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Moderate deviation for the super-Brownian motion with super-Brownian immigration

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Abstract. Moderate deviation principles are established in dimensions $d \geq 3$ for the super Brownian motion with random immigration X_t^ϱ, where the immigration rate is governed by the trajectory of another super-Brownian motion ϱ. It fills in the gap between the central limit theorem and the large deviation principles for this model which obtained by Hong & Li (1999) and Hong (2001).

Key words: Large deviation, super-Brownian motion, random immigration, evolution eqation.

AMS 1991 Subject Classifications: Primary 60J80; Secondary 60F05.

1. Introduction and statement of results

Superprocesses in random medium have been received much attention in recent years, see Dawson & Fleischmann [3], Evans & Perkins [9] etc., Hong and Li [14] considered super-Brownian motion with super-Brownian immigration (SBMSBI, for short) , where the immigration rate is governed by the trajectory of another super-Brownian motion, some interesting properties were revealed, see also Hong [11, 12, 13]. A central limit theorem (CLT) was proved in Hong and Li [14] with the norming

$$a_d(T) = \begin{cases} T^{\frac{3}{4}}, & d = 3 \\ T^{\frac{1}{2}}, & d \geq 4, \end{cases}$$

and a large deviation principle (LDP) was obtained (Hong [13]) with the norming T and speed function

$$c_d(T) = \begin{cases} T^{\frac{1}{2}}, & d = 3 \\ T, & d \geq 4. \end{cases}$$

One of the interesting properties for this model SBMSBI is that there is no the "log" term in our norming and speed functions, which is diferent from the ordinary super-Brownian motion (see Iscoe [15], Iscoe & Lee [17] and Lee [18]). Intuitively, the random immigration "smooth" the critical dimension in our model SBMSBI.

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How about the asymptotic behavior of the SBMSBI with the norming between those of the CLT and LDP ? In the present paper, we will fill in this gap and obtain the so called *moderate deviation principles.*

We first recall the concept of SBMSBI briefly. Let $C(I\!R^d)$ denote the space of continuous bounded functions on $I\!R^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1 + |x|^2)^{-p/2}$ for $x \in I\!R^d$. Let $C_p(I\!R^d) := \{f \in C(I\!R^d) : |f(x)| \leq \text{const}\cdot\phi_p(x)\}$. In duality, let $M_p(I\!R^d)$ be the space of Radon measures μ on $I\!R^d$ such that $\langle \mu, f\rangle := \int f(x)\mu(dx) < \infty$ for all $f \in C_p(I\!R^d)$. We endow $M_p(I\!R^d)$ with the p-vague topology, that is, $\mu_k \to \mu$ if and only if $\langle \mu_k, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_p(I\!R^d)$. Then $M_p(I\!R^d)$ is metrizable. Throughout this paper, λ denotes the Lebesgue measure on $I\!R^d$.

Suppose that $(w_t, t \geq 0)$ is a standard Brownian motion in $I\!R^d$ with semigroup $(P_t)_{t\geq 0}$. Given $\{\varrho_t : t \geq 0\}$ a super Brownian motion with $\varrho_0 = \lambda$, the process $\{X_t^\varrho : t \geq 0\}$ is a super Brownian motion with immigration determined by $\{\varrho_t : t \geq 0\}$ with $X_0^\varrho = \lambda$. we have

$$\begin{aligned}\mathbf{E}\exp\{-\langle X_t^\varrho, f\rangle\} &= \mathbf{E}\left[\mathbf{E}\exp\{-\langle X_t^\varrho, f\rangle\}\Big|\{\sigma(\varrho_s, s\leq t)\}\right]\\ &= \mathbf{E}\exp\{-\langle\lambda, v(t,\cdot)\rangle - \textstyle\int_o^t\langle\varrho_s, v(t-s,\cdot)\rangle ds\}\\ &= \exp\{-\langle\lambda, v(t,\cdot)\rangle - \langle\lambda, u(t,\cdot)\rangle\}\end{aligned} \tag{1.1}$$

where $u(\cdot,\cdot)$ is the unique mild solution of the equation

$$\begin{cases}\dot{u}(s) = \Delta u(s) - u^2(s) + v(s), & 0 \leq s \leq t\\ u(0) = 0\end{cases} \tag{1.2}$$

and $v(\cdot,\cdot)$ is the mild solution of the equation

$$\begin{cases}\dot{v}(t) = \Delta v(t) - v^2(t)\\ v(0) = f\end{cases} \tag{1.3}$$

The process $\{X_t^\varrho : t \geq 0, \mathbf{Q}\}$ is what we call *super-Brownian motion with super-Brownian immigration* (SBMSBI), for details, see Hong & Li [14] and Hong [13], and it may be considered as one kind of multitype superprocesses, see also Dawson, Gorostiza & Li [4], Gorostiza & Lopez-Mimbela [10] and Li [19]. For the general theory on the superprocess, we refer to Dawson [2].

We fix $f \in C_p^+(I\!R^d)$ and let

$$\mathbf{W}(T) := a_d(T)^{-1}[\langle X_T^\varrho, f\rangle - T\langle\lambda, f\rangle],$$

where the norming

$$a_d(T) = \begin{cases}T^{1-\alpha}, & \alpha \in (0, \frac{1}{4}), d = 3\\ T^{1-\beta}, & \beta \in (0, \frac{1}{2}), d \geq 4.\end{cases} \tag{1.4}$$

and

$$\Lambda_d(T,\theta) := c_d(T)^{-1} \log \mathbf{E} \exp[\theta c_d(T)\mathbf{W}(T)], \tag{1.5}$$

where the speed function is defined by

$$c_d(T) = \begin{cases} T^{\frac{1}{2}-2\alpha}, & \alpha \in (0, \frac{1}{4}), d = 3 \\ T^{1-2\beta}, & \beta \in (0, \frac{1}{2}), d \geq 4. \end{cases} \tag{1.6}$$

Then we prove a LDP for $d \geq 3$:

Theorem 1.1 *For $d \geq 3$, $\alpha \in (0, \frac{1}{4})$, $\beta \in (0, \frac{1}{2})$, define*

$$K_d = \begin{cases} \frac{2(4\pi)^{-3/2}}{3} \cdot \langle \lambda, f \rangle, & d = 3 \\ (4\pi)^{-2} \cdot \langle \lambda, f \rangle + \int_0^\infty dr \int f(y) P_r f(y) dy, & d = 4 \\ \int_0^\infty dr \int f(y) P_r f(y) dy, & d \geq 5. \end{cases} \tag{1.7}$$

and $I(x) = \frac{x^2}{4K_d}, |x| < \frac{2K_d}{4a}$. the law of $\mathbf{W}_T$ under $\mathbf{Q}$ admit the LDP with speed function $c_d(T)$ and rate function $I(x)$, i.e. let $O := \{x \in R^d, |x| < \frac{2K_d}{4a}\}$, for any $U \subset O$ is open and C is closed, then

$$\liminf_{T\to\infty} c_d(T)^{-1} \log \mathbf{Q}\{\mathbf{W}(T) \in U\} \geq -\inf_{x\in U} I(x),$$

$$\limsup_{T\to\infty} c_d(T)^{-1} \log \mathbf{Q}\{\mathbf{W}(T) \in C\} \leq -\inf_{x\in C} I(x).$$

□

Remark 1. In other words, we have

(i) For $d = 3$, $\alpha \in (0, \frac{1}{4})$,

$$\liminf_{T\to\infty} T^{2\alpha-\frac{1}{2}} \log \mathbf{Q}\{T^{-1}\langle X_T^\varrho, f\rangle - \langle \lambda, f\rangle \in T^{-\alpha}U\} \geq -\inf_{x\in U} I(x),$$

and

$$\limsup_{T\to\infty} T^{2\alpha-\frac{1}{2}} \log \mathbf{Q}\{T^{-1}\langle X_T^\varrho, f\rangle - \langle \lambda, f\rangle \in T^{-\alpha}U\} \leq -\inf_{x\in C} I(x).$$

(ii) For $d \geq 4$, $\beta \in (0, \frac{1}{2})$,

$$\liminf_{T\to\infty} T^{2\beta-1} \log \mathbf{Q}\{T^{-1}\langle X_T^\varrho, f\rangle - \langle \lambda, f\rangle \in T^{-\beta}U\} \geq -\inf_{x\in U} I(x),$$

and

$$\limsup_{T\to\infty} T^{2\beta-1} \log \mathbf{Q}\{T^{-1}\langle X_T^\varrho, f\rangle - \langle \lambda, f\rangle \in T^{-\beta}U\} \leq -\inf_{x\in C} I(x).$$

where $T^{-b}A := \{T^{-b}x : x \in A\}$.

Remark 2. Corresponding with $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, we arrive at a central limit theorem (CLT) for $\mathbf{W}(T)$ with norming

$$a_d(T) = \begin{cases} T^{\frac{3}{4}}, & d = 3 \\ T^{\frac{1}{2}}, & d \geq 4. \end{cases}$$

See Hong & Li [14]. Similarly, corresponding with $\alpha = 0$, $\beta = 0$, we get a large deviation principle (LDP) for $\mathbf{W}(T)$ with norming T and speed function

$$c_d(T) = \begin{cases} T^{\frac{1}{2}}, & d = 3 \\ T, & d \geq 4. \end{cases}$$

See Hong [13]. Theorem 1.1 fill in the gap between the CLT and LDP, and we call it the moderate deviation principle (MDP).

2. Proof of Theorem 1.1

To obtain the LDP for $\mathbf{W}(t)$, based on the Gärtner-Ellis Theorem ([5]), the key step is to prove the existence of the limit function of $\Lambda_d(T, \theta)$ as $T \to \infty$ and some properties of the limit function for θ in a neighborhood of zero. For this purpose, we need to consider the Laplace transition functional of X_t^ϱ with the test function $f_T := l_d(T)^{-1} f$, where

$$l_d(T) = c_d(T)^{-1} \cdot a_d(T) = \begin{cases} T^{\frac{1}{2}+\alpha}, & \alpha \in (0, \frac{1}{4}), d = 3 \\ T^{\beta}, & \beta \in (0, \frac{1}{2}), d \geq 4. \end{cases} \tag{2.1}$$

Recall (1.1) that for $\theta \leq 0$ we have (in which $-\theta \leftrightarrow \theta$, $-v \leftrightarrow v$, $-u \leftrightarrow u$,)

$$\mathbf{E} \exp\{\langle X_t^\varrho, \theta f_T \rangle\} = \exp\{\langle \lambda, v_T(t, \cdot; \theta) \rangle + \langle \lambda, u_T(t, \cdot; \theta) \rangle\}, \tag{2.2}$$

where $v_T(t, x; \theta)$ and $u_T(t, x; \theta)$ are the mild solutions of the following equations respectively,

$$\begin{cases} \frac{\partial v_T(s)}{\partial s} = \Delta v_T(s) + v_T^2(s), & 0 \leq s \leq T \\ v_T(0) = \theta f_T \end{cases} \tag{2.3}$$

and

$$\begin{cases} \frac{\partial u_T(s)}{\partial s} = \Delta u_T(s) + u_T^2(s) + v_T(s), & 0 \leq s \leq T \\ u_T(0) = 0. \end{cases} \tag{2.4}$$

In what follows, we will firstly to prove the existence and smoothness of the solutions of equations (2.3) and (2.4) for θ in a neighborhood of zero by means of series expansion which was used in Hong [13]; Secondly we extend the Laplace expression (2.2) to θ in a neighborhood of zero; and then we can get the limit of $\Lambda_d(T, \theta)$ as $T \to \infty$.

The following estimation is useful in our proof, for any $f \in C_p^+(I\!R^d)$,

$$P_t f \le c(1 \wedge t^{-d/2}). \tag{2.5}$$

where $c = \max\{(2\pi)^{-d/2}, ||f||\}$ is a positive constant, and then $a := \int_0^\infty c(1 \wedge t^{-d/2})dr < \infty$ when $d \ge 3$.

For any functions $g(t,\cdot), h(t,\cdot) \in C_p(R^d)$, $\forall t \ge 0$, $p > 1$, we define the convolution

$$g(t,x) * h(t,x) := \int_0^t P_s[g(t-s,\cdot)\cdot h(t-s,\cdot)](x)ds. \tag{2.6}$$

Let

$$\begin{cases} g(t,x)^{*1} := g(t,x) \\ g(t,x)^{*n} := \sum_{k=1}^{n-1} g(t,x)^{*k} * g(t,x)^{*(n-k)}, \end{cases} \tag{2.7}$$

and $\{B_n, n \ge 1\}$ is a sequence of positive numbers determined by

$$\begin{cases} B_1 = B_2 = 1 \\ B_n = \sum_{k=1}^{n-1} B_k B_{n-k}, \end{cases} \tag{2.8}$$

see Dynkin [6] and Wang [20]. Recall (2.5) for the positive constant c. Let $0 \le t \le T$, $T > 1$.

Lemma 2.1. *For $d \ge 3$ and $F(t,x) = P_t f_T(x)$,*

$$F(t,x)^{*n} \le B_n a^{n-1} \cdot P_t f_T(x) \tag{2.9}$$

where $a := \int_0^\infty c(1 \wedge t^{-d/2})dr < \infty$ when $d \ge 3$.

Proof. We will prove (2.9) by induction in n. It is trival for $n = 1$. When $n = 2$, from the definition and (2.5), we have

$$\begin{aligned} F(t,x)^{*2} &= \int_0^t P_s[P_{t-s}f_T]^2(x)ds \\ &\le P_t f_T(x) \int_0^t c(1 \wedge (t-s)^{-d/2})ds \\ &= a \cdot P_t f_T(x), \end{aligned}$$

as desired. If (2.9) is true for all $k < n$, by (2.7) and (2.8) we get

$$\begin{aligned} F(t,x)^{*n} &\le \sum_1^{n-1} B_k a^{k-1} \cdot P_t f_T(x) * B_{n-k} a^{n-k-1} \cdot P_t f_T(x) \\ &= B_n a^{n-2} \cdot P_t f_T(x) * P_t f_T(x) \\ &\le B_n a^{n-1} \cdot P_t f_T(x), \end{aligned}$$

and then the proof is complete by induction. □.

Lemma 2.2. *Let $d \geq 3$, $|\theta| < \frac{1}{4a}$, then the equation (2.3) admits an unique mild solution $v_T(t,x;\theta)$, moreover it is analytic in $|\theta| < \frac{1}{4a}$ and*

$$|v_T(t,x;\theta)| \leq b(\theta) \cdot P_t f_T(x), \tag{2.10}$$

where $b(\theta) = (2a)^{-1}[1-(1-4a|\theta|)^{\frac{1}{2}}]$.

Proof. The mild form of equation (2.3) is

$$v_T(t,x;\theta) = \theta P_t f_T(x) + \int_0^t P_s[v_T(t-s,\cdot;\theta)]^2(x)ds, \tag{2.11}$$

i.e.

$$v_T(t,x;\theta) = \theta F(t,x) + v_T(t,x;\theta) * v_T(t,x;\theta). \tag{2.12}$$

Then

$$v_T(t,x;\theta) = \sum_{n=1}^{\infty} F(t,x)^{*n}\theta^n \tag{2.13}$$

by Dynkin [6] (see also Wang [20]) while we prove the convergence of the series on the right hand, where $F(t,x)$ is given in Lemma 2.1. By Lemma 2.1, the series is dominated by

$$|v_T(t,x;\theta)| \leq \sum_{n=1}^{\infty} B_n a^{n-1}|\theta|^n \cdot P_t f_T(x). \tag{2.14}$$

On the other hand, we know (see Dawson [1], also Dynkin [6] and Wang [20]) that the function $g(z) = \frac{1}{2}[1-(1-4z)^{\frac{1}{2}}]$ can be expanded as a power series

$$g(z) = \frac{1}{2}[1-(1-4z)^{\frac{1}{2}}] = \sum_{n=1}^{\infty} B_n z^n,$$

when $|z| < \frac{1}{4}$, where B_n is given in (2.8). So the series (2.13) is uniform absolute convergence for $|\theta| < \frac{1}{4a}$, and from (2.14) we get

$$|v_T(t,x;\theta)| \leq (2a)^{-1}[1-(1-4a|\theta|)^{\frac{1}{2}}] \cdot P_t f_T(x),$$

as desired. □.

The following two Lemmas can be proved by the same method, but we need pay more attention for $d = 3$.

Lemma 2.3. *Let $d \geq 3$, $|\theta| < \frac{1}{4a}$, $v_T(t,x;\theta)$ be the mild solution of equation (2.3), and*

$$G(t,x;\theta) = \int_0^t P_s v_T(t-s,\cdot;\theta)(x)ds,$$

then

$$G(t,x;\theta)^{*n} \leq B_n c^{n-1} b(\theta)^n \cdot tP_t f_T(x) \tag{2.15}$$

where c is given in (2.5) and $b(\theta)$ in Lemma 2.2.

Proof. By Lemma 2.2, it is trival for $n=1$. For $n=2$, if $d \geq 4$

$$\begin{aligned}
G(t,x;\theta)^{*2} &= \int_0^t P_s[\int_0^{t-s} P_r v_T(t-s-r,\cdot;\theta)dr]^2(x)ds \\
&\leq b(\theta)^2 \cdot \int_0^t P_s[\int_0^{t-s} P_r(P_{t-s-r}f_T)dr]^2(x)ds \\
&= b(\theta)^2 \cdot \int_0^t (t-s)^2 P_s(P_{t-s}f_T)^2(x)ds \\
&\leq b(\theta)^2 c \cdot \int_0^t (t-s)^2[1\wedge (t-s)^{-d/2}]ds \cdot P_t f_T(x) \\
&\leq b(\theta)^2 c \cdot tP_t f_T(x),
\end{aligned}$$

we used (1.9) in the fourth step and note that $\int_0^t (t-s)^2[1\wedge(t-s)^{-d/2}]ds \leq t$ when $d \geq 4$. If $d=3$, recall that $f_T = l_d(T)^{-1}f$, and $l_3(T) = T^{\frac{1}{2}+\alpha} > T^{\frac{1}{2}}$, so from the fourth step of the above we have

$$\begin{aligned}
G(t,x;\theta)^{*2} &\leq b(\theta)^2 c \cdot \int_0^t s^2[1\wedge s^{-d/2}]T^{-\frac{1}{2}}ds \cdot P_t f_T(x) \\
&\leq b(\theta)^2 c \cdot tP_t f_T(x),
\end{aligned}$$

If (2.15) is true for all $k < n$ and $d \geq 4$, we get

$$\begin{aligned}
G(t,x;\theta)^{*n} &\leq \sum_{k=1}^{n-1} B_k c^{k-1} b(\theta)^k \cdot [tP_t f_T] * B_{n-k} c^{n-k-1} b(\theta)^{n-k} \cdot [tP_t f_T](x) \\
&= B_n c^{n-2} b(\theta)^n \cdot \int_o^t P_s[(t-s)P_{t-s}f_T]^2(x)ds \\
&\leq B_n c^{n-1} b(\theta)^n \cdot tP_t f_T(x).
\end{aligned}$$

and we can prove it similarly for $d=3$. We are done by induction. □.

Lemma 2.4. *Let $d \geq 3$, $|\theta| < \frac{1}{4a}$, $v_T(t,x;\theta)$ be the mild solution of equation (2.3), then the equation (2.4) admits an unique mild solution $u_T(t,x;\theta)$, moreover it is analytic in $|\theta| < \frac{1}{4a}$ and*

$$|u_T(t,x;\theta)| \leq \beta(\theta) \cdot tP_t f_T(x), \tag{2.16}$$

where $\beta(\theta) = (2c)^{-1}[1-(1-4b(\theta)c)^{\frac{1}{2}}]$.

The proof is similar as Lemma 2.2, we ommit the details. □

Lemma 2.5. *Let $d \geq 3$, X_t^{ϱ} be the SBMSBI, then for $|\theta| < \frac{1}{4a}$, we have*

$$\mathbf{E} \exp\{\langle X_t^{\varrho}, \theta f_T\rangle\} = \exp\{\langle \lambda, v_T(t,\cdot;\theta)\rangle + \langle \lambda, u(t,\cdot;\theta)\rangle\} \tag{2.17}$$

where $v_T(t,x;\theta)$ and $u_T(t,x;\theta)$ are the mild solutions of equations (2.3) and (2.4) respectively.

Proof. From the begining of this section we know that (2.17) is true when $\theta \leq 0$. Note that $v_T(t,x;\theta)$ and $u_T(t,x;\theta)$ is analytic in θ when $|\theta| < \frac{1}{4a}$ by Lemma 2.2 and Lemma 2.4, then (2.17) also holds for $0 < \theta < \frac{1}{4a}$ by properties of Laplace transform of probability measure on $[0,\infty)$ (cf. [21]). $\Box$.

The mild form of equations (2.3) and (2.4) are

$$v_T(t,x;\theta) = \theta P_t f_T(x) + \int_0^t P_s[v_T(t-s,\cdot;\theta)]^2(x)ds, \quad 0 \leq t \leq T \tag{2.18}$$

and

$$u_T(t,x;\theta) = \int_0^t P_s v_T(t-s,\cdot;\theta) + \int_0^t P_s[u_T(t-s,\cdot;\theta)]^2(x)ds, \quad 0 \leq t \leq T. \tag{2.19}$$

By (1.5) and Lemma 2.5, for $|\theta| < \frac{1}{4a}$, we have

$$\begin{aligned} \Lambda_d(T,\theta) &= c_d(T)^{-1} \log \mathbf{E} \exp[\theta c_d(T)\mathbf{W}(T)] \\ &= I + II + III + IV, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} I &= c_d(T)^{-1}\langle \lambda, \theta f_T\rangle, \\ II &= c_d(T)^{-1}\int_0^T \langle \lambda, v_T(s,\cdot;\theta)^2\rangle ds, \\ III &= c_d(T)^{-1}\int_0^T ds \int_0^s \langle \lambda, v_T(r,\cdot;\theta)^2\rangle dr, \\ IV &= c_d(T)^{-1}\int_0^T \langle \lambda, u_T(s,\cdot;\theta)^2\rangle. \end{aligned}$$

Recall $c_d(T)$ and $l_d(T)$ of (1.6) and (2.1), we have

$$I = c_d(T)^{-1}\langle \lambda, \theta f_T\rangle = \theta \cdot c_d(T)^{-1} \cdot l_d(T)^{-1} \cdot \langle \lambda, f\rangle \longrightarrow 0, \tag{2.21}$$

as $T \to \infty$. By Lemma 2.2,

$$II = c_d(T)^{-1}\int_0^T \langle \lambda, v_T(s,\cdot;\theta)^2\rangle ds \leq b(\theta)^2 c_d(T)^{-1} \cdot l_d(T)^{-2}\int_0^T \langle \lambda, (P_s f)^2\rangle ds \longrightarrow 0. \tag{2.22}$$

In what follows we will see that III and IV make contributions to $\Lambda_d(T,\theta)$, we have

Lemma 2.6. *For $d \geq 3$, $|\theta| < \frac{1}{4a}$, as $T \to \infty$,*

$$\mathit{III} \longrightarrow \begin{cases} 0, & d = 3 \\ \theta^2 \int_0^\infty dr \int f(y) P_{2r} f(y) dy, & d \geq 4 \end{cases} \tag{2.23}$$

Proof. (i). When $d = 3$, by Lemma 2.2 as $T \to \infty$,

$$\begin{aligned} \mathit{III} &= c_d(T)^{-1} \int_0^T ds \int_0^s \langle \lambda, v_T(r, \cdot; \theta)^2 \rangle dr \\ &\leq b(\theta)^2 c_d(T)^{-1} \cdot l_d(T)^{-2} \int_0^T ds \int_0^s \langle \lambda, (P_r f)^2 \rangle dr \\ &= b(\theta)^2 T^{-3/2} \int_0^T ds \int_0^s \langle \lambda, (P_r f)^2 \rangle dr \\ &\longrightarrow 0. \end{aligned}$$

(ii).When $d \geq 4$,

$$\begin{aligned} \mathit{III}' : &= c_d(T)^{-1} \int_0^T ds \int_0^s \langle \lambda, (\theta P_r f_T)^2 \rangle dr \\ &= \theta^2 c_d(T)^{-1} \cdot l_d(T)^{-2} \cdot \int_0^T ds \int_0^s \langle \lambda, (P_r f)^2 \rangle dr \\ &= \theta^2 T^{-1} \int_0^T ds \int_0^s \langle \lambda, (P_r f)^2 \rangle dr \\ &\longrightarrow \theta^2 \int_0^\infty dr \int f(y) P_{2r} f(y) dy < \infty, \end{aligned}$$

by l'Hospital's rule. On the other hand, we note that

$$\begin{aligned} \Delta \mathit{III} := \mathit{III}' - \mathit{III} &= c_d(T)^{-1} \int_0^T ds \int_0^s \langle \lambda, (\theta P_r f_T)^2 - v_T(r, \cdot; \theta)^2 \rangle dr \\ &= c_d(T)^{-1} \int_0^T ds \int_0^s \langle \lambda, (\theta P_r f_T)^2 [1 - (\frac{v_T(r, \cdot; \theta)}{\theta P_r f_T})^2] \rangle dr, \end{aligned}$$

If $\theta = 0$, it is evidence that $\Delta \mathit{III} = 0$; for $\theta \neq 0$, from equation (2.18) we know that

$$\frac{v_T(r;\theta)}{\theta P_r f_T} = 1 - \frac{\int_0^r P_{r-h} v_T(h;\theta)^2 dh}{\theta P_r f_T}.$$

By Lemma 2.2 we have

$$\frac{\int_0^r P_{r-h} v_T(h;\theta)^2 dh}{P_r f_T} \leq b(\theta) \frac{\int_0^r P_{r-h} (P_h f_T)^2 dh}{P_r f_T} = b(\theta) T^{-\beta} \frac{\int_0^r P_{r-h} (P_h f)^2 dh}{P_r f} \to 0,$$

as $T \to \infty$. So we get

$$\frac{v_T(r;\theta)}{P_r f_T} \to 1,$$

and as $I\!I\!I'$ is convergence, by dominated convergence theorem we get that $\Delta I\!I\!I \to 0$, complete the proof. $\square$

Lemma 2.7. *For $d \geq 3$, $|\theta| < \frac{1}{4a}$, as $T \to \infty$,*

$$I\!V \longrightarrow \begin{cases} \frac{2(4\pi)^{-3/2}}{3} \cdot \langle \lambda, f \rangle \theta^2, & d = 3 \\ (4\pi)^{-2} \cdot \langle \lambda, f \rangle \theta^2 & d = 4 \\ 0 & d \geq 5 \end{cases} \tag{2.24}$$

The proof is similar as Lemma 2.6, we ommit the details. $\square$

Proof of Theorem 1.1 . Combining (2.21)–(2.24) with (2.20), we get the limit of $\Lambda_d(T, \theta)$ for $|\theta| < \frac{1}{4a}$,

$$\Lambda_d(\theta) := \lim_{T \to \infty} \Lambda_d(T, \theta) = K_d \cdot \theta^2, \tag{2.25}$$

where K_d is given in (1.7). Let $I(x)$ be the Legendre transform of $\Lambda_d(\theta)$ for $|\theta| < \frac{1}{4a}$, i.e.,

$$I(x) := \sup_{|\theta| < \frac{1}{4a}} [\theta x - \Lambda_d(\theta)] = \frac{x^2}{4K_d}, \tag{2.25}$$

where $|x| < \frac{2K_d}{4a}$. Then Theorem 1.1 follows from the general large deviation result Gärtner-Ellis Theorem [cf. Dembo & Zeitouni [5] or Ellis [8]]. The neighborhood O is that of $O := \{x \in R^d, |x| < \frac{2K_d}{4a}\}$. $\square$

Remark 3. It should be interesting to consider the path-valued setting both for the CLT and LDP, at least in lower dimension $d = 3$, see for example Theorem 6.2 of Iscoe [15] for the CLT of the ordinary super-Brownian motion. The tightness of the processes sequence is essential for the path-valued setting limiting behavior, by now the tightness for the super-Brownian motion in higher dimension ($d \geq 4$) is open.

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