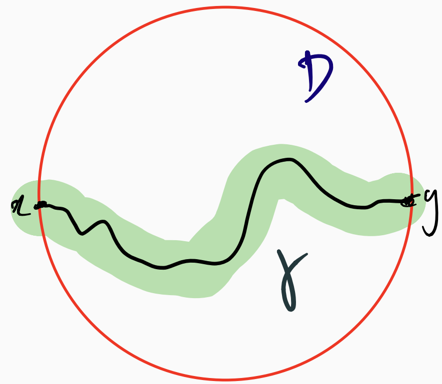
Interplay between Loewner, $\overbrace{}^{\text{Loewner-Kufarev,}}$ and Dirichlet energies:

conformal welding & flow-lines (joint with F. Viklund, KTH)
& foliation by Weil-Petersson quasicircles

Random conformal geometry through the lens of LDP rate functions

SLE_{0+} & Loewner energy



From Lecture 2

For chordal SLE

$$-\lim_{\varepsilon\to 0}\underbrace{\lim_{\kappa\to 0}\kappa\log\mathbb{P}(SLE_\kappa\in O_\varepsilon(\gamma))}_{-\inf_{\tilde\gamma\in O_\varepsilon(\gamma)} I_D(\tilde\gamma)} = I_D(\gamma)$$

Heuristically, $\mathbb{P}(SLE_\kappa$ stays close to $\gamma) \sim \exp\left(-\dfrac{I_D(\gamma)}{\kappa}\right)$

as $\kappa \to 0+$.

Dirichlet energy & GFF

- Similarly, the **Dirichlet energy** of functions φ defined on $D \subset \mathbb{C}$ is the Cameron-Martin norm/large deviation rate function of (a small parameter γ times) the **Gaussian free field** (GFF). $\in H^{-\varepsilon}(D)$
  $(\sqrt{\varepsilon}\,GFF)_{\varepsilon>0}$

$$\text{“}\mathrm{P}(\sqrt{\kappa}\mathit{GFF} \text{ stays close to } 2\varphi) \approx e^{-\mathcal{D}_D(\varphi)/\kappa}, \quad \text{as } \kappa \to 0.\text{”}$$

where $\mathcal{D}_D(\varphi) = \frac{1}{\pi}\int_D |\nabla\varphi(z)|^2 \, |dz|^2$

is the Dirichlet energy of φ in D.

We write $\varphi \in \mathcal{E}(D)$ if $\mathcal{D}_D(\varphi) < \infty$.

Motivation: SLE/GFF coupling

- This lecture: there is a nice interplay between Loewner energy and Dirichlet energy of functions which is reminiscent to SLE/GFF couplings pioneered by Sheffield and Dubédat.
- Our results and proofs are purely analytic (and very short).

Theorem [W. 2019]

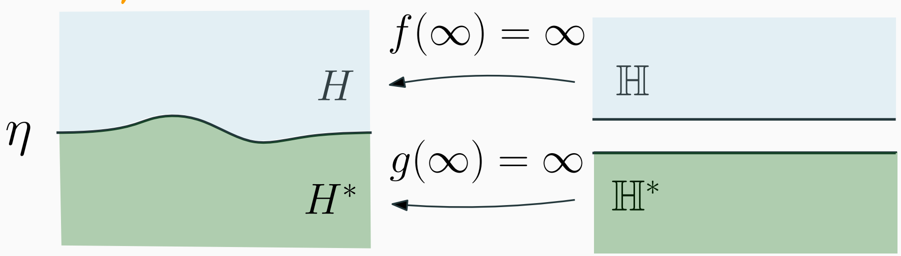
If η passes through ∞, we have the identity

loop

$$I^L(\eta) = \mathcal{D}_{\mathbb{H}}(\log|f'|) + \mathcal{D}_{\mathbb{H}^*}(\log|g'|).$$

Dynamic

static



- $I^L(\eta) < \infty$ iff η is a Weil-Petersson quasicircle
- $I^L(\eta) = 0$ iff η is a circle

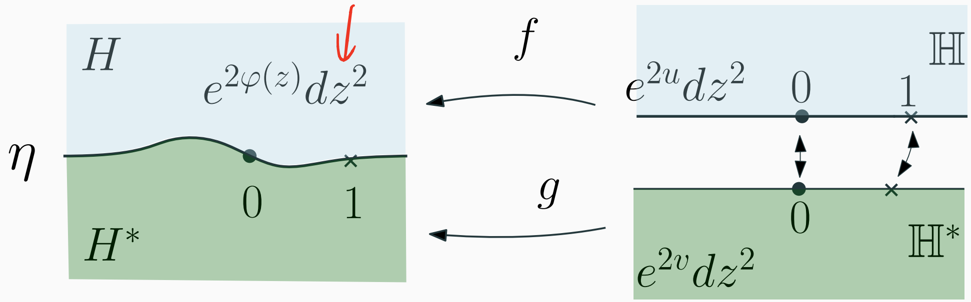
Cutting and Welding identity

Cutting and welding identity

Real-valued

Let $\varphi \in \mathcal{E}(\mathbb{C}) \subset W^{1,2}_{loc}(\mathbb{C}) \subset VMO(\mathbb{C})$, f, g conformal maps from $\mathbb{H}, \mathbb{H}^*$ onto H, H^* fixing ∞.

Euclidean area measure

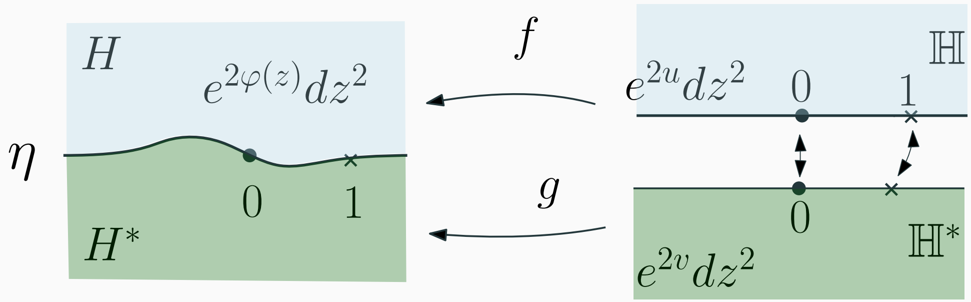


We have $e^{2\varphi} \in L^1_{loc}(\mathbb{C})$ and the transformation law:

$$u(z) = \varphi \circ f(z) + \log|f'(z)|, \quad v(z) = \varphi \circ g(z) + \log|g'(z)|,$$

such that $e^{2u}dz^2 = f^*(e^{2\varphi}dz^2)$, $e^{2v}dz^2 = g^*(e^{2\varphi}dz^2)$.

Cutting and welding identity, cont'd

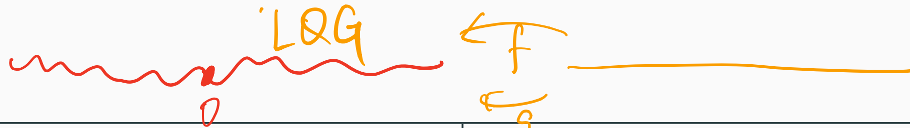


Theorem (cutting)

We have the identity

$$\mathcal{D}_{\mathbb{C}}(\varphi) + I^L(\eta) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v).$$

Large deviation heuristics



SLE/GFF $\gamma := \sqrt{\kappa}$	**Finite energy**
SLE_κ loop.	Finite energy Jordan curve, η.
Free boundary GFF $\gamma\Phi$ on $\mathbb{H}$ (on $\mathbb{C}$).	$2u$, $u \in \mathcal{E}(\mathbb{H})$ $(2\varphi,\ \varphi \in \mathcal{E}(\mathbb{C}))$.
γ-LQG on quantum plane $\approx e^{\gamma\Phi}dz^2$.	$e^{2\varphi}dz^2$, $\varphi \in \mathcal{E}(\mathbb{C})$.
γ-LQG on quantum half-plane on $\mathbb{H}$	$e^{2u}dz^2$, $u \in \mathcal{E}(\mathbb{H})$.
SLE_κ cuts an independent quantum plane $e^{\gamma\Phi}dz^2$ into ind. quantum half-planes $e^{\gamma\Phi_1}, e^{\gamma\Phi_2}$.	Finite energy η cuts $\varphi \in \mathcal{E}(\mathbb{C})$ into $u \in \mathcal{E}(\mathbb{H})$, $v \in \mathcal{E}(\mathbb{H}^*)$ and $I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v)$.

SLE/GFF $\Rightarrow$ one may expect that under appropriate topology and for small κ,

$$\text{“}\mathrm{P}(\text{SLE}_\kappa \text{ loop stays close to } \eta,\ \sqrt{\kappa}\Phi \text{ stays close to } 2\varphi)$$
$$= \mathrm{P}(\sqrt{\kappa}\Phi_1 \text{ stays close to } 2u,\ \sqrt{\kappa}\Phi_2 \text{ stays close to } 2v)\text{”}$$

Large deviation heuristics, cont'd

From the large deviation principle and the independence of SLE and Φ, one expects

$$\begin{aligned}
&\lim_{\kappa\to 0} -\kappa \log \mathrm{P}(\mathrm{SLE}_\kappa \text{ stays close to } \eta,\ \sqrt{\kappa}\Phi \text{ stays close to } 2\varphi) \\
= &\lim_{\kappa\to 0} -\kappa \log \mathrm{P}(\mathrm{SLE}_\kappa \text{ stays close to } \eta) + \lim_{\kappa\to 0} -\kappa \log \mathrm{P}(\sqrt{\kappa}\Phi \text{ stays close to } 2\varphi) \\
= &I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi).
\end{aligned}$$

LDP

Similarly, the independence between Φ_1 and Φ_2 gives

$$\begin{aligned}
&\lim_{\kappa\to 0} -\kappa \log \mathrm{P}(\sqrt{\kappa}\Phi_1 \text{ stays close to } 2u,\ \sqrt{\kappa}\Phi_2 \text{ stays close to } 2v) \\
&= \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v).
\end{aligned}$$

$$\implies I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v).$$

Conversely

One expects the density of an independent couple ($\mathrm{SLE}_\kappa, \sqrt{\kappa}\,\mathrm{GFF}$) has density

$$\begin{aligned}\rho(\eta, 2\varphi) &\propto \exp(-I^L(\eta)/\kappa)\exp(-\mathcal{D}_{\mathbb{C}}(\varphi)/\kappa)\\ &= \exp(-\mathcal{D}_{\mathbb{H}}(2u)/\kappa)\exp(-\mathcal{D}_{\mathbb{H}^*}(2v)/\kappa)\end{aligned}$$

the identity on the action functional also suggests the SLE/GFF coupling.

Converse operation: Isometric welding

Now let $u \in \mathcal{E}(\mathbb{H})$, $v \in \mathcal{E}(\mathbb{H}^*)$. The traces of $u, v \in H^{1/2}(\mathbb{R}) \subset VMO(\mathbb{R})$.

$$\|u\|^2_{H^{1/2}(\mathbb{R})} = \frac{1}{2\pi^2} \iint_{\mathbb{R}\times\mathbb{R}} \frac{|u(z)-u(w)|^2}{|z-w|^2} |dz||dw|$$

We have $e^u, e^v \in L^1_{loc}(\mathbb{R})$ defines two boundary measures $\mu(dx) = e^u dx, \nu(dx) = e^v dx$.

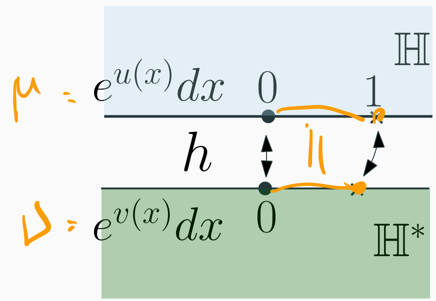
A lemma

$$h : \mathbb{R} = \partial\mathbb{H} \longrightarrow \partial\mathbb{H}^* = \mathbb{R}$$

Lemma

We define $h(0) = 0$, *and* $h(x) :=$

$$\begin{cases} \inf\{y \geq 0 : \mu[0,x] = \nu[0,y]\} & \text{if } x > 0; \\ -\inf\{y \geq 0 : \mu[x,0] = \nu[-y,0]\} & \text{if } x < 0. \end{cases}$$

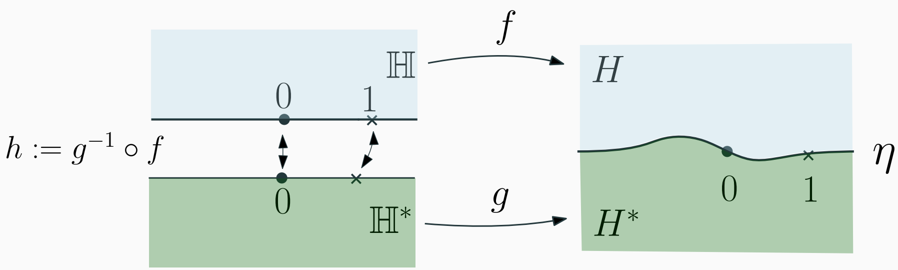


Then h *is a quasisymmetric homeomorphism.*
Moreover, $\log h' \in H^{1/2}(\mathbb{R})$.

Welding problem

We say that the triple (η, f, g) is a **normalized solution to the conformal welding problem** for h if

- η is Jordan curve in $\hat{\mathbb{C}}$ passing through $0, 1, \infty$;
- $f : \mathbb{H} \to H$ is the conformal map fixing $0, 1, \infty$;
- $g : \mathbb{H}^* \to H^*$ is conformal and $g^{-1} \circ f = h$ on $\mathbb{R}$,



Theorem (Shen-Tang-Wu '18)

η is Weil-Petersson quasicircle if and only if $\log h' \in H^{1/2}(\mathbb{R})$.

Why isometric welding: converse of cutting

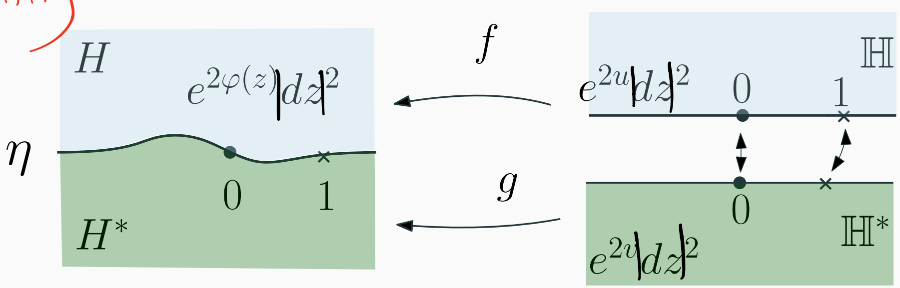
Corollary

There exists a unique normalized solution (η, f, g) to the welding homeomorphism induced by e^u and e^v, and the curve obtained has finite Loewner energy.

Moreover, φ defined from the **transformation law** *is in $\mathcal{E}(\mathbb{C})$, therefore the welding identity holds:*

Cutting · welding identity

$$I^L(\eta) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v) - \mathcal{D}_{\mathbb{C}}(\varphi).$$

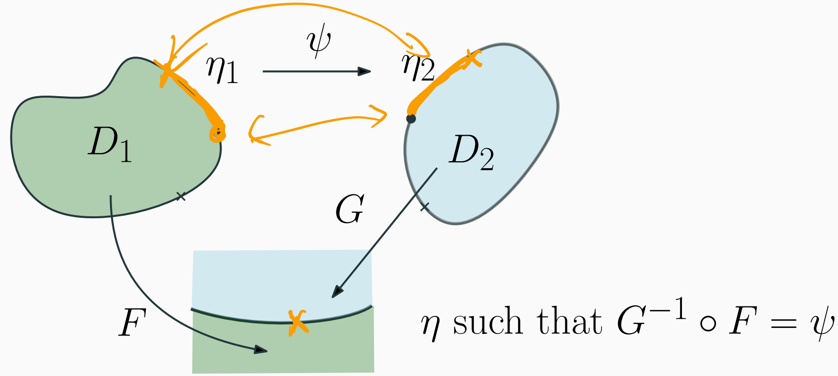


Key: Trace theorem & Sobolev extension theorem for domain bounded by chord-arc curves [Jonsson-Wallin].

Application: arclength conformal welding

Assume η_1, η_2 are rectifiable Jordan curves and $|\eta_1| = |\eta_2|$.

$\psi : \eta_1 \to \eta_2$ preserves arclength.



- *[Huber 1976]* The solution does not always exist.
- *[Bishop 1990]* If the solution exists, η can be a curve of positive area and the solution is not unique. ratio bounded
- *[David 1982, Zinsmeister 1982, Jerison-Kenig 1982]* If η_1 and η_2 are **chord-arc**, then the solution exists and is unique, and is ~~an~~ a quasicircle.
- *[Bishop 1990]* But the Hausdorff dimension of η can take any value in $1 < d < 2 \implies$ not rectifiable.
- Cutting-Welding id. The class of finite energy curves is **closed** under arclength welding.

How does the energy change under the arclength welding operation?

$$I^L(\eta) \quad ?? \quad I^L(\eta_1) + I^L(\eta_2)$$

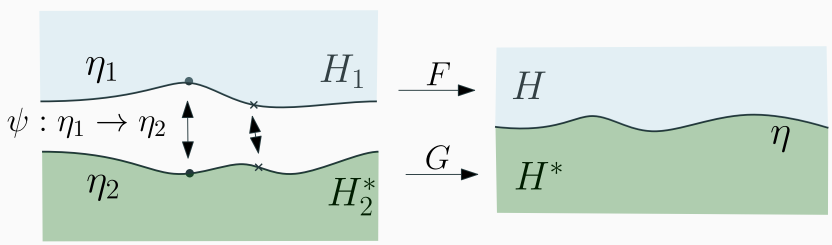
Arclength welding of finite energy domains

Assume $I^L(\eta_1) < \infty, I^L(\eta_2) < \infty$, both passing through ∞. Let H_i, H_i^* be the two connected components of $\mathbb{C} \setminus \eta_i$.

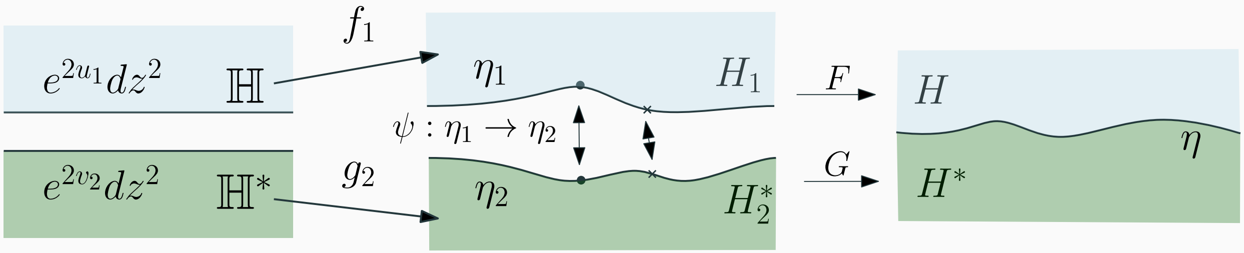
Corollary (sub-additivity)

Let η (resp. $\tilde{\eta}$) be the arclength welding curve of the domains H_1 and H_2^* (resp. H_2 and H_1^*). Then η and $\tilde{\eta}$ have finite energy. Moreover,

$$I^L(\eta) + I^L(\tilde{\eta}) \le I^L(\eta_1) + I^L(\eta_2).$$



Proof of the sub-additivity



In fact, let $u_i = \log |f_i'|$, $v_i = \log |g_i'|$. From the definition of the Loewner energy,

$$I^L(\eta_i) = \mathcal{D}_{\mathbb{H}}(u_i) + \mathcal{D}_{\mathbb{H}^*}(v_i).$$

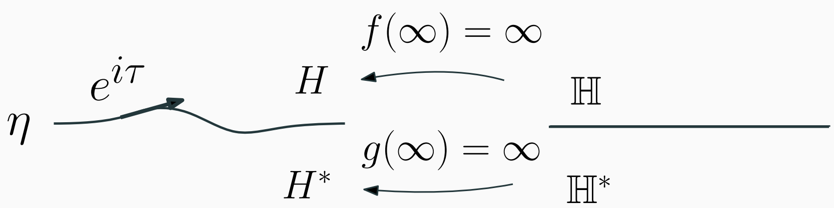
Arclength welding implies that η is the welding curve obtained the isometric welding of e^{u_1} and e^{v_2} and $\tilde{\eta}$ is the isometric welding of e^{u_2} and e^{v_1}. Then, from the welding identity,

$$\begin{aligned} I^L(\eta) + I^L(\tilde{\eta}) &\le \mathcal{D}_{\mathbb{H}}(u_1) + \mathcal{D}_{\mathbb{H}^*}(v_2) + \mathcal{D}_{\mathbb{H}}(u_2) + \mathcal{D}_{\mathbb{H}^*}(v_1) \\ &= I^L(\eta_1) + I^L(\eta_2). \quad \square \end{aligned}$$

Flow-line identity

Winding identity

Assume η is rectifiable.



We denote by

Harmonic extension

$$\mathcal{P}[\tau](z) = \begin{cases} \arg f'(f^{-1}(z)) & z \in H; \\ \arg g'(g^{-1}(z)) & z \in H^* \end{cases}$$

which is the Poisson integral of τ in $\mathbb{C}$.

Flow-line identity

$\operatorname{Im} \log f'$

$\operatorname{Re}_\eta \log f'$

Notice that $\arg(f')$ has the same Dirichlet energy as $\log |f'|$. We have the identity

$$I^L(\eta) = \mathcal{D}_{\mathbb{H}}(\arg f') + \mathcal{D}_{\mathbb{H}^*}(\arg g') = \mathcal{D}_{\mathbb{C}}(\mathcal{P}[\tau]).$$

Consequence: $I^L(\eta) < \infty \Leftrightarrow \eta$ is chord-arc and $\tau \in H^{1/2}(\eta)$.

Flow-line identity, cont'd

Corollary (Flow-line identity)

Conversely, if $\varphi \in \mathcal{E}(\mathbb{C}) \cap C^0(\hat{\mathbb{C}})$, then for all $z_0 \in \mathbb{C}$, there is a unique solution to the differential equation

$$\eta'(t) = e^{i\varphi(\eta(t))},\ \forall t \in \mathbb{R} \quad \text{and} \quad \eta(0) = z_0$$

is an infinite arclength parametrized simple curve and

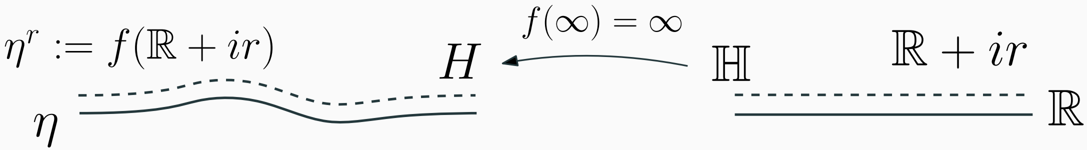
$$\mathcal{D}_{\mathbb{C}}(\varphi) = \underset{\shortparallel}{I^L(\eta)} + \mathcal{D}_{\mathbb{C}}(\varphi_0),$$

where $\varphi_0 = \varphi - \mathcal{P}[\varphi|_\eta]$.

$\mathcal{D}_{\mathbb{C}}(\mathcal{P}(\varphi|_\eta))$

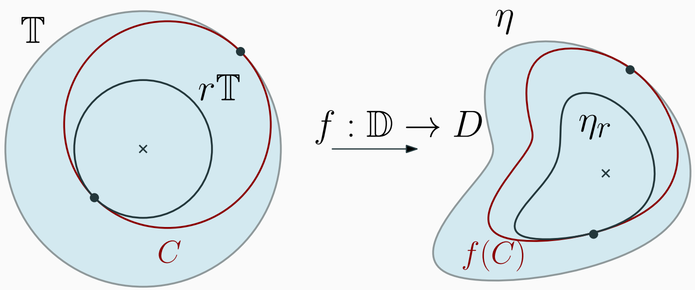
SLE/GFF counterpart (imaginary geometry): The flow-lines of $e^{i\sqrt{\kappa}GFF/2}$ is an SLE$_\kappa$ *curve. Conditioning on the flow-line, φ_0 is an 0-boundary GFF.*

Application: Equipotential energy monotonicity



Corollary [infinite curve]

Let $r > 0$, we have $I^L(\eta^r) \leq I^L(\eta)$.



Corollary [bounded curve]

For $0 < r < 1$, we have $I^L(\eta_r) \leq I^L(f(C)) \leq I^L(\eta)$.

Application: Equipotential energy monotonicity, cont'd

Proposition

The function $r \mapsto I^L(\eta_r)$ (resp. $r \mapsto I^L(\eta^r)$) is continuous and monotone. Moreover,

$$I^L(\eta_r) \xrightarrow{r \to 1-} I^L(\eta); \quad I^L(\eta_r) \xrightarrow{r \to 0+} 0.$$

$$(\text{resp. } I^L(\eta^r) \xrightarrow{r \to 0+} I^L(\eta); \quad I^L(\eta^r) \xrightarrow{r \to \infty} 0.)$$

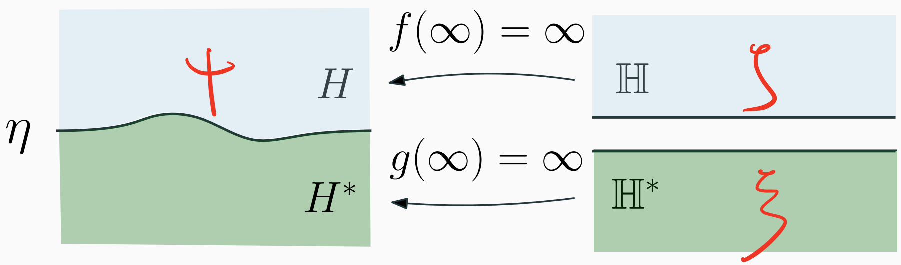
Remark: The vanishing of $I^L(\eta_r)$ as $r \to 0$ can be thought as expressing the fact that conformal maps asymptotically take small circles to circles.

Corollary (Complex identity)

Let ψ be a complex-valued function on $\mathbb{C}$ with finite Dirichlet energy and $\operatorname{Im}\psi \in C^0(\hat{\mathbb{C}})$. Let η be a flow-line of the vector field e^{ψ} and f, g the conformal maps associated to η. Then we have

$$\mathcal{D}_{\mathbb{C}}(\psi) = \mathcal{D}_{\mathbb{H}}(\zeta) + \mathcal{D}_{\mathbb{H}^*}(\xi),$$

where $\zeta = \psi \circ f + \overline{\log f'}$, $\xi = \psi \circ g + \overline{\log g'}$.



Complex function identity, cont'd

It follows from welding and flow-line identities and also implies both identities:

- Taking $\operatorname{Im}\psi = \varphi$ and $\operatorname{Re}(\psi) = 0$
  $\implies$ flow-line identity: $\mathcal{D}_{\mathbb{C}}(\varphi) = I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi_0)$.
- Taking $\operatorname{Re}\psi = \varphi$ and $\operatorname{Im}\psi := \mathcal{P}[\tau]$ where τ is the winding of the curve η
  $\implies$ welding identity: $\mathcal{D}_{\mathbb{C}}(\varphi) + I^L(\eta) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v)$.

SLE/GFF dictionary

A (very loose) dictionary

SLE/GFF with $\gamma = \sqrt{\kappa} \to 0$	**Finite energy**
SLE_κ loop.	Finite energy Jordan curve, η.
Free boundary GFF $\gamma\Phi$ on $\mathbb{H}$ (on $\mathbb{C}$).	$2u$, $u \in \mathcal{E}(\mathbb{H})$ (2φ, $\varphi \in \mathcal{E}(\mathbb{C})$).
γ-LQG on quantum plane $\approx e^{\gamma\Phi} dz^2$.	$e^{2\varphi} dz^2$, $\varphi \in \mathcal{E}(\mathbb{C})$.
γ-LQG on quantum half-plane on $\mathbb{H}$	$e^{2u} dz^2$, $u \in \mathcal{E}(\mathbb{H})$.
γ-LQG boundary measure on $\mathbb{R} \approx e^{\gamma\Phi/2} dx$	$e^{u(x)} dx$, $u \in H^{1/2}(\mathbb{R})$.
SLE_κ cuts an independent quantum plane into independent quantum half-planes.	Finite energy η cuts $\varphi \in \mathcal{E}(\mathbb{C})$ into $u \in \mathcal{E}(\mathbb{H})$, $v \in \mathcal{E}(\mathbb{H}^*)$ and $I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v)$.
Quantum zipper: isometric welding of independent γ-LQG measures on $\mathbb{R}$ produces SLE_κ.	Isometric welding of $e^u dx$ and $e^v dx$, $u, v \in H^{1/2}(\mathbb{R})$ produces a finite energy curve.
γ-LQG chaos w.r.t. Minkowski content equals the pushforward of γ-LQG measures on $\mathbb{R}$.	$e^{\varphi\|_\eta} \|dz\|$, $\varphi\|_\eta \in H^{1/2}(\eta)$, equals the pushforward of $e^u dx$ and $e^v dx$, $u, v \in H^{1/2}(\mathbb{R})$.
Bi-infinite flow-line of $e^{i\Phi/\chi} \approx e^{i\gamma\Phi/2}$ is an SLE_κ loop measurable wrt. Φ.	Bi-infinite flow-line of $e^{i\varphi}$ is a finite energy curve $\mathcal{D}_{\mathbb{C}}(\varphi) = I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi_0)$.
Mating of trees	Complex identity $\Leftrightarrow$ welding+flow-line.

Dirichlet energy

Energy duality

Loewner-Kufarev vs. Loewner

(Work in progress joint w/ Viklund)

SLE duality

$SLE_\kappa \longleftrightarrow SLE_{16/\kappa}$ boundary

Loewner   Loewner-Kufarev

Recall: Loewner-Kufarev equation

$\mathcal{N} := \{ \rho$ Borel measure on $S^1 \times \mathbb{R}$: for interval I,
$\rho(S^1 \times I) = |I| \}$

Disintegration w.r.t t $\rightsquigarrow$ $\rho = \rho_t(d\zeta)\, dt$
where $\rho_t(d\zeta)$ is a probability measure on S^1.
and $t \mapsto \rho_t$ is measurable
$\mathbb{R} \longrightarrow \mathrm{Prob}(S^1)$

$\rho \quad \longleftrightarrow \quad \{\rho_t\}_{t \in \mathbb{R}}$

Whole-Plane Loewner-Kufarev equation.

$z \in \mathbb{C}$

$$(E_z)\quad \begin{cases} \partial_t g_t(z) = -g_t(z)\displaystyle\int_{S^1} \frac{g_t(z)+\zeta}{g_t(z)-\zeta}\,\rho_t(d\zeta) \\[2ex] g_t(z) \sim e^t z \quad \text{as} \quad t \to -\infty \end{cases}$$

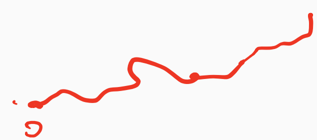
Let $\tau(z)$ solution time of (E_z).

$$D_t := \{ z \in \mathbb{C} \mid \tau(z) > t \} = \mathbb{C} \setminus K_t$$

$\sqsubset \{z \mid \tau(z) \le t\}$

$(D_t)_{t\in\mathbb{R}}$ is a shrinking family of simply connected domain."

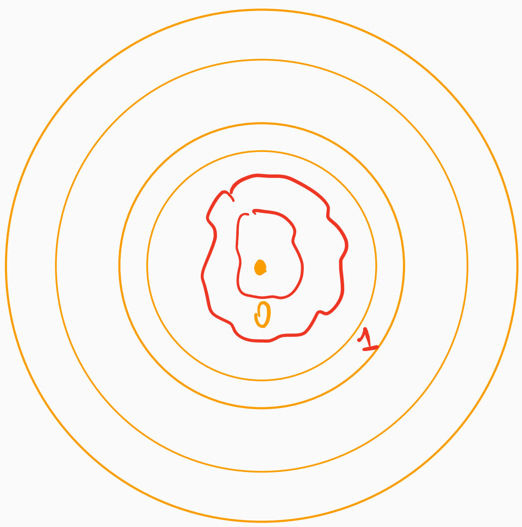
and g_t : conformal $D_t \longrightarrow \mathbb{D}$



Examples

- If $\rho_t = \mathrm{unif}(S^1)$ for all $t \in \mathbb{R}$

  $\Rightarrow D_t = e^{-t}\,\mathbb{D}$ $\rightsquigarrow$ shrinks to $\{0\}$



- If $\rho_t = \mathrm{unif}(S^1)$ for all $t \in \mathbb{R}_-$

  $\Rightarrow D_t = e^{-t}\,\mathbb{D}$ for $t \leq 0$

  $\Rightarrow D_0 = \mathbb{D}$

For $t > 0$ $(D_t)_{t \geq 0}$ is the Loewner chain generated by $(\rho_t)_{t \geq 0}$ in $\mathbb{D}$.

Example cont'd

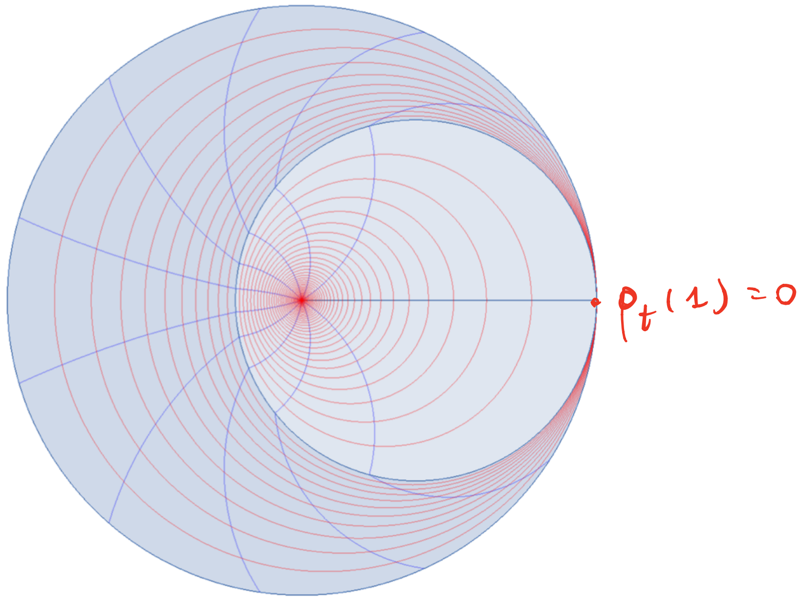


Figure 3.1: The evolution corresponding to the measure that equals $\pi^{-1}\sin^2(\theta/2)\mathrm{d}\theta\mathrm{d}t$ for $0 \leqslant t < 1$ and is uniform for $t \geqslant 1$. The red curves are leaves drawn at equidistant times and the purple lines represent the flow of equidistant points on the unit circle. The winding function is harmonic, but non-zero, in the part foliated after time 1. The Loewner-Kufarev energy of this measure equals 2.

Loewner-Kufarev Energy

For each measure $\mu \in \mathcal{M}_1(S^1)$ we define

$$I^{DV}(\mu) := \frac{1}{2}\int_{S^1} |v'(\theta)|^2 \, d\theta$$

if $d\mu(\theta) = v^2(\theta)\, d\theta$ and $v \in W^{1,2}(S^1)$, ∞ otherwise.

The Loewner-Kufarev energy of $\rho \in \mathcal{N}$

$$S(\rho) := \int_{-\infty}^{\infty} I^{DV}(\rho_t)\, dt$$

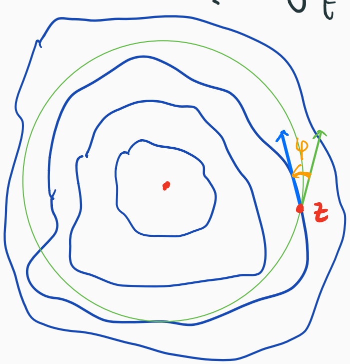
LDP rate function of radial SLE_∞

$$S(\rho) = 0 \quad \text{iff} \quad \rho_t = \text{unif}(S^1) \quad \forall t \text{ a.e.}$$

Winding function

Thm. Let $\rho \in \mathcal{W}$ with $S(\rho) < \infty$, then ρ generates a foliation $(\gamma_t = \partial D_t)_{t\in\mathbb{R}}$ of $\mathbb{C}\setminus\{0\}$ in which every leaf is a Weil-Petersson quasicircle.

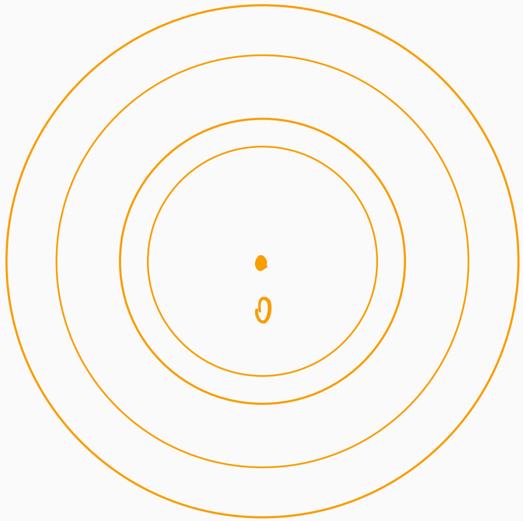
- $t \mapsto \gamma_t$ is continuous
- $\bigcup_{t\in\mathbb{R}} \gamma_t = \mathbb{C}\setminus\{0\}$
- γ_t is differentiable a.e. (Weil-Petersson ok)



Define the winding function $\varphi(z)$ a.e. associated to a foliation.

Energy duality

$S(\rho) = 0 \Rightarrow$



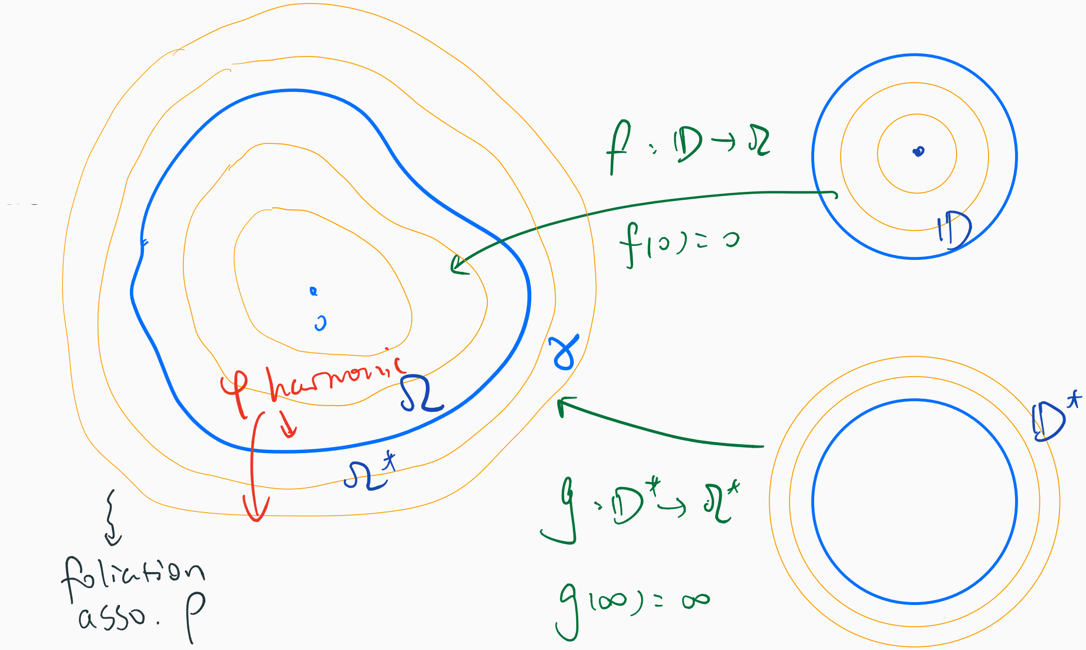
$\Rightarrow \varphi \equiv 0$

Thm. If $S(\rho) < \infty$, then $\varphi \in \mathcal{E}(\mathbb{C})$

and $\boxed{16\, S(\rho) = \mathcal{D}_{\mathbb{C}}(\varphi)}$

$\uparrow$ Dynamic $\qquad \uparrow$ Static

Loewner energy vs Loewner-Kufarev energy



16/k

necessary

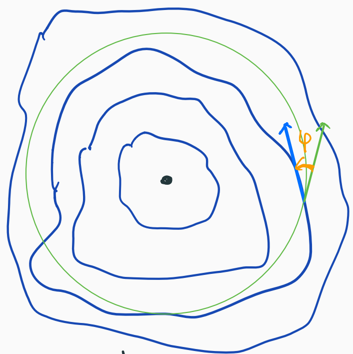
C_{or}

$$I^L(\gamma) = 16 \underbrace{S(\rho)}_{\shortparallel} + \overbrace{2 \log \left| \frac{f'(0)}{g'(\infty)} \right|}$$

$$\inf_{\tilde{\rho} \text{ with } \gamma \text{ a leaf}} S(\tilde{\rho})$$

Reversibility of Loewner-Kufarev energy

$S(\rho)<\infty \rightsquigarrow$



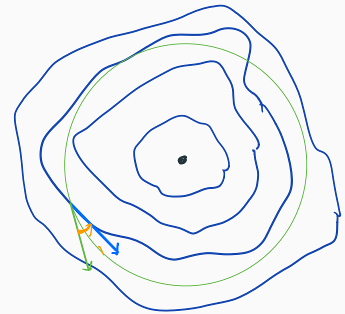
foliation

winding function φ

on $\mathbb{C}\setminus\{0\}$

$z\mapsto \frac{1}{z}$

Conformal



winding $\hat{\varphi}(z)=\varphi(\frac{1}{z})$

$\downarrow$

$\tilde{\rho}\in\mathcal{N}$

SLE$_\infty$

$\downarrow$ $\downarrow$

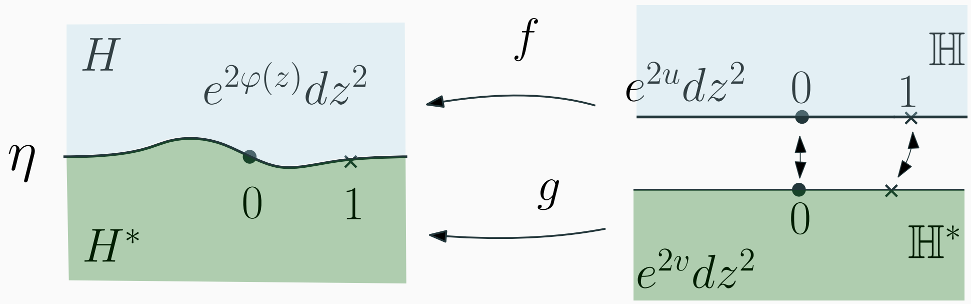
Cor: $S(\rho)=S(\tilde{\rho})$

Proof: $S(\rho)=\frac{1}{16}\mathcal{D}_{\mathbb{C}}(\varphi)=\frac{1}{16}\mathcal{D}_{\mathbb{C}}(\hat{\varphi})=S(\tilde{\rho})$. $\blacksquare$

谢谢！

Thank you !!

Cutting and welding identity, cont'd



Theorem (cutting)

We have the identity

$$\mathcal{D}_{\mathbb{C}}(\varphi) + I^L(\eta) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v).$$

Proof of the welding identity:

Assume that η and φ are smooth.

$$\begin{aligned}\mathcal{D}_{\mathbb{H}}(u) &= \mathcal{D}_{\mathbb{H}}(\varphi\circ f) + \mathcal{D}_{\mathbb{H}}(\log|f'|) + \frac{1}{\pi}\int_{\mathbb{H}} \nabla(\log|f'|)\cdot\nabla(\varphi\circ f)dz^2 \\ &= \mathcal{D}_{H}(\varphi) + \mathcal{D}_{\mathbb{H}}(\log|f'|) + \frac{1}{\pi}\int_{\mathbb{H}} \nabla(\log|f'|)\cdot\nabla(\varphi\circ f)dz^2.\end{aligned}$$

Adding $\mathcal{D}_{\mathbb{H}^*}(v)$ the first two terms sum up to $\mathcal{D}_{\mathbb{C}}(\varphi) + I^L(\eta)$, and the cross terms sum up to 0 since

$$\begin{aligned}\int_{\mathbb{H}} \nabla(\log|f'|)\cdot\nabla(\varphi\circ f)dz^2 &= \int_{\mathbb{R}} (\partial_n \log|f'|)\varphi\circ f(x)dx \\ &= \int_{\mathbb{R}} k(f(x))|f'(x)|\varphi\circ f(x)dx \\ &= \int_{\partial H} k(y)\varphi(y)dy = -\int_{\partial H^*} k(y)\varphi(y)dy. \quad \square\end{aligned}$$

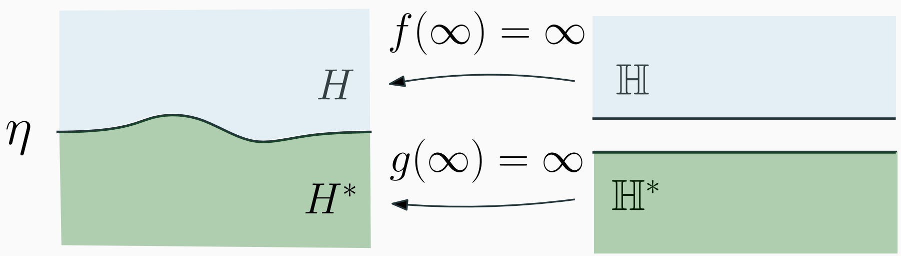
Complex identity

Corollary (Complex identity)

Let ψ be a complex-valued function on $\mathbb{C}$ with finite Dirichlet energy and $\operatorname{Im}\psi \in C^0(\hat{\mathbb{C}})$. Let η be a flow-line of the vector field e^{ψ} and f, g the conformal maps associated to η. Then we have

$$\mathcal{D}_{\mathbb{C}}(\psi) = \mathcal{D}_{\mathbb{H}}(\zeta) + \mathcal{D}_{\mathbb{H}^*}(\xi),$$

where $\zeta = \psi \circ f + \overline{\log f'}$, $\xi = \psi \circ g + \overline{\log g'}$.



Proof of the complex identity

$$\begin{aligned}
\zeta = \psi \circ f + (\log f')^* &= \operatorname{Re}\psi \circ f + \log|f'| + i(\operatorname{Im}\psi \circ f - \arg f') \\
\text{flow-line} \quad &:= u + i\operatorname{Im}\psi_0 \circ f. \\
\xi &= v + i\operatorname{Im}\psi_0 \circ g.
\end{aligned}$$

where $u := \operatorname{Re}\psi \circ f + \log|f'|$, $v := \operatorname{Re}\psi \circ g + \log|g'|$.

We have

$$\begin{aligned}
\mathcal{D}_{\mathbb{C}}(\psi) &= \mathcal{D}_{\mathbb{C}}(\operatorname{Re}\psi) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im}\psi) \\
\text{flow-line id.} \quad &= \mathcal{D}_{\mathbb{C}}(\operatorname{Re}\psi) + I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im}\psi_0) \\
&= \mathcal{D}_{\mathbb{C}}(\operatorname{Re}\psi) + I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im}\psi_0) \\
\text{welding id.} \quad &= \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im}\psi_0) \\
&= \mathcal{D}_{\mathbb{H}}(\zeta) + \mathcal{D}_{\mathbb{H}^*}(\xi). \quad \square
\end{aligned}$$