Activated Random Walks on $\mathbb{Z}^d$

Lectures 7 and 8

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THU-PKU-BNU joint probability webinar – July 2020

Tentative plan

Tentative plan

Recall of previous lectures

Particle-wise construction and mass conservation [§10.1]

Averaged condition for activity [§10.2]

Fixation equivalence [§10.4]

Resampling [§10.3]

Definedness [§11.3] (if time allows)

Recall

Recall

Dynamics and phase space

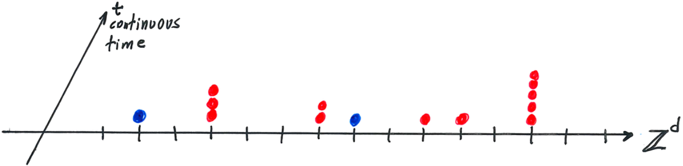
Odometer and toppling procedures

First applications

Exploring instructions in advance

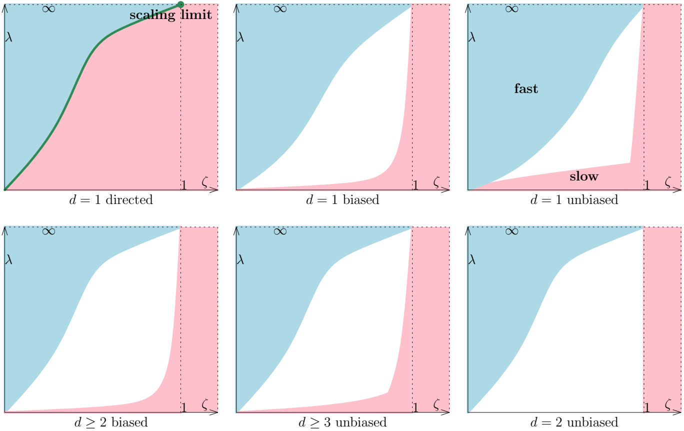
Activity without transience

Definition of the dynamics [§1.1]



	rate	
A → S	λ	
A jumps	1	from x to $x+y$ with y chosen randomly
AS → AA	∞	

Phase space [§1.5]



Main tools

Site-wise representation and Abelian Property

$\rightarrow$ preserving monotonicity, etc

$\rightarrow$ fixation-activity reduced to *toppling procedures*

Particle-wise construction

$\rightarrow$ ergodicity, mass transport, coupling, resampling

$\rightarrow$ a particle stays active $\Rightarrow$ sites stay active

$\rightarrow$ averaged criterion for activity...

...to be used in the site-wise representation!

Odometer and Abelian property [§2.2]

All deterministic

Finite sequences of topplings α, β

$m_\alpha(x) = \#$times x appears in α

$$m_{V,\eta} := \sup_{\beta \subseteq V \text{ legal}} m_\beta.$$

$$m_{V,\eta} \leqslant m_\alpha \text{ if } \alpha \text{ stabilizes } \eta \text{ in } V$$

$$m_{V,\eta} \uparrow m_\eta$$

Criteria for fixation [§2.3]

Corollary 2.8.

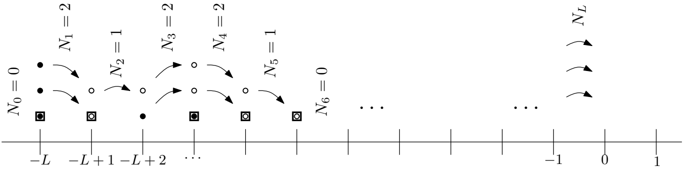
$$\sup_k \inf_V \mathbb{P}[m_V(x) \leqslant k] > 0 \implies \text{Fixation}$$

$$\inf_k \sup_V \mathbb{P}[m_V(x) \geqslant k] > 0 \implies \text{Activity}$$

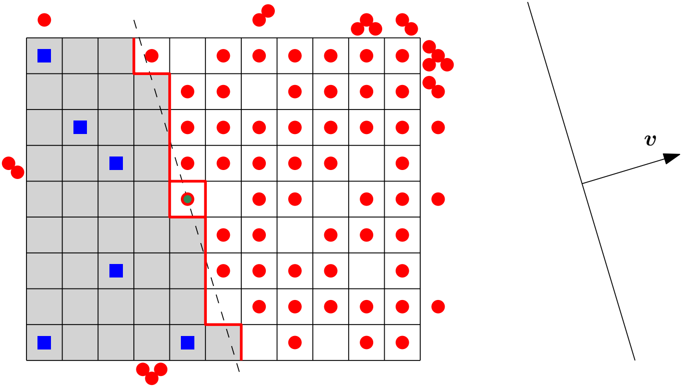
Theorem 2.11.

$$\limsup_n \frac{\mathbb{E}M_n}{|V_n|} > 0 \implies \text{Activity}$$

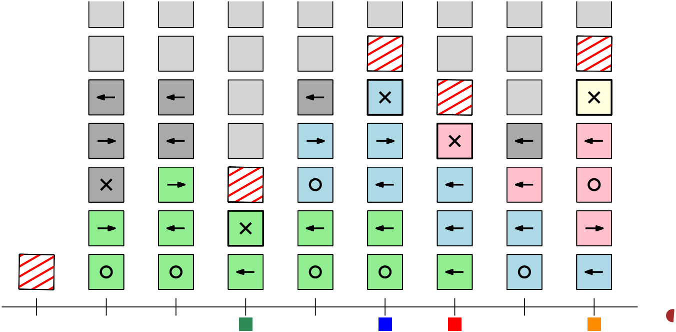
Counting arguments [§3]



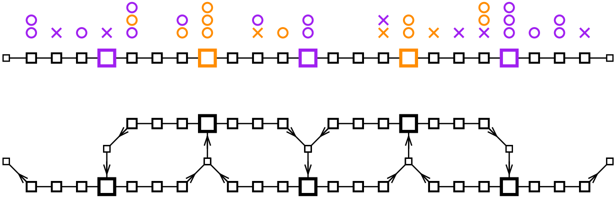
Counting arguments [§3]



Exploring instructions in advance [§4]



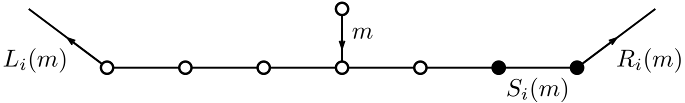
Coarse-grained flow between blocks [§5]



Analyze the odometer $m_1, \ldots, m_n$ at the *buffers*

Single-block dynamics [§5]

Single-block description: $L_i(\cdot),\ R_i(\cdot),\ S_i(\cdot)$



Mass balance equations: $L_i(m_i) = m_{i-1} - R_{i-2}(m_{i-2})$

Single-block estimate: $\sup_\ell \mathbb{E}\Big[\sum_m e^{S_i(m)} \mathbb{1}_{L_i(m)=\ell}\Big] \leqslant 3$

§10.1

The particle-wise construction

System with labeled particles

(Amir, Gurel-Gurevich; R, Tournier)

$Y^{x,j} = (Y_t^{x,j})_{t \geqslant 0}$ position of particle (x, j) at time t

$\gamma^{x,j}(t) \in \{\mathfrak{s}, 1\}$ state of particle (x, j) at time t

The triple $(\eta_0, \boldsymbol{Y}, \boldsymbol{\gamma})$ describes the whole evolution

Construction

$\rightarrow$ mass transport, ergodicity, resampling

$X^{x,j} = (X_t^{x,j})_{t \geqslant 0}$, continuous-time walk

$\mathcal{P}^{x,j} \subseteq \mathbb{R}_+$ Poisson clock

For each **finite** initial configuration ξ, the map

$$(\xi, \boldsymbol{X}, \boldsymbol{\mathcal{P}}) \mapsto (\eta_0, \boldsymbol{Y}, \boldsymbol{\gamma})$$

is a.s. well-defined. Now take $\xi(x) = \eta_0(x) \cdot \mathbb{1}_{B_n^y}(x)$.

Construction (cont)

For a triple $(\eta_0, \boldsymbol{X}, \boldsymbol{\mathcal{P}})$, we we say that $(\eta_0, \boldsymbol{X}, \boldsymbol{\mathcal{P}}) \mapsto (\eta_0, \boldsymbol{Y}, \boldsymbol{\gamma})$ is **well-defined** if:

(i) for each $x, y \in \mathbb{Z}^d$, $j \in \mathbb{N}$ and $t > 0$, both $(Y_s^{x,j})_{s\in[0,t]}$ and $(\gamma_s^{x,j})_{s\in[0,t]}$ are the same in the systems $(\eta_0 \cdot \mathbb{1}_{B_n^y}, \boldsymbol{X}, \boldsymbol{\mathcal{P}})$ for all but finitely many n;

(ii) the limit $(\eta_0, \boldsymbol{Y}, \boldsymbol{\gamma})$ **does not depend on** y.

Construction (cont)

Theorem 10.6. If $\sup_x \mathbb{E}|\eta_0(x)| < \infty$, then the above particle-wise construction is a.s. well-defined.

(R, Tournier)

Fixation equivalence

Theorem 10.7. For η_0 is i.i.d. these are equivalent:

(i) $\mathbf{P}$(some site stays active) > 0;

(ii) $\mathbf{P}$(all sites stay active) $= 1$;

(iii) $\mathbb{P}$(all particles stay active) $= 1$;

(iv) $\mathbb{P}$(some particle stays active) > 0.

(Amir, Gurel-Gurevich)

Mass conservation

Theorem 10.1. Assume particles fixate a.s. Then

$$\mathbb{E}[\text{start at } \mathbf{0}] = \mathbb{E}[\text{settle at } \mathbf{0}].$$

Corollary 10.2. We always have $\zeta_c \leqslant 1$.

Mass conservation (proof)

Consider the random function $f(x, y)$ which counts the number of particles that start at x and end at y.

By the mass transport principle [§2.4],

$$\mathbb{E}\Big[\sum_y f(\mathbf{0}, y)\Big] = \mathbb{E}\Big[\sum_y f(y, \mathbf{0})\Big].$$

This gives

$$\mathbb{E}[\text{start at } \mathbf{0}] = \mathbb{E}[\text{settle at } \mathbf{0}].$$

Coffee break

§10.2
Averaged condition for activity

Averaged condition for activity

Theorem 2.11.

$$\limsup_n \frac{\mathbb{E}M_n}{|V_n|} > 0 \implies \text{Activity}$$

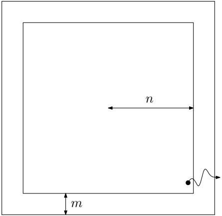
(R, Tournier)

Sketch of proof

Let $A_m^{\mathbf{0},1} =$ "particle $(\mathbf{0}, 1)$ reaches distance m"

$\vdash$ Condition on $\frac{\mathbb{E}M_n}{|V_n|}$ implies that $\mathbb{P}(A_m^{\mathbf{0},1}) \geqslant \delta$

so particles stay active wpp, the use fixation equivalence



$$
\begin{aligned}
\mathbb{E}M_n &\approx \sum_{x\in V_n'} \sum_{i\in\mathbb{N}} \mathbb{P}(\text{particle } Y^{x,i} \text{ exits } V_n) \\
&\leqslant \sum_{x\in V_n'} \sum_{i\in\mathbb{N}} \mathbb{P}(\eta_0(x) \geqslant i \text{ and } A_k^{x,i}) \\
&= |V_n'| \sum_{i\in\mathbb{N}} \mathbb{P}(\eta_0(\mathbf{0}) \geqslant i \text{ and } A_k^{\mathbf{0},1}) \\
&\leqslant |V_n'| \sum_{1\leqslant i\leqslant K} \mathbb{P}(A_k) + |V_n'| \sum_{i>K} \mathbb{P}(\eta_0(\mathbf{0}) \geqslant i) \\
&= |V_n'| K\, \mathbb{P}(A_k) + |V_n'|\, \mathbb{E}[(|\eta_0(\mathbf{0})| - K)^+]
\end{aligned}
$$

§10.4
Fixation equivalence

Idea of proof

$A^{x,j} :=$ "particle (x,j) stays active"

Assume that (iv) holds, so $\mathbb{P}(A^{\mathbf{0},j}) > 0$ for some j.

By interchangeability of particles,
$\mathbb{P}(A^{\mathbf{0},j}) = \mathbb{P}(A^{\mathbf{0},1}, |\eta_0(\mathbf{0})| \geqslant j) \leqslant \mathbb{P}(A^{\mathbf{0},1})$.

Hence, $a := \mathbb{P}(A^{\mathbf{0},1}) > 0$.

Idea of proof

$N_t :=$ number of particles which stay active and are present at site $\mathbf{0}$ at time t.

By the mass transport principle, $\mathbb{E}N_t \geqslant a$, hence $\liminf_t \mathbb{E}N_t > 0$.

However, we need $\limsup_t \mathbb{P}(N_t \geqslant 1) > 0$.

The idea is to introduce extra randomness so as to **spread out** the effect of these particles.

Implementation

Redefine $N_t :=$ number of particles which stay active and visit site $\mathbf{0}$ anytime **after time** t.

Since $(\xi, \boldsymbol{X}, \boldsymbol{\mathcal{P}}) \mapsto (\eta_0, \boldsymbol{Y}, \boldsymbol{\gamma})$ is a **measurable function**, we can approximate $A^{\mathbf{0},1}$ by an event $A^{\mathbf{0}}_{\varepsilon}$ that only depends on $(\eta_0(x), X^x, \mathcal{P}^x)_{\|x\| \leqslant k}$.

Fix a large n and for each x choose a random site $g(x)$ among n sites visited by $X^{x,1}$ after time t.

MTP gives $\mathbb{E} \sum_x \mathbb{1}_{g(x)=\mathbf{0}} \geqslant a - \varepsilon$, now control variance.

§10.3
Resampling

Activity at unit density

Theorem 10.4. For i.i.d. random initial configurations with average $\zeta = 1$, the system a.s. stays active.

(Cabezas, R, Sidoravicius)

A similar model

By monotonicity, we can assume $\lambda = \infty$.

ARW is a particle system with transition τ_{xy} at rate

$$p(y-x)\mathbb{1}_{\eta(x)\geqslant 2}|\eta(x)|.$$

The **particle-hole model** has jump rates

$$p(y-x)[\eta(x)-1]^{+}.$$

Labeled particle description: the **first** particle to arrive at a **hole** will **settle** there, others can go through.

Same site-wise representation, thus **fixation equivalent**

Proof with resampling

Particle-wise construction for the labeled system.

Suppose the system fixates, then show that $\zeta < 1$.

Site $\mathbf{0}$ is a.s. visited by finitely many particles.

So there is a finite $V \subseteq \mathbb{Z}^d$ such that wpp particles starting outside V do not visit $\mathbf{0}$.

Suppose this event occurs, and resample η_0 in V, keeping all other randomness. The new η_0 will have no particles in V with positive probability.

§11.3
The particle-wise construction is well-defined

Overview

$\vdash$ For arbitrary fixed $V_n \uparrow \mathbb{Z}^d$ there is an a.s. limit.

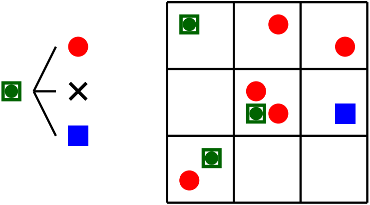
Add particles one by one, updating the whole evolution

$\vdash$ Life of each particle is well-defined through some limit

Main step: $\forall x, T$, the number of particle additions that affect site x by time T has finite expectation

(R, Tournier)

Tracking differences



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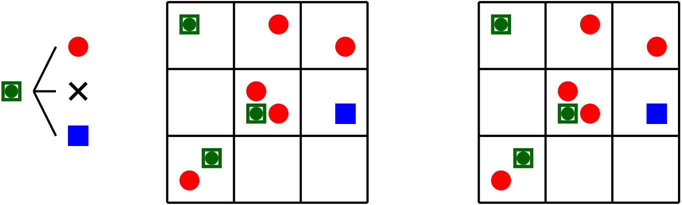

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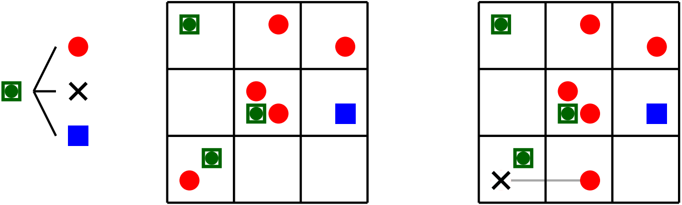

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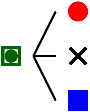

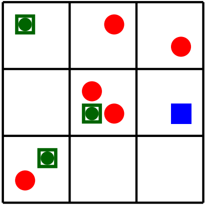

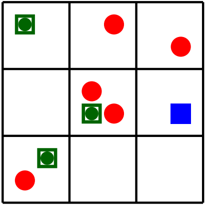

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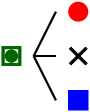

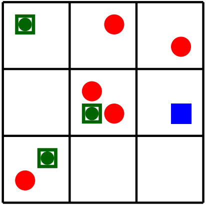

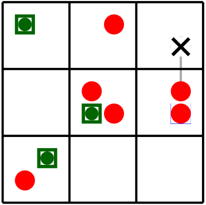

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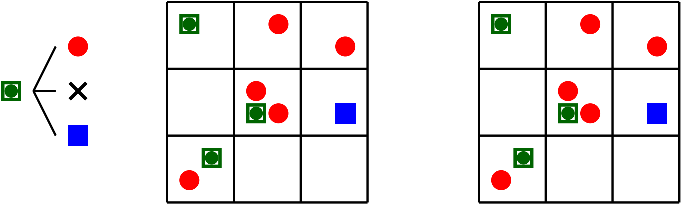

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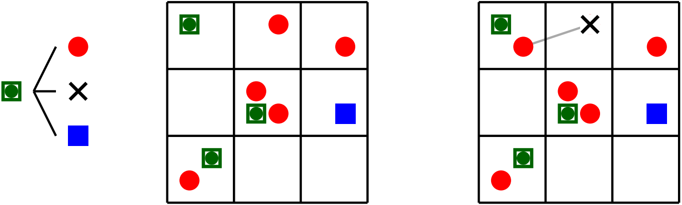

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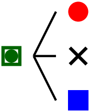

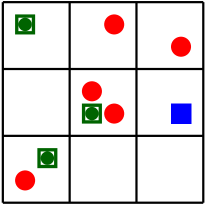

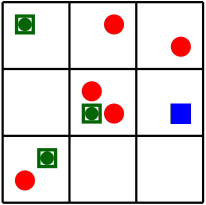

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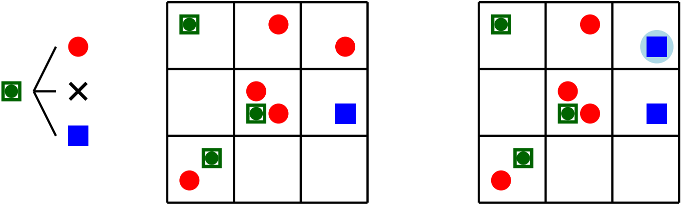

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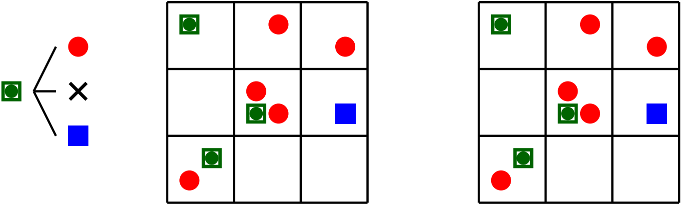

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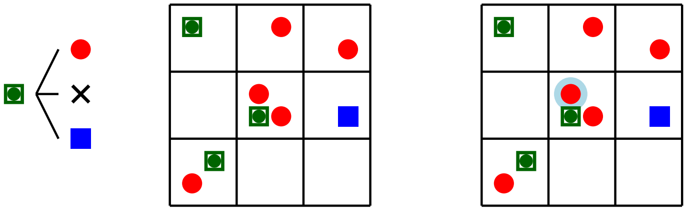

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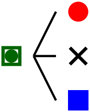

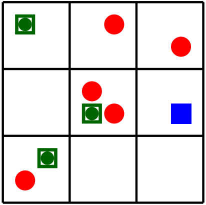

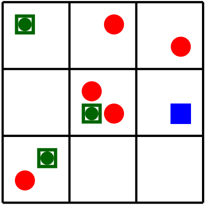

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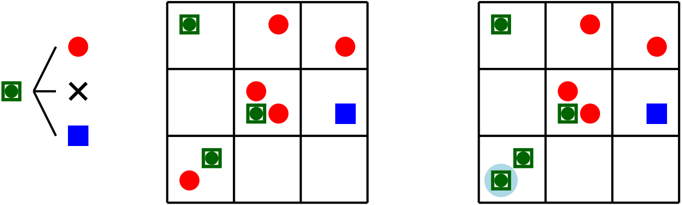

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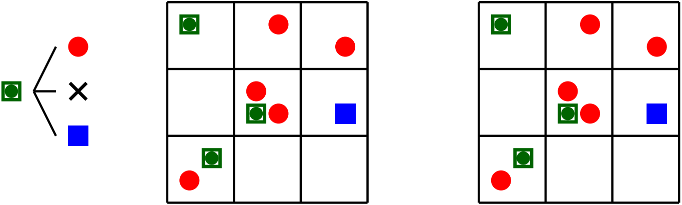

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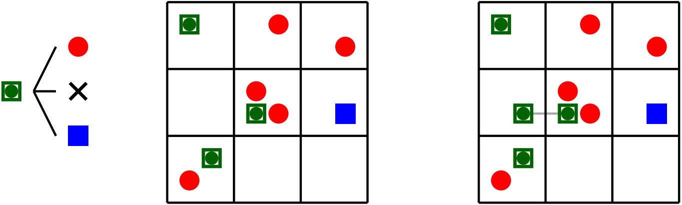

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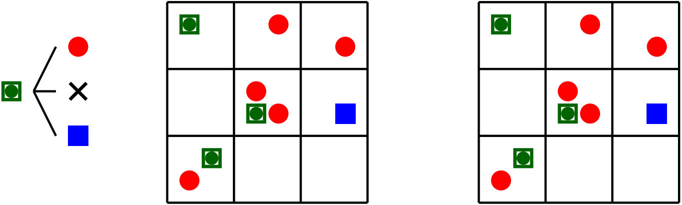

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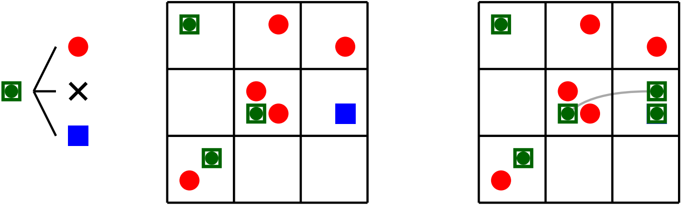

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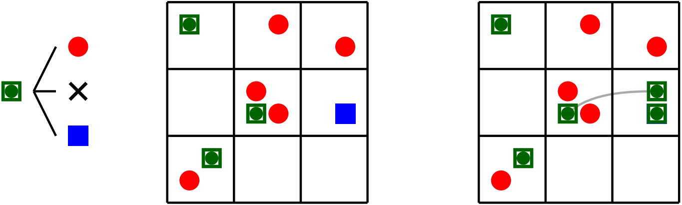

Dominate by a supercritical branching process.

$\mathbb{E}[\text{green}] = e^{(2+\lambda)t}$

Re-index sums etc, Borel-Cantelli...