Immigration process in catalytic medium

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Abstract The longtime behavior of the immigration process associated with a catalytic super-Brownian motion is studied. A large number law is proved in dimension $d\leqslant 3$ and a central limit theorem is proved for dimension $d=3$.

Keywords: immigration process, branching rate functional, Brownian collision local time, catalytic super-Brownian motion.

It is well known that the measure-valued branching process, or superprocess, describes the evolution of a population that evolves according to the law of chance. If we consider a situation where there are some additional source of population from which immigration occurs during the evolution, we need to consider a measure-valued branching process with immigration, or simply immigration process[1,2]. Some limit theorem for the immigration process were obtained in refs. [3, 4]. Recently, much attention is focused on the superprocess in random environment. Randomizing the branching rate functional, Dawson and Fleischmann[5] constructed a super-Brownian motion in catalytic medium, the so-called catalytic super-Brownian motion in dimension $d\leqslant 3$, whose branching rate functional is random and is given by the Brownian collision local time (BCLT). The BCLT is determined by another super-Brownian motion ρ, which is called a catalytic medium (refer ref. [5] for details). A central limit theorem for the occupation time of the catalytic super-Brownian motion is proved in ref. [6].

The situation is also interesting for the immigration process. In this paper, we consider the immigration process associated with catalytic super-Brownian motion (ICSBM) X^{ρ}. And we obtain the weak large number law ($d\leqslant 3$) and the central limit theorem ($d=3$) for the ICSBM X^{ρ} and its occupation time process.

1 Main results

Let $W=[w_t,\ \Pi_{s,a},s,t\geqslant 0,\ a\in R^d]$ denote a standard Brownian motion in R^d with semigroup $\{P_t,\ t\geqslant 0\}$. Let $C(R^d)$ denote the Banach space of continuous bounded functions on R^d equipped with the supreme norm. Let $\phi_p(a):=(1+|a|^2)^{-p/2}$ for $a\in R^d$, and let $C_p(R^d):=\{f\in C(R^d),\ |f(x)|\leqslant C_f\phi_p(x)$ for some constant $C_f\}$. Let $M_p(R^d):=\{$Radon measures μ on R^d such that $\int(1+|x|^p)^{-1}\mu(\mathrm{d}x)<\infty\}$. Suppose that $M_p(R^d)$ is endowed with the p-vague topology. Note $\langle\mu,f\rangle:=\int f(x)\mu(\mathrm{d}x)$. Let λ denote the Lebesgue measure. We shall take $p>d$, so that $\lambda\in M_p(R^d)$.

Suppose that we are given an ordinary $M_p(R^d)$-valued critical branching super-Brownian motion $\rho := [\rho_t, \Omega_1, P_{s,\mu}, t \geq s \geq 0, \mu \in M_p(R^d)]$. (We write P_μ for $P_{0,\mu}$.) For $d \leq 3$ Dawson and Fleischmann[5] proved the existence of the Brownian collision local time (BCLT) $L_{[w,\rho]}(dr)$ of ρ, which is an additive function of W. And for $f \in C_p(R^d)^+$

$$\Pi_{s,a}\int_s^t L_{[w,\rho]}(dr) f(w_r) = \int_s^t dr \int \rho_r(db) p(r-s, a, b) f(b). \tag{1.1}$$

Furthermore, it is the branching rate functional. We refer to ref. [5] for details.

For P_λ-a.s. ρ, the ICSBM starting from μ with the immigration rate ν is denoted by $X^\rho := [X_t^\rho, \Omega_2, P_{\mu,\nu}^\rho, t \geq 0, \mu, \nu \in M_p(R^d)]$. The Laplace functional of its transition probabilities is

$$P_{\mu,\nu}^\rho \exp(-\langle X_t^\rho, f\rangle) = \exp\left\{-\langle \mu, v(0, t, \cdot)\rangle - \int_0^t ds \langle \nu, v(s, t, \cdot)\rangle\right\}, \tag{1.2}$$

where $f \in C_p^+$, $v(\cdot, t, \cdot)$ is the unique positive solution of the evolution equation

$$v(s,t,a) = \Pi_{s,a}\left[f(w_t) - \int_s^t L_{[w,\rho]}(dr) v^2(r,t,w_t)\right]. \tag{1.3}$$

We consider the case $\mu = \nu = \lambda$, and let

$$Q(\cdot) := \int P_{\lambda,\lambda}^\rho(\cdot) P_\lambda(d\rho).$$

The following is the main results.

Theorem 1. Let $d \leq 3$. Then for any $f \in M_p(R^d)$, we have

$$t^{-1}\langle X_t^\rho, f\rangle \to \langle \lambda, f\rangle \text{ in probability under } Q.$$

Let $S(R^d)$ be the space of rapidly decreasing, infinitely differentiable functions of R^d whose all partial derivatives are also rapidly decreasing, and let $S'(R^d)$ be the dual space of $S(R^d)$. Let

$$\langle \overline{X}_t^\rho, f\rangle := t^{-1/2}[\langle X_t^\rho, f\rangle - \langle \lambda, f\rangle - t\langle \lambda, f\rangle], \; f \in S(R^d). \tag{1.4}$$

Theorem 2. Let $d = 3$. Then we have $\overline{X}_t^\rho \to \overline{X}_\infty$ in distribution, where $\overline{X}_\infty$ is a centered Gaussian variable in $S'(R^3)$. Its covariance is

$$\mathrm{Cov}(\langle \overline{X}_\infty, f\rangle, \langle \overline{X}_\infty, g\rangle) = \langle \lambda, fGg\rangle,$$

for f, $g \in S(R^3)$, where G denotes the potential opertor of the Brownian motion.

2 Proofs

Lemma 1. Let $f \in M_p(R^d)$. Then under P_λ there are

(i) $d \leq 3$,

$$a_d(t)^{-1}\left[\int_s^t dr \int \rho_r(db)(P_{t-r}f(b))^2 - \int_s^t dr \int (P_{t-r}f(b))^2 db\right] \to 0,$$

uniformly in s, where $a_1(t) = t^\alpha (\alpha > 3/4)$, $a_2(t) = t^\beta$, $a_3(t) = t^\gamma$, $\beta, \gamma > 0$.

(ii) $d = 3$, $a \in R^3$

$$t^{-1}\left[\int_s^t dr \int \rho_r(db) p(r-s, a, b)(P_{t-r}f(b))^2 - \int_s^t dr \int p(r-s, a, b)(P_{t-r}f(b))^2 db\right] \to 0.$$

Proof. We prove (i) only. By the same method, (ii) can be obtained. Consider the Laplace transition functional of the occupation time of ρ

$$P_\lambda \exp\left\{-\int_0^t \langle \rho_r, f\rangle dr\right\} = \exp\{-\langle \lambda, u(0,t,\cdot)\rangle\}, \tag{2.1}$$

where $u(\cdot, t, \cdot)$ is the solution of the following equation:

$$u(s,t,a) = \int_s^t P_{r-s} f(a) dr - \int_s^t P_{r-s} u(r,t,a)^2 dr. \tag{2.2}$$

Noting that $\| P_s f \| \leqslant (1 \wedge s^{-d/2}) \cdot C$, we can calculate (C denotes a constant; it may have different values in different lines)

$$P_\lambda \int_s^t dr \int \rho_r(db)(P_{t-r} f(b))^2 = \int_s^t dr \int (P_{t-r} f(b))^2 db,$$

$$\mathrm{Var}_\lambda \int_s^t dr \int \rho_r(db)(P_{t-r} f(b))^2$$

$$\leqslant 2\int_0^t dt \int db \left[\int_r^t dh P_{h-r}(P_{t-h} f)^2(b)\right]^2$$

$$\leqslant C \cdot \int_0^t dr \left(\int_r^t dh (1 \wedge (t-h)^{-d/2})\right)^2 \cdot \langle \lambda, (P_{t-r} f)^2\rangle$$

$$\leqslant C \cdot \int_0^t (1 \wedge r^{-d/2}) dr \cdot \left(\int_0^t (1 \wedge r^{-d/2}) dr\right)^2$$

$$\leqslant \begin{cases} C \cdot t^{3/2} & d = 1, \\ C \cdot (\log t)^3 & d = 2,. \\ C & d = 3. \end{cases}$$

By Chebyshev's inequality, for any $\varepsilon > 0$, uniformly in s there is

$$P_\lambda \left\{ a_d(t)^{-1} \left| \int_s^t dr \int \rho_r(db)(P_{t-r} f(b))^2 - \int_s^t dr \int (P_{t-r} f(b))^2 db \right| \geqslant \varepsilon \right\}$$

$$\leqslant \varepsilon^{-2} a_d(t)^{-2} \cdot \mathrm{Var}_\lambda \int_s^t dr \int \rho_r(db)(P_{t-r} f(b))^2$$

$$\leqslant 0(t^{-c(d)}) \to 0 \ (\text{as } t \to \infty),$$

where $c(1) = 2\alpha - 3/2 > 0$, $c(2) = 2\beta - \eta$ $(\beta > \eta > 0)$, $c(3) = 2\gamma$. This completes the proof. Q.E.D.

Proof of Theorem 1. It suffices to prove

$$\lim_{t\to\infty} Q \exp(-t^{-1}\langle X_t^\rho, f\rangle) = \exp(-\langle \lambda, f\rangle). \tag{2.3}$$

Let $f_t := t^{-1} f$. By (1.1)—(1.3), the Laplace transition function of $t^{-1} X_t^\rho$ under Q is

$$Q\exp\{-t^{-1}\langle X_t^\rho, f\rangle\} = \exp\{-\langle \lambda, f_t\rangle - t\langle \lambda, f_t\rangle\}$$

$$\cdot P_\lambda \exp\left\{\int_0^t dr \int \rho_r(db) v(r,t,b)^2 + \int_0^t ds \int_s^t dr \int \rho_r(db) v(r,t,b)^2\right\}, \tag{2.4}$$

where $v(\cdot, t, \cdot)$ is given by (1.3) with f being replaced by f_t. But

$$\int_0^t ds \int_s^t dr \int \rho_r(db) v(r,t,b)^2 \leqslant \int_0^t ds \int_s^t dr \int \rho_r(db)(P_{t-r} f_t(b))^2. \tag{2.5}$$

By the dominated convergence theorem and Lemma 1, under probability P_λ, we have

$$\lim_{t\to\infty} \int_0^t ds \int_s^t dr \int \rho_r(db)(P_{t-r} f_t(b))^2$$

$$= \lim_{t\to\infty}\int_0^t ds\left[t^{-2}\int_s^t dr\int \rho_r(db)(P_{t-r}f(b))^2\right]$$
$$= \lim_{t\to\infty} t^{-2}\int_0^t ds\int_s^t dr\int db(P_{t-r}f_t(b))^2$$
$$\leqslant \lim_{t\to\infty} t^{-2}\cdot\int_0^t\langle\lambda,f\rangle ds\cdot\int_0^t(1\wedge r^{-d/2})dr$$
$$\to 0.$$

That is,

$$\lim_{t\to\infty}\int_0^t ds\int_s^t dr\int\rho_r(db)v(r,t,b)^2 = 0. \tag{2.6}$$

Similarly, we can prove

$$\lim_{t\to\infty}\int_0^t dr\int\rho_r(db)v(r,t,b)^2 = 0. \tag{2.7}$$

Then from (2.4) together with (2.6) and (2.7), (2.3) is obtained. This completes the proof.

Q.E.D.

Proof of Theorem 2. Let $f_t := t^{-1/2}f$. From (1.2) and (1.3), with respect to Q, the Laplace functional of $\overline{X}_t^\rho$ is

$$Q\exp(-\langle\overline{X}_t^\rho, f\rangle)$$
$$= P_\lambda\exp\left\{\int_0^t dr\int\rho_r(db)v(r,t,b)^2 + \int_0^t ds\int_s^t dr\int\rho_r(db)v(r,t,b)^2\right\}, \tag{2.8}$$

where $v(\cdot,t,\cdot)$ is the solution of (1.3) with f being replaced by f_t. Because

$$\int_0^t dr\int(P_{t-r}f_t(b))^2 \leqslant t^{-1}\int_0^t(1\wedge r^{-3/2})\cdot\langle\lambda,f\rangle\to 0 \tag{2.9}$$

as $t\to\infty$. By Lemma 1 and the dominated convergence theorem, under probability P_λ, we have

$$\lim_{t\to\infty}\int_0^t dr\int\rho_r(db)v(r,t,b)^2$$
$$\leqslant \lim_{t\to\infty}\int_0^t dr\int\rho_r(db)(P_{t-r}f_t(b))^2 = 0, \tag{2.10}$$

$$\lim_{t\to\infty}\int_0^t ds\int_s^t dr\int\rho_r(db)(P_{t-r})f_t(b))^2$$
$$= \lim_{t\to\infty}\int_0^t ds\left[t^{-1}\int_s^t dr\int\rho_r(db)(P_{t-r})f(b))^2\right]$$
$$= \lim_{t\to\infty} t^{-1}\int_0^t ds\int_s^t dr\int(P_{t-r})f(b))^2 db$$
$$= \int_0^\infty dr\int(P_rf(b))^2 db$$
$$= \langle\lambda, fGf\rangle, \tag{2.11}$$

where G is the potential operator of Brownian motion. From (1.3)

$$(P_{t-r}f_t(b))^2 - (v(r,t,b))^2 \leqslant 2(P_{t-r}f_t(b))\cdot\int_r^t dh\int\rho_h(dx)v(h,t,x)^2p(h-r,b,x).$$

Using Lemma 1 and Hölder inequality, we get

$$
\begin{aligned}
0 &\leqslant \lim_{t\to\infty}\int_0^t ds\int_s^t dr\int \rho_r(db)[(P_{t-r}f_t(b))^2 - (v(r,t,b))^2] \\
&= \lim_{t\to\infty}\int_0^t ds\int_s^t dr\int db[(P_{t-r}f_t(b))^2 - (v(r,t,b))^2] \\
&\leqslant 2\lim_{t\to\infty}\int_0^t ds\int_s^t dr\int db(P_{t-r}f_t(b)) \cdot \int_r^t dh\int \rho_h(dx)v(h,t,x)^2 p(h-r,b,x) \\
&\leqslant 2\lim_{t\to\infty}\Big[\int_0^t ds\int_s^t dr\int db(P_{t-r}f_t(b))^2\Big]^{1/2} \\
&\quad \cdot \Big\{\int_0^t ds\int_s^t dr\int db\Big[\int_r^t dh\int \rho_h(dx)v(h,t,x)^2 p(h-r,b,x)\Big]^2\Big\}^{1/2} \\
&\leqslant C\cdot \lim_{t\to\infty}\Big\{\int_0^t ds\int_s^t dr\int db\Big[\int_r^t dh\int dx(P_{t-h}f_t(x))^2 p(h-r,b,x)\Big]^2\Big\}^{1/2} \\
&\leqslant C\cdot \lim_{t\to\infty}\Big[\int_0^t ds\int_s^t dr\int db(P_{t-r}f_t(b))^2\Big]^{1/2} \cdot t^{-1/2}\int_0^t dh(1\wedge(t-h)^{-3/2}) \\
&= C\cdot t^{-1/2}\int_0^t dh(1\wedge h^{-3/2}) \\
&\to 0, \qquad\qquad (2.12)
\end{aligned}
$$

as $t\to\infty$. Combining (2.11) and (2.12), we have

$$\lim_{t\to\infty}\int_0^t ds\int_s^t dr\int \rho_r(db)(v(r,t,b))^2 = \langle\lambda, fGf\rangle. \qquad (2.13)$$

From (2.8), (2.10) and (2.13), we prove that

$$\lim_{t\to\infty} Q\exp(-\langle \overline{X}_t^\rho, f\rangle) = \exp(\langle\lambda, fGf\rangle).$$

Then the assertion follows from Iscoe[7]. Q.E.D.

Let $Y_t^\rho := \int_0^t X_r^\rho dr$ be the occupation time process of the ICSBM. By methods similar to Theorems 1 and 2, we can prove the following results.

Theorem 3. Let $d\leqslant 3$. Then for any $f\in M_p(R^d)$,

$$2t^{-2}\langle Y_t^\rho, f\rangle \to \langle\lambda, f\rangle \text{ in probability under } Q.$$

Let

$$\langle \overline{Y}_t^\rho, f\rangle := t^{-5/4}[\langle Y_t^\rho, f\rangle - t\langle\lambda, f\rangle - t^2\langle\lambda, f\rangle],\ f\in S(R^d). \qquad (2.14)$$

Theorem 4. Let $d=3$. Then in $S'(R^3)$,

$$\overline{Y}_t^\rho \to \overline{Y}_\infty \text{ in distribution under } Q,$$

where $\overline{Y}_\infty$ is a centered Gaussian variable in $S'(R^3)$. Its covariance is

$$\mathrm{Cov}(\langle \overline{Y}_\infty, f\rangle, \langle \overline{Y}_\infty, g\rangle) = c\cdot\langle\lambda, f\rangle\langle\lambda, g\rangle,$$

for $f, g\in S(R^3)$, where $c = \dfrac{8(\sqrt{2}-1)}{15\pi^{3/2}}$.

The details for the proof are omitted.

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References

1. Dawson, D. A., Measure-valued Markov processes, in Vol. 1541 of LNM, Berlin: Springer-Verlag, 1993, 1—260.
2. Li, Z. H., Wang, Z. K., Measure-valued branching processes and immigration process, Adv. in Math., 1999, 28(2): 105.
3. Li, Z. H., Li, Z. B., Wang, Z, K., Asymptotic behavior of the measurevalued branching process with immigration, Sci. Chin., Ser. A, 1993, 36: 769.
4. Li, Z. H., Shiga, T., Measure-valued branching diffusions: immigrations, excursions and limit theorems, J. Math. Kyoto Univ., 1995, 35: 233.
5. Dawson, D. A., Fleischmann, K., A continuous super-Brownian motion in a super-Brownian medium, J. Th. Probab., 1997, 10(1): 213.
6. Hong Wenming, Central limit theorem for the occupation time of catalytic super-Brownian motion, Chin. Sci. Bull., 1998, 43(24): 2035.
7. Iscoe, I., A weighted occupation time for a class of measure-valued critical branching Brownian motion, Probab. Theory Relat. Fields, 1986, 71: 85.