Activated Random Walks on $\mathbb{Z}^d$

Lectures 3 and 4

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THU-PKU-BNU joint probability webinar – July 2020

Tentative plan

Tentative plan

Recall of Lectures 1&2, plus leftovers [§1]

Odometer and toppling procedures [§2]

Counting arguments [§3]

Recall

Recall

ARW is interesting and physically relevant [§1.2]

Definition of the dynamics [§1.1]

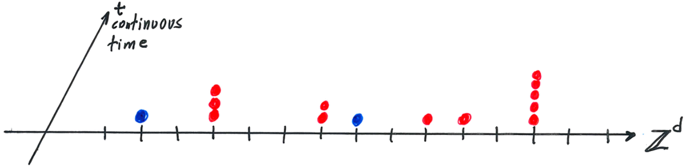
ARW is mathematically challenging [§1.4]

Predictions and results [§1.3,§1.5]

Main tools [§1.6]

Leftover: Open problems [§1.4]

Definition of the dynamics [§1.1]



	rate	
A → S	λ	
A jumps	1	from x to $x+y$ with y chosen randomly
AS → AA	∞	

Constructions

Site-wise construction

stack of instructions at each site

Particle-wise constructions

particles start with a life plan and do pause/resume

Main tools (details later)

Site-wise representation and Abelian Property

$\rightarrow$ preserving monotonicity, etc

$\rightarrow$ fixation-activity reduced to *toppling procedures*

Particle-wise construction (well-defined!):

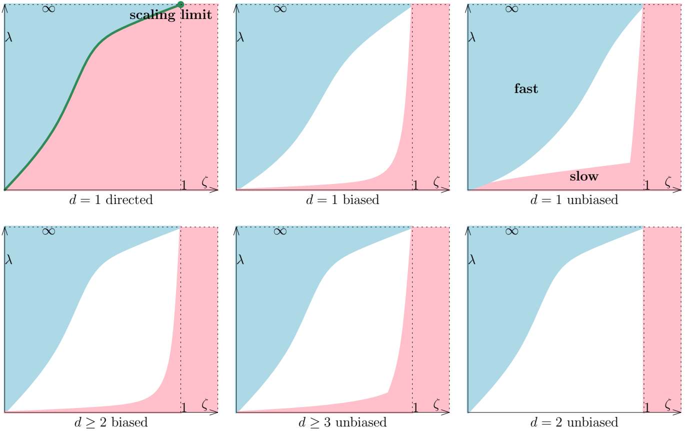
$\rightarrow$ a particle stays active $\Rightarrow$ sites stay active

$\rightarrow$ ergodicity, mass transport, coupling, surgery

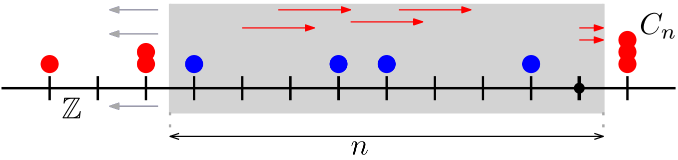
$\rightarrow$ Averaged criterion for activity...

...to be used in the site-wise representation!

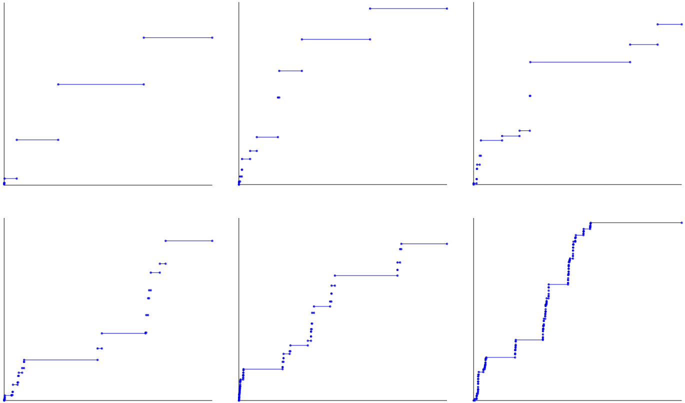
Phase diagram [§1.5]



Critical one-dimensional directed system



(Not in lecture notes)



$\rho = 1.00,\ 0.50,\ 0.30,\ 0.10,\ 0.05,\ 0.00$

Odometer and toppling procedures

Abelian particle-wise construction

$m_V(x) :=$ **odometer** at x when V is stabilized

$$\lim_k \lim_V \mathbb{P}[m_V(x) \geqslant k] = \begin{cases} 0 \Leftrightarrow \text{Fixation} \\ 1 \Leftrightarrow \text{Activity} \end{cases}$$

$M_n :=$ #Particles which quit when B_n is stabilized

$$\limsup_n \frac{\mathbb{E}[M_n]}{|B_n|} > 0 \;\Rightarrow\; \text{Activity}$$

Leftover: **Examples**

$\zeta_c \geqslant \frac{\lambda}{1+\lambda}$ on $d = 1$ — R, Sidoravicius. **Invent. Math.** (2012)

$\zeta_c \leqslant F_p(\lambda)$ — Taggi. **Electron. J. Probab.** (2016); R, Tournier. **Ann. Inst. H. Poincaré Probab. Statist.** (2018)

$\zeta_c > 0$ — Sidoravicius, Teixeira. **Electron. J. Probab.** (2017)

$\zeta_c < 1$ for 1d-SRW — Basu, Ganguly, Hoffman. **Comm. Math. Phys.** (2018)

$\zeta_c \geqslant \frac{\lambda}{1+\lambda}$ — Stauffer, Taggi. **Ann. Probab.** (2018)

$\zeta_c \asymp \sqrt{\lambda}$ for 1d-SRW — Asselah, R, Schapira. *arXiv:1907.12694*

Leftover: **Open problems** [§1.4]

Unbiased walks on $\mathbb{Z}^2$: $\zeta_c < 1$ for some $\lambda < \infty$

Dichotomy for the slow-fast transition on the 1D torus

Sharpness when a fraction of particles start active

Show that ζ_c is strictly increasing in λ

Proofs of fixation/activity that give different insights

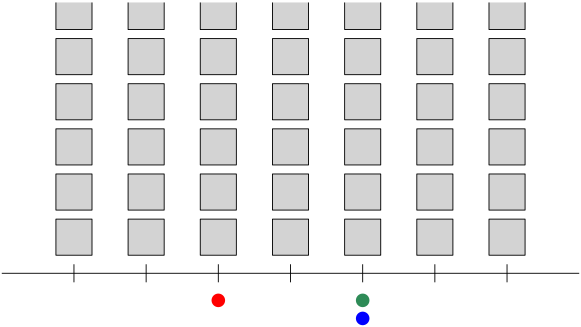
Make sense of scaling limit at criticality for $d \geqslant 2$

Many more...

§2
Odometer and toppling procedures

(painful)

Site-wise representation



Colors here are only to distinguish particles. Active is round, sleepy is square

BGB RRR BBGB GGGR

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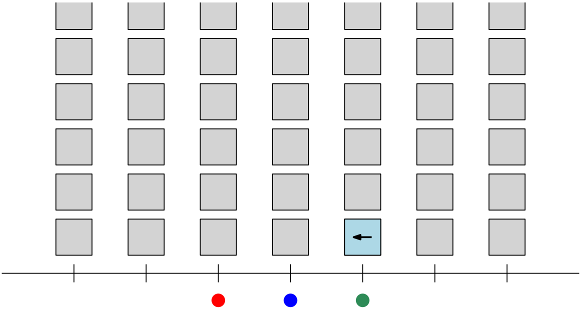

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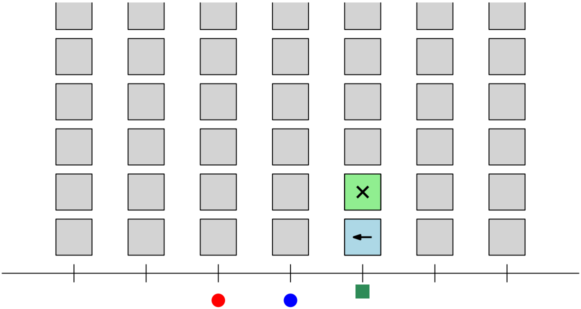

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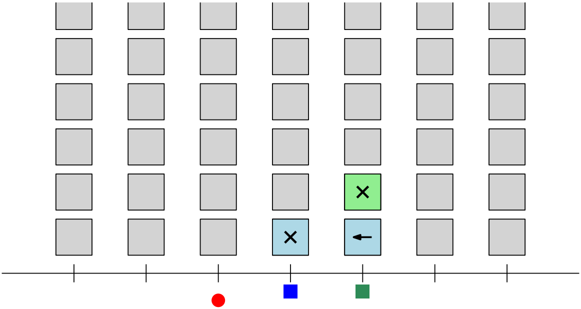

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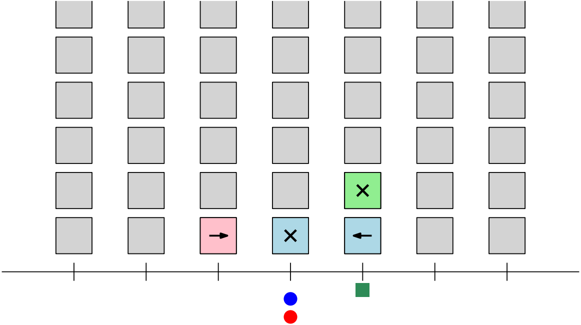

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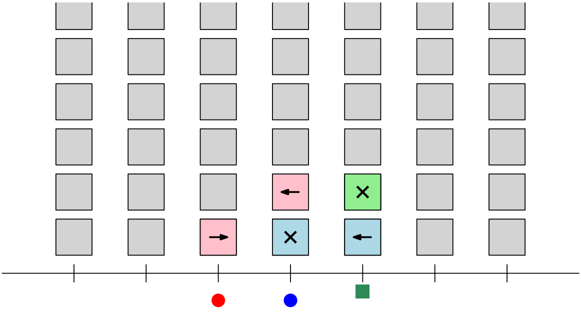

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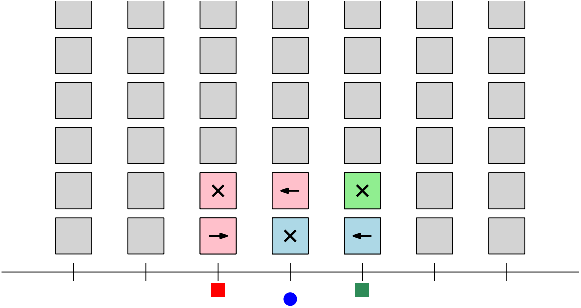

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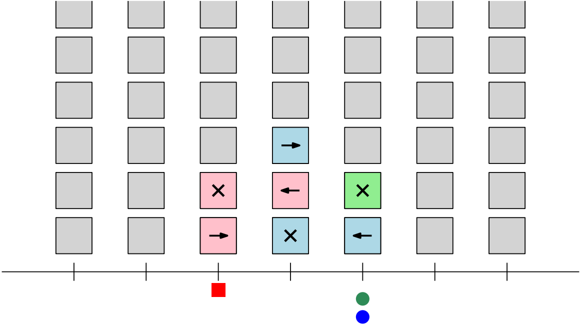

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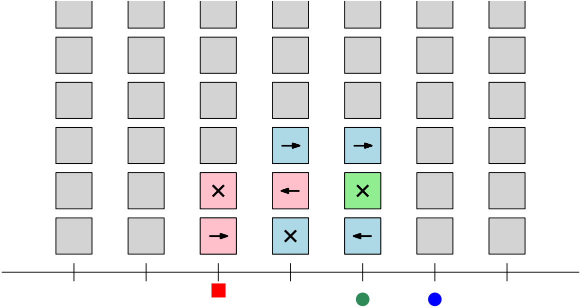

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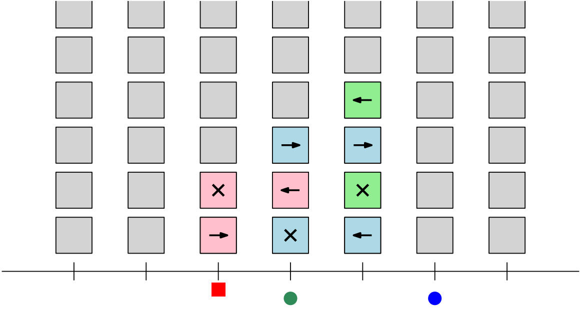

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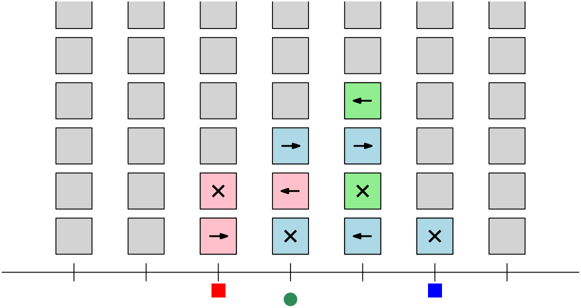

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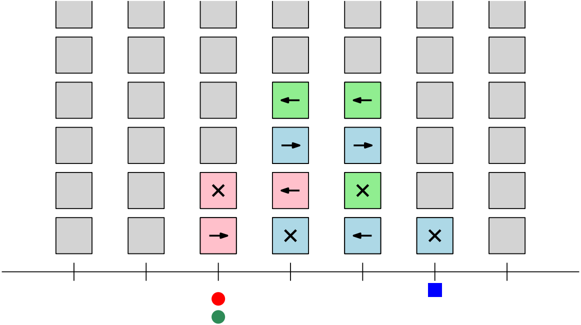

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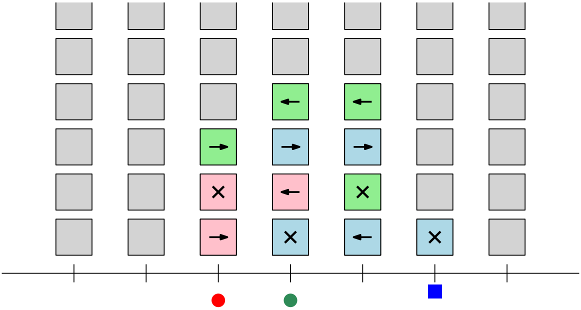

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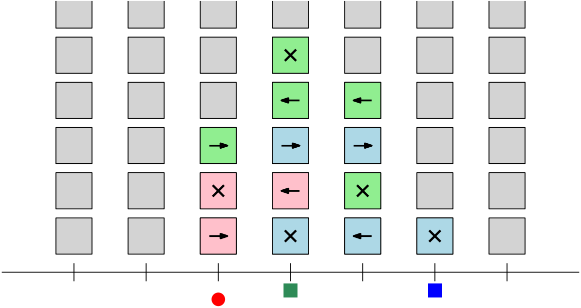

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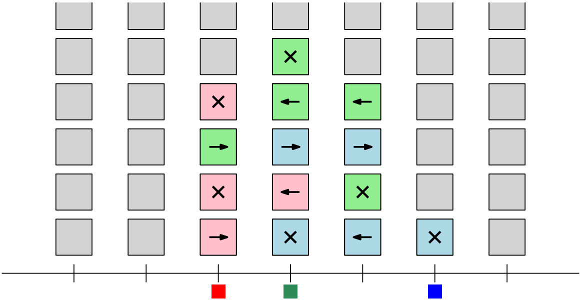

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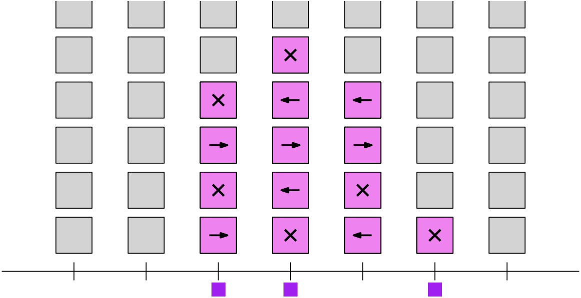

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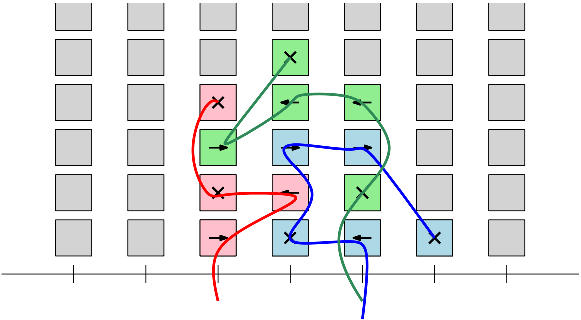

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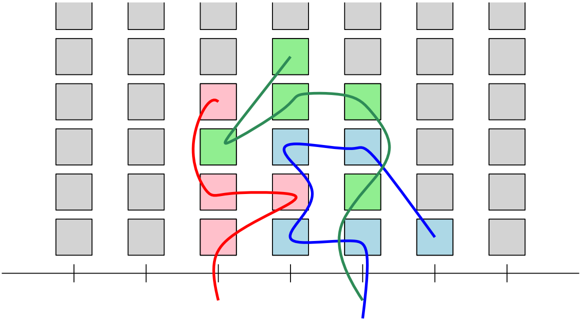

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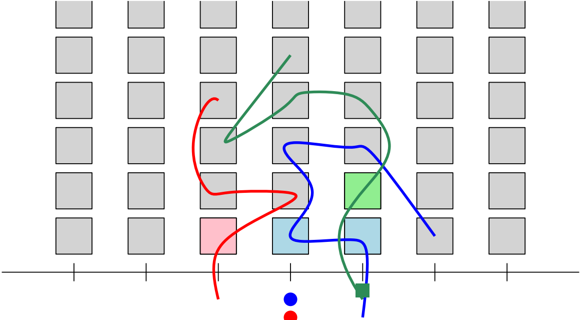

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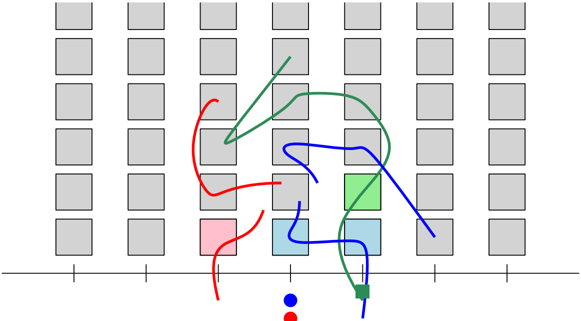

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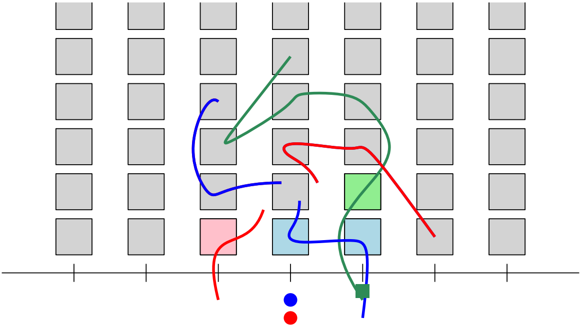

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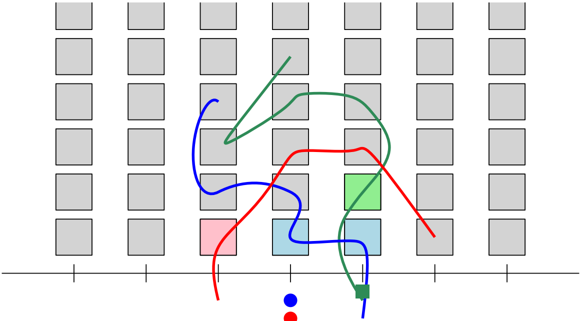

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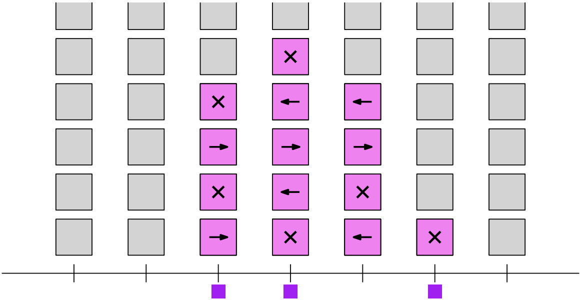

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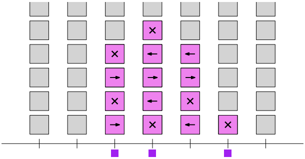


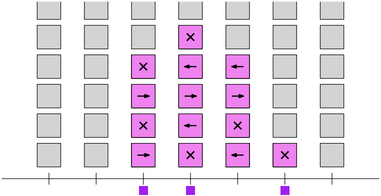
Set of instructions used and the final positions do not depend on the order at which updates are performed

So we can use whatever order is most convenient

Monotonicity

particles added ⇒ more instructions used

forcibly activate particles ⇒ more instructions used



Monotonicity

particles added $\Rightarrow$ more instructions used

forcibly activate particles $\Rightarrow$ more instructions used

To prove fixation:

we can add more particles and wake up particles

To prove activity:

we can ignore some particles or leave them behind

Precise definition [§2.1]

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\mathbb{N}_{\mathfrak{s}} = \mathbb{N}_0 \cup \{\mathfrak{s}\}$.

$\eta_t \in (\mathbb{N}_{\mathfrak{s}})^{\mathbb{Z}^d}$.

$0 < \mathfrak{s} < 1 < 2 < \cdots$

$\mathfrak{s} + 1 = 2$

$1 \cdot \mathfrak{s} = \mathfrak{s}$ and $n \cdot \mathfrak{s} = n$ for $n \geqslant 2$

Precise definition [§2.1]

$\mathbf{P}^{\nu}$ is the law of $(\eta_t)_{t\geq 0}$ subject to transitions:

$$\tau_{x\mathfrak{s}}\eta(z) = \begin{cases} \eta(x)\cdot\mathfrak{s}, & z = x, \\ \eta(z), & z \neq x, \end{cases}$$

$$\tau_{xy}\eta(z) = \begin{cases} \eta(x)-1, & z = x, \\ \eta(y)+1, & z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$$

Odometer and Abelian property [§2.2]

field of instructions $(\tau^{x,j})_{x\in\mathbb{Z}^d, j\in\mathbb{N}}$

odometer field $h = \big(h(x); x \in \mathbb{Z}^d\big)$

x is *unstable for the configuration* η if $\eta(x) \geqslant 1$

toppling operation at x is defined by
$\Phi_x(\eta, h) = \big(\tau^{x,h(x)+1}\eta, h + \delta_x\big)$.

Φ_x is *legal for* (η, h) if $\eta(x) \geqslant 1$
(acceptable if $\eta(x) \geq \mathfrak{s}$)

$\alpha = (x_1, \ldots, x_k) \rightsquigarrow \Phi_\alpha = \Phi_{x_k}\Phi_{x_{k-1}} \cdots \Phi_{x_1}$

Odometer and Abelian property [§2.2]

Lemma 2.1 If α is an acceptable sequence of topplings that stabilizes η in V, and $\beta \subseteq V$ is a legal sequence of topplings for η, then $m_\beta \leqslant m_\alpha$.

$$m_{V,\eta} := \sup_{\beta \subseteq V \text{ legal}} m_\beta.$$

$m_{V,\eta} \leqslant m_\alpha$ if α stabilizes η in V

Odometer and Abelian property [§2.2]

Lemma 2.4 If α is legal for η, contained in V, and stabilizes η in V, then $m_\alpha = m_{V,\eta}$.

Lemma 2.5. If $V \subseteq \tilde{V}$ and $\eta \leqslant \tilde{\eta}$, then $m_{V,\eta} \leqslant m_{\tilde{V},\tilde{\eta}}$.

$$m_\eta := \lim_{V \uparrow \mathbb{Z}^d} m_{V,\eta}$$

Discussions

Criteria for fixation [§2.3]

Sample $\tau^{x,j}$ i.i.d. with the correct distribution

Sample initial configuration $\eta_0 \sim \nu$ independently: $\mathbb{P}^\nu$

Theorem 2.7.
$$\mathbf{P}^\nu\big(\text{fixation of } (\eta_t)_{t\geqslant 0}\big) = \mathbb{P}^\nu(m_{\eta_0}(\mathbf{0}) < \infty) = 0 \text{ or } 1.$$

Corollary 2.8.
$$\sup_k \inf_V \mathbb{P}[m_V(x) \leqslant k] > 0 \implies \text{Fixation}$$
$$\inf_k \sup_V \mathbb{P}[m_V(x) \geqslant k] > 0 \implies \text{Activity}$$

Criteria for fixation [§2.3]

Theorem 2.11. Let M_n count the number of particles that jump out of $V_n = \{-n, \ldots, n\}^d$ when V_n is stabilized via legal topplings, so particles are ignored after leaving V_n. If η_0 is i.i.d. and the condition

$$\limsup_n \frac{\mathbb{E} M_n}{|V_n|} > 0$$

is satisfied, then the system a.s. stays active.

Criteria for fixation [§2.3]

Theorem 2.13. Given the dimension d, sleep rate λ, and jump distribution $p(\cdot)$, there is a number ζ_c such that, for every translation-ergodic distribution ν supported on $(\mathbb{N}_0)^{\mathbb{Z}^d}$ with average density ζ, the ARW dynamics satisfies

$$\mathbf{P}^\nu(\text{system stays active}) = \begin{cases} 0, & \zeta < \zeta_c, \\ 1, & \zeta > \zeta_c. \end{cases}$$

§3
Counting arguments

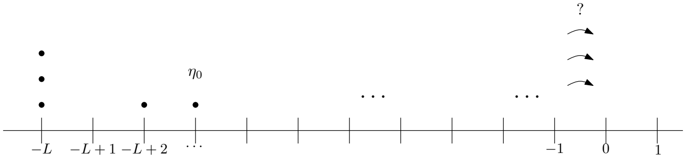
Warm up

Theorem 3.1. For $d = 1$ and initial state i.i.d. with mean $\zeta = 1$ and positive variance, the system a.s. stays active.

The bulldozer

Theorem 3.2. For $d = 1$ and directed walks, $\zeta_c = \frac{\lambda}{1+\lambda}$. If the initial state is i.i.d. with critical density $\zeta = \zeta_c$ and positive variance, then the system a.s. stays active.

The bulldozer



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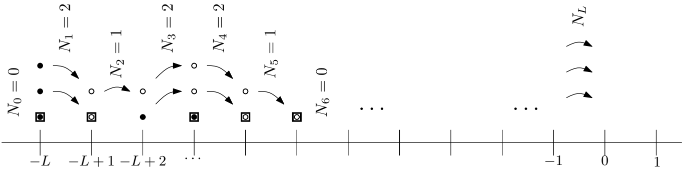
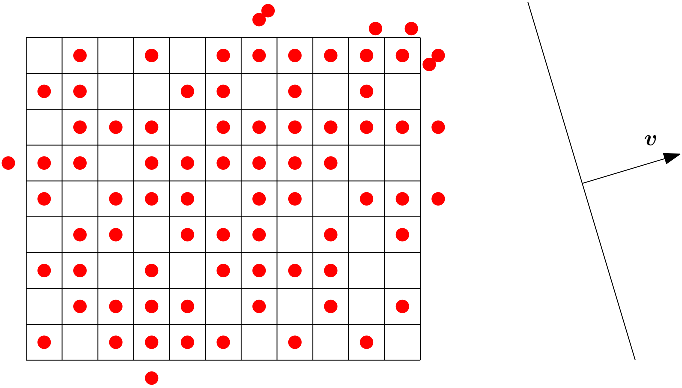


The herder

Theorem 3.3. For $d \geq 1$ and biased walks, $\zeta_c < 1$ for every $\lambda < \infty$ and $\zeta_c \to 0$ as $\lambda \to 0$.

The herder



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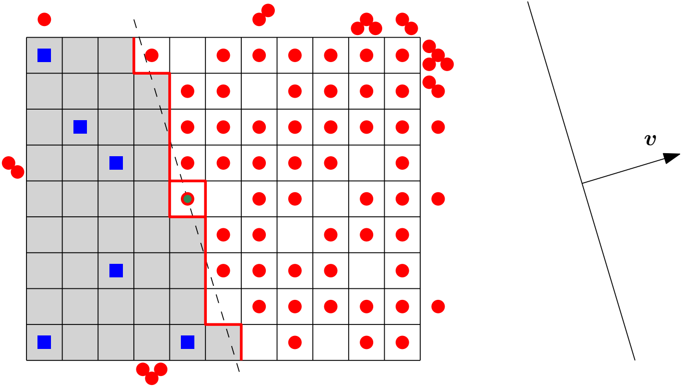


§4
Exploring instructions in advance

Phase transition for $d = 1$

Theorem 4.1. For $d = 1$, for every $\lambda > 0$, we have

$$\zeta_c \geqslant \frac{\lambda}{1+\lambda}.$$

Stabilization strategy for $d = 1$

Set up a trap for each particle

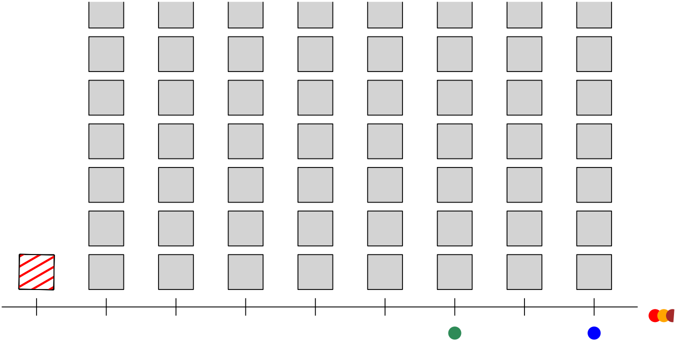
packing the particles to save space

keeping independence

positive probability of success

success $\Rightarrow$ origin never visited

Stabilization strategy for $d = 1$



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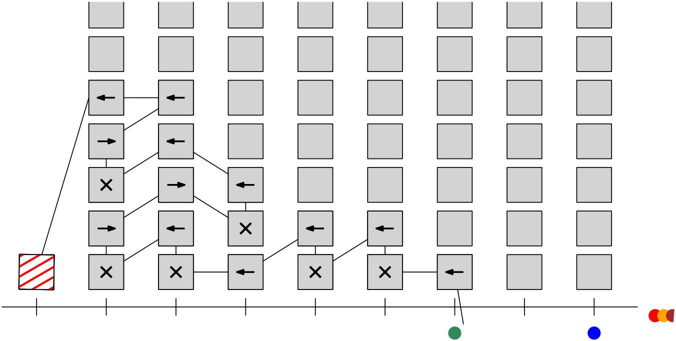

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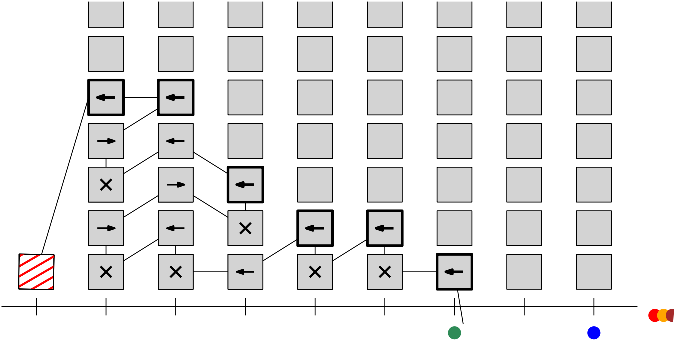

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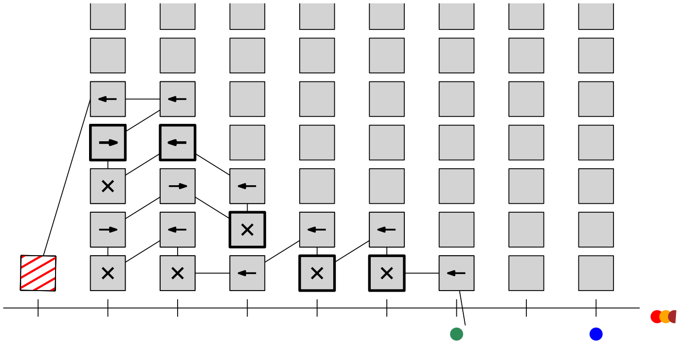

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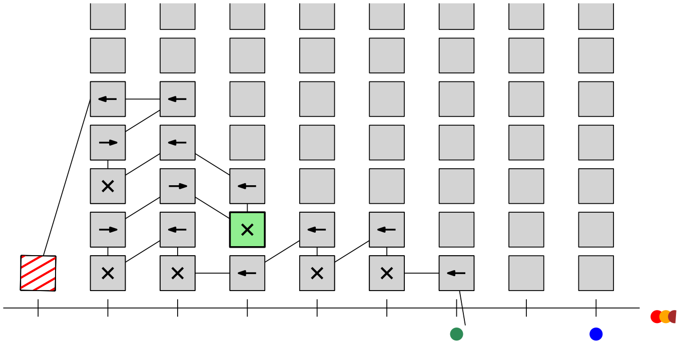

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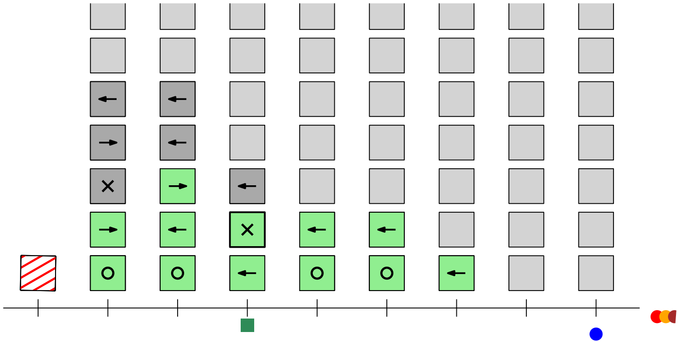

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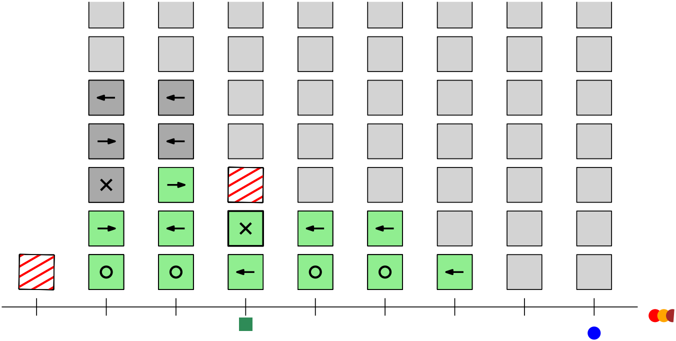

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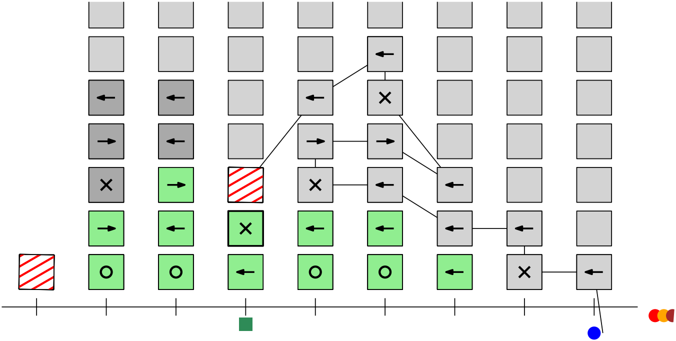

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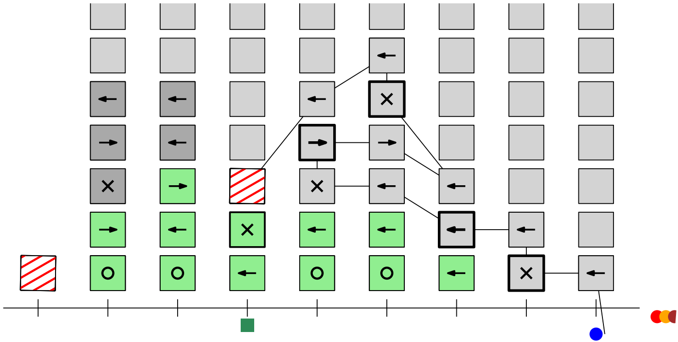

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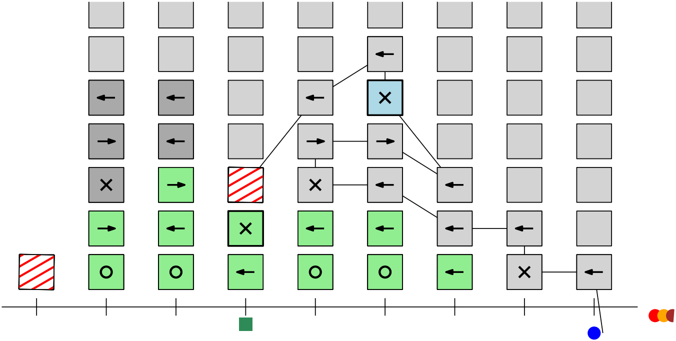

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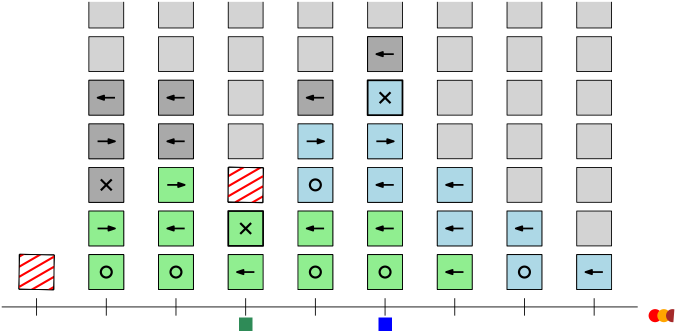
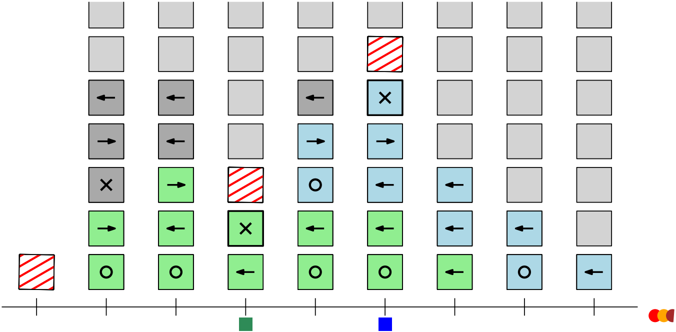
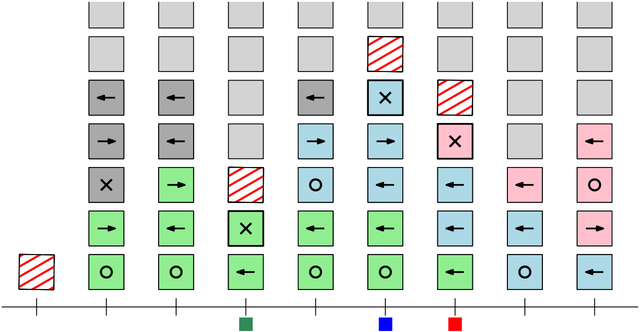


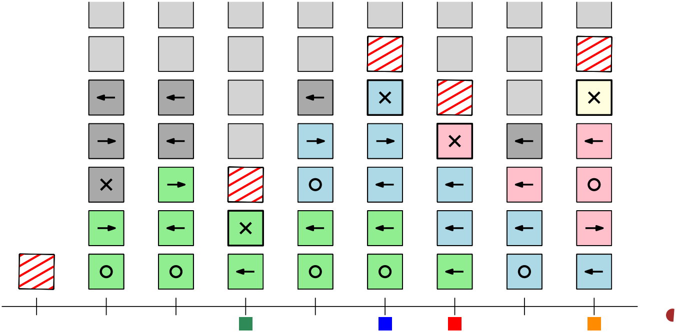
Stabilization strategy for $d = 1$



Stabilization strategy for $d = 1$



Stabilization strategy for $d = 1$



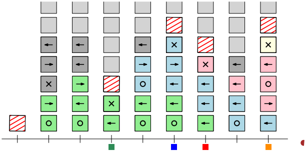
Stabilization strategy for $d = 1$

Average distance between traps: $\frac{1+\lambda}{\lambda}$

Average distance between particles: $\frac{1}{\mu}$

By LLN, if $\mu < \frac{\lambda}{1+\lambda}$ this strategy is successful WPP, thus $\mathbb{P}$(origin never visited) > 0, and by a 0-1 law, $\mathbb{P}$(finitely many visits) $= 1$.

Therefore, $\mu_c \geqslant \frac{\lambda}{1+\lambda}$.



Discussions