ELSEVIER

Statistics & Probability Letters 51 (2001) 285–291

STATISTICS & PROBABILITY LETTERS

www.elsevier.nl/locate/stapro

Fluctuations of a super-Brownian motion with randomly controlled immigration ☆

Wenming Hong[a,*], Zenghu Li[b]

[a] Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

[b] Department of Mathematics, Beijing Normal University, Beijing 100875, People's Republic of China

Received March 2000

Abstract

We study the fluctuations around the mean of a super-Brownian motion with immigration controlled by the trajectory of a stationary immigration process. The main result is a central limit theorem which holds for all dimensions and leads to some Gaussian random fields. © 2001 Elsevier Science B.V. All rights reserved

MSC: primary 60J80; secondary 60F05

Keywords: Super-Brownian motion; Immigration; Stationary process; Central limit theorem

A variety of limit theorems have been proved for Dawson–Watanabe superprocesses. Dawson (1977) obtained a spatial central limit theorem for the stationary state of a (α, d, β)-superprocess with underlying dimension number $d > \alpha/\beta$. Iscoe (1986) proved central limit theorems for the associated weighted occupation time process in the same situation. A central limit theorem of super-Brownian motion was given by Li (1999), which leads to non-degenerate limit distributions for all dimension numbers. Immigration structures associated with Dawson–Watanabe superprocesses have been studied by several authors; see Gorostiza and Lopez-Mimbela (1990), Li (1992a,b, 1996), Li and Wang (1999) and the references therein. Limit theorems for immigration processes were studied by Li and Shiga (1995), where the immigration is governed by a deterministic measure. Hong and Li (1999) considered a super-Brownian motion with immigration governed by the trajectory of another super-Brownian motion and proved a central limit theorem for such process, which lead to Gaussian random fields for dimension numbers $d \geqslant 3$. For $d = 3$ the field is spatially uniform; for $d \geqslant 5$ its covariance is given by the potential operator of the underlying Brownian motion; and for $d = 4$ the limit field involves a mixture of the two kinds of fluctuations mentioned above, which exhibits a departure from the phenomenon by Li (1999) and Li and Shiga (1995). Hong (2000) investigated the asymptotic behavior of the model for $d = 2$.

☆ Supported by China Postdoctoral Science Foundation and the National Natural Science Foundation of China (Grant No. 19671011).

* Corresponding author.

E-mail address: wmhong@fudan.edu.cn (W. Hong).

0167-7152/01/$ - see front matter
PII: S0167-7152(00)00161-9

To find new situations where non-degenerate limit theorems for a superprocess can be obtained, we consider in this paper a super-Brownian motion with immigration controlled by the trajectory of a stationary immigration process. The main result is a central limit theorem for the process. We shall see that the limit theorem gives the same limit laws as the ones in Li (1999) and Li and Shiga (1995), in the contrast to the result of Hong and Li (1999). The study has been stimulated by the work of Dawson and Fleischmann (1997), who studied a super-Brownian motion with random branching mechanism governed by another super-Brownian motion. The process considered here can also be regarded as a special form of the multi-type branching–immigration model studied by Gorostiza and Lopez-Mimbela (1990) and Li (1992a).

1. Super-Brownian motion with immigration

Let $C(\mathbb{R}^d)$ denote the space of continuous bounded functions on $\mathbb{R}^d$. We fix a constant $p>d$ and let $\phi_p(x):=(1+|x|^2)^{-p/2}$ for $x\in\mathbb{R}^d$. Let $C_p(\mathbb{R}^d):=\{f\in C(\mathbb{R}^d)\colon |f(x)|\leqslant \text{const}\cdot\phi_p(x)\}$. In duality, let $M_p(\mathbb{R}^d)$ be the space of Radon measures μ on $\mathbb{R}^d$ such that $\langle\mu,f\rangle:=\int f(x)\mu(\mathrm{d}x)<\infty$ for all $f\in C_p(\mathbb{R}^d)$. We endow $M_p(\mathbb{R}^d)$ with the p-vague topology, that is, $\mu_k\to\mu$ if and only if $\langle\mu_k,f\rangle\to\langle\mu,f\rangle$ for all $f\in C_p(\mathbb{R}^d)$. Then $M_p(\mathbb{R}^d)$ is metrizable. Throughout this note, λ denotes the Lebesgue measure on $\mathbb{R}^d$.

Suppose that $(P_t)_{t\geqslant 0}$ is the semigroup of a standard Brownian motion in $\mathbb{R}^d$. For any $b>0$ we let $P_t^b=\mathrm{e}^{-bt}P_t$. Let $\gamma:=\{\gamma_t\colon t\geqslant 0\}$ be a continuous path from $[0,\infty)$ to $M_p(\mathbb{R}^d)$. In this note, a Markov process $\{X_t^\gamma\colon t\geqslant 0\}$ with state space $M_p(\mathbb{R}^d)$ is called a *subcritical super-Brownian motion with immigration controlled by* γ if it has transition semigroup $(Q_{r,t}^\gamma)_{t\geqslant r\geqslant 0}$ such that

$$\int_{M_p(\mathbb{R}^d)} \mathrm{e}^{-\langle\nu,f\rangle} Q_{r,t}^\gamma(\mu,\,\mathrm{d}\nu)=\exp\left\{-\langle\mu,v(t-r,\cdot)\rangle-\int_r^t\langle\gamma_s,v(t-s,\cdot)\rangle\,\mathrm{d}s\right\} \tag{1}$$

for $f\in C_p^+(\mathbb{R}^d)$, where $v(\cdot,\cdot)$ is the unique solution of the evolution equation

$$v(t,x)=P_t^bf(x)-\int_0^t P_{t-s}^b v(s,\cdot)^2(x)\,\mathrm{d}s,\quad t\geqslant 0; \tag{2}$$

see e.g. Li and Wang (1999).

Let $\mathbf{Q}_\mu^\gamma$ denote the conditional law of $\{X_t^\gamma\colon t\geqslant 0\}$ given that $X_0^\gamma=\mu$. Suppose that $\{\phi(t,\cdot)\colon t\geqslant 0\}$ is a continuous path from $[0,\infty)$ to $C_p^+(\mathbb{R}^d)$ bounded above by const $\cdot\,\phi_p$. By an approximating procedure as Iscoe (1986), one may show

$$\mathbf{Q}_\mu^\gamma \exp\left\{-\int_0^t\langle X_s^\gamma,\phi(s)\rangle\,\mathrm{d}s\right\}=\exp\left\{-\langle\mu,w(t,\cdot)\rangle-\int_0^t\langle\gamma_s,w(s,\cdot)\rangle\,\mathrm{d}s\right\} \tag{3}$$

for $f\in C_p^+(\mathbb{R}^d)$, where $w(\cdot,\cdot)$ satisfies

$$w(r,x)=\int_0^r P_{r-s}^b\phi(t-s)(x)\,\mathrm{d}s-\int_0^r P_{r-s}^b w(s,\cdot)^2(x)\,\mathrm{d}s,\quad 0\leqslant r\leqslant t. \tag{4}$$

In particular, (1) defines a homogeneous semigroup $(Q_t^\lambda)_{t\geqslant 0}$ if $\gamma_t\equiv\lambda$. Observe that, when $b>0$, we have

$$\int_0^\infty\langle\lambda,v(s,\cdot)\rangle\,\mathrm{d}s\leqslant\int_0^\infty\langle\lambda,P_t^bf\rangle\,\mathrm{d}s=\langle\lambda,f\rangle/b<\infty$$

for all $f\in C_p^+(\mathbb{R}^d)$. It follows that $Q_t^\lambda(0,\cdot)\to\mathscr{Q}^\lambda$ as $t\to\infty$, where $\mathscr{Q}^\lambda$ is a stationary distribution for $(Q_t^\lambda)_{t\geqslant 0}$ given by

$$\int_{M_p(\mathbb{R}^d)}\mathrm{e}^{-\langle\nu,f\rangle}\mathscr{Q}^\lambda(\mathrm{d}\nu)=\exp\left\{-\int_0^\infty\langle\lambda,v(s,\cdot)\rangle\,\mathrm{d}s\right\}. \tag{5}$$

Now it is not difficult to construct a probability space $(\Omega, \mathscr{F}, \mathbf{Q})$ on which the two processes $\{\varrho_t\colon t\geqslant 0\}$ and $\{Y_t\colon t\geqslant 0\}$ are defined, where $\{\varrho_t\colon t\geqslant 0\}$ is a stationary subcritical super-Brownian motion with immigration having one-dimensional distribution $\mathscr{Q}^\lambda$, and given $\{\varrho_t\colon t\geqslant 0\}$ the process $\{Y_t\colon t\geqslant 0\}$ is a critical super-Brownian motion with immigration controlled by $\{\varrho_t\colon t\geqslant 0\}$ and $Y_0=0$.

One particular choice for the space $(\Omega, \mathscr{F}, \mathbf{Q})$ is given as follows. Let $C_{[0,\infty)}$ denote the totality of continuous paths $\{w(\cdot)\colon t\geqslant 0\}$ from $[0,\infty)$ to $M_p(\mathbb{R}^d)$, with the Skorokhod topology and the Borel σ-algebra $\mathscr{G}$. Suppose that Q^λ is the distribution on $(C_{[0,\infty)}, \mathscr{G})$ of the stationary immigration process with one-dimensional distribution $\mathscr{Q}^\lambda$ and that Q_0^γ is the distribution of the critical super-Brownian motion with immigration controlled by $\{\gamma_t\colon t\geqslant 0\}$ and $Y_0=0$. Let $\Omega = C_{[0,\infty)}\times C_{[0,\infty)}$ and $\mathscr{F}=\mathscr{G}\times\mathscr{G}$ and define the probability measure $\mathbf{Q}$ on $\mathscr{F}$ by

$$\mathbf{Q}(\mathrm{d}w_1, \mathrm{d}w_2) = Q^\lambda(\mathrm{d}w_1) Q_0^{w_1}(\mathrm{d}w_2), \quad w_1, w_2 \in C_{[0,\infty)}.$$

Let $\varrho_t(w_1,w_2)=w_1(t)$ and $Y_t(w_1,w_2)=w_2(t)$. Then $\{(\varrho_t, Y_t)\colon t\geqslant 0\}$ has the predescribed distribution property.

By (1) we have

$$\mathbf{Q}[\exp\{-\langle Y_t, f\rangle\} | \{\varrho_t\colon t\geqslant 0\}] = \exp\left\{-\int_0^t \langle \varrho_s, u(t-s)\rangle\,\mathrm{d}s\right\}, \tag{6}$$

where $u(\cdot,\cdot)$ is the solution of

$$u(t,x) = P_t f(x) - \int_0^t P_{t-s} u(s,\cdot)^2(x)\,\mathrm{d}s, \quad t\geqslant 0. \tag{7}$$

Taking the expectation of (6) and using (3) and (5) we get

$$\begin{aligned}\mathbf{Q}\exp\{-\langle Y_t, f\rangle\} &= \int_{M_p(\mathbb{R}^d)} \exp\left\{-\langle \mu, w(t,\cdot)\rangle - \int_0^t \langle \lambda, w(r,\cdot)\rangle\,\mathrm{d}r\right\} \mathscr{Q}^\lambda(\mathrm{d}\mu) \\ &= \exp\left\{-\int_0^\infty \langle \lambda, v(r,\cdot)\rangle\,\mathrm{d}r - \int_0^t \langle \lambda, w(r,\cdot)\rangle\,\mathrm{d}r\right\},\end{aligned} \tag{8}$$

where $w(\cdot,\cdot)$ and $v(\cdot,\cdot)$ are defined, respectively, by

$$w(r,x) = \int_0^r P^b_{r-s} u(s,\cdot)(x)\,\mathrm{d}s - \int_0^r P^b_{r-s} w^2(s,\cdot)(x)\,\mathrm{d}s, \quad r\geqslant 0 \tag{9}$$

and

$$v(r,x) = P^b_r w(t,\cdot)(x) - \int_0^r P^b_{r-s} v^2(s,\cdot)(x)\,\mathrm{d}s, \quad r\geqslant 0. \tag{10}$$

2. A central limit theorem

We present here a central limit theorem for the process $\{Y_t\colon t\geqslant 0\}$ defined in the last section. It is not difficult to check by using (7)–(10) that $\mathbf{Q}\{Y_t(f)\} = t\lambda(f)/b$ for $t\geqslant 0$ and $f\in C_p(\mathbb{R}^d)$. Let $\mathscr{S}(\mathbb{R}^d)$ be the space of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$ whose all partial derivatives are also rapidly decreasing, and let $\mathscr{S}'(\mathbb{R}^d)$ be the dual space of $\mathscr{S}(\mathbb{R}^d)$. We define the $\mathscr{S}'(\mathbb{R}^d)$-valued process $\{Z_t\colon t>0\}$ by

$$\langle Z_t, f\rangle := a_d(t)^{-1}[\langle Y_t, f\rangle - t\langle \lambda, f\rangle/b], \quad f\in\mathscr{S}(\mathbb{R}^d), \tag{11}$$

where $a_1(t)=t^{3/4}$, $a_2(t)=(t\log t)^{1/2}$ and $a_d(t)=t^{1/2}$ for $d\geqslant 3$. Then we have

Theorem 1. *As $t \to \infty$, the distribution of Z_t converges to a centered Gaussian random variable Z_∞ in $\mathscr{S}'(\mathbb{R}^d)$ with covariance*

$$\mathbf{Cov}(\langle Z_\infty, f\rangle, \langle Z_\infty, g\rangle) = \begin{cases} 2\langle \lambda, f\rangle\langle \lambda g\rangle/3b\pi^{1/2}, & d=1, \\ \langle \lambda, f\rangle\langle \lambda, g\rangle/4\pi b, & d=2, \\ \langle \lambda, fGg\rangle/2b, & d\geqslant 3, \end{cases}$$

where G denotes the potential operator of the Brownian motion.

Now we proceed to the proof of Theorem 1 by an argument adapted from Li (1999). Let $f_t := a_d(t)^{-1} f$. In the following lemmas and proofs, $u_t(s), w_t(s)$ and $v_t(s)$ are the solutions of Eqs. (7), (9) and (10), respectively, with f being replaced by f_t, and C denotes a constant which may take different values in different lines.

Lemma 2. *For $f \in \mathscr{S}(\mathbb{R}^d)^+$ let*

$$A_d(t, f) := \int_0^t \mathrm{d}r \int_0^r \mathrm{d}s \int_0^s \mathrm{e}^{-b(r-s)} \langle \lambda, (P_{s-q} f_t)^2\rangle \,\mathrm{d}q.$$

Then we have

$$\lim_{t\to\infty} A_d(t, f) = \begin{cases} 2\langle \lambda, f\rangle^2/3b\sqrt{\pi}, & d=1, \\ \langle \lambda, f\rangle^2/4\pi b, & d=2, \\ \langle \lambda, fGf\rangle/2b, & d\geqslant 3. \end{cases}$$

Proof. We have clearly

$$A_d(t, f) = a_d(t)^{-2} \int_0^t \mathrm{e}^{-br}\,\mathrm{d}r \int_0^r \mathrm{e}^{bs}\,\mathrm{d}s \int_0^s \mathrm{d}q \int_{\mathbb{R}^d} P_{s-q} f(x)^2 \,\mathrm{d}x.$$

When $d\geqslant 3$, we use l'Hospital's rule to get

$$\begin{aligned} \lim_{t\to\infty} A_d(t, f) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{e}^{-br}\,\mathrm{d}r \int_0^r \mathrm{e}^{bs}\,\mathrm{d}s \int_0^s \mathrm{d}q \int_{\mathbb{R}^d} P_{s-q} f(x)^2 \,\mathrm{d}x \\ &= \lim_{t\to\infty} \frac{1}{\mathrm{e}^{bt}} \int_0^t \mathrm{e}^{bs}\,\mathrm{d}s \int_0^s \mathrm{d}q \int_{\mathbb{R}^d} P_{s-q} f(x)^2 \,\mathrm{d}x \\ &= \lim_{t\to\infty} \frac{1}{b} \int_0^t \mathrm{d}q \int_{\mathbb{R}^d} P_{s-q} f(x)^2 \,\mathrm{d}x \\ &= \langle \lambda, fGf\rangle/2b. \end{aligned}$$

For $d=1$ we have

$$\begin{aligned} \lim_{t\to\infty} A_1(t, f) &= \lim_{t\to\infty} \frac{1}{t^{3/2}} \int_0^t \mathrm{e}^{-br}\,\mathrm{d}r \int_0^r \mathrm{e}^{bs}\,\mathrm{d}s \int_0^s \mathrm{d}q \int_{\mathbb{R}^d} P_{s-q} f(x)^2 \,\mathrm{d}x \\ &= \lim_{t\to\infty} \frac{2}{3\sqrt{t}\mathrm{e}^{bt}} \int_0^t \mathrm{e}^{bs}\,\mathrm{d}s \int_0^s \mathrm{d}q \int_{\mathbb{R}^d} P_{s-q} f(x)^2 \,\mathrm{d}x \\ &= \lim_{t\to\infty} \frac{2}{3b\sqrt{t}} \int_0^t \mathrm{d}q \int_{\mathbb{R}^d} P_q f(x)^2 \,\mathrm{d}x \end{aligned}$$

$$= \lim_{t\to\infty} \frac{2}{3b\sqrt{t}} \int_0^t \frac{1}{\sqrt{4\pi q}}\,\mathrm{d}q \int_{\mathbb{R}^2} \exp\left\{-\frac{(y-x)^2}{4q}\right\} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \lim_{t\to\infty} \frac{2}{3b} \int_0^1 \frac{1}{\sqrt{4\pi r}}\,\mathrm{d}r \int_{\mathbb{R}^2} \exp\left\{-\frac{(y-x)^2}{4tr}\right\} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= 2\langle\lambda, f\rangle^2/3b\sqrt{\pi},$$

where we used the change of variables $q=t_r$ in the fifth step. Similarly, by setting $q=t^{1-r}$ for $d=2$, one may see that $\lim_{t\to\infty}A_2(t,f)=\langle\lambda,f\rangle^2/4\pi b$. □

Lemma 3. *For* $f\in\mathscr{S}(\mathbb{R}^d)^+$ *let*

$$B_d(t,f) := \int_0^t \mathrm{d}r \int_0^r \mathrm{d}s \int_0^s \mathrm{e}^{-b(r-s)}\langle\lambda, (P_{s-q}f_t)^2 - u_t(s-q,\cdot)^2\rangle\,\mathrm{d}q.$$

Then we have $\lim_{t\to\infty} B_d(t,f)=0$.

Proof. Note that for any $f\in\mathscr{S}(\mathbb{R}^d)^+$ we have

$$\|P_s f\| \leqslant C(1\wedge s^{-d/2}),$$

where $C=C(f)\geqslant 0$. From Eq. (7) we can see that

$$(P_r f_t)^2 - u_t(r)^2 = 2u_t(r)\int_0^r P_{r-s}u_t(s)^2\,\mathrm{d}s + \left(\int_0^r P_{r-s}u_t(s)^2\,\mathrm{d}s\right)^2$$

$$\leqslant 3P_r f_t \int_0^r P_{r-s}(P_s f_t)^2\,\mathrm{d}s$$

$$\leqslant Ca_d(t)^{-3}(P_r f)^2 \int_0^r (1\wedge s^{-d/2})\,\mathrm{d}s.$$

It follows that

$$B_d(t,f) \leqslant Ca_d(t)^{-3}\int_0^t \mathrm{d}r \int_0^r \mathrm{e}^{-b(r-s)}\,\mathrm{d}s \int_0^s \mathrm{d}q \int_{\mathbb{R}^d} P_{s-q}f(x)^2\,\mathrm{d}x \int_0^t (1\wedge l^{-d/2})\,\mathrm{d}l$$

$$\leqslant Ca_d(t)^{-3}\int_0^t \mathrm{d}r \int_0^r \mathrm{e}^{-b(r-s)}\,\mathrm{d}s \int_0^t (1\wedge q^{-d/2})\,\mathrm{d}q \int_0^t (1\wedge l^{-d/2})\,\mathrm{d}l.$$

Then we have for dimension one

$$\limsup_{t\to\infty} B_1(t,f) \leqslant C\limsup_{t\to\infty}\frac{1}{t^{9/4}}\int_0^t \mathrm{d}r \int_0^r \mathrm{e}^{-bs}\,\mathrm{d}s \int_0^t (1\wedge q^{-d/2})\,\mathrm{d}q \int_0^t (1\wedge l^{-d/2})\,\mathrm{d}l$$

$$\leqslant C\limsup_{t\to\infty}\frac{1}{t^{5/4}}\int_0^t (1\wedge q^{-d/2})\,\mathrm{d}q \int_0^t (1\wedge l^{-d/2})\,\mathrm{d}l = 0.$$

The proof for other dimension numbers are similar. □

Proof of Theorem 1. From (7)–(9) and (11) we get the Laplace functional

$$\begin{aligned}
\mathbf{Q}\exp\{-\langle Z_t, f\rangle\} &= \exp\left\{t\langle\lambda, f_t\rangle/b - \int_0^\infty \langle\lambda, v_t(r)\rangle\,\mathrm{d}r - \int_0^t \langle\lambda, w_t(r)\rangle\,\mathrm{d}r\right\} \\
&= \exp\left\{t\langle\lambda, f_t\rangle/b - \int_0^\infty \langle\lambda, v_t(r)\rangle\,\mathrm{d}r - \int_0^t \mathrm{d}r\int_0^r \mathrm{e}^{-b(r-s)}\langle\lambda, u_t(s)\rangle\,\mathrm{d}s\right. \\
&\qquad \left. + \int_0^t \mathrm{d}r\int_0^r \mathrm{e}^{-b(r-s)}\langle\lambda, w_t(s)^2\rangle\,\mathrm{d}s\right\} \\
&= \exp\left\{t\langle\lambda, f_t\rangle/b - \int_0^\infty \langle\lambda, v_t(r)\rangle\,\mathrm{d}r - \int_0^t \mathrm{d}r\int_0^r \mathrm{e}^{-b(r-s)}\langle\lambda, f_t\rangle\,\mathrm{d}s\right. \\
&\qquad + \int_0^t \mathrm{d}r\int_0^r \mathrm{d}s\int_0^s \mathrm{e}^{-b(r-s)}\langle\lambda, u_t(s-q)^2\rangle\,\mathrm{d}q \\
&\qquad \left. + \int_0^t \mathrm{d}r\int_0^r \mathrm{e}^{-b(r-s)}\langle\lambda, w_t(s)^2\rangle\,\mathrm{d}s\right\},
\end{aligned} \tag{12}$$

where

$$t\langle\lambda, f_t\rangle/b - \int_0^t \mathrm{d}r\int_0^r \mathrm{e}^{-b(r-s)}\langle\lambda, f_t\rangle\,\mathrm{d}s = b^{-1}\int_0^t \mathrm{e}^{-br}\langle\lambda, f_t\rangle\,\mathrm{d}s \to 0 \tag{13}$$

as $t\to\infty$. By Eqs. (7), (9) and (10), we have

$$v_t(s) \leqslant P_s^b w_t(t) \leqslant \int_0^t P_{s+t-r}^b u_t(r)\,\mathrm{d}r \leqslant \mathrm{e}^{-bs}\int_0^t \mathrm{e}^{-b(t-r)}P_{s+t}f_t\,\mathrm{d}r \leqslant \mathrm{e}^{-bs}P_{s+t}f_t.$$

It follows that

$$\limsup_{t\to\infty}\int_0^\infty \langle\lambda, v_t(s)\rangle\,\mathrm{d}s \leqslant \lim_{t\to\infty} a_d(t)^{-1}\langle\lambda, f\rangle = 0. \tag{14}$$

Similarly, one may check that

$$\lim_{t\to\infty}\int_0^t \mathrm{d}r\int_0^r \mathrm{e}^{-b(r-s)}\langle\lambda, w_t(s)^2\rangle\,\mathrm{d}s = 0. \tag{15}$$

On the other hand, combining Lemmas 2 and 3, we have

$$\lim_{t\to\infty}\int_0^t \mathrm{d}r\int_0^r \mathrm{d}s\int_0^s \mathrm{e}^{-b(r-s)}\langle\lambda, u_t(s-q)^2\rangle\,\mathrm{d}q = \begin{cases} 2\langle\lambda, f\rangle^2/3b\pi^{1/2}, & d=1, \\ \langle\lambda, f\rangle^2/4\pi b, & d=2, \\ \langle\lambda, fGf\rangle/2b, & d\geqslant 3. \end{cases} \tag{16}$$

Combining (12)–(16) we obtain the desired convergence. □

An immediate consequence of Theorem 1 is the following

Corollary 4. *For $d\geqslant 1$ we have $t^{-1}Y_t\to\lambda$ in probability.*

Acknowledgements

Both authors would like to thank Professor Zikun Wang for his guidance and encouragement.

References

Dawson, D.A., 1977. The critical measure diffusion process. Z. Wahrsch. verw. Geb. 40, 125–145.

Dawson, D.A., Fleischmann, K., 1997. A continuous super-Brownian motion in a super-Brownian medium. J. Theor. Probab. 10, 213–276.

Gorostiza, L.G., Lopez-Mimbela, J.A., 1990. The multitype measure branching process. Adv. Appl. Probab. 22, 49–67.

Hong, W.M., 2000. Ergodic theorem for the two-dimensional super-Brownian motion with super-Brownian immigration. Prog. Natural Sci. 10, 111–116.

Hong, W.M., Li, Z.H., 1999. A central limit theorem for the super-Brownian motion with super-Brownian immigration. J. Appl. Probab. 36, 1218–1224.

Iscoe, I., 1986. A weighted occupation time for a class of measure-valued critical branching Brownian motion. Probab. Theor. Rel. Fields 71, 85–116.

Li, Z.H., 1992a. A note on the multitype measure branching process. Adv. Appl. Probab. 24, 496–498.

Li, Z.H., 1992b. Measure-valued branching processes with immigration. Stoch. Proc. Appl. 43, 249–264.

Li, Z.H., 1996. Immigration structures associated with Dawson–Watanabe superprocesses. Stoch. Proc. Appl. 62, 73–86.

Li, Z.H., 1999. Some central limit theorems for super-Brownian motion. Acta Math. Scientia (Ser. A, Engl. ed.) 19, 121–126.

Li, Z.H., Shiga, T., 1995. Measure-valued branching diffusions: immigrations, excursions and limit theorems. J. Math. Kyoto University 35, 233–274.

Li, Z.H., Wang, Z.K., 1999. Measure-valued branching processes and immigration processes. Adv. Math. (China) 28, 105–134.