Large deviations of Schramm-Loewner Evolutions (SLE_κ)

[Wan19a] Yilin Wang. The energy of a deterministic Loewner chain: reversibility and interpretation via SLE_{0+}. *J. Eur. Math. Soc. (JEMS)*, 21(7):1915–1941, 2019.

[Wan19b] Yilin Wang. Equivalent descriptions of the Loewner energy. *Invent. Math.*, 218(2):573–621, 2019.

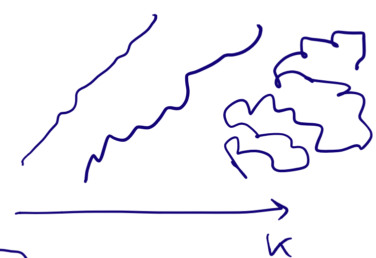
[VW20] Fredrik Viklund and Yilin Wang. Interplay between Loewner and Dirichlet energies via conformal welding and flow-lines. *Geom. Funct. Anal.*, 30(1):289–321, 2020.

[APW20] Morris Ang, Minjae Park, and Yilin Wang. Large deviations of radial SLE_∞. *arXiv preprint: 2002.02654*, 2020.

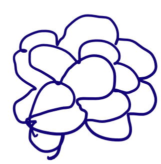
[PW20] Eveliina Peltola and Yilin Wang. Large deviations of multichordal SLE_{0+}, real rational functions, and zeta-regularized determinants of Laplacians. *arXiv preprint arXiv:2006.08574*, 2020.

[VW] Fredrik Viklund and Yilin Wang. Duality of Loewner-Kufarev and Dirichlet energies via foliations by Weil-Petersson quasicircles. *In preparation.*

small $\kappa \in [0,4]$



$\kappa \in (4,8)$



$\kappa \in [8,\infty)$

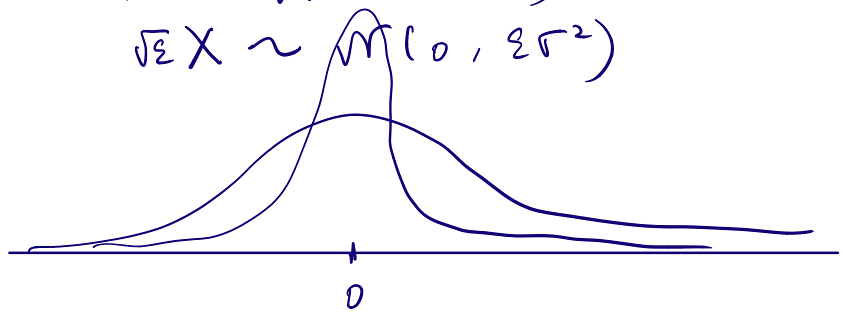
Large deviations principles (LDP) Varadhan 1966

$(\mu_\varepsilon)_{\varepsilon>0}$ measures

1.1 Large deviation principles

$X \sim \mathcal{N}(0, \sigma^2)$ real Gaussian

$\sqrt{\varepsilon} X \sim \mathcal{N}(0, \varepsilon\sigma^2)$



density of $\sqrt{\varepsilon} X = \dfrac{1}{\sqrt{2\pi\varepsilon\sigma^2}} \exp\left(-\dfrac{x^2}{2\varepsilon\sigma^2}\right)$

Rare event :

Let $M > 0$ $\quad \{\sqrt{\varepsilon} X \geq M\}$

$$\varepsilon \log \mathbb{P}(\sqrt{\varepsilon} X \geq M) = \varepsilon \log \left(\frac{1}{\sqrt{2\pi\varepsilon\sigma^2}} \int_M^{\infty} \exp\left(-\frac{x^2}{2\varepsilon\sigma^2}\right) dx \right)$$

$$= \underbrace{-\frac{\varepsilon}{2} \log(2\pi\varepsilon\sigma^2)}_{\xrightarrow{\varepsilon \to 0}\ 0} + \underbrace{\varepsilon \log \left(\int_M^{\infty} \exp\left(-\frac{x^2}{2\varepsilon\sigma^2}\right) dx \right)}_{\xrightarrow{\varepsilon \to 0}\ -\frac{M^2}{2\sigma^2}}$$

$$\xrightarrow[\varepsilon \to 0]{} -\frac{M^2}{2\sigma^2} := -I_X(M)$$

$$= -\inf_{x \geq M} I_X(x)$$

$$\Rightarrow \quad \mathbb{P}(\sqrt{\varepsilon}X \geq M) \underset{\varepsilon \to 0_+}{\sim} \exp\left(-\frac{\inf_{x \geq M} I_X(x)}{\varepsilon}\right)$$

We will say that I_X is the LD rate function of $(\sqrt{\varepsilon}X)_{\varepsilon>0}$

$\mathcal{X}$ Polish space and $\mathcal{B}$ Borel σ-algebra.

Def. 1.1 A **rate function** is a lower-semi-continuous mapping $I : \mathcal{X} \to [0, \infty]$.

the sublevel set $\{x : I(x) \leq \alpha\}$ is a closed set of $\mathcal{X}$, for all $\alpha \geq 0$

A **good rate function** is a rate function whose sublevel sets are all compact.

Def 1.2 We say that a family of probability measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ on $(\mathcal{X}, \mathcal{B})$ **satisfies the LDP of rate function I** if for all open set $O \in \mathcal{B}$ and closed set $F \in \mathcal{B}$, we have

- $$\varliminf_{\varepsilon \to 0_+} \varepsilon \log \mu_\varepsilon(O) \geq -\inf_{x \in O} I(x)$$
- $$\varlimsup_{\varepsilon \to 0_+} \varepsilon \log \mu_\varepsilon(F) \leq -\inf_{x \in F} I(x)$$

Remark: If $A \in \mathcal{B}$ satisfies

$$\inf_{x \in \mathring{A}} I(x) = \inf_{x \in \overline{A}} I(x)$$

then we say that A is a continuity set of I

$$\Rightarrow \lim_{\varepsilon \to 0^+} \varepsilon \log \mu_\varepsilon(A) = \inf_{x \in A} I(x)$$

$$(\Leftarrow) \qquad \mu_\varepsilon(A) \underset{\varepsilon \to 0}{\sim} \exp\left(- \frac{\inf_{x \in A} I(x)}{\varepsilon} \right).$$

Thm (Contraction principle)

If $\mathcal{X}$, $\mathcal{Y}$ are two Polish spaces

$\theta : \mathcal{X} \to \mathcal{Y}$ continuous

$\{\mu_\varepsilon\}_{\varepsilon > 0}$ on $\mathcal{X}$ satisfying an LDP with good rate function $I : \mathcal{X} \to [0, \infty]$.

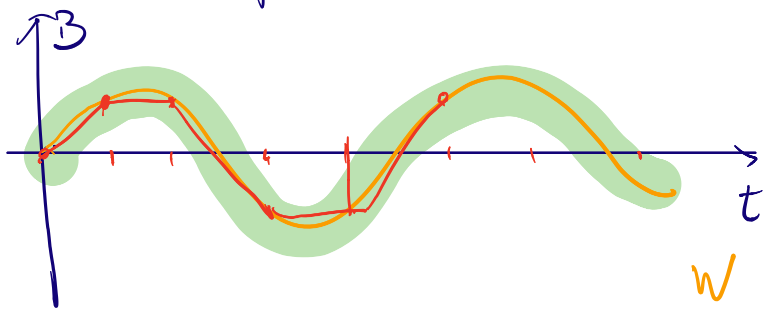
Let $I' : \mathcal{Y} \to [0, \infty]$ be

$$I'(y) := \inf_{x \in \theta^{-1}\{y\}} I(x)$$

Then, the family of push forward proba measures $(\theta_* \mu_\varepsilon)_{\varepsilon > 0}$ on $\mathcal{Y}$ satisfies the LDP with good rate function I'.

Thm 1.6 (Schilder's Theorem)

(LDP for scaled Brownian motion)



$\sqrt{\varepsilon}B$

Def 1.3 Let $T \in (0, \infty]$. $W \in C^0([0,T])$ with $W_0 = 0$ $t \mapsto W_t$

The Dirichlet energy of W is

$$I_T(W) := \frac{1}{2}\int_0^T |\partial_t W_t|^2 \, dt$$

- if W is absolutely continuous $W \in W^{1,1}$
- ∞, otherwise.

$I_T($ W linearly interpolated$)$

Calculus

$$I_T(w) = \sup_{\Pi} \frac{1}{2}\sum_{i=0}^{K-1} \frac{(W_{t_{i+1}} - W_{t_i})^2}{(t_{i+1} - t_i)}$$

where Π is all the partitions $\forall k$

$\{0 = t_0 < t_1 < t_2 \cdots < t_k \le T\}$

(Schilder)

Thm Fix $T \in (0,\infty)$. The process $(\sqrt{\varepsilon} B_t)_{t\in[0,T]}$ is r.v in $(C^0([0,T]), \|\ \|_\infty)$ with good rate function I_T.

Consequence: If $\delta > 0$, $O_\delta(w) := \{\tilde{w} : \|\tilde{w} - w\|_\infty < \delta\}$ is a continuity set

$$\Rightarrow \quad \mathbb{P}(\sqrt{\varepsilon} B \in O_\delta(w)) \underset{\varepsilon\to 0}{\sim} \exp\left(\frac{-\inf_{O_\delta} I_T(\tilde{w})}{\varepsilon}\right)$$

$$\downarrow \delta \to 0$$

by lower-semicontinuity $\quad \exp\left(-\frac{I_T(w)}{\varepsilon}\right)$

Proof sketch.

$$\varepsilon(B_{t_0}, B_{t_1}, \dots, B_{t_k})$$

$$\rightsquigarrow (\varepsilon B_{t_0}, \varepsilon(B_{t_1} - B_{t_0}), \dots, \varepsilon(B_{t_k} - B_{t_{k-1}}))$$

$$\underbrace{\quad}_{\mathcal{N}(0,t_0)} \perp\!\!\!\perp \underbrace{\quad}_{\mathcal{N}(0,t_1-t_0)} \perp\!\!\!\perp \underbrace{\quad}_{\mathcal{N}(0,t_k-t_{k-1})}$$

satisfies the LDP with rate

$$I_\pi(W_{t_0}, \dots, W_{t_k}) = \sum_{i=0}^{k-1} \frac{(W_{t_i} - W_{t_{i-1}})^2}{2(t_i - t_{i-1})}$$

The law of B is the projective limit of its finite dimensional marginals

Gartner – Dawson $\rightsquigarrow$

LDP rate function on projective limit B

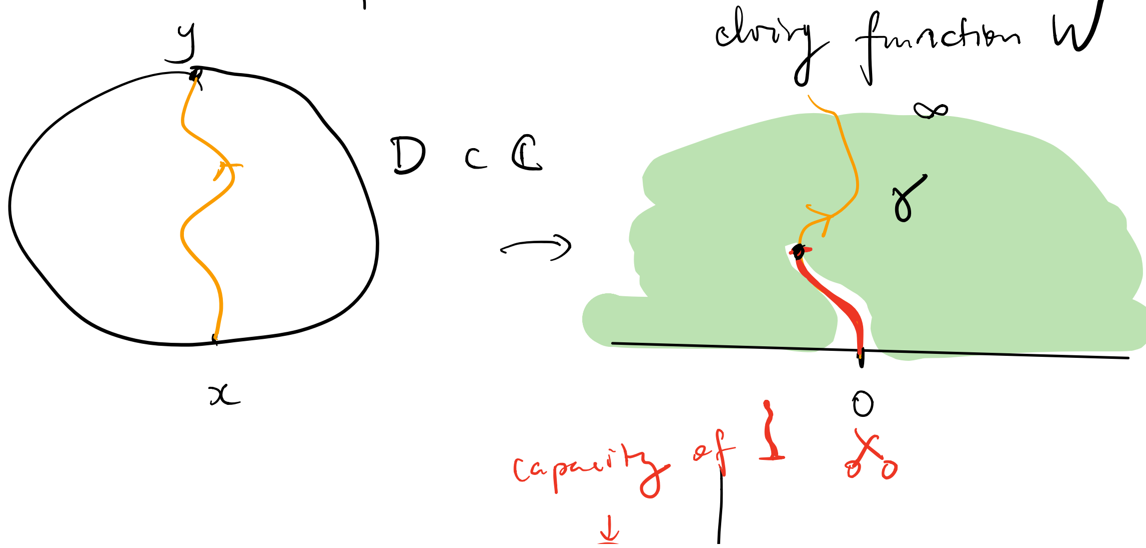
$$= \sup_{\pi} I_{\pi}$$

$$= I_T \qquad \blacksquare$$

Remark. $T = \infty \rightsquigarrow$ Same proof shows that $\left((\varepsilon B_t)_{t \in \mathbb{R}_+}, \|\ \|_{\infty,\text{compacts}} \right)$ satisfies LDP with rate function I_∞

1.2 Chordal Loewner chain & SLE

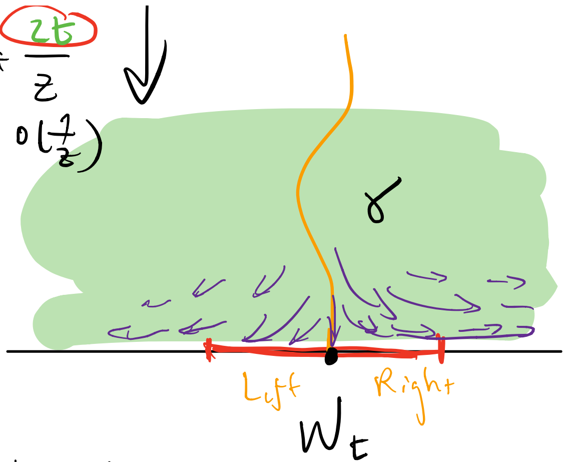
encode simple curve $\rightsquigarrow$ real valued driving function W



$$g_t(z) \sim z + 0 + \frac{2t}{z} + o\left(\frac{1}{z}\right)$$

normalized at ∞

$z \to \infty$



$t \mapsto W_t$ is called the driving function of γ

Conversely, we recover γ from W

$$(E_2) \quad \partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z$$

$\tau(z)$ = maximal survival time of Loewner differential equation

$$K_t = \{ z \in \mathbb{H} : \tau(z) \leq t \}$$

$\leadsto$ (K_t) is a growing family

- g_t : conformal map $\mathbb{H} \setminus K_t \longrightarrow \mathbb{H}$
- If W comes from a simple curve then $K_t = \gamma[0,t]$

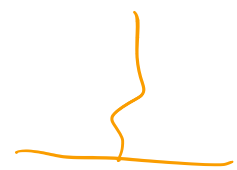
and g_t as described above.

- Not all W comes from a simple chord
- If W is $\frac{1}{2}$-Hölder continuous with Hölder-norm < 4
  $\Rightarrow$ W generates a simple curve.

Definition The chordal SLE_κ in $(\mathbb{H}, 0, \infty)$ is a curve tracing out the growing family of hulls (K_t) driven by $\sqrt{\kappa} B$.

Rohde-Schramm

$\kappa \in [0, 4]$

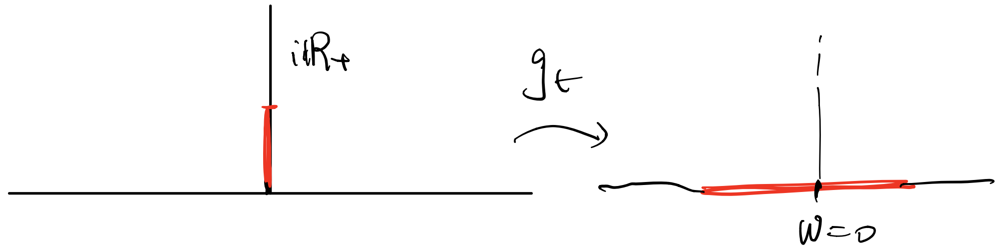


$\kappa \in (4, 8)$

$\kappa \geq 8$ Space filling

$\kappa = 0$ $\quad W \equiv 0$

$\rightsquigarrow g_t(z) = \sqrt{z^2 + 4t}$

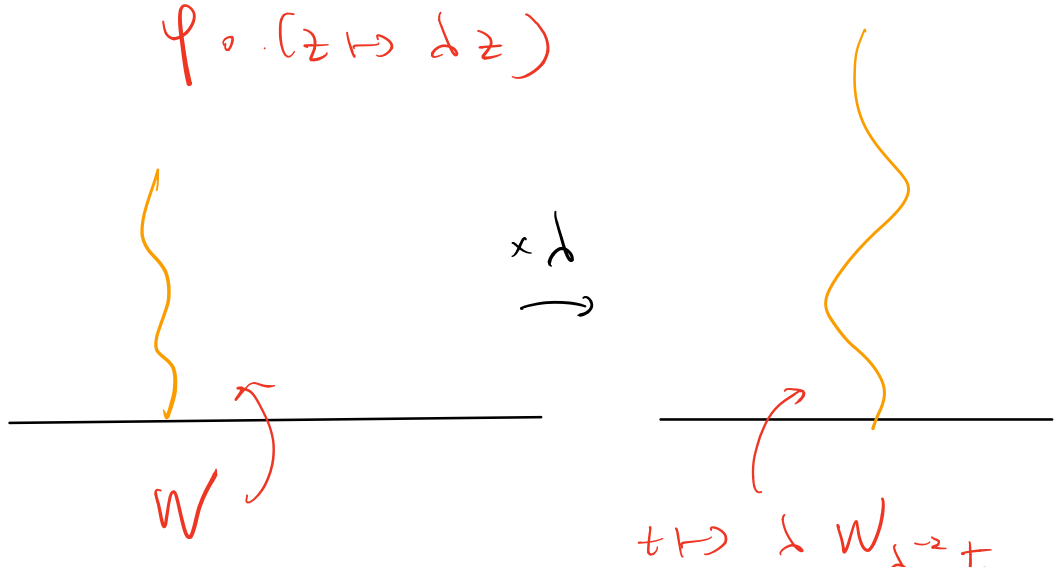


SLE_κ in $(D; x, y)$ is defined as the image of SLE_κ in $(\mathbb{H}; 0, \infty)$ by

$$\varphi : \overset{PSL(2,\mathbb{R})}{\mathbb{H}} \to D$$
$$\underbrace{\varphi(0) = x, \quad \varphi(\infty) = y}$$

Other deg of freedom

$$\varphi \circ (z \mapsto \lambda z)$$



If $W = \sqrt{\kappa} B$ the law is invariant under this transformation

$\Rightarrow$ SLE_κ welldefined in $(D; x, y)$

$\to$ SLE_0 in $(D; x, y)$

$= \varphi(i\mathbb{R})$

$=$ " hyperbolic geodesic in D from x to y "

Next

- LDP of chordal (SLE_κ) $\varepsilon = \kappa$

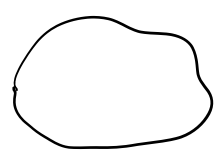
  SLE_{0+}

  $\rightsquigarrow$ Loewner energy

  $\rightsquigarrow$ Reversibility

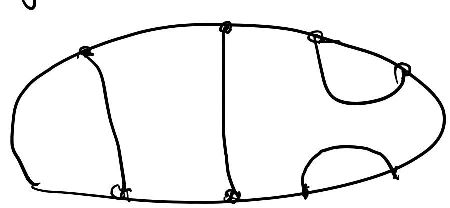
  $\rightsquigarrow$ generalize to Jordan curves

Welding · Cutting flow-lines.



Finite energy Jordan curves $\Leftrightarrow$ Weil-Petersson (26)

$\rightsquigarrow$ generalize to multi chord



Multiple SLE_0

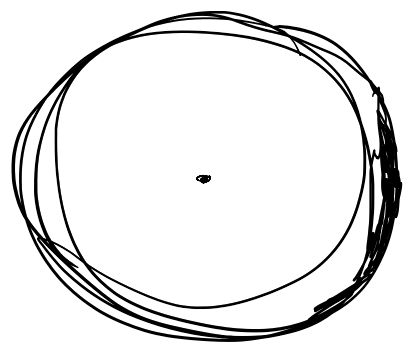
$\Rightarrow$ Rational functions

$\to$ semi classical limit of BPZ equations

$\rightsquigarrow$ $\kappa \to \infty$ LDP $\qquad \varepsilon = \frac{1}{\kappa}$

$\to$ Radial SLE

$\to$ Loewner-Kufarev energy (rate function)



$\rightsquigarrow$ $\kappa \rightsquigarrow 16/\kappa$

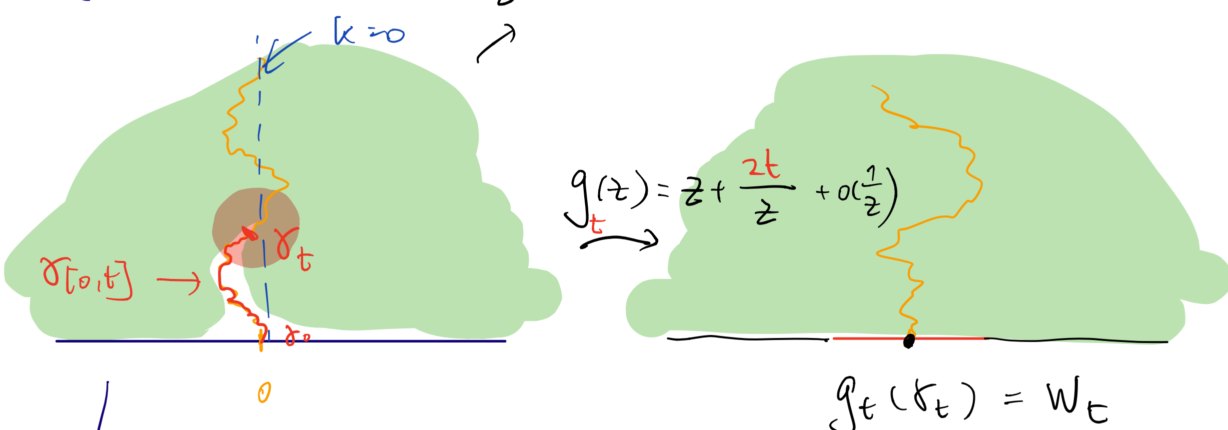
SLE_{0+} $\qquad$ SLE_{∞}

Loewner energy $\leftrightarrow$ L-K energy

Lecture 2

- Recall SLE_κ definition $(\kappa \le 4)$
  - Domain Markov property
  - Conformal invariance
  - Reversibility
- Chordal Loewner energy
  - Large deviation
  - Reversibility
- Loop Loewner energy
  - Equivalent description (Weil-Petersson quasicircles)

$(\mathbb{H}; 0, \infty)$ $\quad i: z \mapsto -\frac{1}{z}$



$t \mapsto W_t$ is the driving function of γ

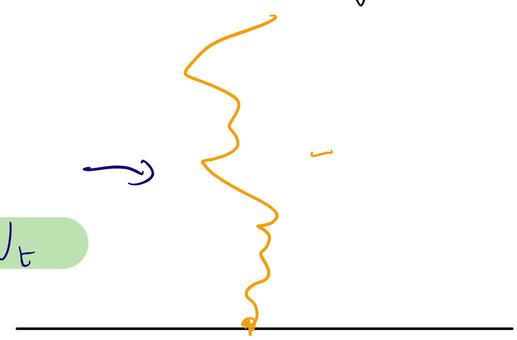
$\mathbb{R}_+ \to \mathbb{R}$

- W is continuous
- $W_0 = 0$

$SLE_\kappa \longleftrightarrow W = \sqrt{\kappa} B$

$z \mapsto z - W_t$

$z \mapsto \lambda z$

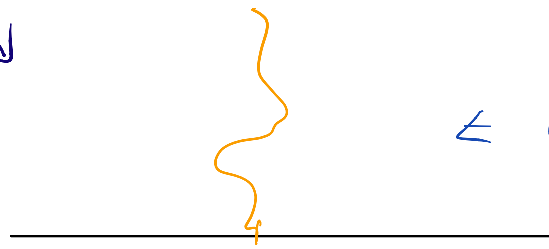


This curve has driving function

$s \mapsto W_{t+s} - W_t \overset{\text{law}}{=} W$

0

→ Domain Markov property of SLE

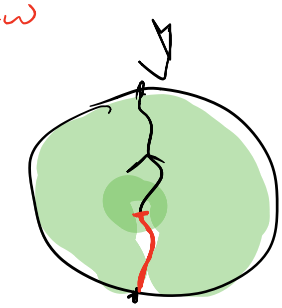


← driving function

$t \mapsto \lambda \cdot W_{\lambda^{-2} t} \overset{\text{law}}{=} W$

Scaling invariance of SLE.

SLE in (D, x, y) is the image of SLE in $(\mathbb{H}, 0, \infty)$ under a conformal map $(\mathbb{H}, 0, \infty) \to (D, x, y)$



x

Thm (Reversibility of SLE) [Zhan '08, Miller–Sheffield '12]

$\kappa \le 4$

The law of the trace of $\underbrace{SLE_\kappa}_{\text{in } (\mathbb{H}, 0, \infty)}$ is invariant under the map $i : z \mapsto -\frac{1}{z}$

Def:

Chordal Loewner energy of a deterministic chord γ in (D, x, y) is defined as

$$I^L_{D,x,y}(\gamma) := I^L_{\mathbb{H},0,\infty}(\varphi(\gamma))$$

$$:= \frac{1}{2}\int_0^\infty |\partial_t W_t|^2 \, dt = I_\infty(W)$$

where $\varphi : (D, x, y) \to (\mathbb{H}, 0, \infty)$ conformal

and W is the driving function of $\varphi(\gamma)$

· Remark: • $I_\infty(t \mapsto \lambda W_{\lambda^{-2} t}) = I_\infty(W)$

$\Rightarrow I^L_{D,x,y}(\gamma)$ is well-defined.

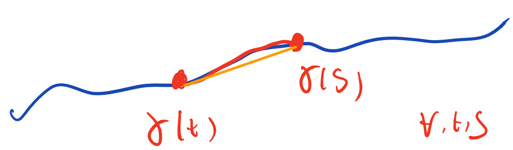
• $I^L_{D,x,y}(\gamma) \geq 0$

$= 0$ iff γ is the hyperbolic geodesic of $(D; x, y)$

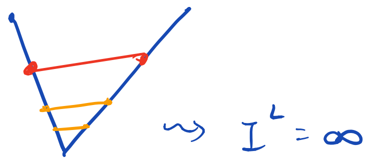
• One can show that $I^L_{D,x,y}(\gamma) < \infty$

$\Rightarrow \gamma$ is asymptotically smooth

$\Updownarrow$



$\forall t, s$ $\quad \dfrac{\text{length}(\gamma(t), \gamma(s))}{|\gamma(t) - \gamma(s)|} \to 1$ as $|t-s| \to 0$



- There are finite energy curves not C^1

⚠ Finite energy curves are smoother than SLE_κ when $\kappa > 0$.

(Beffara $\Rightarrow$ Hausdorff dimension of SLE_κ $= 1 + \frac{\kappa}{8}$ for $0 \le \kappa \le 8$)

From last lecture:

Thm (Schilder). The process $(\sqrt{\varepsilon} B_t)_{t \in [0,T]}$ (SLE_κ) is r.v. in $(C^0([0,T]), \| \ \|_{\infty, C})$ with good rate function I_∞. ← Dirichlet energy

Apply Loewner transform

$I^L_{\mathbb{H}, 0, \infty}(\gamma)$

Thm (Peltola, W. 2020). In (D, x, y), (SLE_κ) satisfies the LDP with good rate function $I^L_{D,x,y}$

- For all open set O, closed set F we have

$$\begin{cases} \varliminf_{\kappa\to 0} \kappa \log \mathbb{P}(SLE_\kappa \in O) \geq -\inf_{\gamma\in O} I^{h}_{D,x,y}(\gamma) \\ \varlimsup_{\kappa\to 0} \kappa \log \mathbb{P}(SLE_\kappa \in F) \leq -\inf_{\gamma\in F} I^{h}_{D,x,y}(\gamma) \end{cases}$$

- $\forall c \geq 0$ $\{\gamma : I^{L}_{D,x,y}(\gamma) \leq c\}$ is compact.

Topology: assume $D = \mathbb{D}$.

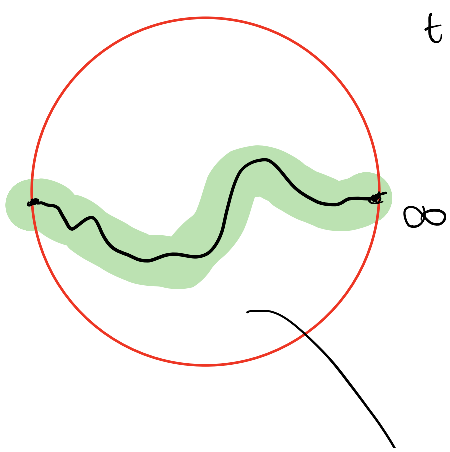
$$d_H(K, \tilde{K}) := \inf\{\varepsilon \geq 0 \mid \tilde{K} \subset \varepsilon\text{-neighbor of } K,\ K \subset \varepsilon\text{-neighbor of } \tilde{K}\}$$

Euclidean distance in $\mathbb{D}$

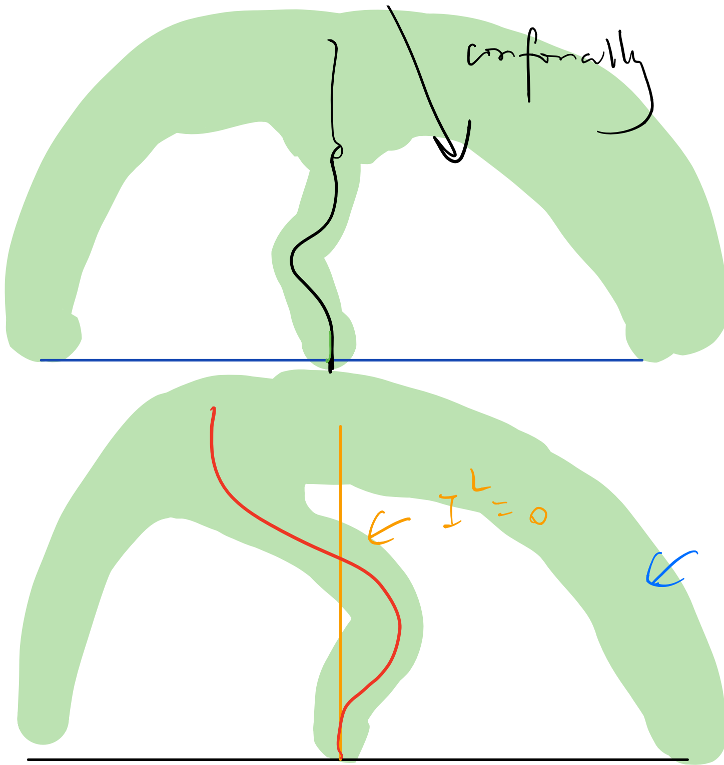
compact sets of $\overline{\mathbb{D}}$

Euclid

We consider SLE_κ by its trace $= \overline{\bigcup_{t\geq 0} \gamma_t}$



In other domain one just use the pushforward metric to define the topology.



ε neighborhood of γ

$O_\varepsilon(\gamma)$

$$\inf_{\tilde\gamma \in O_\varepsilon(\gamma)} I^L(\tilde\gamma) = \inf_{\tilde\gamma \in \overline{O_\varepsilon(\gamma)}} I^L(\tilde\gamma)$$

$$\Rightarrow \quad \mathbb{P}(SLE_\kappa \in O_\varepsilon(\gamma)) \underset{\kappa\to 0}{\sim} \exp\left(-\frac{\inf}{\kappa}\right)$$

Lower semicontinuity $\underset{\varepsilon\to 0}{\Rightarrow}$

$$\inf_{\tilde\gamma \in O_\varepsilon(\gamma)} I^L(\tilde\gamma) \xrightarrow[\varepsilon\to 0]{} I^L(\gamma)$$

$$-\lim_{\varepsilon\to 0}\lim_{\kappa\to 0} \kappa\log \mathbb{P}(SLE_\kappa^R \in O_\varepsilon^R(\gamma)) = I^L(\gamma)$$

$$\mathbb{P}(SLE_\kappa \in O_\varepsilon(\gamma^R)) = I^L(\gamma^R)$$

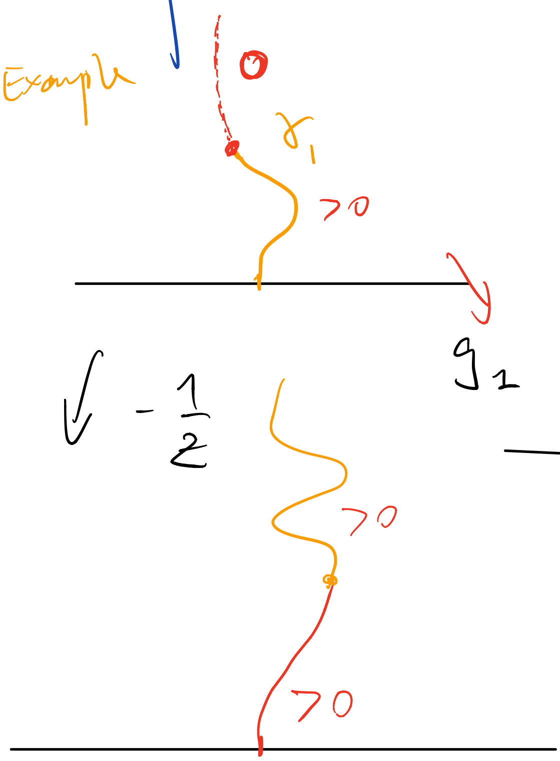
Thm (Reversibility of I^L)

$$\boxed{I^L_{D,x,y}(\gamma) = I^L_{D,y,x}(\gamma)}$$

$$I^L_{\mathbb{H},0,\infty}(\gamma) = I^L_{\mathbb{H},0,\infty}(\gamma^R)$$

Proof: SLE reversibility + LDP

Example

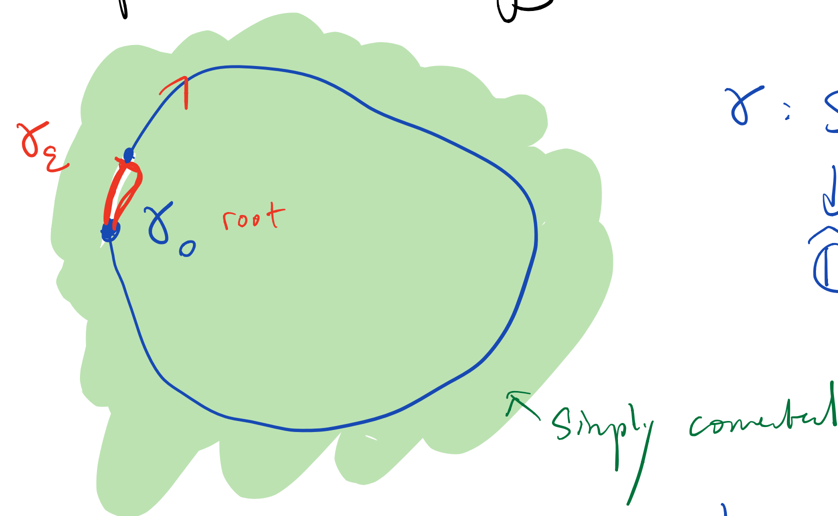


g_1

$\sqrt{-\frac{1}{2}}$

W is constant after time 1

Loop Loewner energy



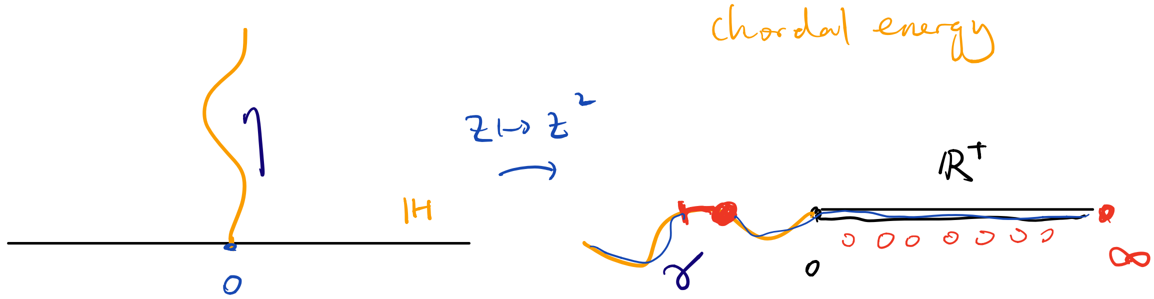
$\gamma : S^1 = [0,1]/_{0\sim 1}$

$\downarrow$ continuous injective

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$$I^L(\gamma, \gamma_0) = \lim_{\varepsilon \to 0} I^L_{\hat{\mathbb{C}} \setminus \gamma[0,\varepsilon]} \left(\gamma[\varepsilon, 1] \right)$$

chordal energy



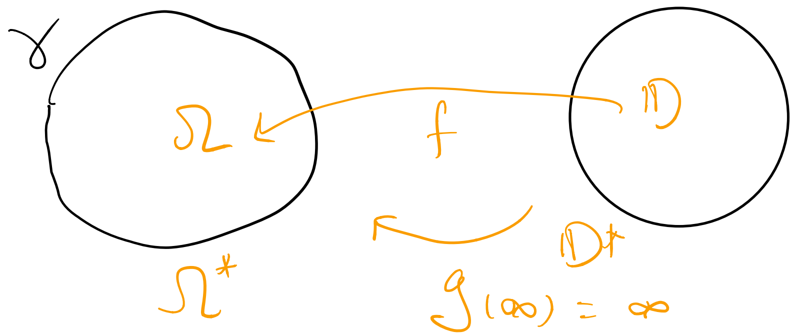
$$I^L(\gamma \cup \mathbb{R}_+, \infty) = I^L_{\mathbb{H}}(\eta) = I^L_{\hat{\mathbb{C}} \setminus \mathbb{R}_+}(\gamma)$$

Thm (Rohde. W.)

$I^L(\gamma, \cancel{\gamma_0})$ does not depend on γ_0

- $I^L(\gamma) = 0 \iff \gamma =$ circle
- $I^L(\gamma)$ Möbius invariant

Thm (W.) γ is bounded Jordan curve



where f, g are conformal maps

$$I^L(\gamma) = \frac{1}{\pi}\int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 |dz|^2$$

$$+ \frac{1}{\pi}\int_{\mathbb{D}^*} \left|\frac{g''}{g'}\right|^2 |dz|^2$$

$$+ 4 \log \frac{|f'(0)|}{|g'(\infty)|}$$

Another proof of Reversibility and root-invariance.

Cor. $I^L(\gamma) < \infty \iff \gamma$ is a Weil-Petersson quasi-circle

$\Updownarrow$

$$\int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 |dz|^2 < \infty$$

Next time

$\rightsquigarrow$ LDP of multiple SLE

$\hookrightarrow$ Rational functions

$\rightsquigarrow$ Proof of theorem $\Leftrightarrow$ WP q.c

$\rightsquigarrow$ Random conformal geometry

$\to$ quantum zipper

$\to$ flow-lines

$\rightsquigarrow$ SLE_∞ LDP

$\rightsquigarrow$ Energy duality $SLE_{0+} \Leftrightarrow SLE_{0+}$ LDP