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Occupation Time Large Deviations for the Super-Brownian Motion with Random Immigration

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Abstract. The occupation time of a super Brownian motion with immigration governed by the trajectory of another super-Brownian motion is considered, a large deviation principle is obtained for dimension $d \geq 7$.

Key words: super-Brownian motion, random immigration, large deviation principle, evolution equation.

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1. Introduction and statement of results

Superprocesses in random medium have been received much attention in recent years, see, for examples, Dawson and Fleischmann [3], Evans and Perkins [8] etc., Hong and Li([10], [11], [13]) considered a super-Brownian motion X with immigration governed by the trajectory of another super-Brownian ϱ motion (SBMSBI, for short), denoted it by X^ϱ. In particles picture, there are two kind of particles in our model: one is the underlying particles governed by X, the other is the immigration particles governed by ϱ which undergo as the underlying particles when they immigrate into the system.

We now recall the model SBMSBI briefly. Let $C(\mathbb{R}^d)$ denote the space of continuous bounded functions on $\mathbb{R}^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1 + |x|^2)^{-p/2}$ for $x \in \mathbb{R}^d$. Let $C_p(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : |f(x)| \leq \text{const}\cdot\phi_p(x)\}$. In duality, let $M_p(\mathbb{R}^d)$ be the space of Radon measures μ on $\mathbb{R}^d$ such that $\langle \mu, f\rangle := \int f(x)\mu(dx) < \infty$ for all $f \in C_p(\mathbb{R}^d)$. We endow $M_p(\mathbb{R}^d)$ with the p-vague topology, that is, $\mu_k \to \mu$ if and only if $\langle \mu_k, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_p(\mathbb{R}^d)$. Then $M_p(\mathbb{R}^d)$ is metrizable. Throughout this paper, λ denotes the Lebesgue measure on $\mathbb{R}^d$.

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Suppose that $W = (w_t, t \geq 0)$ is a standard Brownian motion in $I\!R^d$ with semigroup $(P_t)_{t\geq 0}$. A *super-Brownian motion* $X = (X_t, Q_\mu)$ is an $M_p(I\!R^d)$-valued Markov process with $X_0 = \mu$ and the transition probability given by

$$\mathbf{E}\,\exp\{-\langle X_t, f\rangle\} = \exp\{-\langle \mu, n(t,\cdot)\rangle\}, \quad f \in C_p^+(I\!R^d), \tag{1.1}$$

where $n(\cdot,\cdot)$ is the unique mild solution of the evolution equation

$$\begin{cases} \dot{n}(t) = \Delta n(t) - n^2(t) \\ n(0) = f \end{cases} \tag{1.2}$$

Let $\{g(t,\cdot) : t \geq 0\}$ be a continuous $C_p^+(I\!R^d)$-valued path, for each $a > 0$, it is easy to prove that there is a constant $C_a > 0$ such that $g(t) \leq C_a\phi_p$ for all $t \in [0,a]$. The weighted occupation time of the super Brownian motion is determined by

$$\mathbf{E}\,\exp(-\int_0^t \langle X_s, g(s)\rangle ds) = \exp\{-\langle \mu, m(0,t,\cdot)\rangle\}, \quad f \in C_p^+(I\!R^d), \tag{1.3}$$

where $m(0,\cdot,\cdot)$ is the unique mild solution of

$$\begin{cases} \dot{m}(s) = \Delta m(s) - m^2(s) + g(t-s), \quad 0 \leq s \leq t; \\ m(0) = 0, \end{cases} \tag{1.4}$$

see e.g. Iscoe [15].

Suppose that $\{\gamma_t, t \geq 0\}$ is an $M_p(I\!R^d)$-valued continuous path. A *super-Brownian motion with immigration* determined by $\{\gamma_t, t \geq 0\}$ is an $M_p(I\!R^d)$-valued Markov process $X^\gamma = (X_t^\gamma, Q_\mu^\gamma)$ with transition probability given by

$$\mathbf{E}\,\exp(-\langle X_t^\gamma, f\rangle) = \exp\{-\langle \mu, n(t,\cdot)\rangle - \int_0^t \langle \gamma_s, n(t-s,\cdot)\rangle ds\}, \quad f \in C_p^+(I\!R^d), \tag{1.5}$$

where $n(\cdot,\cdot)$ is given by (1.2); see e.g. Dawson [2], Dynkin [6], Li & Wang [21] and Zhao [24].

Based on (1.3) and (1.5) it is not difficult to construct a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$ on which the processes $\{\varrho_t : t \geq 0\}$ and $\{X_t^\varrho : t \geq 0\}$ are defined, where $\{\varrho_t : t \geq 0\}$ is a super Brownian motion with $\varrho_0 = \lambda$ and, for a given $\{\varrho_t : t \geq 0\}$, the process $\{X_t^\varrho : t \geq 0\}$ is a super Brownian motion with immigration determined by $\{\varrho_t : t \geq 0\}$ with $X_0^\varrho = \lambda$. By (1.3) and (1.5) we have

$$\begin{aligned} \mathbf{E}\,\exp\{-\langle X_t^\varrho, f\rangle\} &= \mathbf{E}\left[\mathbf{E}\,\exp\{-\langle X_t^\varrho, f\rangle\}\Big|\{\sigma(\varrho_s, s \leq t)\}\right] \\ &= \mathbf{E}\,\exp\{-\langle \lambda, n(t,\cdot)\rangle - \int_o^t \langle \varrho_s, n(t-s,\cdot)\rangle ds\} \\ &= \exp\{-\langle \lambda, n(t,\cdot)\rangle - \langle \lambda, m(t,\cdot)\rangle\}, \end{aligned} \tag{1.6}$$

where $m(\cdot,\cdot)$ is the unique mild solution of the equation

$$\begin{cases} \dot{m}(s) = \Delta m(s) - m^2(s) + n(s), & 0 \le s \le t \\ m(0) = 0 \end{cases} \tag{1.7}$$

and $n(\cdot,\cdot)$ is the mild solution of equation (1.2).

The process $\{X_t^\varrho : t \ge 0\}$ is what we call *super-Brownian motion with super-Brownian immigration* (SBMSBI), for details, see Hong & Li [13]. Let

$$Y_t^\varrho := \int_0^t X_s^\varrho ds \tag{1.8}$$

be the occupation time process of SBMSBI in the sense that $\langle Y_t^\varrho, f\rangle := \int_0^t \langle X_s^\varrho, f\rangle ds$, where $f \in C_p^+(R^d)$. By (1.3) and (1.6), we know that the Laplace transition functional of Y_t^ϱ under $\mathbf{Q}$ is given by

$$\mathbf{E}\,\exp\{-\langle Y_t^\varrho, f\rangle\} = \exp\{-\langle \lambda, v(t,\cdot)\rangle - \langle \lambda, u(t,\cdot)\rangle\} \tag{1.9}$$

where $u(\cdot,\cdot)$ is the mild solution of the equation

$$\begin{cases} \dot{u}(s) = \Delta u(s) - u^2(s) + v(s), & 0 \le s \le t \\ u(0) = 0 \end{cases} \tag{1.10}$$

and $v(\cdot,\cdot)$ is the solution of the equation

$$\begin{cases} \dot{v}(t) = \Delta v(t) - v^2(t) + f, \\ v(0) = 0. \end{cases} \tag{1.11}$$

Hong [11] proved the central limit theorem for the occupation time process of the SBMSBI , some interesting properties have been revealed, especially the random immigration "smmoth" the critical dimension. Now we will investigate the LDP of the occupation time process of the SBMSBI. Large deviation principles (LDP) have been proved for the occupation time of the ordinary super-Brownian motion, see, e.g. [17], [18] and [19], etc.. Iscoe and Lee consider for the dimension $d = 3, 4$ in [17] and Lee for $d \ge 5$ in [18], where they proved the speed function is $t^{1/2}$ for $d = 3$, t for $d \ge 5$ and $\log t/t$ for $d = 4$.

Different from Lee [18], we use the series expanding method to obtain the LDP for this model SBMSBI in higher dimension $d \ge 7$.

Let $f \ge 0$ be a Hölder continuous function with compact support in R^d and $\langle \lambda, f\rangle = 1$ and let

$$\mathbf{W}(t) := \frac{1}{t^2}\langle Y_t^\varrho, f\rangle,$$

$$\Lambda_d(t,\theta) := c_d^{-1}(t)\log \mathbf{E}\,\exp[\theta c_d(t)\mathbf{W}(t)], \tag{1.12}$$

where $c_d(t) = t^2$ for $d \geq 7$.

We note that the following estimation is useful in our proof. For any $f \in C_p^+(I\!R^d)$, we have

$$P_t f \leq c(1 \wedge t^{-d/2}). \tag{1.13}$$

where $c = \max\{(2\pi)^{-d/2}, |f|\}$ is a positive constant.

It is proved below that for $d \geq 7$ the following equations

$$\begin{cases} \frac{\partial v(t,x;\theta)}{\partial t} = \Delta v(t,x;\theta) + v^2(t,x;\theta) + \theta f \\ v(0,x;\theta) = 0 \end{cases} \tag{1.14}$$

and

$$\begin{cases} \frac{\partial u(t,x;\theta)}{\partial t} = \Delta u(t,x;\theta) + u^2(t,x;\theta) + v(t,x;\theta) \\ u(0,x;\theta) = 0 \end{cases} \tag{1.15}$$

admit unique mild solutions $v(t,x;\theta)$ and $u(t,x;\theta)$ respectively when $|\theta| < \frac{1}{a}$, where $a = c\cdot\int_0^\infty ds \int_0^s (1\wedge r^{-d/2})dr < \infty$ when $d \geq 7$.

Furthermore,

$$\Lambda(\theta) := \lim_{t\to\infty} \Lambda_d(t,\theta) = \theta/2 + \langle \lambda, [v(\cdot;\theta)]^2\rangle/2 \tag{1.16}$$

exists and is stictly convex, continuously differentiable and $\Lambda'(0) = 1/2$, where $v(x;\theta) := \lim_{t\to\infty} v(t,x;\theta)$ exists

and satisfies the following equation

$$\Delta v(x;\theta) + v^2(x;\theta) + \theta f = 0. \tag{1.17}$$

Let

$$I(\alpha) := \sup_{|\theta|<\frac{1}{4a}} [\alpha\theta - \Lambda(\theta)] \tag{1.18}$$

i.e., the Legendre transform of $\Lambda(\theta)$. Then we obtain a LDP for $d \geq 7$:

Theorem 1.1 *Let $d \geq 7$, then the law of $\mathbf{W}_t$ under $\mathbf{Q}$ admits the LDP with speed function t^2 and rate function $I(\alpha)$, i.e. there exists a neighborhood O of $1/2$ such that if $U \subset O$ is open and $C \subset O$ is closed, then*

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathbf{Q}\{\mathbf{W}(t) \in U\} \geq -\inf_{\alpha\in U} I(\alpha),$$

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbf{Q}\{\mathbf{W}(t) \in C\} \leq -\inf_{\alpha\in C} I(\alpha).$$

Remark. The existence of the solutions of equations (1.14) and (1.15) is well know result when $\theta < 0$. We will use the series expansion to prove the existence of the solution in a neighborhood of 0 and obtain the expression of (1.12) by extending the Laplace transformation. The limit of the solution to eqution (1.14) was considered in Lee [18] when he proved the LDP for the ordinary Super-Brownian motion. Our attention is focused on that of equation (1.15) which is caused by the random immigration.

2. Proof of Theorem 1.1

Firstly, for any functions $g(t,\cdot), h(t,\cdot) \in C_p(R^d)$, $\forall t \geq 0$, $p > 1$, we define the convolution

$$g(t,x) * h(t,x) := \int_0^t P_s[g(t-s,\cdot) \cdot h(t-s,\cdot)](x)ds. \tag{2.1}$$

Let

$$\begin{cases} g^{*1}(t,x) := g(t,x) \\ g(t,x)^{*n} := \sum_{k=1}^{n-1} g(t,x)^{*k} * g(t,x)^{*(n-k)}, \end{cases} \tag{2.2}$$

and $\{B_n, n \geq 1\}$ is a sequence of positive numbers determined by

$$\begin{cases} B_1 = B_2 = 1 \\ B_n = \sum_{k=1}^{n-1} B_k B_{n-k}, \end{cases} \tag{2.3}$$

see Dynkin [5] and Wang [22]. Recall (1.13) for the positive constant c.

Lemma 2.1. *Let $d \geq 7$ and $F(t,x) = \int_0^t P_s f(x)ds$, then*

$$F(t,x)^{*n} \leq B_n a^{n-1} \cdot \int_0^t P_s f(x)ds \tag{2.4}$$

where $a = c \cdot \int_0^\infty ds \int_0^s (1 \wedge r^{-d/2})dr < \infty$ when $d \geq 7$.

Proof. We will prove (2.4) by induction in n. It is trival for $n = 1$. For $n = 2$, from (1.13) and the definition of the convolution, we have

$$\begin{aligned} F(t,x)^{*2} &= \int_0^t P_s[F(t-s,\cdot)]^2(x)ds \\ &= \int_0^t P_s[\int_0^{t-s} P_r f dr]^2(x)ds \\ &\leq \int_0^t ds \int p(s,x,y)[\int_0^{t-s} P_r f(y)dr][\int_0^{t-s} c \cdot (1 \wedge r^{-d/2})dr]dy \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t ds \int_0^s P_{t-r}f(x)dr \cdot [\int_0^s c\cdot(1\wedge r^{-d/2})dr] \\
&\leq \int_0^t P_{t-r}f(x)dr[\int_0^t ds \int_0^s c\cdot(1\wedge r^{-d/2})dr] \\
&\leq a\cdot \int_0^t P_rf(x)dr.
\end{aligned}
$$

Suppose (2.4) is true for all $k<n$, by (2.2) and (2.3) we get

$$
\begin{aligned}
F(t,x)^{*n} &\leq \sum_1^{n-1}[B_k a^{k-1}\cdot \int_0^t P_sf(x)ds]*[B_{n-k}a^{n-k-1}\cdot\int_0^t P_sf(x)ds] \\
&= B_n a^{n-2}\cdot[\int_0^t P_sf(x)ds]*[\int_0^t P_sf(x)ds] \\
&\leq B_n a^{n-1}\cdot \int_0^t P_sf(x)ds
\end{aligned}
$$

and then the proof is complete by induction. □

Lemma 2.2. *Let $d\geq 7$, $|\theta|<\frac{1}{4a}$, then the equation (1.14) admits an unique mild solution $v(t,x;\theta)$, it is analytic in $|\theta|<\frac{1}{4a}$ and*

$$|v(t,x;\theta)|\leq b(\theta)\cdot\int_0^t P_sf(x)ds, \tag{2.5}$$

where $b(\theta)=(2a)^{-1}[1-(1-4a|\theta|)^{1/2}]$. Moreover, $\lim_{t\to\infty}v(t,x;\theta):=v(x;\theta)$ exists and satisfies the following equation

$$\Delta v(x;\theta)+v^2(x;\theta)+\theta f=0. \tag{2.6}$$

Proof. The mild form of equation (1.14) is

$$v(t,x;\theta)=\theta\int_0^t P_sf(x)ds+\int_0^t P_s[v(t-s,\cdot;\theta)]^2(x)ds,$$

i.e.

$$v(t,x;\theta)=\theta F(t,x)+v(t,x;\theta)*v(t,x;\theta). \tag{2.7}$$

Then

$$v(t,x;\theta)=\sum_{n=1}^\infty F(t,x)^{*n}\theta^n \tag{2.8}$$

by Dynkin [5] (see also Wang [22]) and we can prove the convergence of the series on the right hand, by Lemma 2.1 the series is dominated by

$$|v(t, x; \theta)| \leq \sum_{n=1}^{\infty} B_n a^{n-1} |\theta|^n \cdot \int_0^t P_s f(x) ds. \tag{2.9}$$

On the other hand, we know (see Dawson [1], also Dynkin [5] and Wang [22]) that the function $g(z) = \frac{1}{2}[1 - (1 - 4z)^{1/2}]$ could be expanded as a power series

$$g(z) = \frac{1}{2}[1 - (1 - 4z)^{1/2}] = \sum_{n=1}^{\infty} B_n z^n,$$

when $|z| < 1/4$, where B_n is given in (2.3). So the series (2.8) is uniformly absolute convergent for $|\theta| < \frac{1}{4a}$, and from (2.9) we get

$$|v(t, x; \theta)| \leq (2a)^{-1}[1 - (1 - 4a|\theta|)^{1/2}] \cdot \int_0^t P_s f(x) ds,$$

as desired. The last part of the theorem could proved by the similar method to Theorem 3.3 of [15]. □

The following two Lemmas could be proved by the same method, and they reflects the special structure properties of our model SBMSBI.

Lemma 2.3. *Let $d \geq 7$, $|\theta| < \frac{1}{4a}$, $v(t, x; \theta)$ be the mild solution of equation (1.14), and*

$$G(t, x; \theta) = \int_0^t P_s v(t - s, \cdot; \theta)(x) ds,$$

then

$$G(t, x; \theta)^{*n} \leq B_n k^{n-1} b(\theta)^n \cdot \int_0^t dh \int_0^h P_{t-l} f(x) dl \tag{2.10}$$

where $k = c \int_0^\infty dt \int_0^t ds \int_0^s [1 \wedge (t - l)^{-d/2}] dl < \infty$, c is given in (1.13) and $b(\theta)$ is given in Lemma 2.2.

Proof. For $n = 1$, by Lemma 2.2 we have

$$G(t, x; \theta) \leq b(\theta) \cdot \int_0^t P_{t-s}[\int_0^s P_r f dr](x) ds = b(\theta) \cdot \int_0^t ds \int_0^s P_{t-r} f(x) dr.$$

For $n = 2$,

$$G(t, x; \theta)^{*2} \quad = \quad \int_0^t P_{t-s}[G(s, \cdot; \theta)]^2(x) ds$$

$$\begin{aligned}
&\le b(\theta)^2 \cdot \int_0^t P_{t-s}[\int_0^s dh \int_0^h P_{s-l} f dl]^2(x) ds \\
&\le b(\theta)^2 c \cdot \int_0^t ds \int_0^s dh \int_0^h P_{t-l} f(x) dl \cdot [\int_0^s dh \int_0^h (1 \wedge (s-l)^{-d/2}) dl] \\
&\le b(\theta)^2 c \cdot \int_0^t dh \int_0^h P_{t-l} f(x) dl \cdot [\int_0^t ds \int_0^s dh \int_0^h (1 \wedge (s-l)^{-d/2}) dl] \\
&\le k b(\theta)^2 \cdot \int_0^t dh \int_0^h P_{t-l} f(x) dl,
\end{aligned}$$

in which we use (1.13) in the third step and the fact $k = c\int_0^\infty dt \int_0^t ds \int_0^s [1 \wedge (t-l)^{-d/2}] dl < \infty$ when $d \ge 7$. If (2.10) is true for all $k < n$, we get

$$\begin{aligned}
& G(t,x;\theta)^{*n} \\
\le\ & \sum_{m=1}^{n-1} [B_m k^{m-1} b(\theta)^m \cdot \int_0^t dh \int_0^h P_{t-l} f(x) dl] * [B_{n-m} k^{n-m-1} b(\theta)^{n-m} \cdot \int_0^t dh \int_0^h P_{t-l} f(x) dl] \\
=\ & B_n k^{n-2} b(\theta)^n \cdot [\int_0^t dh \int_0^h P_{t-l} f(x) dl] * [\int_0^t dh \int_0^h P_{t-l} f(x) dl] \\
\le\ & B_n k^{n-1} b(\theta)^n \cdot \int_0^t dh \int_0^h P_{t-l} f(x) dl
\end{aligned}$$

as desired by induction. □

Lemma 2.4. *Let $d \ge 7$, $|\theta| < \frac{1}{4a}$, $v(t,x;\theta)$ be the mild solution of equation (1.14), then the equation (1.15) admits an unique mild solution $u(t,x;\theta)$. Moreover, it is analytic in $|\theta| < \frac{1}{4a}$ and*

$$|u(t,x;\theta)| \le \beta(\theta) \cdot \int_0^t dh \int_0^h P_{t-l} f(x) dl, \tag{2.11}$$

where $\beta(\theta) = (2k)^{-1}[1 - (1 - 4b(\theta)k)^{1/2}]$.

Proof. The mild form of equation (1.15) is

$$u(t,x;\theta) = \int_0^t P_s v(t-s,\cdot;\theta)(x) + \int_0^t P_s [u(t-s,\cdot;\theta)]^2(x) ds, \tag{2.12}$$

i.e.

$$u(t,x;\theta) = G(t,x;\theta) + u(t,x;\theta) * u(t,x;\theta). \tag{2.13}$$

Then

$$u(t,x;\theta) = \sum_{n=1}^\infty G(t,x;\theta)^{*n} \tag{2.14}$$

while we prove the convergence of the series on the right hand. By Lemma 2.3, the series is dominated by

$$|u(t,x;\theta)| \le \sum_{n=1}^{\infty} B_n k^{n-1} b(\theta)^n \cdot \int_0^t dh \int_0^h P_{t-l} f(x) dl. \tag{2.15}$$

It is easy to check that $|4b(\theta)k| < 1$ whenever $|\theta| < \frac{1}{4a}$, then the series in (2.14) is uniformly absolute convergent by the same method as in Lemma 2.2, and

$$|u(t,x;\theta)| \le (2k)^{-1}[1-(1-4b(\theta)k)^{1/2}] \cdot \int_0^t dh \int_0^h P_{t-l} f(x) dl. \tag{2.16}$$

This completes the proof. $\square$

Lemma 2.5. *Let $d \ge 7$ and Y_t^{ϱ} be the occupation time process of SBMSBI, then for $|\theta| < \frac{1}{4a}$, we have*

$$\mathbf{E} \exp\{\langle Y_t^{\varrho}, \theta f\rangle\} = \exp\{\langle \lambda, v(t,\cdot;\theta)\rangle + \langle \lambda, u(t,\cdot;\theta)\rangle\} \tag{2.17}$$

where $v(t,x;\theta)$ and $u(t,x;\theta)$ are the mild solutions of equations (1.14) and (1.15) respectively.

Proof. From (1.9), (1.10) and (1.11) (in which $-\theta \leftrightarrow \theta$, $-v \leftrightarrow v$, $-u \leftrightarrow u$), we have

$$\mathbf{E} \exp\{\langle X_t^{\varrho}, \theta f\rangle\} = \exp\{\langle \lambda, v(t,\cdot;\theta)\rangle + \langle \lambda, u(t,\cdot;\theta)\rangle\}, \tag{2.18}$$

where $v(t,x;\theta)$ and $u(t,x;\theta)$ are the mild solutions of the following equations respectively,

$$\begin{cases} \frac{\partial v(t)}{\partial t} = \Delta v(t) + v^2(t) + \theta f \\ v(0) = 0 \end{cases} \tag{2.19}$$

and

$$\begin{cases} \frac{\partial u(t)}{\partial t} = \Delta u(t) + u^2(t) + v(t). \\ u(0) = 0 \end{cases}$$

So (2.17) is true when $\theta \le 0$. Note that $v(t,x;\theta)$ and $u(t,x;\theta)$ is analytic in θ when $|\theta| < \frac{1}{4a}$ by Lemma 2.2 and Lemma 2.4, then (2.17) also holds for $0 < \theta < \frac{1}{4a}$ by properties of Laplace transform of probability measure on $[0,\infty)$ (cf. [23]). $\square$

Lemma 2.6. *Let $d \ge 7$, $|\theta| < \frac{1}{4a}$,*

$$\Lambda(\theta) := \lim_{t\to\infty} \Lambda_d(t,\theta) = \lim_{t\to\infty} t^{-2} \log \mathbf{E} \exp[\theta t^{-2} \mathbf{W}(t)], \tag{2.20}$$

then

$$\Lambda(\theta) = \theta/2 + \langle\lambda, [v(\cdot;\theta)]^2\rangle/2 \tag{2.21}$$

where $v(x;\theta)$ is the solution of equation (2.6) in Lemma 2.2, and $\Lambda(\theta)$ is strictly convex, continuous differentiable in $|\theta| < \frac{1}{4a}$ with $\Lambda'(0) = 1/2$.

Proof. Recall (1.12) and Lemma 2.5 we have

$$\Lambda(\theta) = \lim_{t\to\infty} t^{-2}[\langle\lambda, v(t,\cdot;\theta)\rangle + \langle\lambda, u(t,\cdot;\theta)\rangle], \tag{2.22}$$

where $v(t,x;\theta)$ and $u(t,x;\theta)$ satisfy

$$v(t,x;\theta) = \theta\int_0^t P_s f(x)ds + \int_0^t P_s[v(t-s,\cdot;\theta)]^2(x)ds, \tag{2.23}$$

and

$$u(t,x;\theta) = \int_0^t P_s v(t-s,\cdot;\theta)(x)ds + \int_0^t P_s[v(t-s,\cdot;\theta)]^2(x)ds. \tag{2.24}$$

Then by noticing the fact $\langle\lambda, f\rangle = 1$,

$$\langle\lambda, v(t,\cdot;\theta)\rangle = \theta t + \int_0^t \langle\lambda, [v(s,\cdot;\theta)]^2\rangle ds, \tag{2.25}$$

and

$$\begin{aligned} &\langle\lambda, u(t,\cdot;\theta)\rangle \\ = &\int_0^t \langle\lambda, v(s,\cdot;\theta)\rangle ds + \int_0^t \langle\lambda, [u(s,\cdot;\theta)]^2\rangle ds \\ = &\ \theta t^2/2 + \int_0^t ds \int_0^s \langle\lambda, v^2(h,\cdot;\theta)\rangle dh + \int_0^t \langle\lambda, [u(s,\cdot;\theta)]^2\rangle ds \end{aligned}$$

By L'Hospital's Rule and (2.11), as $t\to\infty$

$$\begin{aligned} t^{-2}\int_0^t \langle\lambda, [u(s,\cdot;\theta)]^2\rangle ds &\le \beta(\theta)^2 t^{-2}\int_0^t \langle\lambda, [\int_0^s dh\int_0^h P_{s-l}f(x)dl]^2\rangle ds \\ &= \beta(\theta)^2 t^{-1}\langle\lambda, [\int_0^t dh\int_0^h P_{t-l}f(x)dl]^2\rangle \\ &= \beta(\theta)^2 t^{-1}\int_0^t dh\int_0^h dl\int_0^t dh'\int_0^{h'} dl'\int\int P_{2t-l-l'}f(x)f(y)dxdy \\ &\le \beta(\theta)^2 t^{-1}\int_0^t dh\int_0^h dl\int_0^t dh'\int_0^{h'} [1\wedge(2t-l-l')^{-d/2}]dl' \\ &\to 0 \end{aligned}$$

and then

$$\begin{aligned}\lim_{t\to\infty} t^{-2}\langle\lambda, u(t,\cdot;\theta)\rangle &= \theta/2 + \lim_{t\to\infty} t^{-2}\int_0^t ds\int_0^s \langle\lambda, v^2(h,\cdot;\theta)\rangle dh \\ &= \theta/2 + \lim_{t\to\infty}\langle\lambda, v^2(t,\cdot;\theta)\rangle/2 \\ &= \theta/2 + \langle\lambda, v^2(\cdot;\theta)\rangle/2\end{aligned}$$

where the second step is according to Cesaro's theorem if the limit in the right-hand side exists and the third step is by lemma 2.2 and the dominated convergence theorem, because $|v(t,x;\theta)| \le b(\theta)\cdot\int_0^t P_s f(x)ds < b(\theta)\cdot\int_0^\infty P_s f(x)ds$ and $\langle\lambda, [b(\theta)\cdot\int_0^\infty P_s f(x)ds]^2\rangle < \infty$ when $d \ge 7$. Note that from (2.5) we have

$$t^{-2}\langle\lambda, v(t,\cdot;\theta)\rangle \le t^{-2}\langle\lambda, b(\theta)\int_0^t P_s f ds\rangle \to 0. \tag{2.26}$$

Combining all the above with (2.22), we get

$$\Lambda(\theta) = \theta/2 + \langle\lambda, [v(\cdot;\theta)]^2\rangle/2.$$

The last part of the theorem could proved by the similar method to Lemma 1.7 of Lee [18]. This completes the proof. □

Proof of Theorem 1.1 . Based on Lemma 2.7, Theorem 1.1 is followed from the general large deviation result Gartner-Ellis Theorem (cf. Dembo and Zeitouni [4], or Ellis [7]). □

Remark. At this moment, we obtain only the local LDP. It is interesting to investigate the full LDP and to consider the LDP for the case of lower dimensions $3 \le d \le 6$. □

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