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A Note On 2-Level Superprocesses

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Abstract. We prove some central limit theorems for a 2-level super-Brownian motion with random immigration, which lead to limiting Gaussian random fields. The covariance of those Gaussian fields are explicitly characterized.

Key words: super-Brownian motion, 2-level superprocess, random immigration, central limit theorem.

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1. Introduction and statement of main results

Measure-valued processes whose dynamics is given by a random reproduction of mass (producing mass fluctuations) together with a mass flow (partly smoothing out the fluctuation) are an interesting object of study. In this paper, as a mass reproduction, we will consider 2-level branching, with a critical reproduction on the individual as well as on the family level; the mass flow is that of Brownian motion, leading to so called 2-level super-Brownian motion. Those 2-level superprocesses have been studied by many authors including Dawson, Gorostiza, Hochberg, Wakobinger and Wu ([5], [7], [12], [13]) etc. Wu ([13]) confirmed Dawson's conjecture about *extinction* in lower dimensions $d \leq 4$; whereas Gorostiza et al ([7]) proved the *persistent* property in higher dimensions $d > 4$ and thus established that $d = 4$ is the critical dimension. For 1-level superprocesses, as we known, this important property was proved by Dawson in the famous paper [1] in which the critical dimension is $d = 2$.

The aim of this note is to consider the central limit behavior of the 2-level super-Brownian motion. The long-time fluctuations of the occupation time of 1-level super-Brownian motion were studied by Iscoe ([11]). Those of (aggregated) 2-level branching particle systems (with a more general class of particle motion) by Dawson et al ([4]). In [4] it was indicated which terms in the covariance functional of the occupation time

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fluctuations drop out in the superprocess limit. We shall actually establish the central limit theorem for the occupation time process of the 2-level super-Brownian motion for $d > 4$ (Theorem 1), which is parallel to the result of [4].

Our interest is focused on the 2-level super-Brownian motion with 2-level super-Brownian immigration, where the random path of a 2-level super-Brownian motion acts as a source of *"family"* level immigration for another 2-level super-Brownian motion. In this model, the immigration *"families"* involve two kind of evolution

a) the variability inherent in the immigration source,

b) the variability produced by the family branching after the immigration.

The fluctuation limit theorems for itself (Theorem 2) and its occupation time (Theorem 3) are proved ($d > 4$), we will see that in lower dimensions a) makes contribution to the fluctuation limits, and in higher dimensions b) makes contribution. Interestingly, in the critical dimension both of them make contributions. A similar phenomenon has been observed for the 1-level super-Brownian motion by Hong & Li ([10]) and Hong ([9]). Moreover, it reveals that the random immigration "smooths" the critical dimension in the sense that there is no " log " term in the normalization for the *critical* dimension.

We first recall some background material and notations. Let $C(I\!\!R^d)$ denote the space of continuous bounded functions on $I\!\!R^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1+|x|^2)^{-p/2}$ for $x \in I\!\!R^d$. Let $C_p(I\!\!R^d) := \{f \in C(I\!\!R^d) : |f(x)| \leq \text{const}\cdot\phi_p(x)\}$. In duality, let $M_p(I\!\!R^d)$ be the space of Radon measures μ on $I\!\!R^d$ such that $\langle \mu, f\rangle := \int f(x)\mu(dx) < \infty$ for all $f \in C_p(I\!\!R^d)$. Throughout this paper, λ denotes the Lebesgue measure on $I\!\!R^d$.

Suppose that $W = (w_t, t \geq 0)$ is a standard Brownian motion in $I\!\!R^d$ with semigroup $(P_t)_{t\geq 0}$. Let $(T_t)_{t\geq 0}$ be the semigroup of the (one-level) super-Brownian motion which was determined by

$$\mathbf{E}\ \exp\{-\langle X_t, f\rangle\} = \exp\{-\langle \mu, n(t,\cdot)\rangle\}, \quad f \in C_p^+(I\!\!R^d), \tag{1}$$

where $n(\cdot,\cdot)$ is the unique mild solution of the evolution equation

$$\begin{cases} \dot{n}(t) = \Delta n(t) - n^2(t) \\ n(0) = f. \end{cases} \tag{2}$$

From this it is easy to obtain the following

Proposition 1. Let $(T_t)_{t\geq 0}$ be the semigroup of the super-Brownian motion. Then for $f \in C_p^+(I\!\!R^d)$ we have

$$T_t(\langle\cdot, f\rangle)(\mu) = \langle \mu, P_t f\rangle \tag{3}$$

and

$$T_t(\langle\cdot,f\rangle^2)(\mu) = \langle\mu, P_tf\rangle^2 + 2\langle\mu, \int_0^t P_s(P_{t-s}f)^2 ds\rangle. \tag{4}$$

□

Let $(T_t)_{t\geq 0}$ be the semigroup of the super-Brownian motion. Let $p > d$ and let

$$M_p^2(I\!R^d) := \left\{\nu \in M(M_p(I\!R^d))) : \int\int \phi_p(x)\mu(dx)\nu(d\mu) < \infty\right\}.$$

The *2-level superprocess* with branching rate $\alpha > 0$ is an $M_p^2(I\!R^d)$-valued continuous homogenous Markov process $\mathcal{X} \equiv \{\mathcal{X}(t), t \geq 0\}$ such that

$$\mathbf{E}\left[\exp(-\langle\langle \mathcal{X}(t), F\rangle\rangle | \mathcal{X}(0) = \nu\right] = \exp(-\langle\langle \nu, v_F(t)\rangle\rangle) \tag{5}$$

for F of the form $F(\mu) = h(\langle\mu, f\rangle)$ with $f \in C_c(I\!R^d)^+$ and $h \in C_b(I\!R^d)$, where $v_F(t,\mu)$ is the solution of the non-linear integral equation

$$v(t,\mu) = T_t v(0,\cdot)(\mu) - \alpha\int_0^t T_{t-s}(v(s,\cdot))^2(\mu)ds, \qquad v(0,\mu) = F(\mu), \tag{6}$$

where $\langle\langle \nu, F\rangle\rangle := \int F(\mu)\nu(d\mu)$. See [7], [13]; and [3] or [12].

As pointed out in [7], the 2-level superprocess $\mathcal{X}$ can be thought of as a "super-superprocess". We will consider the occupation time process $\mathcal{Y} \equiv \{\mathcal{Y}(t), t \geq 0\}$ of the 2-level superprocess $\mathcal{X}$, where $\mathcal{Y}(t) := \int_0^t \mathcal{X}(s)ds$ is determined by the Laplace transition functional

$$\mathbf{E}\left[\exp(-\int_0^t \langle\langle \mathcal{X}(s), F\rangle\rangle ds | \mathcal{X}(0) = \nu\right] = \exp(-\langle\langle \nu, u_F(t)\rangle\rangle), \tag{7}$$

where $u_F(t,\mu)$ is the solution of the non-linear integral equation

$$u(t,\mu) = \int_0^t T_s F(\mu)ds - \alpha\int_0^t T_{t-s}(u(s,\cdot))^2(\mu)ds, \qquad u(0,\mu) = 0. \tag{8}$$

Let $\mathcal{X}(0) = \mathcal{R}_\infty$, which is the canonical equilibrium measure of the 1-level superprocess non-trivial equilibrium state $X(\infty)$ for $d \geq 3$ (see Dawson & Perkins [6], Gorostiza et al [7]). Consider the centered process $\mathcal{Z}(t)$,

$$\langle\langle \mathcal{Z}(t), F\rangle\rangle := a_d(t)^{-1}[\langle\langle \mathcal{Y}(t), F\rangle\rangle - \mathbf{E}\,\langle\langle \mathcal{Y}(t), F\rangle\rangle], \tag{9}$$

where the normalizing factor $a_d(t)$ is given by

$$a_d(t) = \begin{cases} t^{\frac{3}{4}}, & d = 5, \\ (t\log t)^{\frac{1}{2}}, & d = 6, \\ t^{\frac{1}{2}}, & d \geq 7. \end{cases} \tag{10}$$

The *Green potential operator* G of the Brownian motion is defined by

$$Gf = \int_0^\infty P_t f dt, \qquad f \in C_c(I\!R^d)^+,$$

and that the (operator) powers of G are given by

$$G^k f = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} P_t f dt, \quad k \geq 2.$$

Then we obtained the following central limit theorem

Theorem 1 *Let $d \geq 5$ and $\mathcal{X}(0) = \mathcal{R}_\infty$. Then as $t \to \infty$, $\mathcal{Z}(t)$ converges in distribution to a centered Gaussian field $\mathcal{Z}(\infty)$ with covariance*

$$\mathbf{Cov}(\langle \mathcal{Z}(\infty), F\rangle, \langle \mathcal{Z}(\infty), H\rangle) = \begin{cases} \alpha C_d \langle \lambda, f\rangle\langle \lambda, h\rangle, & d = 5, 6 \\ \alpha\langle f, G^3 h\rangle, & d \geq 7. \end{cases}$$

where $C_5 = \frac{16}{9}(2 - \sqrt{2})(2\pi)^{-5/2}$, $C_6 = (2\pi)^{-3}/4$, λ is the Lebesgue measure on $I\!R^d$, $F(\mu) = \langle \mu, f\rangle$ and $H(\mu) = \langle \mu, h\rangle$ for $f \in C_c(I\!R^d)^+$ and $h \in C_c(I\!R^d)^+$. $\square$

This result was indicated by Dawson et al (see 2.6 and Theorem 3.2.1 of [4]), we omit the details of the proof.

Now we consider the *2-level superprocess with random immigration.* Hong and Li ([10]) studied this model in the 1-level situation, where some interesting phenomenon in asymptotic behavior is revealed. Let $\vartheta \equiv (\vartheta_t, t \geq 0)$ be a critical 2-level superprocess with branching rate $\beta > 0$. Taking the path of ϑ as the immigration rate, we get another 2-level superprocess $\mathcal{X}^\vartheta$ with immigration, which is determined by

$$\begin{aligned} \mathbf{E}\exp\left\{-\langle\langle \mathcal{X}_t^\vartheta, F\rangle\rangle\right\} &= \mathbf{E}\left[\mathbf{E}\exp\{-\langle\langle \mathcal{X}_t^\vartheta, F\rangle\rangle\}\Big|\{\sigma(\vartheta_s, s \leq t)\}\right] \\ &= \mathbf{E}\exp\left\{-\langle\langle \mathcal{X}_0, v(t,\cdot)\rangle\rangle - \textstyle\int_o^t \langle\langle \vartheta_s, v(t-s,\cdot)\rangle\rangle ds\right\} \\ &= \exp\left\{-\langle\langle \mathcal{X}_0, v(t,\cdot)\rangle\rangle - \langle\langle \vartheta_0, \overline{v}(t,\cdot)\rangle\rangle\right\}. \end{aligned} \tag{11}$$

where $\overline{v}(\cdot,\cdot)$ is the solution of the non-linear integral equation

$$\overline{v}(t,\mu) = \textstyle\int_0^t T_{t-s} v(s,\mu) ds - \beta \int_0^t T_{t-s}(\overline{v}(s,\cdot))^2(\mu) ds, \qquad \overline{v}(0,\mu) = 0 \tag{12}$$

and $v(\cdot,\cdot)$ is the solution of the equation (6). Consider the centered process $\overline{\mathcal{X}^\vartheta}(t)$,

$$\langle\langle \overline{\mathcal{X}^\vartheta}(t), F\rangle\rangle := b_d(t)^{-1}[\langle\langle \mathcal{X}^\vartheta(t), F\rangle\rangle - \mathbf{E}\langle\langle \mathcal{X}^\vartheta(t), F\rangle\rangle] \quad , \tag{13}$$

where the normalizing factor $b_d(t)$ is given by

$$b_d(t) = \begin{cases} t^{\frac{3}{4}}, & d = 5 \\ t^{\frac{1}{2}}, & d \geq 6. \end{cases} \tag{14}$$

We shall prove the following central limit theorem:

Theorem 2 *Let $d \geq 5$, $\mathcal{X}(0) = a_1\mathcal{R}_\infty$, $\vartheta(0) = a_2\mathcal{R}_\infty$ and $a_1, a_2 > 0$. Then as $t \to \infty$, $\overline{\mathcal{X}^\vartheta}(t)$ converges in distribution to a centered Gaussian field $\overline{\mathcal{X}^\vartheta}(\infty)$ with covariance*

$$\mathbf{Cov}(\langle\langle \overline{\mathcal{X}^\vartheta}(\infty), F\rangle\rangle, \langle\langle \overline{\mathcal{X}^\vartheta}(\infty), H\rangle\rangle)$$
$$= \begin{cases} a_2\beta C_d\langle\lambda, f\rangle\langle\lambda, h\rangle, & d = 5, \\ a_2\beta C_d\langle\lambda, f\rangle\langle\lambda, h\rangle + \frac{1}{2}a_2\alpha\langle f, G^2h\rangle, & d = 6, \\ \frac{1}{2}a_2\alpha\langle f, G^2h\rangle, & d \geq 7. \end{cases}$$

for $F(\mu) = \langle\mu, f\rangle$ and $H(\mu) = \langle\mu, h\rangle$ with $f \in C_c(I\!R^d)^+$ and $h \in C_c(I\!R^d)^+$, where

$$C_d = (4\pi)^{-d/2}\int_0^1 s^2 ds \int_0^\infty (s+r)^{-d/2}dr,$$

for $d = 5, 6$. $\square$

Let $\mathcal{Y}^\vartheta \equiv (\mathcal{Y}^\vartheta_t, t \geq 0)$ be the occupation time of the 2-level process with random immigration. That is, $\mathcal{Y}^\vartheta(t) := \int_0^t \mathcal{X}^\vartheta(s)ds$. We have

$$\mathbf{E}\ \exp\{-\langle\langle \mathcal{Y}^\vartheta_t, F\rangle\rangle\} \ = \exp\{-\langle\langle \mathcal{X}_0, u(t,\cdot)\rangle\rangle - \langle\langle \vartheta_0, \overline{u}(\cdot)\rangle\rangle\} \tag{15}$$

where $\overline{u}(\cdot,\cdot)$ is the solution of the non-linear integral equation

$$\overline{u}(t,\mu) = \textstyle\int_0^t T_{t-s}u(s,\cdot)(\mu)ds - \beta\int_0^t T_{t-s}(\overline{u}(s,\cdot))^2(\mu)ds, \qquad \overline{v}(0,\mu) = 0 \tag{16}$$

and $u(\cdot,\cdot)$ is the solution of the equation (8). Consider the centered process $\overline{\mathcal{Y}^\vartheta}(t)$,

$$\langle\langle \overline{\mathcal{Y}^\vartheta}(t), F\rangle\rangle := c_d(t)^{-1}[\langle\langle \mathcal{Y}^\vartheta(t), F\rangle\rangle - \mathbf{E}\ \langle\langle \mathcal{Y}^\vartheta(t), F\rangle\rangle],, \tag{17}$$

where the normalizing factor $c_d(t)$ is given by

$$c_d(t) = \begin{cases} t^{\frac{12-d}{4}}, & 5 \leq d \leq 7 \\ t, & d \geq 8. \end{cases} \tag{18}$$

Theorem 3 *Let $d \geq 5$, $\mathcal{X}(0) = a_1\mathcal{R}_\infty$ and $\vartheta(0) = a_2\mathcal{R}_\infty$. Then as $t \to \infty$, $\overline{\mathcal{Y}^\vartheta}(t)$ converges in distribution to a centered Gaussian field $\overline{\mathcal{Y}^\vartheta}(\infty)$ with covariance*

$$\mathbf{Cov}(\langle\langle \overline{\mathcal{Y}^\vartheta}(\infty), F\rangle\rangle, \langle\langle \overline{\mathcal{Y}^\vartheta}(\infty), H\rangle\rangle)$$
$$= \begin{cases} a_2\beta C_d\langle\lambda, f\rangle\langle\lambda, h\rangle, & 5 \leq d \leq 7, \\ a_2\beta C_d\langle\lambda, f\rangle\langle\lambda, h\rangle + a_2\alpha\langle f, G^3h\rangle, & d = 8, \\ a_2\alpha\langle f, G^3h\rangle, & d \geq 9. \end{cases}$$

for $F(\mu)=\langle\mu,f\rangle$ *and* $H(\mu)=\langle\mu,h\rangle$ *with* $f\in C_c(I\!\!R^d)^+$ *and* $h\in C_c(I\!\!R^d)^+$, *where*

$$C_d=(2\pi)^{-d/2}\int_0^1 s^4ds\int_0^1 hdh\int_0^1 h'dh'\int_0^1 dr\int_0^1 dr'\int_0^\infty(2l+2s-shr-sh'r')^{-d/2}dl$$

for $5\le d\le 8$. □

Remark. 1. Comparing the normalization in (10), (14) and (18), we found that in the model of the *2-level superprocess with random immigration* there is no “log” term in the *critical* dimension, which is “smoothed” by the random immigration.

2. With a_1,a_2 labelling the amount of the particles and α,β labelling the two kinds of branching in our model, interesting phenomenon is reflected in Theorem 2 and Theorem 3 for the *2-level superprocess with random immigration*: Firstly, only the immigration particles make contributions to the limiting behavior. Secondly, the immigration particles involve two kinds of branching, the branching of the random immigration ϑ and that of the underlying process $\mathcal{X}$. The former makes contributions to the fluctuation limits in low space dimensions; whereas the later makes contributions in higher dimensions. Interestingly, both of the branchings make contributions in the *critical* dimension.

2. Proofs

The method to prove the three theorems is similar. We will prove Theorem 2, and make a simple calculation for Theorem 3 to confirm that $c_d(t)$ is the right normalization for the occupation time process $\overline{\mathcal{Y}^\vartheta}$. First of all, we note that the initial value $\mathcal{R}_\infty$ of the 2-level superprocess is the canonical equilibrium measure of the 1-level superprocess (cf. Dawson and Perkins [6], Gorostiza et al [7]). The properties of $\mathcal{R}_\infty$ is important for our consideration. Let $\nu_0(d\mu)\equiv\int_{I\!\!R^d}\delta_{\delta_x}(d\mu)dx$. Wu ([13]) proved that $\nu_0\in M_p^2(I\!\!R^d)$, $\mathcal{R}_\infty\in M_p^2(I\!\!R^d)$ and

$$T_t^*\nu_0\Rightarrow\mathcal{R}_\infty \tag{19}$$

as $t\to\infty$. The following lemma was proved by Gorostiza et al ([7]), it is a direct calculation based on Proposition 1 and (19).

Lemma 1. *Let* $\mathcal{R}_\infty$ *be the canonical equilibrium measure of the 1-level superprocess. Then*

$$\langle\langle\mathcal{R}_\infty,\langle\cdot,f\rangle\rangle\rangle=\langle\lambda,f\rangle \tag{20}$$

$$\langle\langle \mathcal{R}_\infty, \langle\cdot, f\rangle^2\rangle\rangle = \int_0^\infty \int_{I\!R^d} f(y) P_s f(y) dy ds. \tag{21}$$

□

By (11) and (13) it is easy to get that

$$\mathbf{E} \exp\{-\langle\langle \overline{\mathcal{X}^\vartheta}(t), F\rangle\rangle\} = \exp\{a_1 \alpha I + a_2 \beta I\!I + a_2 \alpha I\!I\!I\}, \tag{22}$$

where

$$\begin{aligned} I &= \langle\langle \mathcal{R}_\infty, \int_0^t T_{t-s} v_t^2(s) ds\rangle\rangle, \\ I\!I &= \langle\langle \mathcal{R}_\infty, \int_0^t T_{t-s} \overline{v}_t^2(s) ds\rangle\rangle, \\ I\!I\!I &= \langle\langle \mathcal{R}_\infty, \int_0^t \int_0^s T_{t-r} v_t^2(r) dr ds\rangle\rangle, \end{aligned}$$

where $v_t(s,\cdot)$ and $\overline{v}_t(s,\cdot)$ are the solutions of equations of (6) and (12) respectively with F replaced by $F_t(\mu) = b_d(t)^{-1} F(\mu) = b_d(t)^{-1}\langle \mu, f\rangle$. Now we will calculate the three limit of the above. First note that by (6) we have

$$v_t(s,\mu) \le T_s F_t(\mu) = b_d(t)^{-1}\langle \mu, P_s f\rangle. \tag{23}$$

Then by Proposition 1 and Lemma 1,

$$\begin{aligned} I &\le \langle\langle \mathcal{R}_\infty, \int_0^t T_{t-s}[b_d(t)^{-1}\langle\cdot, P_s f\rangle]^2 ds\rangle\rangle \\ &= b_d(t)^{-2} \int_0^t \langle\langle \mathcal{R}_\infty, [\langle\cdot, P_s f\rangle]^2\rangle\rangle ds \\ &= b_d(t)^{-2} \int_0^t ds \int_0^\infty dh \int (P_h P_s f)^2(x) dx \\ &\le C \cdot b_d(t)^{-2} \int_0^t ds \int_0^\infty (2s+2h)^{-d/2} dh \\ &\to 0, \end{aligned}$$

for $d \ge 5$ as $t \to \infty$.

Lemma 2. *Let $d \ge 5$. Then as $t \to \infty$*

$$I\!I \longrightarrow \begin{cases} C_d \langle \lambda, f\rangle^2, & d = 5, 6 \\ 0, & d \ge 7. \end{cases}$$

where

$$C_d = (4\pi)^{-d/2} \int_0^1 s^2 ds \int_0^\infty (s+r)^{-d/2} dr$$

for $d=5,6$. $\square$

Proof. Firstly we write $I\!I$ as

$$I\!I = I\!I_1 - I\!I_2 - I\!I_3, \tag{24}$$

where

$$\begin{aligned}
I\!I_1 &= \langle\langle \mathcal{R}_\infty, \int_0^t T_{t-s}[sT_sF_t]^2 ds\rangle\rangle, \\
I\!I_2 &= \langle\langle \mathcal{R}_\infty, \int_0^t T_{t-s}[sT_sF_t]^2 ds\rangle\rangle - \langle\langle \mathcal{R}_\infty, \int_0^t T_{t-s}[\int_0^s T_{s-r}v_t(r)dr]^2 ds\rangle\rangle, \\
I\!I_3 &= \langle\langle \mathcal{R}_\infty, \int_0^t T_{t-s}[\int_0^s T_{s-r}v_t(r)dr]^2 ds\rangle\rangle - \langle\langle \mathcal{R}_\infty, \int_0^t T_{t-s}\overline{v}_t^2(s)ds\rangle\rangle.
\end{aligned}$$

By Proposition 1 and Lemma 1, we have

$$\begin{aligned}
I\!I_1 &= b_d(t)^{-2}\int_0^t \langle\langle \mathcal{R}_\infty, [s\langle\cdot, P_sf\rangle]^2\rangle\rangle ds \\
&= b_d(t)^{-2}\int_0^t s^2 ds \int_0^\infty dh \int (P_hP_sf)^2(x)dx \\
&= b_d(t)^{-2}\int_0^t s^2 ds \int_0^\infty dh \int\int p(2s+2h,y,z)f(y)f(z)dydz \\
&= b_d(t)^{-2}t^{4-d/2}\int_0^1 s^2 ds \int_0^\infty dh \int\int [2\pi(2s+2h)]^{-d/2}\mathrm{e}^{\frac{(y-z)^2}{2t(s+h)}} f(y)f(z)dydz \\
&\longrightarrow \begin{cases} C_d\langle\lambda, f\rangle^2, & d=5,6 \\ 0, & d\ge 7, \end{cases}
\end{aligned}$$

where

$$C_d = (4\pi)^{-d/2}\int_0^1 s^2 ds \int_0^\infty (s+h)^{-d/2}dh$$

for $d=5,6$.

For $I\!I_2$, we note that

$$I\!I_2 = \langle\langle \mathcal{R}_\infty, \int_0^t T_{t-s}[sT_sF_t]^2 \left\{1 - \left[\frac{\int_0^s T_{s-r}v_t(r,\cdot)(\mu)dr}{sT_sF_t(\mu)}\right]^2\right\} ds\rangle\rangle \tag{25}$$

and $I\!I_2 \le I\!I_1$ because by (6)

$$0 \le \frac{\int_0^s T_{s-r}v_t(r,\cdot)(\mu)dr}{sT_sF_t(\mu)} = 1 - \frac{\int_0^s dr \int_0^r T_{s-h}v_t(h,\cdot)^2(\mu)dh}{sT_sF_t(\mu)} \le 1.$$

Moreover,

$$\frac{\int_0^s dr \int_0^r T_{s-h}v_t(h,\cdot)^2(\mu)dh}{sT_sF_t(\mu)} \le b_d(t)^{-1}\frac{\int_0^s dr \int_0^r T_{s-h}[T_hF]^2(\mu)dh}{sT_sF(\mu)} \longrightarrow 0$$

as $t \to \infty$ for all s and $\mathcal{R}_\infty - a.e.\mu$. Since $I\!I_1$ convergence as $t \to \infty$, by dominated convergence theorem we get that

$$I\!I_2 \longrightarrow 0. \tag{26}$$

Based on equations (6) and (12) we can prove that $I\!I_3 \to 0$ similarly. Combining the above with (24) complete the proof. □

Lemma 3. *Let $d \geq 5$. Then as $t \to \infty$*

$$I\!I\!I \longrightarrow \begin{cases} 0, & d = 5, \\ \frac{1}{2}\langle f, G^2 f\rangle, & d \geq 6. \end{cases}$$

□

Proof. Note that

$$I\!I\!I = I\!I\!I_1 - I\!I\!I_2,$$

where

$$\begin{aligned} I\!I\!I_1 &= \langle\langle \mathcal{R}_\infty, \int_0^t \int_0^s T_{t-r}(T_r F_t)^2 dr ds\rangle\rangle, \\ I\!I\!I_2 &= \langle\langle \mathcal{R}_\infty, \int_0^t \int_0^s T_{t-r}(T_r F_t)^2 dr ds\rangle\rangle - \langle\langle \mathcal{R}_\infty, \int_0^t \int_0^s T_{t-r} v_t^2(r) dr ds\rangle\rangle. \end{aligned}$$

By Proposition 1 and Lemma 1, we have

$$\begin{aligned} I\!I\!I_1 &= b_d(t)^{-2} \int_0^t ds \int_0^s \langle\langle \mathcal{R}_\infty, [\langle \cdot, P_r f\rangle]^2\rangle\rangle dr \\ &= 2b_d(t)^{-2} \int_0^t ds \int_0^s dr \int_0^\infty dh \int (P_h P_r f)^2(x) dx \\ &= 2b_d(t)^{-2} \int_0^t ds \int_0^s dr \int_0^\infty dh \int \int p(2r + 2h, y, z) f(y) f(z) dy dz \\ &\longrightarrow \begin{cases} 0, & d = 5, \\ \frac{1}{2} \int_0^\infty dr \int_0^\infty dh \int f P_{r+h} f dy & d \geq 6. \end{cases} \\ &= \begin{cases} 0, & d = 5, \\ \frac{1}{2}\langle f, G^2 f\rangle & d \geq 6. \end{cases} \end{aligned}$$

Similar as in Lemma 2, we can prove that $I\!I\!I_2 \to 0$. Those complete the proof. □

Proof of Theorem 2. Combining Lemma 2 and Lemma 3 with (22), we get the results from (cf. Iscoe [11]). □

For Theorem 3, by (15) and (17) we get that

$$\mathbf{E}\exp\{-\langle\langle\overline{\mathcal{X}^{\vartheta}}(t),F\rangle\rangle\} \quad = \quad \exp\{a_1\alpha J_1+a_2\beta J_2+a_2\alpha J_3\}, \tag{27}$$

where

$$\begin{aligned} J_1 &= \langle\langle\mathcal{R}_\infty,\int_0^t T_{t-s}u_t^2(s)ds\rangle\rangle,\\ J_2 &= \langle\langle\mathcal{R}_\infty,\int_0^t T_{t-s}\overline{u}_t^2(s)ds\rangle\rangle,\\ J_3 &= \langle\langle\mathcal{R}_\infty,\int_0^t ds\int_0^s T_{t-r}u_t^2(r)dr\rangle\rangle, \end{aligned}$$

where $u_t(s,\cdot)$ and $\overline{u}_t(s,\cdot)$ are the solutions of equations of (8) and (16) respectively with F replaced by $F_t(\mu)=c_d(t)^{-1}F(\mu)=c_d(t)^{-1}\langle\mu,f\rangle$. It is easy to verify that $J_1\to 0$, and then we can calculate the limits of J_2 and J_3 similar as in lemma 2 and lemma 3 respectively. For example we consider J_3 as follows.

Lemma 4. *Let $d\geq 5$. Then as $t\to\infty$*

$$J_3 \longrightarrow \begin{cases} 0, & 5\leq d\leq 7,\\ \langle f,G^3f\rangle, & d\geq 8. \end{cases}$$

□

Proof. To calculate the limit of J_3, it is enough to consider that of the J_3',

$$J_3'=\langle\langle\mathcal{R}_\infty,\int_0^t ds\int_0^s T_{t-r}[\int_0^r T_hF_tdh]^2dr\rangle\rangle,$$

By Proposition 1 and Lemma 1, we have

$$\begin{aligned} J_3' &= c_d(t)^{-2}\int_0^t ds\int_0^s\langle\langle\mathcal{R}_\infty,[\langle\cdot,\int_0^r P_hfdh\rangle]^2\rangle\rangle dr\\ &= 2c_d(t)^{-2}\int_0^t ds\int_0^s dr\int_0^\infty dl\int(\int_0^r P_{h+l}fdh)^2(x)dx\\ &= c_d(t)^{-2}\int_0^t ds\int_0^s dr\int_0^\infty dl\int_0^r dh\int_0^r dh'\int\int p(l+h+h',y,z)f(y)f(z)dydz\\ &\longrightarrow \begin{cases} 0, & 5\leq d\leq 7,\\ \frac{1}{2}\int_0^\infty dl\int_0^\infty dh\int_0^\infty dh'\int fP_{l+h+h'}fdy, & d\geq 8. \end{cases}\\ &= \begin{cases} 0, & 5\leq d\leq 7,\\ \langle f,G^3f\rangle, & d\geq 8 \end{cases} \end{aligned}$$

as $t\to\infty$. Then we can prove that $\Delta J_3:=J_3-J_3'\to 0$. □

The remaining proof is similar as Theorem 2. We omit the details.

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References

[1] Dawson, D.A., 1977. *The critical measure diffusion process*, Z. Wahrsch. verw. Geb. 40, 125-145.

[2] Dawson, D.A., 1993. *Measure-valued Markov processes*, In: Lect. Notes. Math. 1541, 1-260. Springer-Verlag, Berlin.

[3] Dawson, D.A., Gorostiza, L.G., Li, Z.H., 2002. *Non-local branching superprocesses and some related models*,Acta Applicandae Mathematicae, 74, 93–112.

[4] Dawson, D. A.; Gorostiza, L. G.; Wakolbinger, A. 2001. *Occupation time fluctuations in branching systems*. J. Theoret. Probab. 14, 729–796.

[5] Dawson, D.A., Hochberg, K.J., 1991. *A multilevel branching model*, Adv. Appl. Prob. 23, 701-715.

[6] Dawson, D.A., Perkins,E.A., 1991. *Historical process*. Mem. Amer. Math.Soc. 454, Providence, RI.

[7] Gorostiza, L.G., Hochberg, K.J. and Wakolbinger, A., 1995. *Persistence of a critical super-2 process*, J. Appl. Prob. 32, 534-540.

[8] Hong, W.M., 2000. *Ergodic theorem for the two-dimensional super-Brownian motion with super-Brownian immigration*, Progress in Natural Science 10, 111-116.

[9] Hong, W.M., 2002. *Longtime behavior for the occupation time of super-Brownian motion with random immigration*, Stochastic Process. Appl. 102, no. 1, 43–62. .

[10] Hong, W.M. and Li, Z.H., 1999. *A central limit theorem for the super-Brownian motion with super-Brownian immigration*, J. Appl. Probab. 36, 1218-1224.

[11] Iscoe, I. A., 1986. *Weighted occupation time for a class of measure-valued branching processes*. Probab. Theory Relat. Fields 71, 85–116.

[12] Wu, Y., 1992. *Dynamic Particle Systems and Multilevel Measure Branching Processes*, Carleton U. thesis.

[13] Wu, Y., 1994. *Asymptotic behavior of the two level maesure branching process*, Ann. Prob. 22, 854-874.