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Limiting behavior of the super-Brownian motion with super-Brownian immigration

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Abstract. Large deviation and moderate deviation principles are established in dimensions $d \geq 3$ for the super Brownian motion with super Brownian immigration X_t^{ϱ}.

Résumé. Nous montrons des principes de déviations grandes et modérées pour super mouvement brownien et super mouvement brownien avec immigration, X_t^{ϱ}, en dimension $d \geq 3$.

Superprocesses in random medium have been received much attention in recent years, see Evans & Perkins [9], Dawson & Fleischmann [3] etc., Hong and Li [14] considered super-Brownian motion with super-Brownian immigration (SBMSBI, for short) , where the immigration rate is governed by the trajectory of another super-Brownian motion, some interesting properties were revealed, see also Hong [11, 12, 13]. We first recall the concept of SBMSBI briefly. Let $C(\mathbb{R}^d)$ denote the space of continuous bounded functions on $\mathbb{R}^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1+|x|^2)^{-p/2}$ for $x \in \mathbb{R}^d$. Let $C_p(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : |f(x)| \leq \text{const}\cdot\phi_p(x)\}$. In duality, let $M_p(\mathbb{R}^d)$ be the space of Radon measures μ on $\mathbb{R}^d$ such that $\langle \mu, f\rangle := \int f(x)\mu(dx) < \infty$ for all $f \in C_p(\mathbb{R}^d)$. We endow $M_p(\mathbb{R}^d)$ with the p-vague topology, that is, $\mu_k \to \mu$ if and only if $\langle \mu_k, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_p(\mathbb{R}^d)$. Then $M_p(\mathbb{R}^d)$ is metrizable. Throughout this paper, λ denotes the Lebesgue measure on $\mathbb{R}^d$.

Suppose that $(w_t, t \geq 0)$ is a standard Brownian motion in $\mathbb{R}^d$ with semigroup $(P_t)_{t\geq 0}$. Given $\{\varrho_t : t \geq 0\}$ a super-Brownian motion with $\varrho_0 = \lambda$, the process $\{X_t^{\varrho} : t \geq 0\}$ is a super Brownian motion with immigration determined by $\{\varrho_t : t \geq 0\}$ with $X_0^{\varrho} = \lambda$. we have

$$\begin{aligned} \mathbf{E}\exp\{-\langle X_t^{\varrho}, f\rangle\} &= \mathbf{E}\left[\mathbf{E}\exp\{-\langle X_t^{\varrho}, f\rangle\}\Big|\{\sigma(\varrho_s, s \leq t)\}\right] \\ &= \mathbf{E}\exp\{-\langle \lambda, v(t,\cdot)\rangle - \textstyle\int_o^t \langle \varrho_s, v(t-s,\cdot)\rangle ds\} \\ &= \exp\{-\langle \lambda, v(t,\cdot)\rangle - \langle \lambda, u(t,\cdot)\rangle\} \end{aligned} \tag{1}$$

where $u(\cdot,\cdot)$ is the unique mild solution of the equation

$$\begin{cases} \dot{u}(s) = \Delta u(s) - u^2(s) + v(s), & 0 \leq s \leq t \\ u(0) = 0 \end{cases} \tag{2}$$

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and $v(\cdot,\cdot)$ is the mild solution of the equation

$$\begin{cases} \dot{v}(t) = \Delta v(t) - v^2(t) \\ v(0) = f \end{cases} \tag{3}$$

The process $\{X_t^\varrho : t \geq 0, \mathbf{Q}\}$ is what we call *super-Brownian motion with super-Brownian immigration* (SBMSBI), for details, see Hong & Li [14] and Hong [13], and it may be considered as one kind of multitype superprocesses, see also Dawson, Gorostiza & Li [4], Gorostiza & Lopez-Mimbela [10] and Li [19]. For the general theory of superprocesses, we refer to Dawson [2]. The central limit theorems for SBMSBI and its occupation time were proved in Hong & Li [14] and Hong [12]. Now we will focus on the large deviation principles for the SBMSBI. A full description of this work will appear elsewhere. (Preprints may be obtained from the author.)

1. Large deviations We fix $f \in C_p^+(I\!R^d)$ satisfying $\langle \lambda, f\rangle = 1$. Let

$$\mathbf{W}(t) := \frac{1}{t}\langle X_t^\varrho, f\rangle,$$

and

$$\Lambda_d(t,\theta) := c_d^{-1}(t)\log \mathbf{E}\ \exp[\theta c_d(t)\mathbf{W}(t)], \tag{4}$$

where the speed function is defined by

$$c_d(t) = \begin{cases} t^{1/2}, & d = 3 \\ t, & d \geq 4. \end{cases}$$

To obtain the LDP, based on the Gartner-Ellis Theorem ([5]), the key step is to prove the existence of the limit function of $\Lambda_d(t,\theta)$ as $t \to \infty$ and some properties of the limit function.

For this purpose, we will prove that for $d \geq 4$ the following equations

$$\begin{cases} \frac{\partial v(t,x;\theta)}{\partial t} = \Delta v(t,x;\theta) + v^2(t,x;\theta) \\ v(0,x;\theta) = \theta f \end{cases} \tag{5}$$

and

$$\begin{cases} \frac{\partial u(t,x;\theta)}{\partial t} = \Delta u(t,x;\theta) + u^2(t,x;\theta) + v(t,x;\theta) \\ u(0,x;\theta) = 0 \end{cases} \tag{6}$$

admit unique mild solutions $v(t,x;\theta)$ and $u(t,x;\theta)$ respectively when $|\theta| < \frac{1}{4a}$, where a is a positive constant. Furthermore, for $d \geq 5$, there is $\delta > 0$ such that

$$\Lambda(\theta) := \lim_{t\to\infty} \Lambda_d(t,\theta) = \theta + \int_0^\infty \langle \lambda, [v(s,\cdot;\theta)]^2\rangle ds, \tag{7}$$

exists and is stictly convex, continuously differentiable in $|\theta| < \delta < \frac{1}{4a}$ with $\Lambda'(0) = 1$. For $d = 4$, we have

$$\limsup_{t\to\infty} \Lambda_4(t,\theta) \leq \theta + \int_0^\infty \langle \lambda, [v(s,\cdot;\theta)]^2\rangle ds + c\beta(\theta)^2 := \Lambda_4(\theta), \tag{8}$$

and $\Lambda_4(\theta)$ is finite, strictly convex, continuously differentiable in $|\theta| < \frac{1}{4a}$. we can obtain an upper large deviation bound for $d = 4$. For $d = 3$, we will prove that the equation

$$\begin{cases} \frac{\partial \overline{u}(t)}{\partial t} = \Delta \overline{u}(t) + \overline{u}^2(t) + \theta p(t) \qquad 0 \le t \le 1 \\ u(0) = 0 \end{cases} \tag{9}$$

admit unique mild solutions $\overline{u}(t, \cdot; \theta) \in C([0,1], L^2(R^3))$ for $|\theta| < \frac{3}{16c_3}$, where $c_3 = (2\pi)^{-3/2}$, $p(t) = p(t,x)$ is the transition density function of the Brownian motion. Moreover we will prove that there is $\delta_3 > 0$ such that

$$\Lambda_3(\theta) := \lim_{t\to\infty} \Lambda_d(t,\theta) = \langle \lambda, \overline{u}(1, \cdot; \theta)\rangle,$$

which is continuous differential and strictly convex in $|\theta| < \delta_3 < \frac{3}{16c_3}$ with $\Lambda_3'(0) = 1$. Let $I(\alpha)$ be the Legendre transform of $\Lambda(\theta)$, i.e.

$$I(\alpha) := \sup_{|\theta|<\delta} [\alpha\theta - \Lambda(\theta)] \tag{10}$$

and $I_d(\alpha)$ be the Legendre transform of $\Lambda_d(\theta)$. Then we have

Theorem 1 *(1) For $d \ge 5$, the law of $\mathbf{W}(t)$ under $\mathbf{Q}$ admit the LDP with speed function t and rate function $I(\alpha)$, i.e. there exists a neighborhood O of 1 such that if $U \subset O$ is open and $C \subset O$ is closed, then*

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbf{Q}\{\mathbf{W}(t) \in U\} \ge -\inf_{\alpha\in U} I(\alpha),$$

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{Q}\{\mathbf{W}(t) \in C\} \le -\inf_{\alpha\in C} I(\alpha).$$

(2) For $d = 4$, the law of $\mathbf{W}(t)$ under $\mathbf{Q}$ admit the upper large deviation bound with speed function t and rate function $I_4(\alpha)$.

(3) For $d = 3$, the law of $\mathbf{W}(t)$ under $\mathbf{Q}$ admit the LDP with speed function $t^{1/2}$ and rate function $I_3(\alpha)$. □

Remark.(1).The result is a local large deviation, for the steepness of the function $\Lambda_d(\theta)$ and in turn to obtain the full LDP is still open.

(2). At this moment, we only obtain the upper large deviation bound for $d = 4$, but it is enough to ensure the speed function is in fact t. It is an interesting question to look for the lower bound. □

In contrast to Lee [18] and Iscoe & Lee [17], where they use the partial differential equation method to get the result, our technique is based on Dynkin's moment formula and the structure of this model to prove the existence of the solutions of the correspondence equations and to get some useful estimates for the solutions, which play a key role in the proofs. For $d = 3$, to prove the L^2-convergence of the evolution equation, with some estimates in hand, the technique is adapted from Iscoe [16].

2. Moderate deviations In the previous section, we obtained a large deviation principle (LDP) with the norming T and speed function

$$c_d(T) = \begin{cases} T^{\frac{1}{2}}, & d = 3 \\ T, & d \geq 4. \end{cases}$$

and we recall that a central limit theorem (CLT) was proved in Hong and Li [14] with the norming

$$a_d(T) = \begin{cases} T^{\frac{3}{4}}, & d = 3 \\ T^{\frac{1}{2}}, & d \geq 4, \end{cases}$$

What can be said about the asymptotic behavior of the SBMSBI with the norming between those of the CLT and LDP ? We will fill in this gap and obtain the so called *moderate deviation principles.* We fix $f \in C_p^+(I\!R^d)$ and let

$$\overline{\mathbf{W}}(T) := a_d(T)^{-1}[\langle X_T^\varrho, f\rangle - T\langle \lambda, f\rangle],$$

where the norming

$$a_d(T) = \begin{cases} T^{1-\alpha}, & \alpha \in (0, \frac{1}{4}), d = 3 \\ T^{1-\beta}, & \beta \in (0, \frac{1}{2}), d \geq 4. \end{cases} \tag{11}$$

and

$$\Lambda_d(T, \theta) := c_d(T)^{-1} \log \mathbf{E} \exp[\theta c_d(T)\overline{\mathbf{W}}(T)], \tag{12}$$

where the speed function is defined by

$$c_d(T) = \begin{cases} T^{\frac{1}{2}-2\alpha}, & \alpha \in (0, \frac{1}{4}), d = 3 \\ T^{1-2\beta}, & \beta \in (0, \frac{1}{2}), d \geq 4. \end{cases} \tag{13}$$

Then we prove a LDP for $d \geq 3$:

Theorem 2 *For $d \geq 3$, $\alpha \in (0, \frac{1}{4})$, $\beta \in (0, \frac{1}{2})$, define*

$$K_d = \begin{cases} 2(4\pi)^{-3/2}/3 \cdot \langle \lambda, f\rangle, & d = 3 \\ (4\pi)^{-2} \cdot \langle \lambda, f\rangle + \int_0^\infty dr \int f(y) P_r f(y) dy, & d = 4 \\ \int_0^\infty dr \int f(y) P_r f(y) dy, & d \geq 5. \end{cases} \tag{14}$$

and $I(x) = \frac{x^2}{4K_d}, |x| < \frac{2K_d}{4a}$. the law of $\overline{\mathbf{W}}(T)$ under $\mathbf{Q}$ satisfies the LDP with speed function $c_d(T)$ and rate function $I(x)$, i.e. let $O := \{x \in R^d, |x| < \frac{2K_d}{4a}\}$, for any $U \subset O$ is open and C is closed, then

$$\liminf_{T\to\infty} c_d(T)^{-1} \log \mathbf{Q}\{\overline{\mathbf{W}}(T) \in U\} \geq -\inf_{x\in U} I(x),$$

$$\limsup_{T\to\infty} c_d(T)^{-1} \log \mathbf{Q}\{\overline{\mathbf{W}}(T) \in C\} \leq -\inf_{x\in C} I(x).$$

□

Remark 1. In other words, we have (i) For $d = 3$, $\alpha \in (0, \frac{1}{4})$,

$$\liminf_{T\to\infty} T^{2\alpha-\frac{1}{2}} \log \mathbf{Q}\{T^{-1}\langle X_T^\varrho, f\rangle - \langle \lambda, f\rangle \in T^{-\alpha}U\} \geq -\inf_{x\in U} I(x),$$

and

$$\limsup_{T\to\infty} T^{2\alpha-\frac{1}{2}} \log \mathbf{Q}\{T^{-1}\langle X_T^{\varrho}, f\rangle - \langle \lambda, f\rangle \in T^{-\alpha}U\} \le -\inf_{x\in C} I(x).$$

(ii) For $d \ge 4$, $\beta \in (0, \frac{1}{2})$,

$$\liminf_{T\to\infty} T^{2\beta-1} \log \mathbf{Q}\{T^{-1}\langle X_T^{\varrho}, f\rangle - \langle \lambda, f\rangle \in T^{-\beta}U\} \ge -\inf_{x\in U} I(x),$$

and

$$\limsup_{T\to\infty} T^{2\beta-1} \log \mathbf{Q}\{T^{-1}\langle X_T^{\varrho}, f\rangle - \langle \lambda, f\rangle \in T^{-\beta}U\} \le -\inf_{x\in C} I(x).$$

where $T^{-b}A := \{T^{-b}x : x \in A\}$.

Remark 2. Corresponding to $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, we arrive at a central limit theorem (CLT) for $\overline{\mathbf{W}}(T)$ with norming

$$a_d(T) = \begin{cases} T^{\frac{3}{4}}, & d = 3 \\ T^{\frac{1}{2}}, & d \ge 4. \end{cases}$$

See Hong & Li [14]. Similarly, corresponding to $\alpha = 0$, $\beta = 0$, we got a large deviation principle (LDP) in last section for $\overline{\mathbf{W}}(T)$ with norming T and speed function

$$c_d(T) = \begin{cases} T^{\frac{1}{2}}, & d = 3 \\ T, & d \ge 4. \end{cases}$$

Theorem 1.1 fill in the gap between the CLT and LDP, and we call it the moderate deviation principle (MDP).

Remark 3. It should be pointed out that there is no the "log" term in our norming and speed functions, which is different from the ordinary super-Brownian motion (see Iscoe [15], Iscoe & Lee [17] and Lee [18]). Intuitively, the random immigration "smooth" the critical dimension in our model SBMSBI.

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References

[1] Dawson, D.A., 1977. The critical measure diffusion process, *Z. Wahrsch. verw. Geb.* 40, 125-145.

[2] Dawson, D.A., 1993. Measure-valued Markov processes, In: *Lect. Notes. Math.* 1541, 1-260. Springer-Verlag, Berlin.

[3] Dawson, D. A. and Fleischmann, K., 1997. A continuous super-Brownian motion in a super–Brownian medium, *J. Th. Probab.* 10, 213–276.

[4] Dawson, D.A., Gorostiza, L.G., Li, Z.H., 2002. *Non-local branching superprocesses and some related models*, Acta Applicandae Mathematicae, to appear.

[5] Dembo,A. and Zeitouni,O., 1998. *Large Deviations Techniques and Applications*. Springer.

[6] Dynkin, E.B., 1989. Superprocesses and their linear additive functionals, *Trans. Amer. Math. Soc.* 314, 255-282.

[7] Dynkin, E.B., 1994. *An Introduction to Branching Measure-valued Processes*, Amer. Math. Soc., Providence, RI.

[8] Ellis, R. S., 1985. *Entropy, Large Deviations and Statistical Mechanics.* Springer, New York.

[9] Evans, S. N. and Perkins, P. A., 1994. Measure-valued branching diffusions with singular interactions, *Can. J. Math.* 46, 120-168.

[10] Gorostiza, L.G. and Lopez-Mimbela, J.A., 1990. The multitype measure branching process, *Adv. Appl. Probab.* 22, 49-67.

[11] Hong, W.M., 2000. Ergodic theorem for the two-dimensional super-Brownian motion with super-Brownian immigration, *Progress in Natural Science* 10, 111-116.

[12] Hong, W.M.,2002. Longtime behavior for the occupation time of super-Brownian motion with random immigration, *Stoch. Process. Appl.*, to appear.

[13] Hong, W.M.,2001. Large deviations for super-Brownian motion with super-Brownian immigration, *Submitted.*

[14] Hong, W.M. and Li, Z.H., 1999. A central limit theorem for the super-Brownian motion with super-Brownian immigration, *J. Appl. Probab.* 36, 1218-1224.

[15] Iscoe, I., 1986. A weighted occupation time for a class of measure-valued critical branching Brownian motion, *Probab. Th. Rel. Fields* 71, 85-116.

[16] Iscoe,I., 1986. Ergodic theory and a local occupation time for measure -valued critical branching Brownian motion, *Stochastics* 18, 197–243.

[17] Iscoe,I., Lee, T.Y., 1993. Large deviations for occupation times of measure-valued branching Brownian motions, *Stoch. Stoch. Rep.* 45, 177-209.

[18] Lee, T.Y., 1993. Some limit theorems for super-Brownian motion and semilinear differential equations. *Ann. Probab.* 21, 979-995.

[19] Li, Z.H., 1992. A note on the multitype measure branching process, *Adv. Appl. Probab.* 24, 496-498.

[20] Wang, Z.K., 1990. Power series expansion for superprocesses, *Math. Acta. Scientia.* (in chinese), 10, 361-364.

[21] Widder, D.V., 1941. *The Laplace Transform.* Princeton University Press.

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