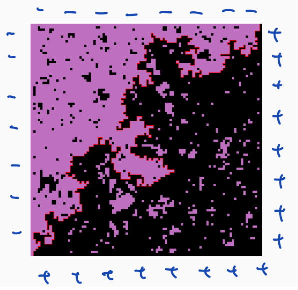
Today's plan:

- Multichordal SLE_{0+}
    - ↳ $\kappa=0$ Rational function & Shapiro's conjecture
    - ↳ LDP Multichordal energy

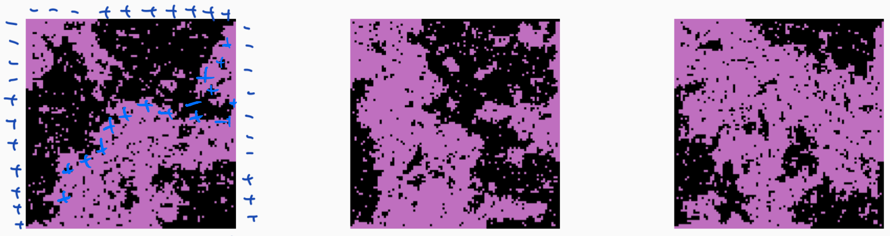
- LDP of SLE_∞ $\kappa \to \infty$ $\left(\varepsilon = \frac{1}{\kappa}\right)$
    - ↳ Radial Loewner chain
    - ↳ Donsker-Varadhan's theorem
    - ↳ Loewner-Kufarev energy

Peltola

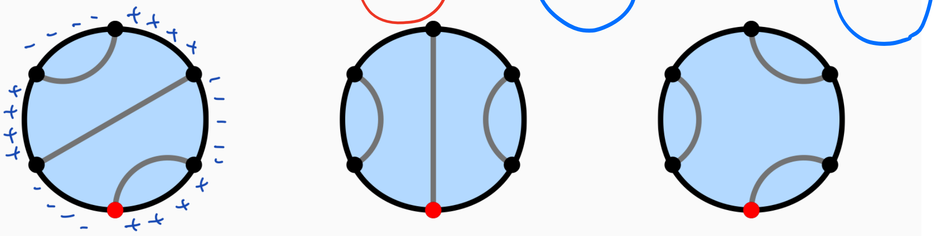


A simulation of interface in critical Ising lattice model, which approximates the SLE_3.

Simulations of multichordal SLE_3:



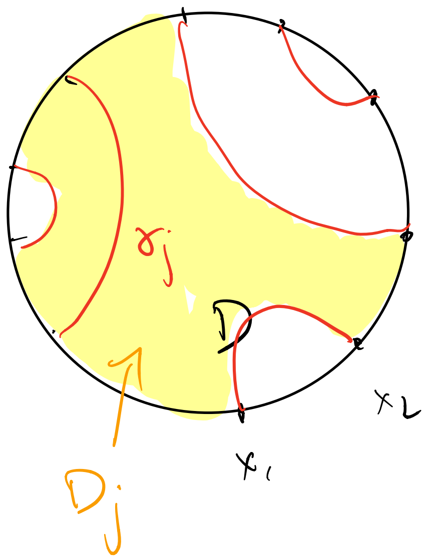
The corresponding link pattern α:



Cardy, Werner, Dubédat, Lawler, Kozdron, Bauer, Bernard, Kytölä, Sheffield, Miller, Wu, Peltola, Beffara, etc.

Definition (Theorem (Beffara-Peltola-Wu))

For $0<\kappa\leq 4$, the multichordal SLE_κ in $\mathcal{X}_\alpha(D, x_1, \ldots, x_{2n})$ is the unique measure on multichords $(\gamma_1, \ldots, \gamma_n)$ $\mathcal{X}_\alpha(D, x_1, \ldots, x_{2n})$ such that γ_j is the single SLE_κ in D_j.



simply c. domain

$$\mathcal{X}_\alpha(D, x_1, \ldots, x_{2n})$$

link pattern of 2n points

$= \{(\gamma_1, \ldots, \gamma_n)$. disjoint, connecting $x_1, \ldots, x_{2n}$ according to $\alpha\}$

- By Radon-Nikodym derivative.

(assume that $0<\kappa<\frac{8}{3}$)

$$c(\kappa) = \frac{(3\kappa-8)(6-\kappa)}{2\kappa} < 0$$

↑ central charge of SLE_κ

$$\frac{d\mathbb{P}_\alpha^{\kappa}}{d\mathbb{P}_\alpha^{\kappa,\mathrm{ind}}}(\gamma_1,\dots,\gamma_n) \leftarrow \text{multichordal SLE}_\kappa$$

$$= \frac{\exp\left(\frac{c(\kappa)}{2}\, m_D(\gamma_1,\dots,\gamma_n)\right)}{\mathbb{E}^{\kappa,\mathrm{ind}}\left(\exp\left(\frac{c(\kappa)}{2}\, m_D(\gamma_1,\dots,\gamma_n)\right)\right)}$$

where $m_D(\gamma_1,\dots,\gamma_n)$ $\overset{\infty}{\vee}$

$$= \sum_{p=2}^{n} \underset{\uparrow}{\mu_D^{\mathrm{loop}}}\left(\ell \mid \ell\cap\gamma_i \neq \emptyset \text{ for at least } p \text{ chords } \gamma_i\right)$$

$\underset{0}{\vee}$

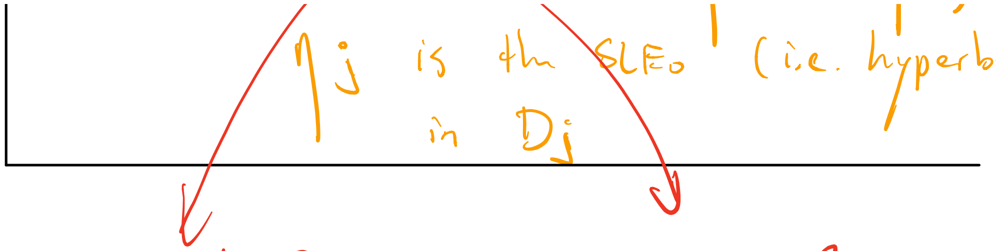
Brownian loop measure

(Lawler, Schramm, Werner)

- conformally invariant
- $\kappa > 0$, not for $\kappa = 0$.

Def: It is natural to define Multichordal SLE_0 in $\mathcal{X}_\alpha(D; x_1,\dots,x_{2n})$ to be the geodesic multichord in $\mathcal{X}_\alpha$.

$(\eta_1,\dots,\eta_n)$



η_j is the SLE$_0$ (i.e. hyperbolic geod.) in D_j

exists ?

minimizer of the multichordal Loewner energy

unique ?

(Algebraic method)

(Results jointly with E. Peltola)

Thm: Let $(\eta_1, \dots \eta_n)$ be a geodesic multichord in $\mathbb{H}$, connecting $x_1, \dots, x_{2n} \in \mathbb{R}$ (don't fix α yet)

Then, there exists a rational function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$

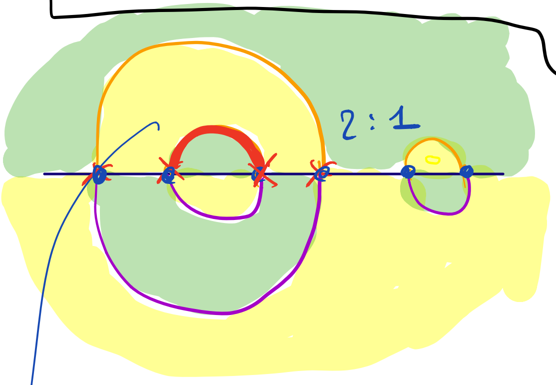
$$h_\eta = \frac{P}{Q} \leftarrow \text{polynomials}$$

of degree $n+1$. ← # preimage of a regular value of h_η

such that

$$\eta_1 \cup \eta_2 \cup \dots \cup \eta_n \cup \eta_1^* \cup \eta_2^* \dots \cup \eta_n^* \cup \mathbb{R} = h_\eta^{-1}\{\mathbb{R}\}.$$

Moreover, h_η has critical points at $\{x_1, \dots, x_{2n}\}$ all of index 2.



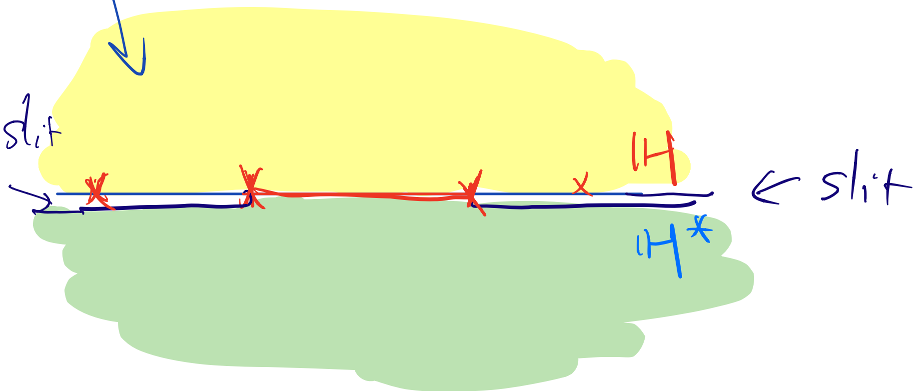
x_0 is critical part of h of index k

if $h(x) = h(x_0) + (x-x_0)^k c$
$\quad$ - - - -

h is k-to 1 cover in a neighborhood of x_0.

Construction of h_η.

h_η conformal



$h_\eta : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ $\quad$ deg = $n+1$

□

Classifying g. multichord $\iff$ Rational function with critical points $(x_1, \dots, x_{2n}) /\sim$

h_η is unique up to post-composition
by $PSL(2,\mathbb{R})$ $\in$ conformal autom of $\mathbb{H}$
or by $z \longmapsto -z$ ↖ switch $\mathbb{H}$ and $\mathbb{H}^*$

Thm (L. Goldberg '91)

For a given set $(\overset{\in\,\mathbb{C}}{z_1}, \dots, z_{2n})$ of critical points of index 2, there are at most a Catalan number C_n of rational functions of degree $n+1$ with these critical points $/\sim PSL(2,\mathbb{C})$

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \{\#\,\alpha\}$$

Assuming existence of geo. multichord for each α $\rightsquigarrow$ $\overline{\eta}_\alpha$ ↖ multi

$[h_\alpha]$ $PSL(2,\mathbb{R})$, $z \to -\bar{z}$
also distinct for $PSL(2,\mathbb{C})$ equivalence

gives C_n rational functions $/PSL(2,\mathbb{C})$ ←

Cor. There is a unique geodesic multich. in $X_\alpha(\mathbb{H}, x_1, \ldots, x_{2n})$

Cor When $z_1 = x_1 \ \ldots \ z_{2n} = x_{2n}$ all real then $\exists \ C_n$ rational functions with these critical points.

Moreover, these rational functions are $PSL(2,\mathbb{C})$-equivalent to a REAL rational function.

Shapiro's conjecture.

(Eremenko - Gabrielov '02 Annals)

$\Rightarrow$ Multichordal SLE_0 well-defined.

Part II SLE$_\infty$ $\kappa \to \infty$

$\kappa \geq 8$

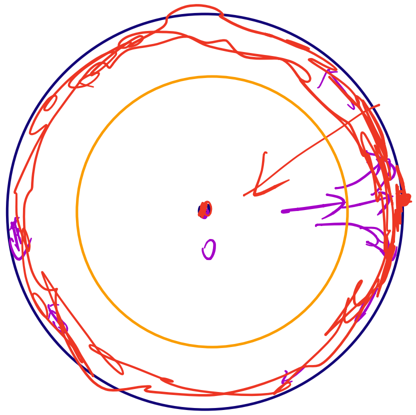
$\cdot z$ $g_t \xrightarrow[\text{unif. compact}]{\kappa \to \infty} \mathrm{Id}_{\mathbb{H}}$ t is fixed

↑ Not interesting

$W_t = \sqrt{\kappa} B_t$

- Radial SLE ($\mathbb{D}$, target at 0)

noradized at 0



$\zeta_t \in S^1$

$$\begin{cases} \partial_t g_t(z) = -g_t(z) \dfrac{g_t(z) + \zeta_t}{g_t(z) - \zeta_t} \\ g_0(z) = z \end{cases}$$

$$-z \frac{z + \zeta_t}{z - \zeta_t}$$

ζ_t = BM on S^1 with diffusitity κ

$= e^{iB_{\kappa t}} \overset{\text{law}}{=} e^{i\sqrt{\kappa} B_t}$

⇝ $\tau(z)$ = maximal solution time

$K_t = \{ z \in \mathbb{D} \mid \tau(z) \leq t \}$

→ Same phase transition as Chordal SLE.

$\kappa \to \infty$

$$\boxed{\text{Thm: As } k \to \infty \text{ . } SLE_k \ (K_t) \text{ converges to } (\mathbb{D} \setminus e^{-t}\mathbb{D})}$$

$t \rightsquigarrow t+\Delta t$ $\qquad z \in \mathbb{D}$

$$\Delta g_t(z) = \int_t^{t+\Delta t} \underbrace{-g_t(z)}_{\text{almost constant}} \frac{g_s(z)+B_s^k}{g_s(z)-B_s^k}\, ds$$

small

$$\int_{S^1} \frac{g_t(z)+\zeta}{g_t(z)-\zeta}\, \delta_{B_s^k}$$

$$= \int_t^{t+\Delta t} \int_{S^1} -g_t(z) \frac{g_t(z)+\zeta}{g_t(z)-\zeta}\, \delta_{B_s^k}\, ds$$

$$= \int_{S^1} -g_t(z) \frac{g_t(z)+\zeta}{g_t(z)-\zeta} \underbrace{\left(L^k_{t+\Delta t}(\zeta) - L^k_t(\zeta) \right)}_{\text{Local time of } BM^k \text{ on } S^1}$$

Definition. "Loewner-Kufarev energy"

$$\mathcal{N}_1 := \{ \rho \text{ measure on } S^1 \times [0,1] ; \ \rho(S^1 \times I) = |I| \text{ for all intervals } I \subset [0,1] \}$$

$$\rho \in \mathcal{N}_1 \rightsquigarrow \rho(d\xi\, dt) = \rho_t(d\xi) \cdot dt$$

$$S(\rho) = \int_0^1 I^{DV}(\rho_t)\, dt$$

where $I^{DV}(\rho_t) = \frac{1}{2} \int_{S^1} (\nu_t')^2 \, d\xi$ if $\rho_t(d\xi) \overset{\uparrow}{=} \nu_t^2(\xi)\, d\xi$

finite a.c. w.r.t $d\xi$

∞ otherwise

driving measures of Loewner chain.

$\hookrightarrow_{k\to\infty}$ uniform measure.

$\uparrow$ ρ_t = uniform on S^1

$\nu_t = \sqrt{\rho_t} \to$ constant

[Ang. Park. W] $\Rightarrow$ $S(\rho) = 0$

Thm. $\mathcal{N}_1$ endowed with the weak topology

$\{\delta_{B^k_t}\}_{t\in[0,1]} \in \mathcal{N}_1$

satisfies the LDP with rate function

$S(\rho)$

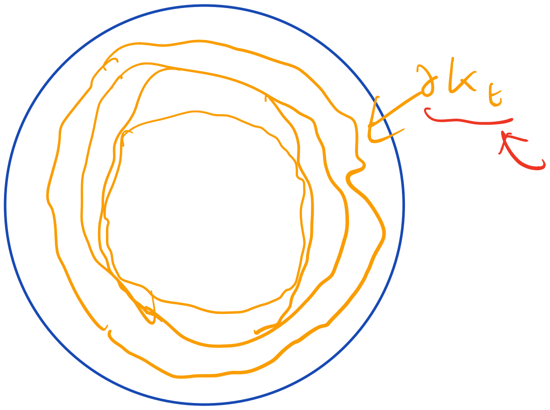
$\hookrightarrow$ regularizing effect

For each $\rho \in \mathcal{N}_1$ $\Leftrightarrow$ (K_t) growing hulls

$\nearrow$ homeomorphism (unif Car

Contraction principle

$\Rightarrow$ L-K energy is the LDP of Radial SLE_k as $k\to\infty$.



$\text{finite } \overbrace{\text{Loewner energy}}^{\text{SLE}_{0+} \text{ rate function}}$