Activated Random Walks on $\mathbb{Z}^d$

Lectures 5 and 6

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THU-PKU-BNU joint probability webinar – July 2020

Tentative plan

Tentative plan

Recall of Lectures 1–4

Exploring instructions in advance [§4]

Activity without transience [§5]

Particle-wise construction [§10] (if time allows)

Recall

Recall

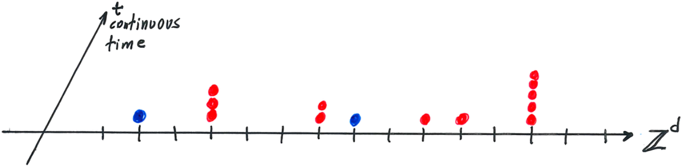
Definition of the dynamics

Phase diagram

Odometer and toppling procedures

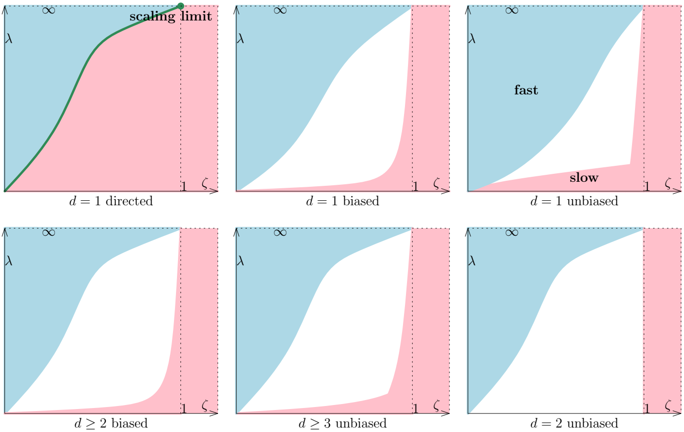
First applications

Definition of the dynamics [§1.1]

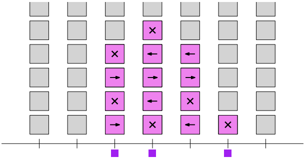


	rate	
A → S	λ	
A jumps	1	from x to $x+y$ with y chosen randomly
AS → AA	∞	

Phase diagram [§1.5]



Site-wise representation [§2.2]



Set of instructions used and the final positions do not depend on the order at which updates are performed

So we can use whatever order is most convenient

Monotonicity [§2.2]

particles added $\Rightarrow$ more instructions used

forcibly activate particles $\Rightarrow$ more instructions used

To prove fixation:

we can add more particles and wake up particles

To prove activity:

we can ignore some particles or leave them behind

Odometer and Abelian property [§2.2]

All deterministic

Finite sequences of topplings α, β

$m_\alpha(x) = \#$times x appears in α

$$m_{V,\eta} := \sup_{\beta \subseteq V \text{ legal}} m_\beta.$$

$$m_{V,\eta} \leqslant m_\alpha \text{ if } \alpha \text{ stabilizes } \eta \text{ in } V$$

$$m_{V,\eta} \uparrow m_\eta$$

Criteria for fixation [§2.3]

Corollary 2.8.

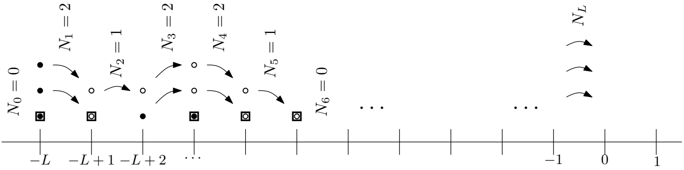
$$\sup_k \inf_V \mathbb{P}[m_V(x) \leqslant k] > 0 \implies \text{Fixation}$$

$$\inf_k \sup_V \mathbb{P}[m_V(x) \geqslant k] > 0 \implies \text{Activity}$$

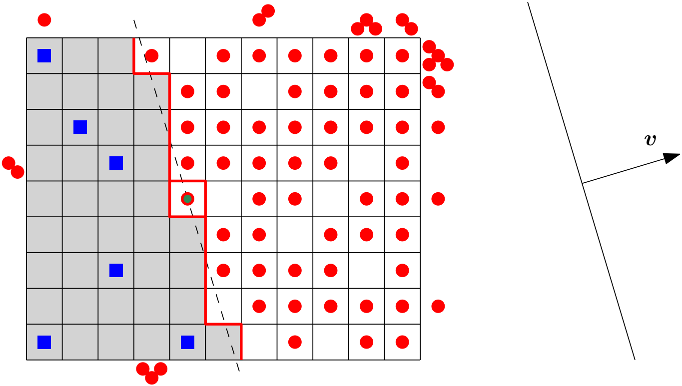
Theorem 2.11.

$$\limsup_n \frac{\mathbb{E} M_n}{|V_n|} > 0 \implies \text{Activity}$$

Counting arguments [§3]



Counting arguments [§3]



§4
Exploring instructions in advance

Phase transition for $d = 1$

Theorem 4.1. For $d = 1$, for every $\lambda > 0$, we have

$$\zeta_c \geqslant \frac{\lambda}{1+\lambda}.$$

(R, Sidoravicius)

Stabilization procedure for $d = 1$

Set up a trap for each particle

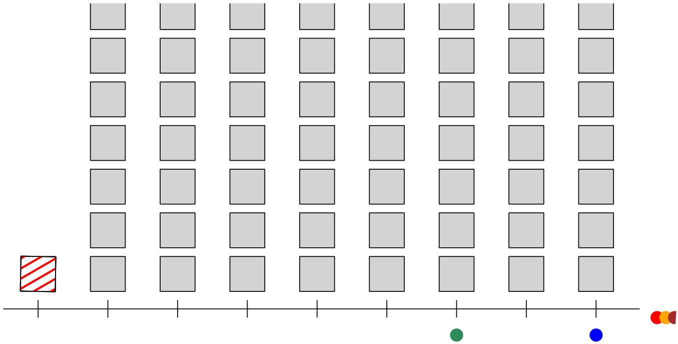
packing the particles to save space

keeping independence

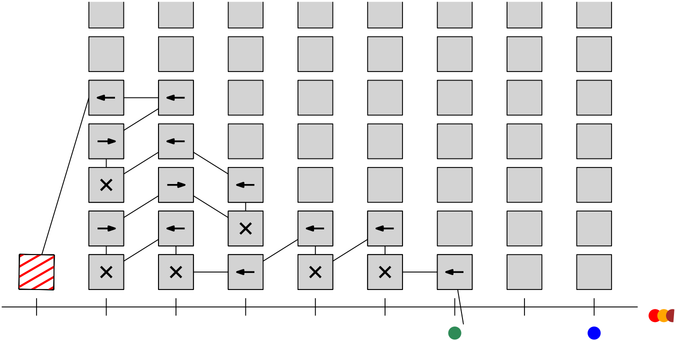
positive probability of success

success $\Rightarrow$ origin never visited

Stabilization procedure for $d = 1$



Stabilization procedure for $d = 1$



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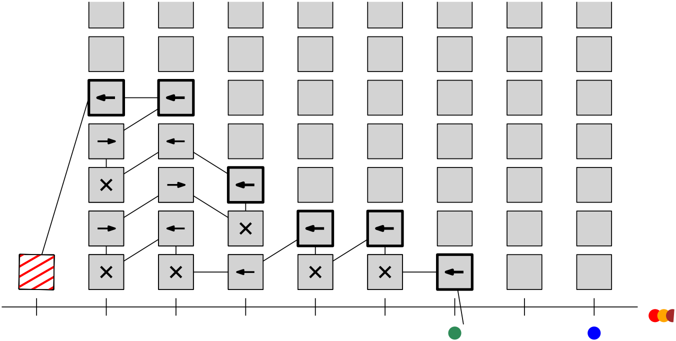

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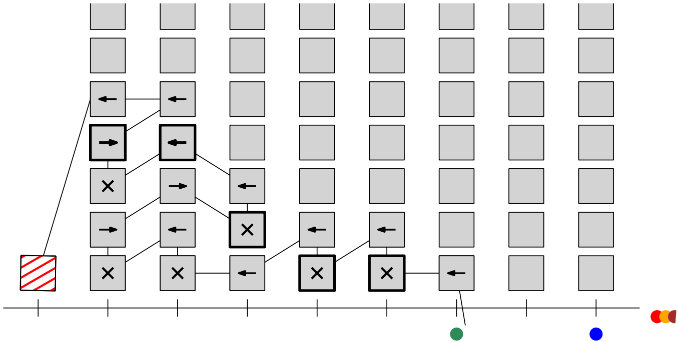

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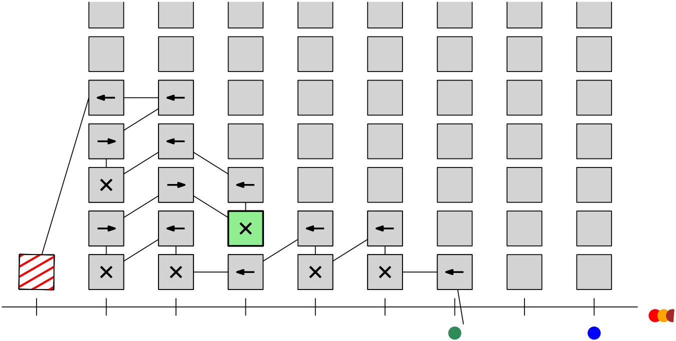

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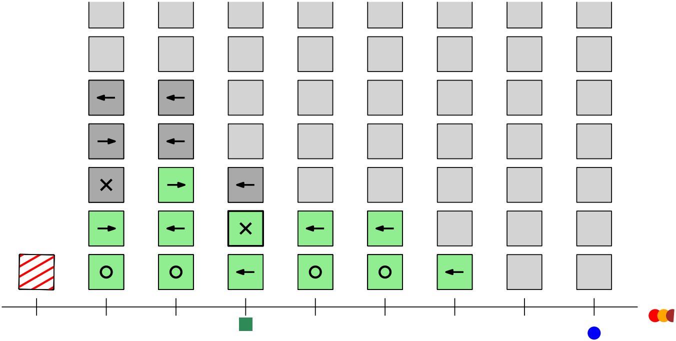

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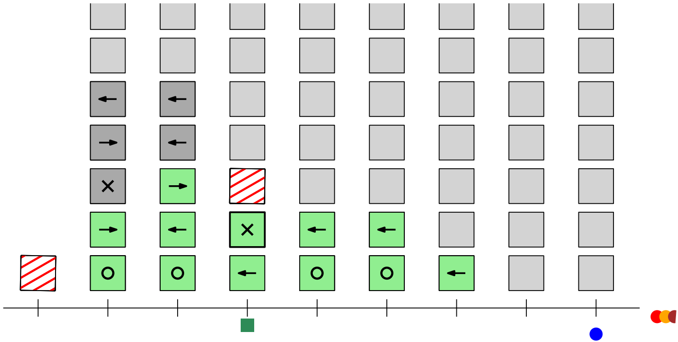

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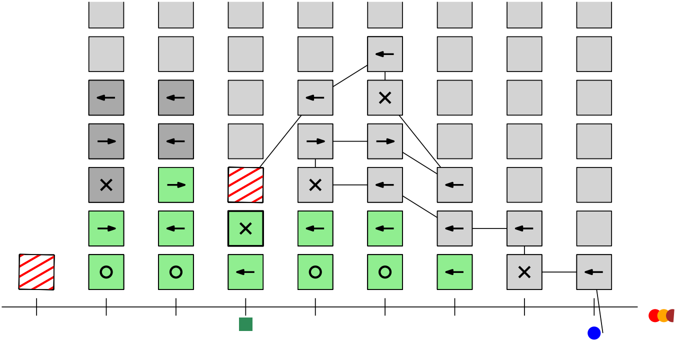
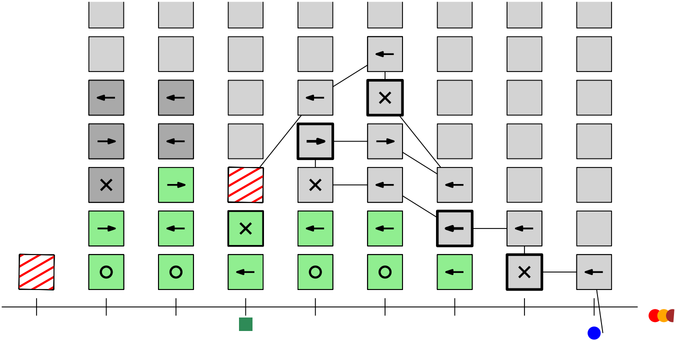


Stabilization procedure for $d = 1$



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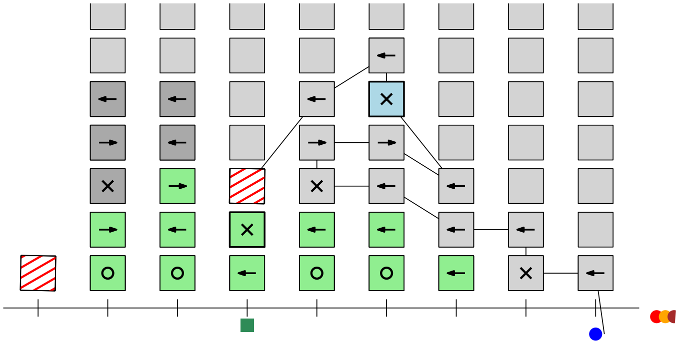

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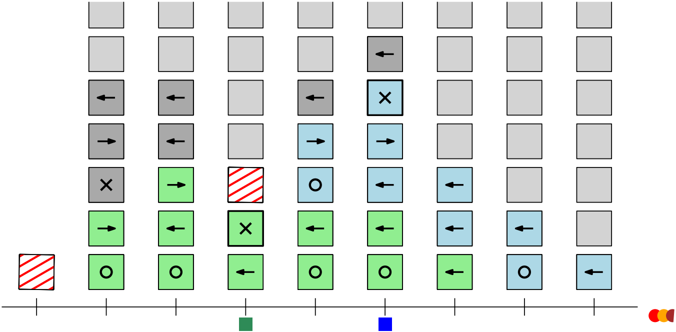

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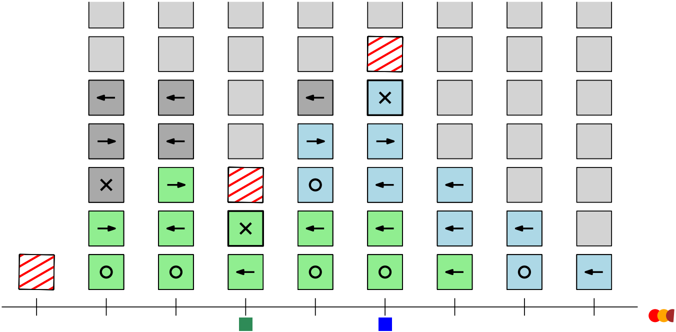

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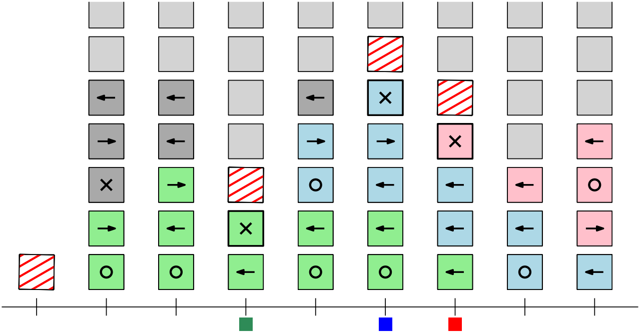
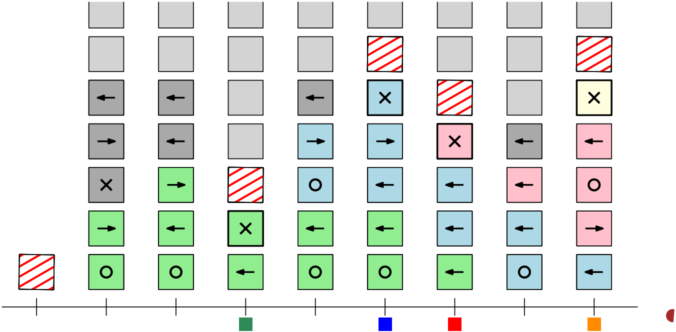


Stabilization procedure for $d = 1$



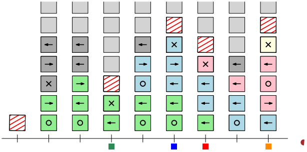
Stabilization procedure for $d = 1$

Average distance between traps: $\frac{1+\lambda}{\lambda}$

Average distance between particles: $\frac{1}{\mu}$

By LLN, if $\mu < \frac{\lambda}{1+\lambda}$ this strategy is successful WPP, thus $\mathbb{P}(\text{origin never visited}) > 0$, and by a 0-1 law, $\mathbb{P}(\text{finitely many visits}) = 1$.

Therefore, $\mu_c \geqslant \frac{\lambda}{1+\lambda}$.



Coffee break

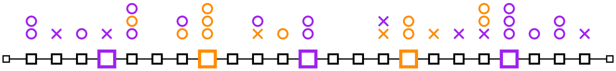
§5
Activity without transience

Activity without transience

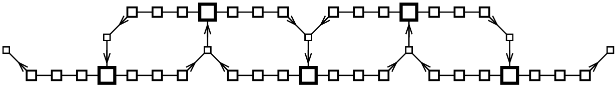
Theorem. For $d = 1$, SSRW, $\zeta > 0$, we have $\lambda_c > 0$.

(Basu, Ganguly, Hoffman)

Toppling procedure



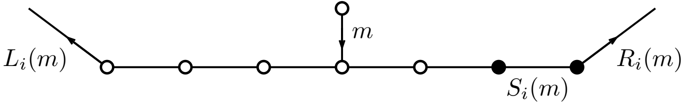
Abelian equivalent with buffers



Analyze the odometer $m_1, \ldots, m_n$ at the buffers

Single-block dynamics

Single-block description: $L_i(\cdot),\ R_i(\cdot),\ S_i(\cdot)$



Mass balance equations: $L_i(m_i) = m_{i-1} - R_{i-2}(m_{i-2})$

Bound using only MBE

Consider exponential moment $\mathbb{E}\left[e^{\sum S_i}\right]$ and MBE to factorize, then sum over $(m_i, \ldots, m_n)$ to decouple:

$$\leqslant \mathbb{E} \sum_{m_1} \cdots \sum_{m_n} \prod_{i=1}^{n} e^{S_i(m_i)} \mathbb{1}_{\{L_i(m_i) = m_{i-1} - R_{i-2}(m_{i-2})\}}$$

Single-block estimate: $\sup_{\ell} \mathbb{E}\Big[\sum_m e^{S_i(m)} \mathbb{1}_{L_i(m)=\ell}\Big] \leqslant 3$

Valid for any block size, provided λ is small enough

Particle-wise construction

Particle-wise construction [§10.1]

Description

Labeled particles $\rightarrow$ mass transport, ergodicity, surgery

Construction: assign to each particle a CTRW+beep

(Amir, Gurel-Gurevich; R, Tournier)

Mass conservation [§10.1]

Theorem 10.1. Assume particles fixate a.s. Then

$$\mathbb{E}[\text{start at } \mathbf{o}] = \mathbb{E}[\text{settle at } \mathbf{o}].$$

Corollary 10.2. We always have $\zeta_c \leqslant 1$.

(Amir, Gurel-Gurevich)

Resampling [§10.3]

Theorem 10.4. Site fixation $\Rightarrow \zeta < 1$.

(Cabezas, R, Sidoravicius)

Definedness [§11.3]

Theorem 10.6.

The particle-wise construction is well-defined

Add particles one by one, updating the whole evolution

$\vdash$ Life of each particle is well-defined through some limit

Main step: $\forall x, T$, the number of particle additions that affect site x by time T has finite expectation