obtained by the branching structure in (2.5). The stationary measure $\{\overline{\nu}_n,\ n\in\mathbb{Z}\}$ can be expressed as

$$\overline{\nu}_n = \begin{cases} \breve{\mu}_0 P_0 A_1^- A_2^- \cdots A_{n-1}^- (I - P_n\zeta_{n+1}^- - R_n)^{-1} & n>0, \\ \breve{\mu}_0 & n=0. \end{cases}$$

Note that

$$\sum_{n,i} \overline{\nu}_n(i) = \sum_{m=0}^{+\infty} P_{\breve{\mu}_0}(T_0 > m) = E_{\breve{\mu}_0} T_0^+.$$

The condition $\varrho_1^+ < \infty$ ensures that

$$\sum_{n,i} \overline{\nu}_n(i) = E_0(T_0^+) = \breve{\mu}_0 P_0 \left(\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^- \right) + \breve{\mu}_0 \mathbf{1} = \varrho^+ < \varrho_1^+ < \infty.$$

As a consequence the stationary distribution equals

$$\nu_n(i) = \frac{\overline{\nu}_n(i)}{E_0 T_0^+} = \frac{\breve{\mu}_0 P_0 A_1^- A_2^- \cdots A_{n-1}^- \widetilde{u}_n^-}{\breve{\mu}_0 P_0 (\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^-) + \breve{\mu}_0 \mathbf{1}} \quad n>0,$$

where $\widetilde{u}_n^- = (I - P_n\zeta_{n+1}^- - R_n)^{-1}$, and

$$\nu_0(i) = \frac{\breve{\mu}_0}{\breve{\mu}_0 P_0 (\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^-) + \breve{\mu}_0 \mathbf{1}}.$$

$\square$

3.4 Light-tailed behavior—*Proof of Theorem 1.5*

It is well-known that the stationary distribution for the state-independent random walk (or a QBD process) on a half-strip is matrix-geometric. Therefore, the tail has a geometric (or exponential) decay. For the state-dependent random walk on a half-strip, the stationary tail does not always have an exponential decay. In this paper, we provide a criterion for this case, which is proved here.

3.4.1 Preliminaries

Let $B = (b_{i,j}) > 0$ (which is called a positive matrix) if all $b_{i,j} > 0$; and $B \geq 0$ if all $b_{i,j} \geq 0$. The spectrum of $n\times n$ matrix B is denoted as $\sigma(B) = \{\lambda_1, \lambda_2, \cdots \lambda_n\}$, where $\sigma(B)$ is the set of all eigenvalues $\lambda_i \in \mathbb{C}$. Define the spectral radius of B as $\rho(B) = \max\{|\lambda_i| : \ \lambda_i \in \sigma(B),\ 1\leq i\leq n\}$.

Proposition 3.3 *(Perron's Theorem in [9]) If $B > 0$ is an $n \times n$ matrix, then*

(1) $\rho(B) > 0$;

(2) $\rho(B)$ *is an eigenvalue of B, and it is the unique eigenvalue of maximum modulus;*

(3) $\rho(B)$ *is algebraically (and hence geometrically) simple;*

(4)

$$\lim_{m\to\infty} \left(\frac{B}{\rho(B)}\right)^m = L > 0.$$

Define $\|\cdot\|$ as the maximum column sum matrix norm, i.e. $\|B\| = \max_{1\le i\le n} \sum_{j=1}^{n} |b_{i,j}|$ for $B = (b_{i,j})$.

Proposition 3.4 *(Krause ([11], 94), Ostrowski ([14], 73)) Denote $\sigma(A) = \{\lambda_1, \lambda_2, \cdots \lambda_n\}$ where λ_i are eigenvalues of A, and $\sigma(B) = \{\mu_1, \mu_2, \cdots \mu_n\}$, where μ_i are eigenvalues of B. Define $d(\sigma(A), \sigma(B))$ as the optimal matching distance between the spectrums $\sigma(A)$ and $\sigma(B)$, that is,*

$$d(\sigma(A), \sigma(B)) = \min_{\theta\in S_n} \max_{1\le i\le n} |\lambda_i - \mu_{\theta_i}|,$$

where S_n is denoted as the group of all permutations on sets $\{1, 2, \cdots, n\}$. Then, for any two matrices $A, B \in \mathbb{R}^{n\times n}$, we have

$$d(\sigma(A), \sigma(B)) \le 4(2K)^{1-\frac{1}{n}} \|A - B\|^{\frac{1}{n}},$$

where $K = \max\{\|A\|, \|B\|\}$.

Simply speaking, Proposition 3.4 tells us that there exists a permutation $\theta \in S_n$, such that the maximum distance between the corresponding eigenvalues is small enough.

3.4.2 Spectral radius

Consider the state-independent random walk $\{\overline{X}_n,\ n \ge 0\}$ with transition probability block (P, Q, R), starting from layer 0 with an initial distribution $\bar{\mu}$. Let ζ^- be the unique sequence of stochastic matrices satisfying

$$\zeta^- = (I - P\zeta^- - R)^{-1}Q,$$

and

$$A^- = (I - P\zeta^- - R)^{-1}P, \qquad \mathbf{u}^- = (I - P\zeta^- - R)^{-1}\mathbf{1}.$$

Denote the spectral radius of A^- as $\rho(A^-)$, and the maximum eigenvalues of A^- as λ_{A^-}. Assume that the random walk is positive recurrent ($\bar{\varrho}_1^+ < \infty$).

Proof of (2) in Corollary 1.3 By *Perron's Theorem* in Proposition 3.3, we have $\rho(A^-) = \lambda_{A^-}$. The condition says $\bar{\varrho}_1^+ = \mathbf{1}'P\ (\sum_{k\ge1}\ (A^-)^{k-1}\ \mathbf{u}^-) + d < \infty$, i.e.,

$$\mathbf{1}'P \cdot \left(\sum_{k\ge1} (\lambda_{A^-})^{k-1} (\frac{A^-}{\lambda_{A^-}})^{k-1} \right) \cdot \mathbf{u}^- < \infty. \tag{3.7}$$

On the other hand, from (4) of Proposition 3.3, we know

$$\lim_{k\to\infty} (\frac{A^-}{\lambda_{A^-}})^{k-1} = L > 0 \tag{3.8}$$

which, together with (3.7), leads to $\lambda_{A^-} < 1$. $\square$

Let

$$E = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ & & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{d\times d}.$$

Denote the maximum eigenvalues of $(A^- - \varepsilon E)$ as λ_ε^-, and the maximum eigenvalues of $(A^- + \varepsilon E)$ as λ_ε^+. We then have

$$\rho(A^- - \varepsilon E) = \lambda_\varepsilon^- \quad \text{and} \quad \rho(A^- + \varepsilon E) = \lambda_\varepsilon^+.$$

Lemma 3.5 *Suppose that $\lambda_{A^-} < 1$, and for $A^- = (a_{i,j}^-)$,*

$$\varepsilon < \min\{\min_{i,j} a_{i,j}^-,\ \frac{1}{C^d}(1-\lambda_{A^-})^d\}.$$

Let $C = 4(2\|A^- + E\|)^{1-\frac{1}{d}} d^{\frac{1}{d}}$. Then,

$$\lambda_{A^-} - C\varepsilon^{\frac{1}{d}} < \lambda_\varepsilon^- < 1, \quad \textit{and} \quad \lambda_\varepsilon^+ < \lambda_{A^-} + C\varepsilon^{\frac{1}{d}} < 1. \tag{3.9}$$

Proof. For such $\varepsilon > 0$, both $A^- - \varepsilon E$ and $A^- + \varepsilon E$ are positive matrices, and the definitions of λ_ε^- and λ_ε^+ are meaningful. By Proposition 3.3, λ_ε^- and λ_ε^+ are real-valued. By Proposition 3.4, it is not hard to get that

$$d(\sigma(A^-), \sigma(A^- - \varepsilon E)) \le 4(2K)^{1-\frac{1}{d}} d^{\frac{1}{d}} \varepsilon^{\frac{1}{d}} \le C\varepsilon^{\frac{1}{d}},$$

and

$$d(\sigma(A^-), \sigma(A^- + \varepsilon E)) \le 4(2K)^{1-\frac{1}{d}} d^{\frac{1}{d}} \varepsilon^{\frac{1}{d}} \le C\varepsilon^{\frac{1}{d}},$$

where $C = 4(2K)^{1-\frac{1}{d}} d^{\frac{1}{d}}$, $K = \|A^- + E\|$.

Note that $0 < A^- - \varepsilon E \le A^- \le A^- + \varepsilon E$, then

$$1 > \lambda_{A^-} = \rho(A^-) \ge \rho(A^- - \varepsilon E) = \lambda_\varepsilon^- \quad \text{and} \quad \lambda_{A^-} = \rho(A^-) \le \rho(A^- + \varepsilon E) = \lambda_\varepsilon^+,$$

—colorredand $\lambda_{A^-} + C\varepsilon^{\frac{1}{d}} < 1$ for such $\varepsilon > 0$.

It is obvious that $\lambda_{A^-} - C\varepsilon^{\frac{1}{d}} < \lambda_\varepsilon^- < 1$. Otherwise if $\lambda_\varepsilon^- < \lambda_{A^-} - C\varepsilon^{\frac{1}{d}}$, then for λ_{A^-}, there exists no permutation such that $d(\sigma(A^-), \sigma(A^- - \varepsilon E)) \le C\varepsilon^{\frac{1}{d}}$, due to the fact that λ_ε^- is the largest eigenvalue of $A^- - \varepsilon E$.

Similarly, $\lambda_\varepsilon^+ < \lambda_{A^-} + C\varepsilon^{\frac{1}{d}} < 1$ holds. Otherwise if $\lambda_\varepsilon^+ > \lambda_{A^-} + C\varepsilon^{\frac{1}{d}}$, then for $\lambda_{A^-}^{\varepsilon+}$, there exists no permutation such that $d(\sigma(A^-), \sigma(A^- - \varepsilon E)) \le C\varepsilon^{\frac{1}{d}}$ holds, due to the fact that λ_{A^-} is the largest eigenvalue of A^-. □

3.4.3 Light-tailed behavior— proof of Theorem 1.5

Recall $D = \{(P, Q, R) : (P + Q + R)\mathbf{1} = \mathbf{1},\ \bar{\varrho}_1^+ < \infty\}$.

Proof of Theorem 1.5 Part (1) of the theorem has been proved in (2) of Corollary 1.3. Now, we focus on part (2). The random walk $\{X_n,\ n \in \mathbb{Z}\}$ starts from layer 0 with an censored measure $\breve{\mu}_0$, and the transition probabilities: $(P_n, Q_n, R_n) \to (P, Q, R) \in D$ as $n \to \infty$. It is easy to find that the random walk $\{X_n,\ n \in \mathbb{Z}\}$ is positive recurrent. To this end, recall that $A_n^- = (I - P_n\zeta_{n+1}^- - R_n)^{-1}P_n$ and $A^- = (I - P\zeta^- - R)^{-1}P$. Then, $A_n^- \to A^-$ as $n \to +\infty$ because that $(P_n, Q_n, R_n) \to (P, Q, R)$ as $n \to \infty$; and $\varrho_1^+ < \infty$ follows from $\bar{\varrho}_1^+ < \infty$ as $(P, Q, R) \in D$.

Also we have $\lambda_{A^-} < 1$ as $(P, Q, R) \in D$. For each $\varepsilon > 0$ defined in Lemma 3.5, there exists N, such that when $n > N$,

$$0 < A^- - \varepsilon E \le A_n^- \le A^- + \varepsilon E,$$

and then

$$(A^- - \varepsilon E)^k \le A_{N+1}^- A_{N+2}^- \cdots A_{N+k}^- \le (A^- + \varepsilon E)^k.$$

Let

$$\Phi_n(i) = \mu P_0 A_1^- A_2^- \cdots A_{n-1}^- \widetilde{u}_n^-(i). \tag{3.10}$$

Now we consider the first inequality. Notice

$$A_{N+1}^- A_{N+2}^- \cdots A_{N+k}^- \ge (\lambda_\varepsilon^-)^k (\frac{A^- - \varepsilon E}{\lambda_\varepsilon^-})^k,$$

for the given N, therefore we have

$$\begin{aligned} \Phi_{N+k}(i) &= \mu P_0 A_1^- A_2^- \cdots A_{N+k-1}^- \mathbf{u}_{N+k}^- = \mu D_1(N) \cdot \left(A_{N+1}^- A_2^- \cdots A_{k+N-1}^-\right) \cdot \widetilde{u}_{N+k}^-(i) \\ &\ge \mu D_1(N)(\lambda_\varepsilon^-)^k (\frac{A^- - \varepsilon E}{\lambda_\varepsilon^-})^k\ \widetilde{u}_{N+k}^-(i), \end{aligned}$$

where $D_1(N) = P_0 A_1^- A_2^- \cdots A_N^-$ and $\widetilde{u}_{N+k}^- = (I - P_{N+k}\zeta_{N+k+1}^- - R_{N+k})^{-1}$. Hence,

$$\frac{\log \Phi_{N+k}(i)}{N+k} \ge \frac{k \log \lambda_\varepsilon^-}{N+k} + \frac{\log \mu D_1(N)(\frac{A^- - \varepsilon E}{\lambda_\varepsilon^-})^k\ \widetilde{u}_{N+k}^-(i)}{N+k}. \tag{3.11}$$

By Proposition 3.3, there exists a positive matrix W_ε^-, such that

$$\lim_{k\to\infty}(\frac{A^- - \varepsilon E}{\lambda_\varepsilon^-})^k = W_\varepsilon^-.$$

Together with $\lim_{k\to\infty} \widetilde{u}_{N+k}^-(i) = \widetilde{u}^-(i) = (I - P\zeta^- - R)^{-1}(i)$, as a consequence, $\log \mu D_1(N)(\frac{A^- - \varepsilon E}{\lambda_\varepsilon^-})^k \ \widetilde{u}_{N+k}^-(i)$ is bounded in k. Thus from (3.11),

$$\liminf_{k\to\infty} \frac{\log \Phi_{N+k}(i)}{N+k} \geq \log \lambda_\varepsilon^-.$$

Note that from Lemma 3.5, $\lambda_{A^-} - C\varepsilon^{\frac{1}{d}} < \lambda_\varepsilon^- < 1$, so

$$\liminf_{k\to\infty} \frac{\log \Phi_{N+k}(i)}{N+k} \geq \log(\lambda_{A^-} - C\varepsilon^{\frac{1}{d}}). \tag{3.12}$$

Similarly, for the second inequality, notice

$$A_{N+1}^- A_{N+2}^- \cdots A_{N+k}^- \leq (\lambda_\varepsilon^+)^k (\frac{A^- - \varepsilon E}{\lambda_\varepsilon^+})^k,$$

therefore we have

$$\limsup_{k\to\infty} \frac{\log \Phi_{N+k}(i)}{N+k} \leq \log \lambda_\varepsilon^+.$$

Note that from Lemma 3.5, $\lambda_\varepsilon^+ < \lambda_{A^-} + C\varepsilon^{\frac{1}{d}} < 1$, so

$$\limsup_{k\to\infty} \frac{\log \Phi_{N+k}(i)}{N+k} \leq \log(\lambda_{A^-} + C\varepsilon^{\frac{1}{d}}). \tag{3.13}$$

Combine (3.12) and (3.13) to have

$$\log(\lambda_{A^-} - C\varepsilon^{\frac{1}{d}}) \leq \liminf_{k\to\infty} \frac{\log \Phi_k(i)}{k} \leq \limsup_{k\to\infty} \frac{\log \Phi_k(i)}{k} \leq \log(\lambda_{A^-} + C\varepsilon^{\frac{1}{d}}).$$

Let $\varepsilon \to 0$, we get

$$\lim_{k\to\infty} \frac{\log \Phi_k(i)}{k} = \log \lambda_{A^-}.$$

If $(P_n, Q_n, R_n) \to (P, Q, R)$, $(P, Q, R) \in D$, the stationary distribution $\{\nu_n, \ n \geq 0\}$ is given by

$$\nu_n = \frac{\breve{\mu}_0 P_0 A_1^- A_2^- \cdots A_{n-1}^- \widetilde{u}_n^-}{\breve{\mu}_0 P_0 (\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^-) + \breve{\mu}_0 \mathbf{1}}, \quad n > 0,$$

where the denominator $\breve{\mu}_0 P_0 (\sum_{k\geq 1} A_1^- A_2^- \cdots A_{k-1}^- \mathbf{u}_k^-) + \breve{\mu}_0 \mathbf{1} < \varrho_1^+ < +\infty$.

Thus for the stationary distribution $\{\nu_n, \ n \geq 0\}$, we have

$$\lim_{n\to\infty} \frac{\log \nu_n(i)}{n} = \lim_{k\to\infty} \frac{\log \Phi_k(i)}{k} = \log \lambda_{A^-},$$

i.e., the stationary distribution is light-tailed, with the decay rate $0 \leq \lambda_{A^-} < 1$ along the layer direction. □

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