# Explicit stationary distribution of the $(L,1)$-reflecting random walk on the half line*

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**Abstract**

In this paper, we consider the $(L,1)$ state-dependent reflecting random walk (RW) on the half line, which is a RW allowing jumps to the left at a maxial size $L$. For this model, we provide an explicit criterion for (positive) recurrence and an explicit expression for the stationary distribution. As an application, we prove the geometric tail asymptotic behavior of the stationary distribution under certain conditions. The main tool employed in the paper is the intrinsic branching structure within the $(L,1)$-random walk.

**Key words and phrases**: random walk, multi-type branching process, recurrence, positive recurrence, stationary distribution, tail asymptotic.

**AMS 2000 Subject Classifications**: Primary 60K37; Secondary 60J85

## 1 Introduction and main results

### 1.1 The background and motivation

We consider the $(L,1)$-reflecting random walk on the half line, i.e., a Markov chain $\{X_n\}_{n\geq 0}$ on $\mathbb{Z}^+ = \{0,1,2,\dots\}$ with $X_0 = 0$ and the transition probabilities $P_{ij}$ specified by : $P_{01} = p(0) = 1$; for $0 < i < L$,

$$P_{ij} = \begin{cases} p(i), & \text{for } j = i+1, \\ q_{i-j}(i), & \text{for } 0 < j < i, \\ \sum_{k=i}^{L} q_k(i), & \text{for } j = 0, \\ 0, & \text{otherwise}, \end{cases}$$

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and for $i \geq L$,

$$P_{ij} = \begin{cases} p(i), & \text{for } j = i+1, \\ q_{i-j}(i), & \text{for } i-L \leq j < i, \\ 0, & \text{otherwise}, \end{cases}$$

where $q_1(i)+q_2(i)+\cdots+q_L(i)+p(i) = 1$, and $q_1(i), q_2(i), \cdots, q_L(i) \geq 0, 0 < p(i) < 1$. Obviously, this Markov chain is irreducible.

For example, when $L = 2$, the transition matrix $P$ is given by

$$P = \begin{pmatrix} 0 & 1 & & & & \\ q_1(1)+q_2(1) & 0 & p(1) & & & \\ q_2(2) & q_1(2) & 0 & p(2) & & \\ & q_2(3) & q_1(3) & 0 & p(3) & \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

in which all unspecified entries are zero.

It is well-known that for the $(1,1)$-RW, the criteria for the (positive) recurrence and the expression for the stationary distribution have been given explicitly, for example see [8] and [9]). However, for $L > 1$, no such explicit expressions are found to our best knowledge. The aim of the present paper is to give explicit criteria of the (positive) recurrence and explicit expressions of the stationary distribution for the $(L,1)$-RW. Our method is probabilistic by using the intrinsic branching structures hidden in the $(L,1)$-RW ([5] and [6]). The results obtained in this paper can be applied to various state-dependent queueing systems.

## 1.2 Main results

First, we recall two results on the (positive) recurrence and the stationary distribution of a general Markov chain $X_n$ (e.g., [9] or [3]). For this purpose, let $N(i) = \sum_{n=1}^{\infty} 1_{\{X_n=i\}}$ be the number of visits to state $i$ by the chain, $T_i = \inf\{n > 0 : X_n = i\}$ is the first time for the chain to be in state $i$, and $E^i$ is the expected value when the walk starts at $X_0 = i$.

**Fact 1** $\{X_n\}_{n\geq 0}$ is recurrent $\Longleftrightarrow E^0 N(0) = \infty \Longleftrightarrow \sum_{j=0}^{\infty} P_{ij} y_j = y_i$ $(i \geq 0)$ have no bounded nonconstant solution; and

**Fact 2** $\{X_n\}_{n\geq 0}$ is positive recurrent $\Longleftrightarrow E^0 T_0 < \infty$. In this case, the stationary distribution $\pi(i)$ is given by $\pi(i) = \frac{1}{E^i T_i}$.

The idea of the present paper is to express $E^0 N(0)$ and $E^i T_i$ explicitly through using the intrinsic branching structure hidden in the $(L,1)$-RW.

Three results are obtained: the first one is a characterization for recurrence; the second is an expression for the stationary probability distribution; and the last one is a criterion for the stationary probabilities to have a geometric decay. For the first two results, we present them for $L = 2$ since the notation for a general $L$ is very demanding. The last result is presented for a general $L$.

#### 1.2.1 *Criteria for the recurrence*

Let $e_1 = (1,0)$, $e_2 = (0,1)$, $u = e_1' + e_2'$, the sum of the transposes of $e_1$ and $e_2$,

$$M_1 = \begin{pmatrix} \frac{q_1(1)+q_2(1)}{p(1)} & 0 \\ \frac{1}{p(1)} & 0 \end{pmatrix}, \quad M_i = \begin{pmatrix} \frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} \\ 1+\frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} \end{pmatrix}, \; i > 1.$$

**Theorem 1.1.** *Let*

$$\kappa := \sum_{k=1}^{\infty} e_1 M_k M_{k-1} \dots M_1 u,$$

*if $\kappa = \infty$, then the walk $\{X_n\}_{n\geq 0}$ is recurrence.*

**Remark** Actually, $\kappa$ in the theorem is the expectation number of the visiting times by the random walk at position 0, i.e., $\kappa = E^0 N(0)$ , which can be calculated by the means of the intrinsic branching structure within the walk and the $M_i$ is the offspring mean matrix.

#### 1.2.2 *Criteria for the positive recurrence and stationary distribution*

Next, we will give the criteria for the positive recurrence and explicit formula of the stationary distribution based on the Fact 2. Let $P^i$ denote the probability when the walk start at $X_0 = i$. Define

$$\tau_0 = 0, \qquad \tau_i = \inf\{n > 0 : X_n < i\}, \quad i \geq 1 \tag{1.1}$$

**Theorem 1.2.** *Assume $E^1\tau_1 < \infty$, then the walk $\{X_n\}_{n\geq 0}$ is positive recurrence. Furthermore the stationary distribution is given by, for $i \geq 0$,*

$$\pi(i) = \frac{1}{E^i T_i} = \frac{1}{p(i)E^{i+1}\tau_{i+1} + (q_1(i) + q_2(i) + p(i)P^{i+1}[(i+1,+\infty), i-1])E^{i-1}T_i + q_2(i)E^{i-2}T_{i-1} + 1},$$

*where explicit expressions for $P^{i+1}[(i+1,+\infty), i-1]$, $E^i\tau_i$ and $E^iT_{i+1}$ are given in (2.2), (2.4) and (2.1), respectively.*

#### 1.2.3 *Tail behavior of the stationary distribution*

It is well know that when the transition probabilities are state-independent, the tail of the stationary distribution of the walk has a geometric decay. Here with the help of the explicit expression of the stationary distribution in given in the previous theorem, we can consider the tail behavior for the state-dependent case. We notice that the expression of the stationary distribution in Theorem 1.2 is given in terms of the decomposition of the trajectory, from which we find out that the dominant contribution to the tail asymptotic behavior of $\pi(i)$ is from $E^{i-1}T_i$ ( because of $E^1\tau_1 < \infty$). With this observation, characterize the tail behavior of the state-dependent $(L,1)$-RW as follows.

Let $D := \{(p, q_1, q_2, \cdots q_L) : p + \sum_{j=1}^{L} q_j = 1; \quad \sum_{j=1}^{L} jq_j > p; \quad \forall j,\ q_j \geq 0, p > 0\}$. $\rho(M)$ is the spectral radius of $M$, and $\lambda_M$ is the maximum eigenvalue of it, where

$$M = \begin{pmatrix} \frac{q_1}{p} & \frac{q_2}{p} & \cdots & \frac{q_{L-1}}{p} & \frac{q_L}{p} \\ 1+\frac{q_1}{p} & \frac{q_2}{p(i)} & \cdots & \frac{q_{L-1}}{p} & \frac{q_L}{p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{q_1}{p} & \frac{q_2}{p} & \cdots & \frac{q_{L-1}}{p} & \frac{q_L}{p} \\ \frac{q_1}{p} & \frac{q_2}{p} & \cdots & 1+\frac{q_{L-1}}{p} & \frac{q_L}{p} \end{pmatrix}_{L\times L} . \tag{1.2}$$

**Theorem 1.3.** *(1) If $P \in D$, we have $\lambda_M = \rho(M) > 1$.*
*(2) If $P(i) := (p(i), q_1(i), q_2(i), \cdots, q_L(i)) \to P$, as $i \to \infty$, and $P \in D$, then the stationary distribution exist; and*

$$\lim_{i\to\infty} \frac{\log \pi(i)}{i} = -\log \lambda_M.$$

**Remark** Here $D$ is the positive recurrence district for the state-independent $(L, 1)$-RW, and $M$ is the correspond offspring mean matrix. The theorem says that if the transition probability of the state-dependent $(L, 1)$-RW goes to a point in $D$ as the state goes to infinite then the walk is positive recurrent and the tail of the stationary distribution is geometric decay.

## 1.3 Examples

Three special cases are provided here as examples.

### 1.3.1 *Degenerate to the case of* $(1, 1)$ *state-dependent RW*

The $(1, 1)$-RW is the special case in which $q_j(i) \equiv 0$ for $j > 1$. We denote $q_1(i)$ by $q(i)$. In this case, $p(i), q(i) > 0$. Let

$$\mu_0 = 1, \quad \mu_i = \frac{p(0)p(1)\cdots p(i-1)}{q(1)q(2)\cdots q(i)} \quad \text{for } i > 0, \quad \mu = \sum_{i=0}^{\infty} \mu_i.$$

Then, Theorem 1.1 and Theorem 1.2 lead to the following corollaries, which are known literature results (e.g. [2] and [9]).

**Corollary 1.1.** *The chain is recurrent iff* $\sum_{i=0}^{\infty} \frac{1}{\mu_i p(i)} = \infty$.

**Corollary 1.2.** *The stationary distribution exists iff $\mu < \infty$. In this case, $\pi_i = \mu_i/\mu$.*

For the $(1, 1)$-RW, the tail of the stationary distribution has a very simple form as follows.

**Corollary 1.3.** *If $p(i) \to p$ as $i \to \infty$, and $p < \frac{1}{2}$, then the stationary distribution exists and*

$$\lim_{i\to\infty} \frac{\log \pi(i)}{i} = -\log \frac{1-p}{p}.$$

#### 1.3.2 *Degenerate to the case of* $(2,1)$ *state-independent RW*

We consider the classical state-independent $(2,1)$-RW on the positive line, for which the transition matrix is given by

$$P = \begin{pmatrix} 0 & 1 & & & & \\ q_1+q_2 & 0 & p & & & \\ q_2 & q_1 & 0 & p & & \\ & q_2 & q_1 & 0 & p & \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $q_1+q_2+p=1$, and $q_1, q_2 \geq 0$ and $0<p<1$. In this case, the recurrence criterion and all quantities in (2.2), (2.4) and (2.1) can be calculated directly from our results as follows, which lead to a calculation of the stationary distribution.

**Corollary 1.4.**

$$\kappa = \infty \Longleftrightarrow q_1 + 2q_2 \geq p.$$

**Corollary 1.5.** *If $q_1+2q_2>p$, then the walk $\{X_n\}_{n\geq 0}$ is positive recurrence, and*

$$P^{i+1}[(i+1,+\infty), i-1] = \frac{\Delta - q_1 - q_2}{2p},$$

$$E^i\tau_i = \frac{2\Delta}{1-p-pq_1+3pq_2+(1-3p)\Delta},$$

$$E^iT_{i+1} = 1 + \frac{1}{\lambda_1-\lambda_2}\Big(\frac{\lambda_1^2(1-\lambda_1^{i-1})(\lambda_2+2)}{1-\lambda_1} - \frac{\lambda_2^2(1-\lambda_2^{i-1})(\lambda_1+2)}{1-\lambda_2}\Big) + 2\frac{\lambda_1^{i+1}-\lambda_2^{i+1}}{\lambda_1-\lambda_2},$$

*where $\Delta = \sqrt{(q_1+q_2)^2+4pq_2}$, and $\lambda_1$ and $\lambda_2$ are the eigenvalues of*

$$M = \begin{pmatrix} \frac{q_1}{p} & \frac{q_2}{p} \\ 1+\frac{q_1}{p} & \frac{q_2}{p} \end{pmatrix}.$$

**Remark** As a consequence, we can get $\pi(i) = C(\frac{1}{\lambda_M})^i$, where $C$ is determined by the transition probabilities, and $\lambda_M = max\{|\lambda_1|, |\lambda_2|\}$.

#### 1.3.3 State-dependent queue with bulk service and impatient customers

This is a special case of the $(L,1)$-RW, in which $p(k)>0$ is deceasing (arriving customers with impatience since the probability $p(k)$ for a customer to join the system is deceasing as the number of customers in the system increases); and the server can simultaneously serve multiple customers up to size $L$ (bulk service). Assume $p(k) \searrow p$ and $q_j(k) \to q_j$ for $j=1,2,\ldots,L$. Then, according to Theorem 1.3 if $\sum_j jq_j > p$, then the stationary probability distribution $\pi_k$ exists and its tail decays geometrically with rate $1/\lambda_M$.

We arrange the remainder of this paper as follows. As the main tool, the intrinsic branching structure within the $(L,1)$-RW will be briefly reviewed in Section 2 for $L=2$; and then the proofs of the theorems and corollaries will be detailed in Section 3 except Theorem 1.3, which will be proved in Section 4 for a general $L$.

## 2 A brief review for the intrinsic branching structure

The intrinsic branching structure within a random walk has been studied by many authors. For the $(1,1)$-RW, Dwass ([4], 1975) and Kesten *et al.* ([10], 1975) observed a Galton-Watson process with the geometric offspring distribution hidden in the nearest random walk. The branching structure is a powerful tool in the study of random walks in a random environment (RWRE, for short). In [10], Kesten *et al.*, proved a stable law for the nearest RWRE by using this branching structure. The key point is that the hitting time $T_i$ can be calculated accurately by the branching structure.

However, if the random walk is allowed to have jumps, even to a bounded range, referred to as the $(L,R)$-RW, the situation will become much more complicated.

For the $(L,1)$-RW, when the walk starts at 0 and $\limsup_{n\to\infty} X_n = \infty$, a multi-type branching process has been revealed by Hong *et al.*([5], 2009). for calculating the hitting time $T_1 = \inf\{n > 0, X_n > 0\}$. A similar work has been done for the $(1,R)$-RW ([6], 2010). It must be emphasized that these two branching structures are not symmetric, instead they are essentially different.

For the purpose of calculating the stationary distribution in this paper, both structures will be used. More specifically, for calculating $E^iT_i$ for $i > 0$, if the first step is down (possible at $i-1$ or $i-2$ when $L = 2$), the branching structure within the $(L,1)$-RW is used, and alternatively if the first step is up (to state $i+1$), the branching structure within the $(1,R)$-RW is used. Here we call them the "lower" and the "upper" branching structures respectively, which we will briefly introduce below. Note that if we assume $q_2(i) \equiv 0$, both branching structures degenerate to the case of the $(1,1)$-RW.

### 2.1 The "lower" branching structure

The following discussion is based on $L = 2$. The general case can be similarly discussed, which is much more complicated. Assume that $X_0 = i$, if the first step is down (possible at $i-1$ or $i-2$) we can calculate $E^iT_i$ by using the branching structure within the $(L,1)$-RW ([5], 2009). Define

$$U_0^1 = \#\{0 < j < T_{i+1} : X_{j-1} = 1, X_j = 0\} \quad \text{and} \quad U_0^2 = 0;$$

and

$$U_k^l = \#\{0 < j < T_{i+1} : X_{j-1} > k, X_j = k-l+1\} \quad \text{for } 1 \le k < i+1,\ l = 1,2.$$

Setting

$$U_k = (U_k^1, U_k^2) \quad \text{for } 0 \le k < i+1.$$

We then have the following property.

**Theorem A** (Hong and Wang [5]) *(1) The process* $\{U_k\}_{k=i}^0$ is a 2-type branching process whose branching mechanism is given by:

$$P(U_0 = (a,0) \big| U_1 = e_1) = \big(q_1(1) + q_2(1)\big)^a p(1),$$
$$P(U_0 = (1+a,0) \big| U_1 = e_2) = \big(q_1(1) + q_2(1)^a p(1);$$

and for $k > 1$,

$$P(U_{k-1} = (a,b) \big| U_k = e_1) = \frac{(a+b)!}{a!b!} q_1(k)^a q_2(k)^b p(k),$$

$$P(U_{k-1}=(1+a,b)\big|U_k=e_2)=\frac{(a+b)!}{a!b!}q_1(k)^a q_2(k)^b p(k).$$

*(2) For the process* $\{U_k\}_{k=i}^0$*, let* $M_k$ *be the* $2\times 2$ *mean matrix whose* $l$*-th row is* $E(U_{k-1}|U_k=e_l)$*. Then, one has that*

$$M_1=\begin{pmatrix} \frac{q_1(1)+q_2(1)}{p(1)} & 0 \\ \frac{1}{p(1)} & 0 \end{pmatrix},\quad M_k=\begin{pmatrix} \frac{q_1(k)}{p(k)} & \frac{q_2(k)}{p(k)} \\ 1+\frac{q_1(k)}{p(k)} & \frac{q_2(k)}{p(k)} \end{pmatrix},\quad k>1.$$

*(3)* $E^0T_1=1$*, and for* $i>0$*,* $T_{i+1}=1+\sum_{k=0}^i U_k\cdot(2,1)'$ *and*

$$E^iT_{i+1}=1+\sum_{k=0}^{i}EU_k\cdot(2,1)'=1+\sum_{k=1}^{i}e_1M_iM_{i-1}\cdots M_{i-k+1}(2,1)'. \tag{2.1}$$

**Remark** i) The positions of the walk correspond to the time of the branching process. For example, in our notation, $U_k$ is indeed the $(i-k)$-th generation of the branching process.

ii) The condition $\limsup_{n\to\infty}X_n=+\infty$ in [5] is obviously satisfied in our reflecting model.

iii) It is not difficult to understand that the branching structure from $U_1$ to $U_0$ is different from others because of reflecting. We omit the proof here.

## 2.2 The "upper" branching structure

Assume that $X_0=i$, $i>0$. If the first step is up (at $i+1$), we can calculate $E^{i+1}T_i$ by using the branching structure within $(1,R)$-RW ([6], 2010). If the $(2,1)$-reflecting random walk is recurrent, $\tau_k$, $k\ge i$, is defined the same as in Section 1. Note that $\tau_k<\infty$ $P$-a.s. To calculate $\tau_i$ accurately, Hong and Zhang ([6], 2010) defined a multi-type branching process by decomposing the path of the walk. Intuitively, if the walk from $k\ge i$ takes a step to $k+1$, it must across back to $k$ or jump over $k$ (to $k-1$) because of $\tau_k<\infty$ $P$-a.s., in which there are only three ways of moving down: from $k+1$ to $k$, from $k+2$ to $k$ and from $k+1$ to $k-1$. So we divide all the steps from $k$ to $k+1$ into three kinds of steps according to the above three ways of moving down. Let $A(k)$, $B(k)$ and $C(k)$ be the numbers of steps from $k$ to $k+1$ before time $\tau_i$ corresponding to moving from $k+1$ to $k$, from $k+2$ to $k$ and from $k+1$ to $k-1$, respectively. As for the last step of $\tau_i$, we can consider it as a immigration for the multi-type branching processes.

Consider integers $n\ge i>1$, and define the exit probabilities:

$$P^i[(i,n),i-1]=P^i\{X_n \text{ leaving } \{i,i+1,\dots,n-1,n\} \text{ at the point } i-1\},$$
$$P^i[(i,n),i-2]=P^i\{X_n \text{ leaving } \{i,i+1,\dots,n-1,n\} \text{ at the point } i-2\}.$$

In Hong and Zhang [6] (see Lemma 2.1), it has been calculated that (see also [1])

$$P^i[(i,n),i-1]=\frac{\langle e_1,[\tilde{M}_i+\cdots+\tilde{M}_n\cdots\tilde{M}_i]v\rangle}{1+\langle e_1,[\tilde{M}_i+\cdots+\tilde{M}_n\cdots\tilde{M}_i]e_1\rangle},$$
$$P^i[(i,n),i-2]=\frac{\langle e_1,[\tilde{M}_i+\cdots+\tilde{M}_n\cdots\tilde{M}_i]e_2\rangle}{1+\langle e_1,[\tilde{M}_i+\cdots+\tilde{M}_n\cdots\tilde{M}_i]e_1\rangle}, \tag{2.2}$$

where $v=e_1'-e_2'$, and $\tilde{M}_i=\begin{pmatrix} \frac{q_1(i)+q_2(i)}{p(i)} & \frac{q_2(i)}{p(i)} \\ 1 & 0 \end{pmatrix}$, $i\ge 1$.

If $\kappa = \infty$, let

$$\begin{aligned}
\gamma(i) &= p(i) \cdot P^{i+1}[(i+1,+\infty), i-1], \\
\alpha(i) &= p(i) \cdot P^{i+1}[(i+1,+\infty), i] \cdot \frac{q_1(i+1)}{q_1(i+1)+\gamma(i+1)}, \\
\beta(i) &= p(i) \cdot P^{i+1}[(i+1,+\infty), i] \cdot \frac{\gamma(i+1)}{q_1(i+1)+\gamma(i+1)}.
\end{aligned} \tag{2.3}$$

Set for $k \geq i$,

$$V(k) = [A(k), B(k), C(k)].$$

Then we have the following branching structure within the $(2,1)$-RW.

**Theorem B** ( Hong and Zhang [6]) *Assume $\kappa = \infty$. Then,*

*(1) $\Big(V(k) = [A(k), B(k), C(k)]\Big)_{k \geq i}$ is an inhomogeneous multi-type branching process with immigration*

$$\begin{aligned}
V(i-1) &= [1,0,0], \quad \textit{with probability } \frac{q_1(i)}{1-\alpha(i)-\beta(i)}, \\
V(i-1) &= [0,1,0], \quad \textit{with probability } \frac{\gamma(i)}{1-\alpha(i)-\beta(i)}, \\
V(i-1) &= [0,0,1], \quad \textit{with probability } \frac{q_2(k)}{1-\alpha(i)-\beta(i)}.
\end{aligned}$$

*The offspring distribution is given by*

$$\begin{aligned}
P\Big(V(k) = [a,b,0] \,\Big|\, V(k+1) = [1,0,0]\Big) &= [1-\alpha(k)-\beta(k)] C_{a+b}^{a} \alpha(k)^a \beta(k)^b, \\
P\Big(V(k) = [a,b,1] \,\Big|\, V(k+1) = [0,1,0]\Big) &= [1-\alpha(k)-\beta(k)] C_{a+b}^{a} \alpha(k)^a \beta(k)^b, \\
P\Big(V(k) = [a,b,0] \,\Big|\, V(k+1) = [0,0,1]\Big) &= [1-\alpha(k)-\beta(k)] C_{a+b}^{a} \alpha(k)^a \beta(k)^b,
\end{aligned}$$

*where $\alpha(k)$, $\beta(k)$ and $\gamma(i)$ are defined in (2.2).*

*(2) The offspring mean matrix of the $(k-i+1)$-st generation of the multi-type branching process is*

$$N_k = \begin{pmatrix} \frac{\alpha(k)}{1-\alpha(k)-\beta(k)} & \frac{\beta(k)}{1-\alpha(k)-\beta(k)} & 0 \\ \frac{\alpha(k)}{1-\alpha(k)-\beta(k)} & \frac{\beta(k)}{1-\alpha(k)-\beta(k)} & 1 \\ \frac{\alpha(k)}{1-\alpha(k)-\beta(k)} & \frac{\beta(k)}{1-\alpha(k)-\beta(k)} & 0 \end{pmatrix}.$$

*(3) $\tau_i = 1 + \sum_{k=i}^{\infty} \big[2A(i) + 2B(i) + C(i)\big] = 1 + \big\langle (2,2,1), \sum_{k=i}^{\infty} V(i) \big\rangle$, and*

$$E^i \tau_i = 1 + \Big\langle (2,2,1),\ \frac{1}{1-\alpha(i)-\beta(i)} \Big(q_1(i),\ \gamma(i),\ q_2(i)\Big) \cdot \sum_{k=1}^{+\infty} N_1 \cdots N_k \Big\rangle. \tag{2.4}$$

**Remark** i) In [6], Hong and Zhang considered the branching structure within the $(1,2)$-RW on the line starting at 0 before the ladder time $\inf\{k > 0, X_k > 0\}$. It corresponds to the "upper" part of our model.

ii) $E^1 \tau_1 < \infty$ is the sufficient condition of $\kappa = \infty$.

## 3 Proofs

*Proof of Theorem 1.1* According to Fact 1, we can prove the result by two different methods. Here we prove it through calculating $E^0 N(0)$ directly by using the branching structure; and in the appendix we provide an analytical proof by solving the system of infinite linear equations. Actually, we find the solution to the system of infinite linear equations also in terms of the observation of the branching structure.

Recall $N(0) = \sum_{n=1}^{\infty} 1_{\{X_n=0\}}$ is the occupation time of position 0. Because the walk is reflected at 0 with probability 1, the walk goes to $+\infty$ a.s. Note that the walk goes up skip freely. Therefore, we can decompose the whole trajectory of the walk as the combination of the pieces from position $i$ to $i+1$ for $i = 0, 1, 2, \ldots$, i.e., $\{X_n, n \geq 0\} = \bigcup_{i=0}^{\infty} \{X_n, T_i \leq n < T_{i+1}\}$, where $T_0 = 0$. So, to calculate $N(0)$ the occupation time of position 0, we need only to calculate the occupation time of position 0 in each piece $\{X_n, T_i \leq n < T_{i+1}\}$ of the trajectory, in which a "lower" multi-type process with an immigration at position $i$ is hidden (Theorem A). To this end, let $\theta_1 = 1$ and for $i > 1$,

$$\theta_i = \sharp\{k : T_{i-1} \leq k < T_i, X_k = 0\};$$

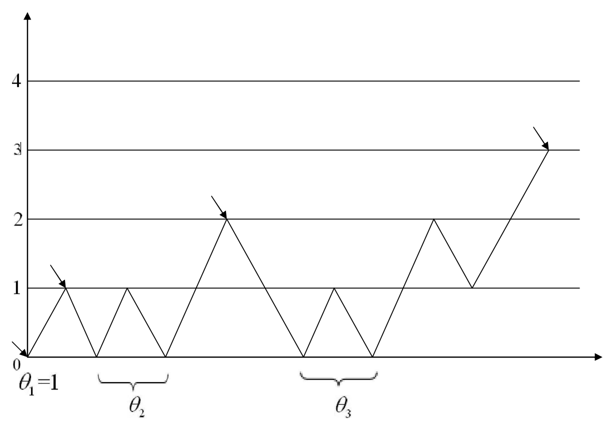


Figure 1: Immigration structure for the branching process

which is the population of the $i$-th generation of the multi-type branching process with a immigration at $i$. By Theorem A, we have

$$E^0 \theta_i = e_1 M_{i-1} M_{i-2} \cdots M_1 u, \quad i \geq 1.$$

From the decomposition we know that $N(0) := \sum_{n=1}^{\infty} 1_{\{X_n=0\}} = \sum_{i=1}^{\infty} \theta_i$. As a consequence we get

$$E^0 N(0) = E^0 \sum_{i=1}^{\infty} \theta_i = \sum_{i=1}^{\infty} e_1 M_{i-1} M_{i-2} \cdots M_1 u := \kappa.$$

The proof is complete by Fact 1. $\square$

**Remark** Let $\xi_0 = 0$, $\xi_1 = 1$ and $\xi_n = \sum_{i=1}^{n} \theta_i$ for $n \geq 1$, the occupation time at position 0 before the walk hitting position $n$. If $y_n = E^0 \xi_n$, we find that $y_n$ is a solution of the system of infinite linear equations $\sum_{j=0}^{+\infty} P_{ij}y_j = y_i \ i \geq 0$. See the appendix, in which an analytical version proof of the present theorem is provided.

*Proof of Theorem 1.2* In our model, $X_n$ is irreducible and note that $E^0T_0 = 1 + E^1\tau_1$, where the explicit expression of $E^i\tau_i$ is given in (2.4). So $E^1\tau_1 < \infty$ assures that the walk $X_n$ is positive recurrent by Fact 2 and has the stationary distribution $\pi$ with $\pi(i) = \frac{1}{E^iT_i}$. To calculate $E^iT_i$, we consider the first step of the walk starting at $X_0 = i$. There are four possible types of the trajectory of the walk from position $i$ back to position $i$ as shown in the following graph.

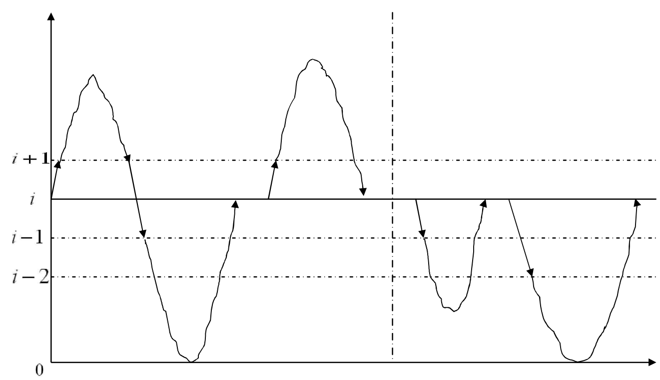


Figure 2:Four types of trajectories from i to i

By the Markov property one can get, for $i > 0$,

$$\begin{aligned} E^iT_i &= p(i)(E^{i+1}T_i + 1) + q_1(i)(E^{i-1}T_i + 1) + q_2(i)(E^{i-2}T_i + 1) \\ &= p(i)(E^{i+1}T_i + 1) + q_1(i)(E^{i-1}T_i + 1) + q_2(i)(E^{i-2}T_{i-1} + E^{i-1}T_i + 1) \\ &= p(i)E^{i+1}T_i + (q_1(i) + q_2(i))E^{i-1}T_i + q_2(i)E^{i-2}T_{i-1} + 1, \end{aligned} \tag{3.1}$$

and $E^0T_0 = E^1T_0 + 1$.

We now calculate $E^{i+1}T_i$. For the walk starting at $i+1$, there are only two kinds of ways to hit $i$ for the first time: from $i+1$ or $i+2$ to $i$ with probability $P^{i+1}[(i+1, +\infty), i]$; or first from $i+1$ to $i-1$, then from $i-1$ to $i$ with the probability $P^{i+1}[(i+1, +\infty), i-1]$. Consequently,

$$\begin{aligned} E^{i+1}T_i &= P^{i+1}[(i+1, +\infty), i]E^{i+1}\tau_{i+1} + P^{i+1}[(i+1, +\infty), i-1](E^{i+1}\tau_{i+1} + E^{i-1}T_i) \\ &= E^{i+1}\tau_{i+1} + P^{i+1}[(i+1, +\infty), i-1]E^{i-1}T_i. \end{aligned}$$

By (3.1), we get

$$E^iT_i = p_iE^{i+1}\tau_{i+1} + (q_1(i) + q_2(i) + p(i)P^{i+1}[(i+1, +\infty), i-1)]E^{i-1}T_i + q_2(i)E^{i-2}T_{i-1} + 1.$$

□

*Proof of Corollary 1.1* In this case, we have for $i \geq 1$, $M_i := \begin{pmatrix} \frac{q(i)}{p(i)} & 0 \\ \frac{1}{p(i)} & 0 \end{pmatrix}$. Therefore,

$$\kappa = \sum_{k=1}^{\infty} e_1 M_k M_{k-1} \ldots M_1 u = \sum_{k=1}^{\infty} (1,\ 0) \begin{pmatrix} \frac{q_kq_{k-1}\cdots q_i}{p_kp_{k-1}\cdots p_i} & 0 \\ \frac{q_{k-1}q_{k-2}\cdots q_i}{p_kp_{k-1}\cdots p_i} & 0 \end{pmatrix} (1,\ 1)' = \sum_{i=1}^{\infty} \frac{1}{\mu_i p_i}.$$

We also have $\frac{1}{\mu_0 p_0} = 1$. As a result, $\kappa = \infty \Leftrightarrow \sum_{i=0}^{\infty} \frac{1}{\mu_i p_i} = \infty$. □

*Proof of Corollary 1.2* By (2.2), (2.3), (2.4) and (2.1), one can get

$$E^i \tau_i = \frac{1}{q(i)} + \frac{p(i)}{q(i)q(i+1)} + \frac{p(i)p(i+1)}{q(i)q(i+1)q(i+2)} + \cdots , \qquad i \geq 1,$$

$$E^i T_{i+1} = \frac{1}{p(i)} + \frac{q(i)}{p(i-1)p(i)} + \frac{q(i-1)q(i)}{p(i-2)p(i-1)p(i)} + \cdots + \frac{q(1)q(2)\dots q(i)}{p(0)p(1)\dots p(i)}, \qquad i \geq 0,$$

So $\mu = E^1 \tau_1$, and $\mu < \infty \Longleftrightarrow E^1 \tau_1 < \infty$. By Theorem 1.2,

$$\pi(i) = \frac{1}{p(i)E^{i+1}\tau_{i+1} + q(i)E^{i-1}T_i + 1} = \frac{\mu_i}{\mu}.$$

□

*Proof of Corollary 1.4* We define

$$M_1 = \begin{pmatrix} \frac{q_1+q_2}{p} & 0 \\ \frac{1}{p} & 0 \end{pmatrix} := A, \quad M_i = \begin{pmatrix} \frac{q_1}{p} & \frac{q_2}{p} \\ 1 + \frac{q_1}{p} & \frac{q_2}{p} \end{pmatrix} := B, \quad i > 1$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 + \lambda_2 & 0 \\ 1 & 1 \end{pmatrix} := C, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} := D.$$

Let $\lambda_{1,2} = \frac{q_1+q_2 \pm \sqrt{(q_1+q_2)^2+4pq_2}}{2p}$ be the eigenvalues of $M_i, i > 1$,and let $\tilde{\lambda}_{1,2}$ be the eigenvalues of $M_1$. We can see that $\tilde{\lambda}_1 = \lambda_1 + \lambda_2$ and $\tilde{\lambda}_2 = 0$. Then, we can decompose $M_i$ as

$$M_1 = C \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot C^{-1}, \quad M_i = D \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot D^{-1}, \quad i > 1$$

Then, if $\lambda_1 \neq 1$

$$\begin{aligned} &\sum_{k=1}^{n} e_1 M_k M_{k-1} \dots M_1 u = e_1 (E + B + B^2 \dots B^n) A u \\ = \;& e_1 \cdot D \cdot \begin{pmatrix} \frac{1-\lambda_1^n}{1-\lambda_1} & 0 \\ 0 & \frac{1-\lambda_2^n}{1-\lambda_2} \end{pmatrix} \cdot D^{-1} \cdot C \cdot \begin{pmatrix} \lambda_1 + \lambda_2 & 0 \\ 0 & 0 \end{pmatrix} \cdot C^{-1} \cdot (1,1)' \\ = \;& \frac{1}{\lambda_1 - \lambda_2} \Big( \frac{1-\lambda_1^n}{1-\lambda_1} \lambda_1^2 - \frac{1-\lambda_2^n}{1-\lambda_2} \lambda_2^2 \Big), \end{aligned}$$

if $\lambda_1 = 1$

$$\sum_{k=1}^{n} e_1 M_k M_{k-1} \dots M_1 u = \frac{1}{\lambda_1 - \lambda_2} \Big( \lambda_1^2 n - \frac{1-\lambda_2^n}{1-\lambda_2} \lambda_2^2 \Big).$$

It is easy to see that $\lambda_2 = \frac{q_1+q_2-\sqrt{(q_1+q_2)^2+4pq_2}}{2p} \in (-1, 0]$. We get $\lambda_2^n \to 0$ as $n \to \infty$.

Hence, $\sum_{k=1}^{\infty} e_1 M_k M_{k-1} \dots M_1 u = \infty \Leftrightarrow \lambda_1 \geq 1$. By some calculation,

$$\lambda_1 \geq 1 \Leftrightarrow q_1 + 2q_2 \geq p.$$

□

*Proof of Corollary 1.5* Define $A, B, C, D, \lambda_1$ and $\lambda_2$ the same as before. $P^{i+1}[(i+1, +\infty), i-1]$ and $E^i\tau_i$ are calculated in [6], we need only to calculate $E^iT_{i+1}$. By Theorem A, we get

$$\begin{aligned}
E^iT_{i+1} &= 1 + e_1 M_i (2,1)' + e_1 M_i M_{i-1}(2,1)' + \dots + e_1 M_i M_{i-1} \dots M_1 (2,1)' \\
&= 1 + e_1(B + B^2 + \dots + B^{i-1})(2,1)' + e_1(B_{i-1}A)(2,1)' \\
&= 1 + e_1 \cdot D \cdot \begin{pmatrix} \frac{\lambda_1(1-\lambda_1^{i-1})}{1-\lambda_1} & 0 \\ 0 & \frac{\lambda_2(1-\lambda_2^{i-1})}{1-\lambda_2} \end{pmatrix} \cdot D^{-1} \cdot (2,1)' \\
&\quad + e_1 \cdot D \cdot \begin{pmatrix} \lambda_1^{i-1} & 0 \\ 0 & \lambda_2^{i-1} \end{pmatrix} \cdot D^{-1} \cdot C \cdot \begin{pmatrix} \lambda_1 + \lambda_2 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 + \lambda_2 & 0 \\ 0 & 0 \end{pmatrix} \cdot C^{-1} \cdot (2,1)' \\
&= 1 + \frac{1}{\lambda_1 - \lambda_2}\left(\frac{\lambda_1^2(1-\lambda_1^{i-1})(\lambda_2+2)}{1-\lambda_1} - \frac{\lambda_2^2(1-\lambda_2^{i-1})(\lambda_1+2)}{1-\lambda_2}\right) + 2\frac{\lambda_1^{i+1} - \lambda_2^{i+1}}{\lambda_1 - \lambda_2}.
\end{aligned}$$

□

## 4 Tail asymptotic of $\pi(i)$ — proof of Theorem 1.3

It is well known that when the random walk is state-independent the tail of the stationary distribution of the walk has a geometric decay. With the help of the explicit expression of the stationary distribution, we can consider the tail behavior in the state-dependent case. We note that the expression of the stationary distribution in Theorem 1.2 is given in terms of the decomposition of the trajectory with different parts. This enables us to find out that the key factor to determine the tail asymptotic behavior of $\pi(i)$ is $E^iT_{i+1}$ ( because of $E^1\tau_1 < \infty$). In this section, we consider the case with a general $L$ (not only for $L = 2$), i.e., the $(L, 1)$-RW.

Suppose $L \geq 1$, to express $E^iT_{i+1}$ we need only the "lower" branching structure hidden in the random walk which we have introduced in Section 2. Recall for $i > 0$,

$$E^iT_{i+1} = 1 + \sum_{k=0}^{i} EU_k \cdot w_L' = 1 + \sum_{k=1}^{i} e_1 M_i M_{i-1} \dots M_{i-k+1} w_L',$$

where $e_L = (1, 0, \cdots, 0)$, $w_L = (2, 1, \cdots, 1)$. The offspring mean matrices of the multi-type branching processes are given by:

$$M_i = \begin{pmatrix} \frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} & \cdots & \frac{q_{L-1}(i)}{p(i)} & \frac{q_L(i)}{p(i)} \\ 1 + \frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} & \cdots & \frac{q_{L-1}(i)}{p(i)} & \frac{q_L(i)}{p(i)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} & \cdots & \frac{q_{L-1}(i)}{p(i)} & \frac{q_L(i)}{p(i)} \\ \frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} & \cdots & 1 + \frac{q_{L-1}(i)}{p(i)} & \frac{q_L(i)}{p(i)} \end{pmatrix}_{L \times L}, \quad i \geq L,$$

with a little attention to the reflect effect we notice the difference for $1 \leq i < L$,

$$M_{L-1} = \begin{pmatrix} \frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} & \cdots & \frac{q_{L-1}(i)+q_L(i)}{p(i)} & 0 \\ 1+\frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} & \cdots & \frac{q_{L-1}(i)+q_L(i)}{p(i)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} & \cdots & \frac{q_{L-1}(i)+q_L(i)}{p(i)} & 0 \\ \frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} & \cdots & 1+\frac{q_{L-1}(i)+q_L(i)}{p(i)} & 0 \end{pmatrix}_{L\times L}, \quad \ldots\ldots,$$

and

$$M_1 = \begin{pmatrix} \frac{q_1(i)+\cdots+q_L(i)}{p(i)} & 0 & \cdots & 0 & 0 \\ 1+\frac{q_1(i)+\cdots+q_L(i)}{p(i)} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{q_1(i)+\cdots+q_L(i)}{p(i)} & 0 & \cdots & 0 & 0 \\ \frac{q_1(i)+\cdots+q_L(i)}{p(i)} & 0 & \cdots & 0 & 0 \end{pmatrix}_{L\times L}.$$

The following two propositions about matrix analysis are needed for our proof. Recall, for a matrix $A$, $\rho(A)$ is the spectral radius of $A$, and $\lambda_A$ is the maximum eigenvalue of it.

**Proposition 4.1.** *(Perron's Theorem, [7]) If $A = (a_{ij})_{L\times L} \in \mathbb{R}^{L\times L}$. and $A > 0$(which means $\forall i, j > 0, a_{ij} > 0$), then*
*(a)$\rho(A) > 0$;*
*(b) $\rho(A)$ is an eigenvalue of $A$, and it is the unique eigenvalue of maximum modulus;*
*(c) $\rho(A)$ is an algebraically (and hence geometrically) simple value of $A$;*
*(d) $[\rho(A)^{-1}A]^k \to R$ as $k \to \infty$, where $R \in \mathbb{R}^{L\times L}$, and $R > 0$.* □

A measure for the distance between the spectra $\sigma(A) = \{\lambda_1, \lambda_2, \ldots \lambda_L\}$ and $\sigma(B) = \{\mu_1, \mu_2, \ldots \mu_L\}$ is defined below, which is the optimal matching distance:

$$d(\sigma(A), \sigma(B)) = \min_{\theta\in S_L} \max_{i\in\{1,2,\cdots L\}} |\lambda_i - \mu_{\theta_i}|,$$

where $S_L$ denotes the group of all permutations of the numbers $1, 2, \cdots L$

**Proposition 4.2.** *([11],[12])*

$$d(\sigma(A), \sigma(B)) \leq 4(2K)^{1-\frac{1}{n}} \|A - B\|^{\frac{1}{n}}$$

*where $K = \max\{\|A\|, \|B\|\}$.*

This result says that there exists a permutation such that the maximum of the distance between the corresponding eigenvalues is small enough.

Let $D := \{(p, q_1, q_2, \cdots q_L) : p + \sum_{j=1}^{L} q_j = 1; \quad \sum_{j=1}^{L} jq_j > p; \quad \forall j,\ q_j \geq 0, p > 0\}$. A point $P = (p, q_1, q_2, \cdots q_L)$ in $D$ corresponds a state-independent transition probability of the $(L, 1)$-RW, and the offspring mean matrix of the correspond multi-type process is denoted as $M$ (it is independent of the position $i$).

**Lemma 4.1.** *If $P \in D$, the offspring mean matrix of the lower branching process $M = (m_{ij})_{L\times L}$ has the unique eigenvalue of maximum modulus. That is*

$$\rho(M) > 1,$$

*and $\rho(M) = \lambda_M$. Let $C = 4(\frac{2}{p} + 2L)^{1-\frac{1}{L}} L^{\frac{1}{L}}$. For $\varepsilon < \min\{\frac{1}{C^L}(\lambda_M - 1)^L, \min_{i,j} m_{ij}\}$, we have*

$$\lambda_M^{\varepsilon-} := \rho(M - \varepsilon E) > \lambda_M - C\varepsilon^{\frac{1}{L}} > 1, \quad 1 < \lambda_M^{\varepsilon+} := \rho(M + \varepsilon E) < \lambda_M + C\varepsilon^{\frac{1}{L}},$$

where $E = (1, 1, \cdots 1)' \cdot (1, 1, \cdots 1)$.

*Proof* First, to prove $\rho(M) > 1$, let $\hat{M} = \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$, by some calculations, we get

$$|\lambda I - \hat{M}| = \begin{pmatrix} \lambda - 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda - \frac{q_1}{p} & \cdots & -\frac{q_{L-1}}{p} & -\frac{q_L}{p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 - \frac{q_1}{p} & \cdots & \lambda - \frac{q_{L-1}}{p} & -\frac{q_L}{p} \\ 0 & -\frac{q_1}{p} & \cdots & -1 - \frac{q_{L-1}}{p} & \lambda - \frac{q_L}{p} \end{pmatrix} = \begin{pmatrix} \lambda - \frac{1}{p} & \frac{q_1}{p} & \cdots & \frac{q_{L-1}}{p} & \frac{q_L}{p} \\ -1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & \cdots & -1 & \lambda \end{pmatrix}$$

.

Define $f(x) = x^{L+1} - \frac{1}{p}x^{L-1} + \cdots + \frac{q_{L-1}}{p}x + \frac{q_L}{p}$. We have $|\lambda I - \hat{M}| = f(x)$. Let $F(x) = f(\frac{1}{x})$, then,

$$F(x) = q_L x^{L+1} + q_{L-1}x^L + \cdots + q_1 x^2 - x + p.$$

Note that $P \in D$, it is easy to see that $F(x) = 0$ has a unique real root in $(0, 1)$. As a consequence $|\lambda I - \hat{M}| = 0$ has only one real root larger than 1, so the largest eigenvalue of $\hat{M}$ is larger than one. One can see that 1 is not an eigenvalue of $M$, and the set of eigenvalues of $\hat{M}$ is the union of the set of eigenvalues of $M$ and $\{1\}$. So we get $\rho(M) > 1$. By Proposition 4.1, $\rho(M) = \lambda_M$ is obvious.

Next, for $A \in \mathbb{R}^{L\times L}$, define $\|\cdot\|$ to be the maximum column sum matrix norm, that is: $\|A\| = \max_{1\le j\le L} \sum_{i=1}^{L} |a_{ij}|$. Choose an $\varepsilon$ such that both $M - \varepsilon E$ and $M + \varepsilon E$ are positive matrices. By Proposition 4.1, both $\lambda_M^{\varepsilon-}$ and $\lambda_M^{\varepsilon+}$ are meaningful. By Proposition 4.2, we have

$$d(\sigma(M), \sigma(M - \varepsilon E)) \le 4(\frac{2}{p})^{1-\frac{1}{L}} L^{\frac{1}{L}} \varepsilon^{\frac{1}{L}} \le C\varepsilon^{\frac{1}{L}},$$
$$d(\sigma(M), \sigma(M + \varepsilon E)) \le 4(\frac{2}{p} + 2L)^{1-\frac{1}{L}} L^{\frac{1}{L}} \varepsilon^{\frac{1}{L}} \le C\varepsilon^{\frac{1}{L}}.$$

For such $\varepsilon$, it is obvious that $\lambda_M - C\varepsilon^{\frac{1}{L}} > 1$. If $\lambda_M^{\varepsilon-} < \lambda_M - C\varepsilon^{\frac{1}{L}}$, for $\lambda_M$, then there exists no permutation such that $d(\sigma(M), \sigma(M - \varepsilon E)) \le C\varepsilon^{\frac{1}{L}}$, because $\lambda_M^{\varepsilon-}$ is the largest eigenvalue of $M - \varepsilon E$.

On the other hand, because $M + \varepsilon E > M > 0$, we have $\rho(M + \varepsilon E) \ge \rho(M) > 1$ ([7], P491, Corollary 8.1.19). If $\lambda_M^{\varepsilon+} > \lambda_M + C\varepsilon^{\frac{1}{L}}$, then there exists no permutation such that $d(\sigma(M), \sigma(M + \varepsilon E)) \le C\varepsilon^{\frac{1}{L}}$, because $\lambda_M$ is the largest eigenvalue of $M$. □

*Proof of Theorem 1.3* The first part of the Theorem 1.3 has been proved in Lemma 4.1. Now we focus on the second part.

*Step 1* To prove $\lim_{k\to\infty} \frac{\log E^k T_{k+1}}{k} = \log \lambda_M$. Let

$$\Phi(i) = \sum_{j=1}^{i} e_L M_i M_{i-1} \dots M_{i-j+1} w'_L,$$

$$\varphi(i) = \sum_{j=1}^{i} M_i M_{i-1} \dots M_{i-j+1}.$$

By formula (2.1), $E^i T_{i+1} = 1 + \Phi(i)$.

Because $P(i) \to P$, we have $M_i \to M$. By Lemma 4.1, $\lambda_M > 1$. For each $\varepsilon$ we defined in Lemma 4.1. $\exists N$, when $n > N$,

$$0 < M - \varepsilon E \leq M_n \leq M + \varepsilon E.$$

Then we have

$$\begin{aligned}
&0 < (M - \varepsilon E)^2 \leq M_{N+2} M_{N+1} \leq (M + \varepsilon E)^2, \\
&\dots\dots \\
&0 < (M - \varepsilon E)^k \leq M_{N+k} M_{N+k-1} \cdots M_{N+1} \leq (M + \varepsilon E)^k.
\end{aligned}$$

Summarizing these formulas leads to:

$$\begin{aligned}
&(M - \varepsilon E)^k + (M - \varepsilon E)^{k-1} + \dots + (M - \varepsilon E) \\
&\leq M_{N+k} M_{N+k-1} \cdots M_{N+1} + M_{N+k-1} \cdots M_{N+1} + \dots + M_{N+1} \\
&\leq (M + \varepsilon E)^k + (M + \varepsilon E)^{k-1} + \dots + (M + \varepsilon E).
\end{aligned}$$

For the given $N$, Let $A_N = M_N M_{N-1} \cdots M_1 + M_{N-1} \cdots M_1 + \dots + M_1$, $B_N = M_N M_{N-1} \cdots M_1$. Then

$$\begin{aligned}
&(M - \varepsilon E)^k B_N + (M - \varepsilon E)^{k-1} B_N + \dots + (M - \varepsilon E) B_N + A_N \\
&\leq \varphi(N + k) \leq (M + \varepsilon E)^k B_N + (M + \varepsilon E)^{k-1} B_N + \dots + (M + \varepsilon E) B_N + A_N. \quad (4.1)
\end{aligned}$$

Now we first consider the "$\leq$" part of the result. From (4.1), we have

$$\varphi(N + k) \geq (M - \varepsilon E)^k B_N.$$

Then

$$\begin{aligned}
&\Phi(N + k) \geq (\lambda_M^{\varepsilon-})^k e_L \frac{(M - \varepsilon E)^k}{(\lambda_M^{\varepsilon-})^k} B_N w_L, \\
&\log(\Phi(N + k)) \geq k \log \lambda_M^{\varepsilon-} + \log e_L \frac{(M - \varepsilon E)^k}{(\lambda_M^{\varepsilon-})^k} B_N w_L. \quad (4.2)
\end{aligned}$$

By *Proposition 4.1*, as $k \to \infty$,

$$\frac{(M - \varepsilon E)^k}{(\lambda_M^{\varepsilon-})^k} \to R_M^{\varepsilon-} > 0,$$

as a consequence $e_L\frac{(M-\varepsilon E)^k}{(\lambda_M^{\varepsilon -})^k}B_N w_L$ is bounded in $k$. Thus from (4.2) as $k\to\infty$

$$\underline{\lim}_{k\to\infty}\frac{\log\Phi(k)}{k}\geq\log\lambda_M^{\varepsilon -}.$$

Note that from Lemma 4.1, $\lambda_M^{\varepsilon -}>\lambda_M-C\varepsilon^{\frac{1}{L}}>1$. So

$$\underline{\lim}_{k\to\infty}\frac{\log\Phi(k)}{k}\geq\log(\lambda_M-C\varepsilon^{\frac{1}{L}}). \tag{4.3}$$

For the right "$\leq$" part, define $\psi(k)=e_L\frac{(M+\varepsilon E)^k}{(\lambda_M^{\varepsilon +})^k}B_N w_L$. From (4.1), we have

$$\Phi(N+k)\leq(\lambda_M^{\varepsilon +})^k(\psi(k)+\psi(k-1)+\cdots+\psi(1)+A_N),$$
$$\log\Phi(N+k)\leq k\log(\lambda_M^{\varepsilon +})+\log(\psi(k)+\psi(k-1)+\cdots+\psi(1)+A_N).$$

It's easy to see that $\frac{\log(\psi(k)+\psi(k-1)+\cdots+\psi(1)+A_N)}{N+k}\to 0$ as $k\to\infty$ (because $\lim_{k\to\infty}\psi(k)$ exists). Therefore,

$$\overline{\lim}_{k\to\infty}\frac{\log\Phi(k)}{k}\leq\log\lambda_M^{\varepsilon +}.$$

By Lemma 4.1, $1<\lambda_M^{\epsilon +}<\lambda_M+C\epsilon^{\frac{1}{L}}$. So

$$\overline{\lim}_{k\to\infty}\frac{\log\Phi(k)}{k}\leq\log(\lambda_M+C\varepsilon^{\frac{1}{L}}). \tag{4.4}$$

Combine (4.3) and (4.4) to have

$$\log(\lambda_M-C\varepsilon^{\frac{1}{L}})\leq\underline{\lim}_{k\to\infty}\frac{\log\Phi(k)}{k}\leq\overline{\lim}_{k\to\infty}\frac{\log\Phi(k)}{k}\leq\log(\lambda_M+C\varepsilon^{\frac{1}{L}}).$$

Let $\varepsilon\to 0$, we get ( recall $E^iT_{i+1}=1+\Phi(i)$)

$$\lim_{k\to\infty}\frac{\log E^kT_{k+1}}{k}=\lim_{k\to\infty}\frac{\log\Phi(k)}{k}=\log\lambda_M. \tag{4.5}$$

*Step 2* It's easy to see that when $P(i)\to P$ and $P\in D$, the walk is positive recurrence. So the stationary distribution exists and $\pi(i)=\frac{1}{E^iT_i}$. By the same method we have used in the proof of Theorem 1.2, we get

$$E^iT_i=p(i)E^{i+1}\tau_{i+1}+\sum_{l=1}^{L-1}\vartheta_l E^{i-l}T_{i-l+1}+q_L(i)E^{i-L}T_{i-L+1}+1 \tag{4.6}$$

where $\vartheta_l=\sum_{k=l}^{L}q_k(i)+p(i)\sum_{j=l}^{L-1}P^{i+1}[(i+1,+\infty),i-j]$. Because the walk is positive recurrence, $E^{i+1}\tau_{i+1}<E^1\tau_1<\infty$; on the other hand, under our condition, $0<p(i)<1$, $0\leq q_l(i)<1$, $0<\vartheta_l<L+1$. Then (4.6) tells us that $E^{i-1}T_i$ is the dominated term for $E^iT_i$. By (4.5),

$$\lim_{k\to\infty}\frac{\log\pi(k)}{k}=-\lim_{k\to\infty}\frac{\log E^kT_k}{k}=-\lim_{k\to\infty}\frac{\log E^{k-1}T_k}{k}=-\log\lambda_M.$$

□

**Appendix:** *Analytical proof of Theorem 1.1.*

Let $\xi_0 = 0$, $\xi_1 = 1$, and $\xi_n = \sum_{i=1}^{n} \theta_i$ for $n \geq 1$, the occupation time at position 0 before the walk hitting position $n$. By Theorem A, we have

$$E^0\theta_i = e_1 M_{i-1} M_{i-2} \cdots M_1 u, \quad i \geq 1.$$

For the system of equations $\sum_{j=0}^{\infty} P_{ij} y_j = y_i, i \geq 0$, we will prove $y_n = E^0\xi_n$ is the solution, the probabilistic meaning of which is the expectation of the visiting time at position 0 before the walk hitting the position $n$. The system of equations can be rewritten as,

$$\begin{aligned}
y_1 &= p(1)y_2 + q_1 y_0, \\
y_2 &= p(2)y_3 + q_1(2)y_1 + q_2(2)y_0, \\
&\vdots \\
y_n &= p(n)y_{n+1} + q_1(n)y_{n-1} + q_2(n)y_{n-2}, \\
&\vdots
\end{aligned}$$

It is not hard to show that the solution spans a two dimensional linear space. We can prescribe $y_0$ and $y_1$ arbitrarily for the initial values, and then all the other $y_i$ are determined by these equations. Trivially $y_i \equiv 1$ is a solution. We show that If $y_n = E^0\xi_n$, for $n \geq 0$ is also a solution now.

Setting $x_0 = 0, x_1 = 1$, and for $n \geq 2$, $x_n = y_n - y_{n-1}$, we obtain:

$$\begin{aligned}
x_2 &= \left(\frac{q_1(1)}{p_1(1)} + \frac{q_2(1)}{p_1(1)}\right)x_1, \\
x_3 &= \frac{q_2(2)}{p_2(2)}x_1 + \left(\frac{q_1(2)}{p_2(2)} + \frac{q_2(2)}{p_2(2)}\right)x_2, \\
&\vdots \\
x_{n+1} &= \frac{q_2(n)}{p_2(n)}x_{n-1} + \left(\frac{q_1(n)}{p_2(n)} + \frac{q_2(n)}{p_2(n)}\right)x_n \\
&\vdots
\end{aligned}$$

We just need to prove that $x_n = E^0\theta_n$ is the correspond solution. In probabilistic meaning, $x_n$ is the expectation of the visiting time at position 0 in the $n$-th immigration structure. For the first equation,

$$E^0\theta_2 = e_1 M_1 u = \frac{q_1(1)}{p_1(1)} + \frac{q_2(1)}{p_1(1)} = \left(\frac{q_1(1)}{p_1(1)} + \frac{q_2(1)}{p_1(1)}\right)E^0\theta_1.$$

For the $n$-th equation, let $K_n = M_{n-2}M_{n-3}\cdots M_1 u, b_n(1) = \frac{q_1(n)}{p(n)}, b_n(2) = \frac{q_2(n)}{p(n)}$.

$$E^0\theta_{n+1} \quad = \quad e_1 M_n M_{n-1} K_n$$

$$= \Big(b_n(1)b_{n-1}(1)+b_n(2)b_{n-1}(2)+b_n(2),\ b_n(1)b_{n-1}(2)+b_n(2)b_{n-1}(2)\Big)K_n.$$

On the other hand,

$$\begin{aligned}
& \frac{q_2(n)}{p_2(n)}E^0\theta_{n-1}+\Big(\frac{q_1(n)}{p_2(n)}+\frac{q_2(n)}{p_2(n)}\Big)E^0\theta_n \\
= \ & (b_n(2)e_1+(b_n(1)+b_n(1))e_1M_{n-1})K_n \\
= \ & ((b_n(2),0)+(b_n(1)+b_n(2),0)M_{n-1})K_n \\
= \ & \Big((b_n(2),0)+(b_n(1)+b_n(2),0)\begin{pmatrix} b_n(1) & b_n(2) \\ 1+b_n(1) & b_n(2)\end{pmatrix}\Big)K_n \\
= \ & \Big(b_n(1)b_{n-1}(1)+b_n(2)b_{n-1}(2)+b_n(2), b_n(1)b_{n-1}(2)+b_n(2)b_{n-1}(2)\Big)K_n,
\end{aligned}$$

which means that $x_n=E^0\theta_n$ is the correspond solution of the equations above. So $y_n=E^0\xi_n$ is the solution of $\sum_{j=0}^{\infty}P_{ij}y_j=y_i, i\geq 0$. It is evidence that $y_n\equiv 1$ are another solutions of the system. Therefore, the general solution is given by

$$(y_0,y_1,\ldots,y_n\ldots)=\alpha\cdot(1,1,\ldots,1,\ldots)'+\beta\cdot(E^0\xi_0,E^0\xi_1,\ldots,E^0\xi_n\ldots)',$$

where $\alpha,\beta,\in\mathbb{R}$, and a nonconstant bounded solution exists if and only if $E^0\xi_n$ is bounded, i.e., $\sum_{k=1}^{\infty}e_1M_kM_{k-1}\cdots M_1u<\infty$ □

**Remark** The probability meaning of $\theta_i$, $i>0$, is the expectation of the local time at zero of the walk starting at $i-1$ before $T_i$, which is the motivation of the proof.

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