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QUENCHED MEAN LIMIT THEOREMS FOR THE SUPER-BROWNIAN MOTION WITH SUPER-BROWNIAN IMMIGRATION*

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The limiting behavior of the expectation of the super-Brownian motion with super-Brownian immigration under the *quenched* probability is considered: A central limit theorem is proved for $d \geq 3$, an ergodic property is considered for $d = 2$ and a local large deviation is obtained for $d = 3$.

Keywords: Central limit theorem; ergodic; large deviation; immigration; super-Brownian motion.

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1. Introduction and Statement of Results

Let $C(\mathbb{R}^d)$ denote the space of continuous bounded functions on $\mathbb{R}^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1 + |x|^2)^{-p/2}$ for $x \in \mathbb{R}^d$. Let $C_p(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : |f(x)| \leq \text{const}\cdot\phi_p(x)\}$. In duality, let $M_p(\mathbb{R}^d)$ be the space of Radon measures μ on $\mathbb{R}^d$ such that $\langle \mu, f \rangle := \int f(x)\mu(dx) < \infty$ for all $f \in C_p(\mathbb{R}^d)$. We endow $M_p(\mathbb{R}^d)$ with the p-vague topology, i.e. $\mu_k \to \mu$ if and only if $\langle \mu_k, f \rangle \to \langle \mu, f \rangle$ for all $f \in C_p(\mathbb{R}^d)$. Then $M_p(\mathbb{R}^d)$ is metrizable. Throughout this paper, λ denotes the Lebesgue measure on $\mathbb{R}^d$.

Suppose that $W = (w_t, t \geq 0)$ is a standard Brownian motion in $\mathbb{R}^d$ with semigroup $(S_t)_{t\geq 0}$. A *super-Brownian motion* $\varrho = (\varrho_t, P_\mu)$ is an $M_p(\mathbb{R}^d)$-valued Markov process with $\varrho_0 = \mu$ and the transition probabilities given by

$$P_\mu \exp\{-\langle \varrho_t, f \rangle\} = \exp\{-\langle \mu, v(t, \cdot) \rangle\}\,, \qquad f \in C_p^+(\mathbb{R}^d)\,, \tag{1.1}$$

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where $v(\cdot,\cdot)$ is the unique mild solution of the evolution equation

$$\begin{cases} \dot{v}(t) = \Delta v(t) - v^2(t)\,, \\ v(0) = f\,. \end{cases} \tag{1.2}$$

Given a super-Brownian motion $\varrho = (\varrho_t, P_\mu)$, we will consider another super-Brownian motion with the immigration rate controled by the trajectory of ϱ, the so-called *super-Brownian motion with super-Brownian immigration* (SBMSBI, for short) $X^\varrho = (X_t^\varrho, P_\nu^\varrho)$ with $X_0^\varrho = \nu$, whose transition probabilities is given by

$$P_\nu^\varrho \exp\{-\langle X_t^\varrho, f\rangle\} = \exp\left\{-\langle \nu, v(t,\cdot)\rangle - \int_0^t \langle \varrho_s, v(t-s,\cdot)\rangle ds\right\}. \tag{1.3}$$

In the following we take $\mu = \nu = \lambda$, and write P^ϱ (resp. P) for P_λ^ϱ (resp. P_λ). This model was considered by Hong and Li[9] and Hong,[6–8] where some interesting and new phenomena were revealed under the *annealed* probability law:

$$\mathbb{P}(\cdot) := \int P^\varrho(\cdot) P(d\varrho)\,,$$

i.e. the random immigration "smooth" the critical dimension in the sense that the "log" term does not appear in the norming for the longtime behavior.

In this paper, we will study the limiting behavior of the process SBMSBI under the *quenched* probability law P_λ^ϱ. For any $f \in C_p(R^d)$, the expectation of the process X_t^ϱ under the *quenched* law $P^\varrho\langle X_t^\varrho, f\rangle$ is the functional of ϱ. To consider the central limit behavior, we define the centered functional $F(\varrho.,T;f)$ by

$$F(\varrho.,T;f) := a_d(T)^{-1}\{P^\varrho\langle X_T^\varrho, f\rangle - P[P^\varrho\langle X_T^\varrho, f\rangle]\}\,, \tag{1.4}$$

where

$$a_d(T) = \begin{cases} T^{(6-d)/4}\,, & 3 \le d \le 5\,, \\ (\log T)^{1/2}\,, & d = 6\,, \\ 1\,, & d \ge 7\,. \end{cases}$$

Then we have

Theorem 1.1. *For $d \ge 3$, $f \in C_p^+(\mathbb{R}^d)$, as $T \to \infty$, $F(\varrho.,T;f) \Rightarrow \xi(f)$ in distribution under the law P, $\xi(f)$ is determined by*

$$P\exp\{-\theta\xi(f)\} = \exp\{F(\theta,f)\}\,, \qquad \theta \ge 0$$

where

$$F(\theta,f) = \begin{cases} C_d\theta^2\langle\lambda,f\rangle^2\,, & 3 \le d \le 6 \\ \displaystyle\int_0^\infty \langle\lambda, u_\theta^2(s,\cdot)\rangle ds\,, & d \ge 7\,. \end{cases}$$

$C_d = (4\pi)^{-d/2}\int_0^1 s^{2-d/2}ds$ *for* $3 \le d \le 5$, $C_6 = (4\pi)^{-3}$; *and* $u_\theta(t,x)$ *is the mild solution of equation*

$$u_\theta(t,x) = \theta t S_t f(x) - \int_0^t S_{t-s}u_\theta^2(s,\cdot)(x)ds\,.$$

Remark 1.1. (a) The limiting variable is Gaussian for $3 \le d \le 6$, whereas it is non-Gaussian in higher dimensions $d \ge 7$.

(b) Recall that under the *annealed* probability[9] $\mathbb{P}$ the norm is

$$\bar{a}_d(T) = \begin{cases} T^{3/4}\,, & d = 3\,, \\ T^{1/2}\,, & d \ge 4\,, \end{cases}$$

and both of them are different from that of Iscoe,[10] where the occupation time process of the ordinary super-Brownian motion was considered.

For $d = 2$, we prove a weak ergodic theorem.

Theorem 1.2. *$d = 2$, as $T \to \infty$*

$$T^{-1}P^\varrho\langle X_T^\varrho, f\rangle \longrightarrow \xi\cdot\langle\lambda, f\rangle \quad \textit{weakly} \quad (\textit{with respect to } P)\,,$$

where ξ is a non-negative, infinitely divisible random variable whose Laplace transform is given by

$$P\exp\{-\theta\xi\} = \exp\{-\langle\lambda, w(1,\cdot;\theta)\rangle\}\,, \qquad \theta \ge 0 \tag{1.5}$$

where $w \equiv w(t,x;\theta)$ is the mild solution of the evolution equation

$$w(t,x;\theta) = \theta t p(t,x) - \int_0^t S_{t-s}w^2(s,\cdot;\theta)ds\,,$$

where $p(s,x)$ is the transition density function of the standard Brownian motion.

For $d = 3$, we will prove a local large deviation. Fix $f \in C_p^+(\mathbb{R}^d)$ satisfying $\langle\lambda, f\rangle = 1$ and let

$$\mathbf{W}(t) := \frac{1}{t}P^\varrho\langle X_t^\varrho, f\rangle \tag{1.6}$$

and

$$\Lambda(T,\theta) := T^{-1/2}\log P\exp[\theta T^{1/2}\mathbf{W}(T)]\,. \tag{1.7}$$

For $d = 3$, we will proved in Lemma 3.8 that the equation

$$w(t,x;\theta) = \theta t p(t,x) + \int_0^t S_{t-s}w^2(s,\cdot;\theta)ds$$

admit unique mild solutions $w(t,\cdot;\theta) \in C([0,1], L^2(\mathbb{R}^3))$ for $|\theta| < \frac{1}{4a}$, where $a := \frac{2}{3}(2\pi)^{-3/2}$, $p(t) = p(t,x)$ is the transition density function of the Brownian motion. Moreover, we will prove that there is $\delta > 0$ such that

$$\Lambda(\theta) := \lim_{T\to\infty}\Lambda(T,\theta) = \langle\lambda, w(1,\cdot;\theta)\rangle\,,$$

which is continuous differential and strictly convex in $|\theta| < \delta < \frac{1}{4a}$ with $\Lambda'(0) = 1$. Let $I(\alpha)$ be the Legendre transform of $\Lambda(\theta)$, i.e.

$$I(\alpha) := \sup_{|\theta|<\delta} [\alpha\theta - \Lambda(\theta)] . \tag{1.8}$$

Then we have

Theorem 1.3. *For $d = 3$, the law of $\mathbf{W}(T)$ under P admit the LDP with speed function $T^{-1/2}$ and rate function $I(\alpha)$, i.e. there exists a neighborhood O of 1 such that if $U \subset O$ is open and $C \subset O$ is closed, then*

$$\liminf_{T\to\infty} T^{-1/2} \log P\{\mathbf{W}(T) \in U\} \geq - \inf_{\alpha\in U} I(\alpha) ,$$

$$\limsup_{T\to\infty} T^{-1/2} \log P\{\mathbf{W}(t) \in C\} \leq - \inf_{\alpha\in C} I(\alpha) .$$

Remark 1.2. It should be interesting to consider the LDP for $d > 3$, but for now we have only obtained this result for $d = 3$ and the steepness has not been given yet. We leave it as an open problem and will be considered further.

Theorem 1.1 will be proved in Sec. 2. By considering the convergence of the solutions of the evolution equation, one proves Theorem 1.2 in Sec. 3. Using Dynkin's moment method to prove the existence of the solution of the equation and extend the Laplace transformation to some positive range, we obtained a local LDP for $d = 3$ in Sec. 4.

2. Proof of Theorem 1.1

The mild solution of Eq. (1.2) is

$$v(t,x) = S_t f(x) - \int_0^t S_{t-s} v(s,\cdot)^2(x) ds , \qquad t \geq 0 \quad f \in C_p^+(\mathbb{R}^d) . \tag{2.1}$$

From Eq. (1.3) we get

$$P^{\varrho}\langle X_T^{\varrho}, f\rangle = \langle \lambda, f\rangle + \int_0^T \langle \varrho_s, S_{T-s} f\rangle ds , \tag{2.2}$$

which is the functional of the process $\varrho = (\varrho_t, P)$ and is determined by the Laplace functional

$$P \exp\{-P^{\varrho}\langle X_t^{\varrho}, f\rangle\} = \exp\{-\langle \lambda, f\rangle - \langle \lambda, u(t,\cdot)\rangle\} , \tag{2.3}$$

where $u(t,x)$ is the mild solution of the following evolution equation,

$$\begin{cases} \dot{u}(t) = \Delta u(t) - u^2(t) + S_t f & 0 < t \leq T \\ u(0) = 0 , \end{cases} \tag{2.4}$$

i.e.

$$u(t,x) = t S_t f(x) - \int_0^t S_{t-s} u^2(s,\cdot)(x) ds , \qquad t \geq 0 , \tag{2.5}$$

see Iscoe.[10] Let $f_T := a_d(T)^{-1}f$, from (1.4) and (2.3) we get

$$P\exp\{-F(\varrho.,T;f)\} = \exp\left\{\int_0^T \langle\lambda, u_T^2(t,\cdot)\rangle dt\right\}, \tag{2.6}$$

where $u_T(t,x)$ is the mild solution of Eq. (2.5) with f replaced by f_T.

Lemma 2.1. *Let* $3 \le d \le 6$,

$$A_d(T,f) := \int_0^T \langle\lambda, (tS_t f_T)^2\rangle dt\,. \tag{2.7}$$

We have

$$\lim_{T\to\infty} A_d(T,f) = C_d\langle\lambda, f\rangle^2\,, \tag{2.8}$$

where $C_d = (4\pi)^{-d/2}\int_0^1 s^{2-d/2}ds$ *for* $3 \le d \le 5$, $C_6 = (4\pi)^{-3}$.

Proof. From (2.7)

$$\begin{aligned} A_d(T,f) &= a_d(T)^{-2}\int_0^T \langle\lambda, (tS_t f)^2\rangle dt \\ &= a_d(T)^{-2}\int_0^T t^2 dt\int\int p(2t,y,z)f(y)f(z)dydz\,. \end{aligned}$$

When $3 \le d \le 5$,

$$\begin{aligned} \lim_{T\to\infty} A_d(T,f) &= \lim_{T\to\infty} a_d(T)^{-2}T^{3-d/2}\int_0^1 t^2 dt\int\int (4\pi t)^{-d/2}e^{-\frac{|y-z|^2}{2Tt}}f(y)f(z)dydz \\ &= C_d\langle\lambda, f\rangle^2 \end{aligned}$$

by dominated convergence theorem. When $d = 6$,

$$\begin{aligned} \lim_{T\to\infty} A_d(T,f) &= \lim_{T\to\infty} a_d(T)^{-2}\int_1^T t^2 dt\int\int p(2t,y,z)f(y)f(z)dydz \\ &= \lim_{T\to\infty} a_d(T)^{-2}\log T\int_0^1 T^{3s}ds\int\int (4\pi T^s)^{-3}e^{-\frac{|y-z|^2}{2T^s}}f(y)f(z)dydz \\ &= C_d\langle\lambda, f\rangle^2 \end{aligned}$$

by dominated convergence theorem, where we have taken the transformation $t = T^s$ at the second step. Completes the proof. □

Lemma 2.2. *Let* $3 \le d \le 6$,

$$\varepsilon_d(T,f) := \int_0^T \langle\lambda, (tS_t f_T)^2\rangle dt - \int_0^T \langle\lambda, u_T^2(t,\cdot)\rangle dt\,. \tag{2.9}$$

We have

$$\lim_{T\to\infty} \varepsilon_d(T,f) = 0\,. \tag{2.10}$$

Proof. From Eq. (2.7),

$$0 \le [tS_t f_T(x)]^2 - u_T^2(t,x) \le 2[tS_t f_T(x)] \int_0^t S_{t-s} u_T^2(s,\cdot)(x) ds\,.$$

Let C denotes a positive constant and it may be different values at different line. Recall the useful inequality $S_t f(x) \le C(1 \wedge t^{-d/2})$,

$$\begin{aligned} 0 \le \lim_{T\to\infty} \varepsilon_d(T,f) &\le \lim_{T\to\infty} 2\int_0^T \langle \lambda, [tS_t f_T] \int_0^t S_{t-s}(sS_s f_T)^2 ds\rangle dt \\ &\le C \lim_{T\to\infty} a_d(T)^{-3} \int_0^T t\langle \lambda, (S_t f)^2 \int_0^t s^2(1\wedge s^{-d/2}) ds\rangle dt \\ &\le C \lim_{T\to\infty} a_d(T)^{-3} \int_0^T t(1\wedge t^{-d/2}) dt \cdot \int_0^T s^2(1\wedge s^{-d/2}) dt \\ &= 0\,, \end{aligned}$$

completes the proof. □

Proof of Theorem 1.1. By Lemmas 2.1 and 2.2, when $3 \le d \le 6$,

$$\lim_{T\to\infty} \int_0^T \langle \lambda, u_T^2(t,\cdot)\rangle dt = C_d \langle \lambda, f\rangle^2\,. \tag{2.11}$$

When $d \ge 7$, $\int_0^T \langle \lambda, u_T^2(t,\cdot)\rangle dt = \int_0^T \langle \lambda, u^2(t,\cdot)\rangle dt$ is increasing in T and note that

$$\int_0^\infty \langle \lambda, u^2(t,\cdot)\rangle dt \le \int_0^\infty \langle \lambda, (tS_t f)^2\rangle dt \le C\int_0^\infty t^2(1\wedge t^{-d/2}) dt < \infty\,.$$

So

$$\lim_{T\to\infty} \int_0^T \langle \lambda, u_T^2(t,\cdot)\rangle dt = \int_0^\infty \langle \lambda, u^2(t,\cdot)\rangle dt\,. \tag{2.12}$$

Combining (2.11) and (2.12) with (2.6), and the bilateral Laplace transform discussed by Iscoe (Theorem 5.4 of Ref. 10), the proof is complete. □

3. Proof of Theorem 1.2

In this section, we assume $1 \le d \le 3$. Let $b_d(T) = T^{2-d/2}$, $f_T = b_d^{-1}(T) f$, and $u_T(t,x)$ be the mild solution of Eq. (2.4) with f being replaced by f_T (for simplicity, we consider $f \in C_p(\mathbb{R}^d)^+$ such that $\langle \lambda, f\rangle = 1$),

$$\begin{cases} \dot{u_T}(t) = \Delta u_T(t) - u_T^2(t) + S_t f_T\,, & 0 < t \le T \\ u_T(0) = 0\,. \end{cases} \tag{3.1}$$

Let $w_T(t,x) := T u_T(Tt, T^{1/2}x)$, it is easy to verify that $w_T(t,x)$ satisfy the following equation

$$\begin{cases} \dot{w_T}(t) = \Delta w_T(t) - w_T^2(t) + T^2 S_{Tt} f_T(T^{1/2}\cdot)\,, & 0 < t \le 1 \\ w_T(0) = 0\,, \end{cases} \tag{3.2}$$

i.e.

$$w_T(t,x) = tT^2 S_t f_T(T^{1/2}\cdot)(x) - \int_0^t S_{t-s} w_T^2(s,\cdot)(x)ds\,. \tag{3.3}$$

In what follows, we will prove that $w_T(t,x)$ converges in $L^2(\mathbb{R}^d,\lambda)$ uniformly in $t\in[0,1]$, and pointwise for each $t\in[0,1]$ to $w(t,x)$ as $T\to\infty$, which is the mild solution of the following equation,

$$\begin{cases} \dot{w}(t,x) = \Delta w(t,x) - w^2(t,x) + p(s,x)\,, & 0 < t \le 1 \\ w(0,x) = 0\,, \end{cases} \tag{3.4}$$

where $p(s,x,y) = p(s,x-y)$ is the transition density function of Brownian motion, i.e.

$$w(t,x) = tp(t,x) - \int_0^t S_{t-s} w^2(s,\cdot)(x)ds\,. \tag{3.5}$$

Lemma 3.1. *If* $1\le d\le 3$, *then*

$$\lim_{T\to\infty} tT^2 S_t f_T(T^{1/2}\cdot)(x) = tp(t,x)\,,$$

in $L^2(\mathbb{R}^d,\lambda)$ *uniformly in* $t\in[0,1]$, *and pointwise for each* $t\in[0,1]$.

Proof. First of all, we note that

$$\begin{aligned} tT^2 S_t f_T(T^{1/2}\cdot)(x) &= tT^2 b_d^{-1}(T)\int p(t,x,z) f(T^{1/2}z)dz \\ &= tT^2 b_d^{-1}(T) T^{-d/2}\int p(s,x,T^{-1/2}z) f(z)dz \\ &= t\int p(t,x,T^{-1/2}z) f(z)dz\,, \end{aligned}$$

which converges pointwise to $tp(t,x)$ as $T\to\infty$ by Lebesgue's dominated convergence theorem.

Noting $\langle\lambda, f\rangle = 1$ we have

$$\begin{aligned} &\left\| t\int p(t,x,T^{-1/2}z)f(z)dz - tp(t,x)\right\|_{L^2}^2 \\ &\quad = t^2\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}[p(t,x-T^{-1/2}z) - p(t,x)]f(z)dz\right]^2 dx \\ &\quad = t^2\int_{\mathbb{R}^d} f(y)dy\int_{\mathbb{R}^d} f(z)\Big[p(2t,T^{-1/2}y - T^{-1/2}z) - p(2t,T^{-1/2}y) \\ &\qquad - p(2t,T^{-1/2}z) + p(2t,0)\Big]dz \end{aligned}$$

$$= t^{2-d/2} \int_{\mathbb{R}^d} f(y)dy \int_{\mathbb{R}^d} f(z) \Big[p(2, T^{-1/2}y - T^{-1/2}z) - p(2, T^{-1/2}y)$$

$$- p(2, T^{-1/2}z) + p(2,0) \Big] dz$$

$$\leq \int_{\mathbb{R}^d} f(y)dy \int_{\mathbb{R}^d} f(z) \Big[p(2, T^{-1/2}y - T^{-1/2}z) - p(2, T^{-1/2}y)$$

$$- p(2, T^{-1/2}z) + p(2,0) \Big] dz$$

$$\longrightarrow 0\,,$$

uniformly in $t \in [0,1]$ as $T \to \infty$ by Lebesgue's dominated convergence theorem, because the integrand is dominated by $4p(2,0)$. □

Lemma 3.2. *If* $1 \leq d \leq 3$, $w_T(t,x)$ *be the mild solution of Eq.* (3.3), *then* $w(t,x) := \lim_{T\to\infty} w_T(t,x)$ *exists in* $C([0,+\infty), L^2(\lambda))$, *and pointwise for each* $t \in [0,1]$ *and* $w(t,x)$ *is the mild solution of Eq.* (3.5).

Proof. From Eq. (3.3), it is easy to see that

$$\begin{aligned} \|w_T(t,x)\|_{L^2}^2 &\leq \int_{\mathbb{R}^d} \left[t \int_{\mathbb{R}^d} p(t,x,T^{-1/2}z) f(z)dz \right]^2 dx \\ &= t^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(2t, T^{-1/2}y, T^{-1/2}z) f(y) f(z) dydz \\ &\leq C \cdot t^{2-d/2}\,, \end{aligned}$$

also note that

$$\begin{aligned} S_{t-s} w_T^2(s,\cdot)(x) &\leq \int_{\mathbb{R}^d} p(t-s,x,y) \left[s \int_{\mathbb{R}^d} p(s,y,T^{-1/2}z) f(z)dz \right]^2 dy \\ &\leq C \cdot s^{2-d/2} \int_{\mathbb{R}^d} p(t,x,T^{-1/2}z) f(z)dz \\ &\leq C \cdot s^{2-d/2} p(t,0)\,, \end{aligned}$$

which is integrable on $[0,t]$ when $1 \leq d \leq 3$. With these two estimations and Lemma 3.1 in hand, the remaining proof is similar to that of Proposition 3.9 in Iscoe[11]: firstly, we can prove that the limit $w(t,x)$ exists in $C([0,+\infty), L^2(\lambda))$. Then, the limit is taken in pointwise and satisfies (3.5). Finally, the mild solution of (3.5) is unique. We omit the details here. □

Lemma 3.3. $\lim_{T\to\infty} \langle \lambda, w_T(t,\cdot) \rangle = \langle \lambda, w(t,\cdot) \rangle$, *for* $t \geq 0$.

Proof. From Eqs. (3.3) and (3.5), we have

$$\langle \lambda, w_T(t,\cdot)\rangle = t - \int_0^t \langle \lambda, w_T^2(s,\cdot)\rangle ds \tag{3.6}$$

and

$$\langle \lambda, w(t,\cdot)\rangle = t - \int_0^t \langle \lambda, w^2(s,\cdot)\rangle ds\,. \tag{3.7}$$

By Lemma 3.2, the second term on the right-hand side of (3.6) converge to that of (3.7), we are done. □

Proof of Theorem 1.2. From (2.3) we know the the Laplace functional of $T^{-1}P^{\varrho}\langle X_T^{\varrho}, f\rangle$ is

$$P\exp\{-T^{-1}P^{\varrho}\langle X_T^{\varrho}, f\rangle\} = \exp\{-\langle \lambda, f_T\rangle - \langle \lambda, u_T(T,\cdot)\rangle\} \tag{3.8}$$

and $u_T(t,x)$ is the mild solution of Eq. (2.4) with f being replaced by $f_T = T^{-1}f$.

By time and space transformation, $w_T(t,x) := Tu_T(Tt, T^{1/2}x)$, we have for $d = 2$

$$\langle \lambda, u_T(T,\cdot)\rangle = \langle \lambda, w_T(1,x)\rangle \tag{3.9}$$

and $w_T(t,x)$ satisfies Eq. (3.3). By Lemma 3.3, as $T\to\infty$, we get

$$\langle \lambda, u_T(T,\cdot)\rangle \longrightarrow \langle \lambda, w(1,\cdot)\rangle\,,$$

where $w(\cdot,\cdot)$ is the mild solution of (3.5), i.e.

$$\begin{cases} \dot w(t,x) = \Delta w(t,x) - w^2(t,x) + p(s,x)\,, & 0 < t \le 1 \\ w(0,x) = 0\,. \end{cases} \tag{3.10}$$

For the unnormalized case, we can replace f with θf, where $\theta > 0$ and $\langle \lambda, f\rangle = 1$, and obtain

$$\begin{cases} \dot w(t,x;\theta) = \Delta w(t,x) - w^2(t,x;\theta) + \theta p_t(x)\,, \\ w(0) = 0\,. \end{cases} \tag{3.11}$$

It follows that

$$\lim_{T\to\infty} P\exp\{-T^{-1}P^{\varrho}\langle X_T^{\varrho}, f\rangle\} = \exp(-\langle \lambda, w(1,\cdot;\theta)\rangle)\,,$$

where $w(t,x;\theta)$ is given by (3.11), and the rest of the proof is similar to Iscoe.[11] □

4. Proof of Theorem 1.3

From (1.6) and (2.3) we know the Laplace transition functional of $\mathbf{W}(t)$ (in which $-\theta \leftrightarrow \theta$, $-u \leftrightarrow u$, $\theta \le 0$) is given by

$$P\exp[\theta T^{1/2}\mathbf{W}(t)] = \exp\{\theta\langle \lambda, f_T\rangle + \langle \lambda, u_T(t,\cdot;\theta)\rangle\}\,, \tag{4.1}$$

where $f_T := T^{-1/2}f$ (let $\langle\lambda, f\rangle = 1$) and $u_T(t,\cdot;\theta)$ is the mild solution of the following evolution equation,

$$\begin{cases} \dot{u}_T(t) = \Delta u_T(t) + u_T^2(t) + \theta S_t f_T\,, & 0 < t \le T \\ u(0) = 0\,. \end{cases} \tag{4.2}$$

Let $w_T(t,x;\theta) := Tu_T(Tt, T^{1/2}x;\theta)$, it is easy to verify that $w_T(t,x;\theta)$ satisfy the following equation

$$\begin{cases} \dot{w_T}(t) = \Delta w_T(t) + w_T^2(t) + \theta T^2 S_{Tt} f_T(T^{1/2}\cdot)\,, & 0 < t \le 1 \\ w_T(0) = 0\,, \end{cases} \tag{4.3}$$

i.e.

$$w_T(t,x;\theta) = \theta t T^2 S_t f_T(T^{1/2}\cdot)(x) + \int_0^t S_{t-s} w_T^2(s,\cdot;\theta)(x)ds\,. \tag{4.4}$$

The key step in this section is to prove that the mild solution of (4.3) converges to that of

$$\begin{cases} \dot{w}(t,x) = \Delta w(t,x) + w^2(t,x) + \theta p(t,x)\,, & 0 < t \le 1 \\ w(0,x) = 0\,, \end{cases} \tag{4.5}$$

as $t \to \infty$, where $p(t,x)$ is the transition density function of Brownian motion.

The existence of the solutions of Eqs. (4.3) and (4.5) is well known when $\theta \le 0$. Here we need the existence of the solutions in $|\theta| < \delta$ for some $\delta > 0$, and we will use the Dynkin's moment method as in Hong[7,8] to prove the existence and some properties. Once the existence is established with some estimation for the solutions, the convergence is followed as we in Sec. 3. And by the properties of the solutions of (4.3) and (4.5), we can extend (4.1) to $0 < \theta < \delta$.

For any functions $g(t,\cdot)$, $h(t,\cdot) \in C_p(\mathbb{R}^d)$, $\forall\ t \ge 0$, $p > 1$, we define the convolution

$$g(t,x) * h(t,x) := \int_0^t S_s[g(t-s,\cdot)\cdot h(t-s,\cdot)](x)ds\,. \tag{4.6}$$

Let

$$\begin{cases} g^{*1}(t,x) := g(t,x)\,, \\ g(t,x)^{*n} := \displaystyle\sum_{k=1}^{n-1} g(t,x)^{*k} * g(t,x)^{*(n-k)} \end{cases} \tag{4.7}$$

and $\{B_n, n \ge 1\}$ is a sequence of positive numbers determined by

$$\begin{cases} B_1 = B_2 = 1\,, \\ B_n = \displaystyle\sum_{k=1}^{n-1} B_k B_{n-k}\,, \end{cases} \tag{4.8}$$

see Dynkin[5] and Wang.[14]

Lemma 4.1. *Let $d = 3$ and $F_T(t,x) = tT^2 S_t f_T(T^{1/2}\cdot)(x)$, $0 \le t \le 1$, then*

$$F_T(t,x)^{*n} \le B_n a^{n-1} \cdot t \int p(t,x,T^{-1/2}z)f(z)dz \tag{4.9}$$

where $a := \frac{2}{3}(2\pi)^{-3/2}$.

Proof. We will prove (4.9) by induction in n. As we have seen in Lemma 3.1

$$F_T(t,x) = t\int p(t,x,T^{-1/2}z)f(z)dz\,.$$

It is trivial for $n = 1$. When $n = 2$, from the definition we have

$$\begin{aligned} F_T(t,x)^{*2} &= \int_0^t S_s\left[(t-s)\int p(t-s,\cdot,T^{-1/2}z)f(z)dz\right]^2(x)ds \\ &\le \int_0^t (t-s)^2 p(t-s,0)S_s\left[\int p(t-s,\cdot,T^{-1/2}z)f(z)dz\right](x)ds \\ &= a\cdot t\int p(t,x,T^{-1/2}z)f(z)dz\,, \end{aligned}$$

as desired. If (4.9) is true for all $k < n$, by (4.7) and (4.8) we get

$$\begin{aligned} F_T(t,x)^{*n} \le \sum_1^{n-1} & B_k a^{k-1}\cdot t\int p(t,x,T^{-1/2}z)f(z)dz * B_{n-k}a^{n-k-1} \\ & \cdot t\int p(t,x,T^{-1/2}z)f(z)dz \\ = {} & B_n a^{n-2}\cdot F_T(t,x)^{*2} \\ \le {} & B_n a^{n-1}\cdot t\int p(t,x,T^{-1/2}z)f(z)dz\,, \end{aligned}$$

and then the proof is complete by induction. □

Lemma 4.2. *Let $d = 3$, $|\theta| < \frac{1}{4a}$, then Eq.* (4.3) *admits a unique mild solution $w_T(t,x;\theta)$, moreover it is analytic in $|\theta| < \frac{1}{4a}$ and*

$$|w_T(t,x;\theta)| \le b(\theta)\cdot t\int p(t,x,T^{-1/2}z)f(z)dz\,, \tag{4.10}$$

where $b(\theta) = (2a)^{-1}[1-(1-4a|\theta|)^{1/2}]$.

Proof. We can rewrite Eq. (4.4) by convolution as

$$w_T(t,x;\theta) = \theta F_T(t,x) + w_T(t,x;\theta) * w_T(t,x;\theta)\,. \tag{4.11}$$

Then one gets the solution of (4.11)

$$w_T(t,x;\theta) = \sum_{n=1}^{\infty} F_T(t,x)^{*n}\theta^n \tag{4.12}$$

by Dynkin[5] (see also Wang[14]) while the convergence of the series on the right is proved, where $F_T(t,x)$ is given in Lemma 4.1. By Lemma 4.1, the series is dominated by

$$|w_T(t,x;\theta)| \le \sum_{n=1}^{\infty} B_n a^{n-1}|\theta|^n \cdot t \int p(t,x,T^{-1/2}z)f(z)dz\,. \tag{4.13}$$

On the other hand, we know (see Dawson,[1] also Dynkin[5] and Wang[14]) that the function $g(z) = \frac{1}{2}[1-(1-4z)^{1/2}]$ can be expanded as a power series

$$g(z) = \frac{1}{2}[1-(1-4z)^{1/2}] = \sum_{n=1}^{\infty} B_n z^n\,,$$

when $|z| < 1/4$, where B_n is given in (4.8). So the series (4.12) is absolutely convergence for $|\theta| < \frac{1}{4a}$, and from (4.13) we get

$$|w_T(t,x;\theta)| \le (2a)^{-1}[1-(1-4a|\theta|)^{1/2}] \cdot t \int p(t,x,T^{-1/2}z)f(z)dz\,,$$

as desired. □

By the same method as above, we can also prove the existence of the solution of Eq. (4.5) for $|\theta| < \frac{1}{4a}$ when $d = 3$.

Lemma 4.3. *Let $d = 3$, $|\theta| < \frac{1}{4a}$, $w_T(t,x;\theta)$ and $w(t,x;\theta)$ are the mild solutions of Eq.* (4.3) *and that of* (4.5) *respectively, then*

$$w_T(t,x;\theta) \to w(t,x;\theta)\,, \tag{4.14}$$

pointwise and in $L^2(R^3,\lambda)$ uniformly for $0 \le t \le 1$ as $T \to \infty$. Moreover,

$$\lim_{t\to\infty} \langle \lambda, w_T(t,\cdot;\theta)\rangle = \langle \lambda, w(t,\cdot;\theta)\rangle\,. \tag{4.15}$$

Proof. By Lemma 4.2 and similar result for Eq. (4.5), the convergence result (4.14) and (4.15) can be proved similar as in Sec. 3, we omit the details. □

Lemma 4.4. *Let $d = 3$, $|\theta| < \frac{1}{4a}$,*

$$\Lambda(\theta) := \lim_{T\to\infty} \Lambda(T,\theta) = \lim_{T\to\infty} T^{-1/2} \log P \exp[\theta T^{1/2}\mathbf{W}(T)]\,, \tag{4.16}$$

then

$$\Lambda(\theta) = \langle \lambda, w(1,\cdot;\theta)\rangle\,, \tag{4.17}$$

where $w(s,x;\theta)$ is the mild solution of Eq. (4.5). *And there is $\delta > 0$ such that $\Lambda(\theta)$ is strictly convex, continuous differentiable in $|\theta| < \delta < \frac{1}{4a}$ with $\Lambda'(0) = 1$.*

Proof. Recall (4.1) we know

$$P\exp[\theta T^{1/2}\mathbf{W}(T)] = \exp\{\theta\langle\lambda, f_T\rangle + \langle\lambda, u_T(T,\cdot;\theta)\rangle\} \tag{4.18}$$

for $\theta \leq 0$, where $f_T := T^{-1/2}f$ and $u_T(t,\cdot;\theta)$ is the mild solution of the evolution Eq. (4.2). By the transformation $w_T(t,x;\theta) := Tu_T(Tt, T^{1/2}x;\theta)$, one gets (note that $d = 3$),

$$\langle \lambda, u_T(T,\cdot;\theta)\rangle = T^{1/2}\langle \lambda, w_T(1,\cdot;\theta)\rangle \tag{4.19}$$

and $w_T(t,x;\theta)$ is the mild solution of Eq. (4.3), it is analytic in $|\theta| < \frac{1}{4a}$ by Lemma 4.2, then (4.18) also holds for $0 < \theta < \frac{1}{4a}$ by properties of Laplace transform of probability measure on $[0,\infty)$ (cf. Ref. 15).

From (4.16), (4.18) and (4.19), it follows that

$$\begin{aligned}\Lambda(\theta) := \lim_{T\to\infty} \Lambda(T,\theta) &= \lim_{T\to\infty} T^{-1/2} \log P \exp[\theta T^{1/2}\mathbf{W}(T)]\,,\\ &= \lim_{T\to\infty} T^{-1/2}[\theta\langle \lambda, f_T\rangle + T^{1/2}\langle \lambda, w_T(1,\cdot;\theta)\rangle]\\ &= \langle \lambda, w(1,\cdot;\theta)\rangle\,,\end{aligned}$$

by Lemma 4.3, where $w(t,x;\theta)$ is the mild solution of (4.5).

The mild solution of Eq. (4.5) is

$$w(t,x;\theta) = \theta t p(t,x) + \int_0^t P_{t-r}w(r,\cdot;\theta)^2(x)dr\,,$$

then

$$\Lambda(\theta) = \theta + \int_0^1 \langle \lambda, w(r,\cdot;\theta)^2\rangle dr\,.$$

We have $\Lambda'(0) = 1$ and $\Lambda''(0) = (\pi)^{-3/2}/6$, and then there is an $\delta > 0$ such that $\Lambda(\theta)$ is strictly convex in $|\theta| < \delta < \frac{1}{4a}$. □

Proof of Theorem 1.3. Based on Lemma 4.4, Theorem 1.3 followed from the general large deviation result Gärtner–Ellis Theorem (cf. Dembo & Zeitouni[4]). The neighborhood O is that of $\{\Lambda'(\theta) : |\theta| < \delta\}$. □

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