Activated Random Walks on $\mathbb{Z}^d$

Lectures 9 and 10

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THU-PKU-BNU joint probability webinar – July 2020

Tentative plan

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Recall of previous lectures

Particle-wise construction is well-defined [§11.3]

Uniqueness of the critical density [§8]

Weak and strong stabilization [§7]

Recall

Recall

Dynamics and phase space

Odometer and toppling procedures

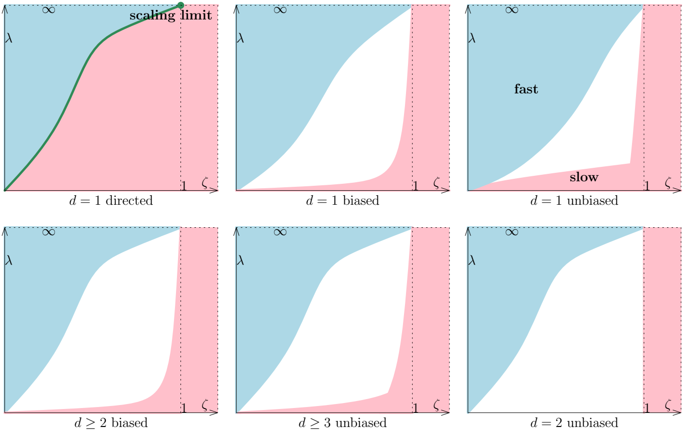
Counting arguments

Exploring instructions in advance

Coarse-grained flow between blocks

The particle-wise construction and applications

Phase space [§1.5]



Odometer and Abelian property [§2.2]

All deterministic: finite sequences of topplings α, β

$$m_\alpha(x) := \#\text{times } x \text{ appears in } \alpha$$

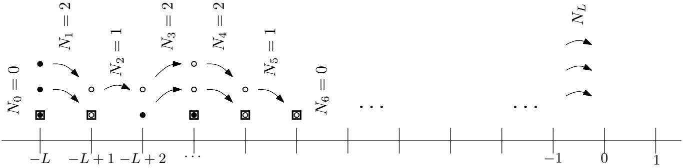
$$m_{V,\eta} := \sup_{\beta \subseteq V \text{ legal}} m_\beta .$$

$$m_{V,\eta} \leqslant m_\alpha \text{ if } \alpha \text{ stabilizes } \eta \text{ in } V$$

$$m_{V,\eta} \uparrow m_\eta$$

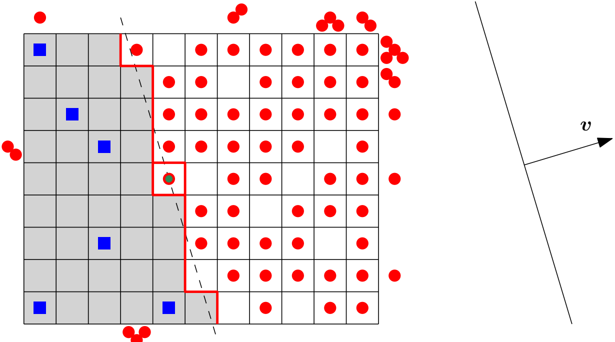
Now make it random

Counting arguments [§3]



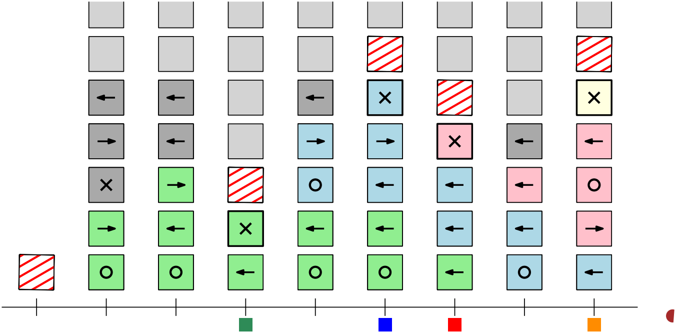
(folklore)

Counting arguments [§3]



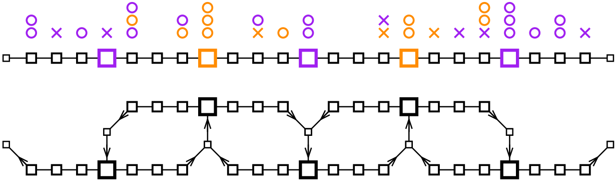
(Taggi; R, Tournier)

Exploring instructions in advance [§4]



(R, Sidoravicius)

Coarse-grained flow between blocks [§5]



Analyze the odometer $m_1, \ldots, m_n$ at the *buffers*

Mass Balance Equations and Single-Block Dynamics

(Basu, Ganguly, Hoffman)

The particle-wise construction [§10.1]

Labeled particles

Constructed from η_0, CTRW, Clocks

Defined through a **limit** (see below)

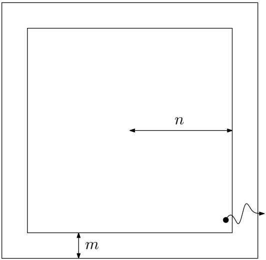
Fixation equivalence: Sites fixate $\Leftrightarrow$ Particles fixate

Conservation: If fixate, $\mathbb{E}[\text{start at } \mathbf{0}] = \mathbb{E}[\text{settle at } \mathbf{0}]$

Corollary: $\zeta_c \leqslant 1$

Averaged condition for activity [§10.2]

$$\limsup_n \frac{\mathbb{E} M_n}{|V_n|} > 0 \implies \text{Activity}$$



(R, Tournier)

Fixation equivalence [§10.4]

Theorem 10.7. Sites fixate $\Leftrightarrow$ Particles fixate

Idea: extra randomness so as to **spread out** the effect of non-fixating particles and control variance.

Implementation: for each particle that stays active, tag at random one of the n first sites visited after time t.

(Amir, Gurel-Gurevich)

Resampling [§10.3]

Theorem 10.4. For i.i.d. random initial configurations with average $\zeta = 1$, the system a.s. stays active.

Take $\lambda = \infty$, change rates to **particle-hole model**.

Suppose finitely many particles visit $\mathbf{0}$.

Resample η_0 on the sites where they may start, so **wpp** site $\mathbf{0}$ is never visited. **Conclude** that $\zeta < 1$.

(Cabezas, R, Sidoravicius)

§11.3
The particle-wise construction is well-defined

Construction [§10.1]

For a triple $(\eta_0, \boldsymbol{X}, \boldsymbol{\mathcal{P}})$, we we say that $(\eta_0, \boldsymbol{X}, \boldsymbol{\mathcal{P}}) \mapsto (\eta_0, \boldsymbol{Y}, \boldsymbol{\gamma})$ is **well-defined** if:

(i) for each $x, y \in \mathbb{Z}^d$, $j \in \mathbb{N}$ and $t > 0$, both $(Y_s^{x,j})_{s\in[0,t]}$ and $(\gamma_s^{x,j})_{s\in[0,t]}$ are the same in the systems $(\eta_0 \cdot \mathbb{1}_{B_n^y}, \boldsymbol{X}, \boldsymbol{\mathcal{P}})$ for all but finitely many n;

(ii) the limit $(\eta_0, \boldsymbol{Y}, \boldsymbol{\gamma})$ **does not depend on** y.

Statement

Theorem 10.6. If $\sup_x \mathbb{E}|\eta_0(x)| < \infty$, then the above particle-wise construction is a.s. well-defined.

(R, Tournier)

Overview of the proof

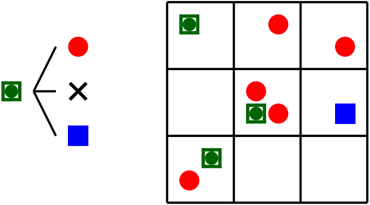
$\vdash$ For arbitrary fixed $V_n \uparrow \mathbb{Z}^d$ there is an a.s. limit.

Add particles one by one, updating the whole evolution

$\vdash$ Life of each particle is well-defined through some limit

Main step: $\forall x, T$, the number of particle additions that affect site x by time T has finite expectation

Tracking differences



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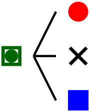

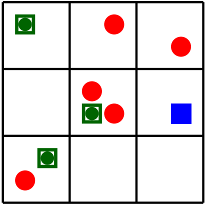

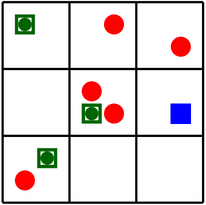

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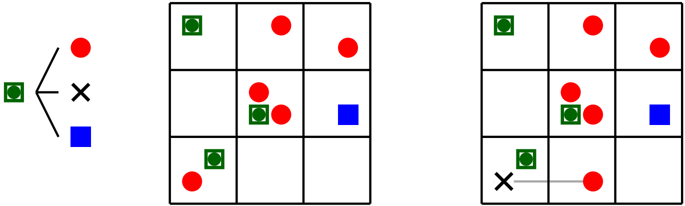

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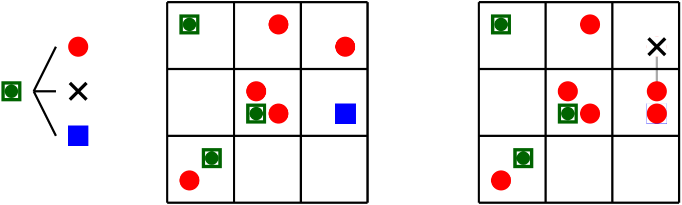

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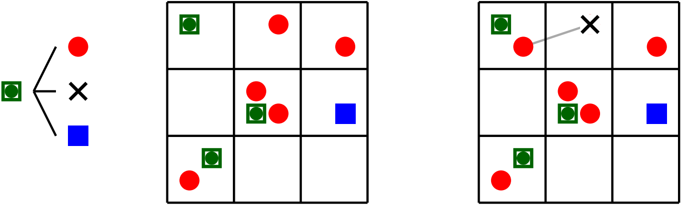

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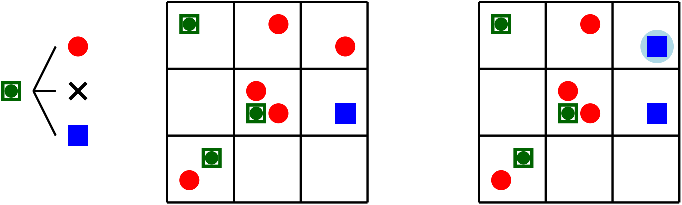

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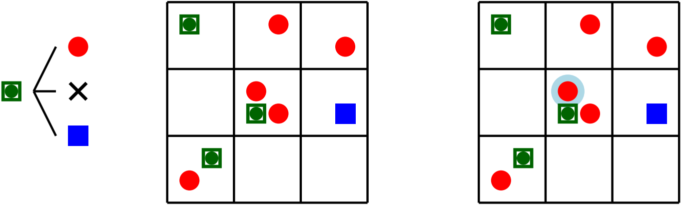

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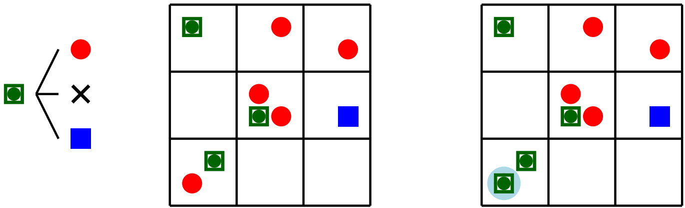

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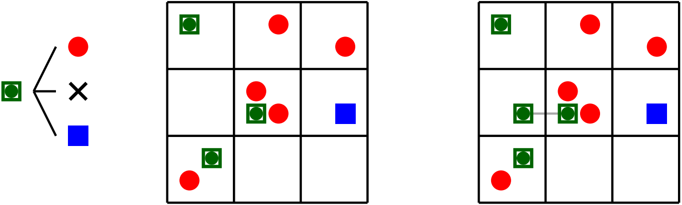

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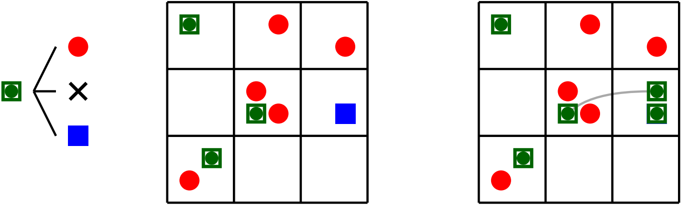

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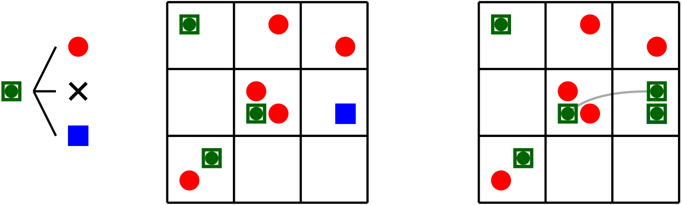


Dominate by a supercritical branching process.

$\mathbb{E}[\text{green}] = e^{(2+\lambda)t}$

Re-index sums etc, Borel-Cantelli...

§8
Uniqueness of the critical density

Uniqueness of the critical density

Theorem 2.13. Given the dimension d, sleep rate λ, and jump distribution $p(\cdot)$, there is a number ζ_c such that, for every translation-ergodic distribution ν supported on $(\mathbb{N}_0)^{\mathbb{Z}^d}$ with average density ζ, the ARW dynamics satisfies

$$\mathbf{P}^\nu(\text{system stays active}) = \begin{cases} 0, & \zeta < \zeta_c, \\ 1, & \zeta > \zeta_c. \end{cases}$$

(R, Sidoravicius, Zindy)

Equivalent statement

Theorem 8.1. Let d, λ and $p(\cdot)$ be given. Let ν_1 and ν_2 be two spatially ergodic distributions on $(\mathbb{N}_0)^{\mathbb{Z}^d}$, with respective densities $\zeta_1 < \zeta_2$. If the ARW system is a.s. fixating with initial state ν_2, then it is also a.s. fixating with initial state ν_1.

The more general version

Open Problem. Suppose ν is a translation-ergodic active state (active means ν is supported on $(\mathbb{N}_{\mathfrak{s}})^{\mathbb{Z}^d} \setminus \{0, \mathfrak{s}\}^{\mathbb{Z}^d}$) with density $\zeta > \zeta_c$. Show that the ARW with initial state ν a.s. stays active.

Idea of the proof

- Embedding the initial configuration into another one with higher density (decoupling)

- Stabilization of the embedded configuration

- Stabilization of the original configuration

Remark. Not a sequential procedure like previous ones

Decoupling

Sample $\eta_0 \sim \nu_1$, $\xi_0 \sim \nu_2$ and $\mathcal{I}$ independently $\to \omega$.

Assume wlog ν_1 or ν_2 mixing, hence ω **ergodic**.

$\vdash$ A doubly-infinite procedure which is a **factor** of ω.

Let $A_0 = \{x : \eta_0(x) > \xi_0(x)\}$. Topple every site in A_0.
Result η_1 is insensitive to the order, hence a **factor**.

Repeat for $\eta_0, \eta_1, \eta_2, \ldots$. **Limit** $\eta_\infty = \eta_0'$ exists.

Each site is toppled **finitely often**, otherwise $\zeta_1 \geqslant \zeta_2$.

Stabilization of the larger configuration

Delete the instructions used in the previous stage.

Zero out odometer.

Conditioning on the outcome of the first step, the remaining instructions are again i.i.d. with the correct distribution.

Stabilize ξ_0. Odometer $m_{\xi_0}(x) < +\infty$ by assumption.

Stabilization of the original configuration

Since $\eta'_0 \leqslant \xi_0$, we also have $m_{\eta'_0}(x) < +\infty$.

Two stages: embedding and then stabilizing.

Some topplings in the first stage were not legal (because we made forced sleepy particles to wake up and jump).

Hence, the sum of the (locally finite) odometers obtained in these two stages is an upper bound for the odometer of η_0.

§7
Weak and strong stabilization

Results

Theorem 7.1. For any jump distribution in any dimension, $\zeta_c \geqslant \frac{\lambda}{1+\lambda}$.

Theorem 7.2. If $d > 2$, then $\zeta_c < 1$ for every $\lambda < \infty$ and $\zeta_c \to 0$ as $\lambda \to 0$.

(Stauffer, Taggi)

Open Problem. Prove a similar statement for unbiased walks on $\mathbb{Z}^2$.

Weak and strong stabilization

We say that $\mathbf{0}$ is *w-stable* if $\eta(\mathbf{0}) \leqslant 1$, and we say that $\mathbf{0}$ is *s-stable* if $\eta(\mathbf{0}) = 0$. Otherwise we say that $\mathbf{0}$ is *w-unstable* or *s-unstable*. For $y \neq \mathbf{0}$ we say that y is stable, w-stable, and s-stable if $\eta(y) \leqslant \mathfrak{s}$.

Comparison:

$$m^w_{V,\eta} \leqslant m_{V,\eta} \leqslant m^s_{V,\eta}.$$

η^w_V and η'_V: configuration after (weakly) stabilizing

Proof of Theorem 7.1

Using Abelian Property, one way to stabilize η_0 on B_n is to first weakly stabilize it and then stabilize it.

$\vdash m_{V,\eta_0}(\mathbf{0}) \geqslant 1 \quad \Longrightarrow \quad \eta_V^w(\mathbf{0}) = 1$

$\vdash \mathbb{P}\left(\eta_V'(\mathbf{0}) = \mathfrak{s}\right) \geqslant \frac{\lambda}{1+\lambda}\, \mathbb{P}\left(\eta_V^w(\mathbf{0}) = 1\right)$

Hence, $\mathbb{P}\left(\eta_V'(\mathbf{0}) = \mathfrak{s}\right) \geqslant \frac{\lambda}{1+\lambda}\, \mathbb{P}\left(m_{V,\eta_0}(\mathbf{0}) \geqslant 1\right)$

Using monotonicity and amenability... non-fixation implies $\zeta \geqslant \frac{\lambda}{1+\lambda}$.

Jump odometer and extra particles

Define the "jump odometer" $\bar{m}_{V,\eta}$ by counting only the number of jump instructions performed at each site when η is stabilized in V.

Define $\bar{m}^s_{V,\eta}$ and $\bar{m}^w_{V,\eta}$ similarly.

Let $\eta^+ = \eta + \delta_{\mathbf{0}}$ denote the result of adding an active particle at $\mathbf{0}$ to a configuration η.

Strong − weak = extra particle

Lemma 7.5. We have $\bar{m}^s_{V,\eta} = \bar{m}^w_{V,\eta^+}$.

Proof. A sequence of topplings β is w-legal for η^+ if and only if it is s-legal for η.

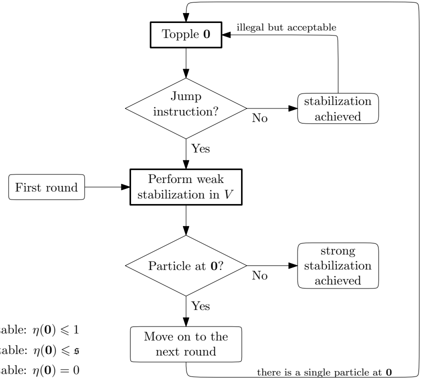
Getting rid of the extra particle

Lemma 7.6. We have $\mathbb{E}\Big[\bar{m}^w_{V,\eta^+}(\mathbf{0})\Big] \leqslant G + \mathbb{E}\Big[\bar{m}^w_{V,\eta}(\mathbf{0})\Big]$.

Sketch. Force the particle to move.

Corollary 7.7. $\mathbb{E}\left[\bar{m}^s_{V,\eta}(\mathbf{0}) - \bar{m}^w_{V,\eta}(\mathbf{0})\right] \leqslant G$.

Successive weak stabilizations



w-stable: $\eta(\mathbf{0}) \leqslant 1$

stable: $\eta(\mathbf{0}) \leqslant \mathfrak{s}$

s-stable: $\eta(\mathbf{0}) = 0$

Successive weak stabilizations (cont)

Let T_V and T_V^s count the number of rounds needed for stabilization and strong stabilization to be achieved, respectively (weak stabilization is always achieved in the first round). From this definition we have

$$T_V = 1 \iff \eta_V^w(\mathbf{0}) = 0 \iff T_V^s = 1.$$

Successive weak stabilizations (cont)

Lemma 7.8. $\bar{m}^s_{V,\eta}(\mathbf{0}) \geqslant \bar{m}^w_{V,\eta}(\mathbf{0}) + T^s_V - 1.$

Corollary 7.9. $\mathbb{E}T_V \leqslant \mathbb{E}T^s_V \leqslant 1 + G.$

Proof of the main theorems (overview)

$$\mathbb{P}\left(\eta'_V(\mathbf{0}) = \mathfrak{s}\right) = \sum_{n=2}^{\infty} \mathbb{P}\left(\eta'(\mathbf{0}) = \mathfrak{s}, T_V = n\right)$$

$$\mathbb{P}(\eta'_V(\mathbf{0}) = \mathfrak{s}, T_V = n) \leqslant \tfrac{\lambda}{1+\lambda}\left(\tfrac{1}{1+\lambda}\right)^{n-2}$$

$$\mathbb{P}\left(\eta'_V(\mathbf{0}) = \mathfrak{s}, T_V = n \,\middle|\, T_V \geqslant n\right) = \tfrac{\lambda}{1+\lambda}$$