Large deviations of Schramm-Loewner Evolutions (SLE_κ)

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[Wan19b] Yilin Wang. Equivalent descriptions of the Loewner energy. *Invent. Math.*, 218(2):573–621, 2019.

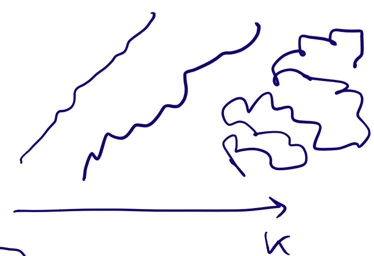
[VW20] Fredrik Viklund and Yilin Wang. Interplay between Loewner and Dirichlet energies via conformal welding and flow-lines. *Geom. Funct. Anal.*, 30(1):289–321, 2020.

[APW20] Morris Ang, Minjae Park, and Yilin Wang. Large deviations of radial SLE_∞. *arXiv preprint: 2002.02654*, 2020.

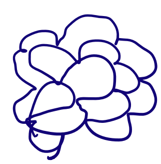
[PW20] Eveliina Peltola and Yilin Wang. Large deviations of multichordal SLE_{0+}, real rational functions, and zeta-regularized determinants of Laplacians. *arXiv preprint arXiv:2006.08574*, 2020.

[VW] Fredrik Viklund and Yilin Wang. Duality of Loewner-Kufarev and Dirichlet energies via foliations by Weil-Petersson quasicircles. *In preparation.*

small $\kappa \in [0,4]$



$\kappa \in (4,8)$



$\kappa \in [8,\infty)$

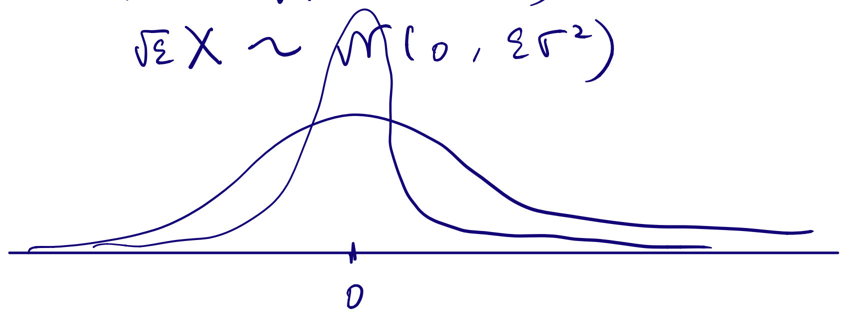
Large deviations principles (LDP) Varadhan 1966

$(\mu_\varepsilon)_{\varepsilon>0}$ measures

1.1 Large deviation principles

$X \sim \mathcal{N}(0, \sigma^2)$ real Gaussian

$\sqrt{\varepsilon} X \sim \mathcal{N}(0, \varepsilon\sigma^2)$



density of $\sqrt{\varepsilon} X$ $= \dfrac{1}{\sqrt{2\pi\varepsilon\sigma^2}} \exp\left(-\dfrac{x^2}{2\varepsilon\sigma^2}\right)$

Rare event :

Let $M > 0$ $\quad \{\sqrt{\varepsilon} X \geq M\}$

$$\varepsilon \log \mathbb{P}(\sqrt{\varepsilon} X \geq M) = \varepsilon \log \left(\frac{1}{\sqrt{2\pi\varepsilon\sigma^2}} \int_M^\infty \exp\left(-\frac{x^2}{2\varepsilon\sigma^2}\right) dx \right)$$

$$= \underbrace{-\frac{\varepsilon}{2} \log(2\pi\varepsilon\sigma^2)}_{\xrightarrow{\varepsilon \to 0}\ 0} + \underbrace{\varepsilon \log\left(\int_M^\infty \exp\left(-\frac{x^2}{2\varepsilon\sigma^2}\right) dx \right)}_{\xrightarrow{\varepsilon \to 0}\ -\frac{M^2}{2\sigma^2}}$$

$$\xrightarrow[\varepsilon \to 0]{} -\frac{M^2}{2\sigma^2} := -I_X(M)$$

$$= -\inf_{x \geq M} I_X(x)$$

$$\Rightarrow \quad \mathbb{P}(\sqrt{\varepsilon} X \geq M) \underset{\varepsilon \to 0_+}{\sim} \exp\left(-\frac{\inf_{x \geq M} I_X(x)}{\varepsilon}\right)$$

We will say that I_X is the LD rate function of $(\sqrt{\varepsilon} X)_{\varepsilon > 0}$

$\mathcal{X}$ Polish space and $\mathcal{B}$ Borel σ-algebra.

Def. 1.1 A rate function is a lower-semi-continuous mapping $I : \mathcal{X} \to [0, \infty]$.

the sublevel set $\{x : I(x) \leq \alpha\}$ is a closed set of $\mathcal{X}$, for all $\alpha \geq 0$

A good rate function is a rate function whose sublevel sets are all compact.

Def 1.2 We say that a family of probability measures $\{\mu_\varepsilon\}_{\varepsilon > 0}$ on $(\mathcal{X}, \mathcal{B})$ satisfies the LDP of rate function I if for all open set $O \in \mathcal{B}$ and closed set $F \in \mathcal{B}$, we have

- $$\underline{\lim}_{\varepsilon \to 0_+} \varepsilon \log \mu_\varepsilon(O) \geq -\inf_{x \in O} I(x)$$
- $$\overline{\lim}_{\varepsilon \to 0_+} \varepsilon \log \mu_\varepsilon(F) \leq -\inf_{x \in F} I(x)$$

Remark: If $A \in \mathcal{B}$ satisfies

$$\inf_{x \in \mathring{A}} I(x) = \inf_{x \in \overline{A}} I(x)$$

then we say that A is a continuity set of I

$$\Rightarrow \lim_{\varepsilon \to 0^+} \varepsilon \log \mu_\varepsilon(A) = \inf_{x \in A} I(x)$$

$$(\Leftarrow) \qquad \mu_\varepsilon(A) \underset{\varepsilon \to 0}{\sim} \exp\left(- \frac{\inf_{x \in A} I(x)}{\varepsilon} \right).$$

Thm (Contraction principle)

If $\mathcal{X}, \mathcal{Y}$ are two Polish spaces

$\theta : \mathcal{X} \to \mathcal{Y}$ continuous

$\{\mu_\varepsilon\}_{\varepsilon > 0}$ on $\mathcal{X}$ satisfying an LDP with good rate function $I : \mathcal{X} \to [0, \infty]$.

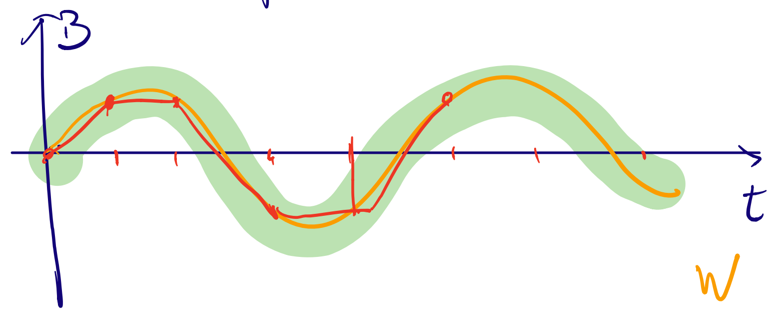
Let $I' : \mathcal{Y} \to [0, \infty]$ be

$$I'(y) := \inf_{x \in \theta^{-1}\{y\}} I(x)$$

Then, the family of pushforward proba measures $(\theta_* \mu_\varepsilon)_{\varepsilon > 0}$ on $\mathcal{Y}$ satisfies the LDP with good rate function I'.

Thm 1.6 (Schilder's Theorem)

(LDP for scaled Brownian motion)



Def 1.3 Let $T \in (0, \infty]$. $W \in C^0([0,T])$ with $W_0 = 0$ $\quad t \mapsto W_t$

The Dirichlet energy of W is

$$I_T(W) := \frac{1}{2}\int_0^T |\partial_t W_t|^2 \, dt$$

- if W is absolutely continuous $W \in W^{1,1}$
- ∞, otherwise.

$I_T($ W linearly interpolated $)$

Calculus

$$I_T(w) = \sup_{\pi} \frac{1}{2}\sum_{i=0}^{k-1} \frac{(W_{t_{i+1}} - W_{t_i})^2}{(t_{i+1} - t_i)}$$

where π is all the partitions $\forall k$

$\{ 0 = t_0 < t_1 < t_2 \cdots < t_k \le T \}$

(Schilder)

Thm Fix $T \in (0,\infty)$. The process $(\sqrt{\varepsilon} B_t)_{t\in[0,T]}$ is r.v in $(C^0([0,T]), \|\ \|_\infty)$ with good rate function I_T.

Consequence: If $\delta > 0$, $O_\delta(w) := \{\tilde{w} : \|\tilde{w} - w\|_\infty < \delta\}$ is a continuity set

$$\Rightarrow \quad \mathbb{P}(\sqrt{\varepsilon} B \in O_\delta(w)) \underset{\varepsilon\to 0}{\sim} \exp\left(\frac{-\inf_{O_\delta} I_T(\tilde{w})}{\varepsilon}\right)$$

$$\downarrow \delta \to 0$$

by lower-semicontinuity

$$\exp\left(-\frac{I_T(w)}{\varepsilon}\right)$$

Proof sketch.

$$\varepsilon(B_{t_0}, B_{t_1}, \dots, B_{t_k})$$

$$\rightsquigarrow \left(\varepsilon B_{t_0}, \varepsilon(B_{t_1} - B_{t_0}), \dots, \varepsilon(B_{t_k} - B_{t_{k-1}})\right)$$

$$\mathcal{N}(0, t_0) \perp\!\!\!\perp \mathcal{N}(0, t_1 - t_0) \perp\!\!\!\perp \mathcal{N}(0, t_k - t_{k-1})$$

satisfies the LDP with rate

$$I_\pi(W_{t_0}, \dots, W_{t_k}) = \sum_{i=0}^{k-1} \frac{(W_{t_i} - W_{t_{i-1}})^2}{2(t_i - t_{i-1})}$$

The law of B is the projective limit of its finite dimensional marginals

Gartner - Dawson $\rightsquigarrow$

LDP rate function on projective limit B

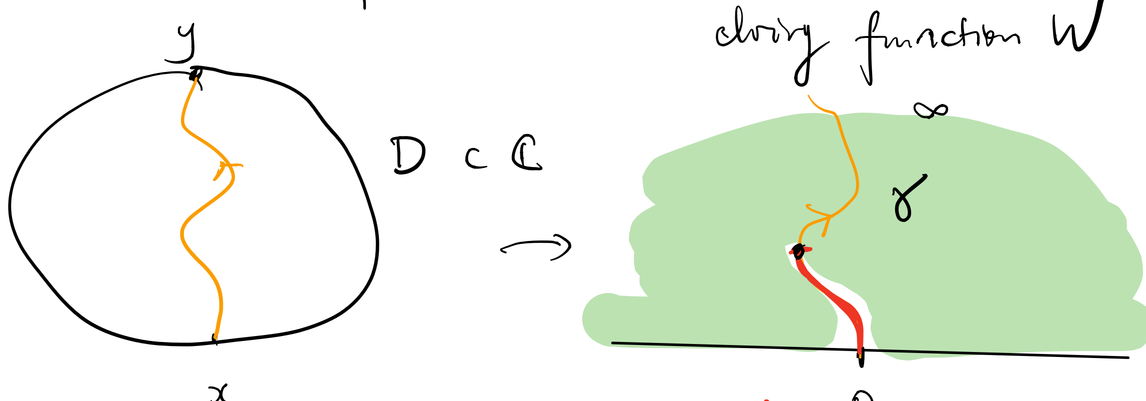
$$= \sup_{\pi} I_{\pi}$$

$$= I_T \qquad \square$$

Remark. $T = \infty \rightsquigarrow$ Same proof shows that $((\varepsilon B_t)_{t \in \mathbb{R}_+}, \|\ \|_{\infty, \text{compacts}})$ satisfies LDP with rate function I_∞

1.2 Chordal Loewner chain & SLE

encode simple curve $\rightsquigarrow$ real valued driving function W



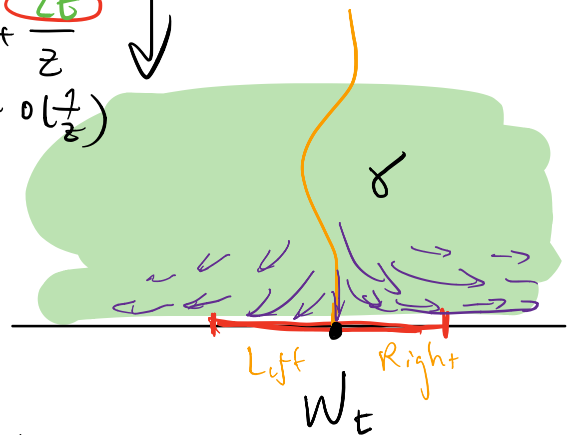
capacity of γ_0

$\downarrow$

$$g_t(z) \sim z + 0 + \frac{2t}{z} + o\left(\frac{1}{z}\right)$$

normalized at ∞

$z \to \infty$



$t \mapsto W_t$ is called the driving function of γ

Conversely, we recover γ from W

$$(E_2) \quad \partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \, . \quad g_0(z) = z$$

$T(z)$ = maximal survival time of Loewner differential equation

$$K_t = \{ z \in \mathbb{H} : T(z) \le t \}$$

$\leadsto$ (K_t) is a growing family

- g_t : conformal map $\mathbb{H} \setminus K_t \longrightarrow \mathbb{H}$
- If W comes from a simple curve then $K_t = \gamma[0,t]$

and g_t as described above.

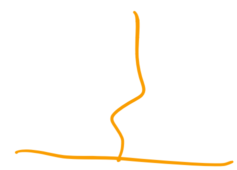
- Not all W comes from a simple chord
- If W is $\frac{1}{2}$-Hölder continuous with Hölder-norm < 4

  $\Rightarrow$ W generates a simple curve.

Definition The chordal SLE_κ in $(\mathbb{H}, 0, \infty)$ is a curve tracing out the growing family of hulls (K_t) driven by $\sqrt{\kappa} B$.

Rohde-Schramm

$\kappa \in [0, 4]$

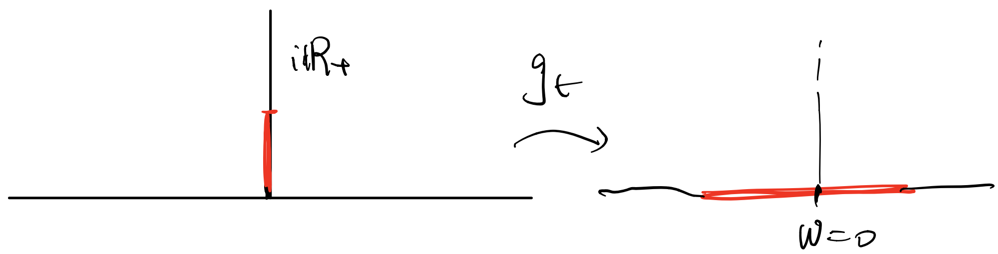


$\kappa \in (4, 8)$

$\kappa \geq 8$ Space filling

$\kappa = 0$ $\quad W \equiv 0$

$\rightsquigarrow g_t(z) = \sqrt{z^2 + 4t}$



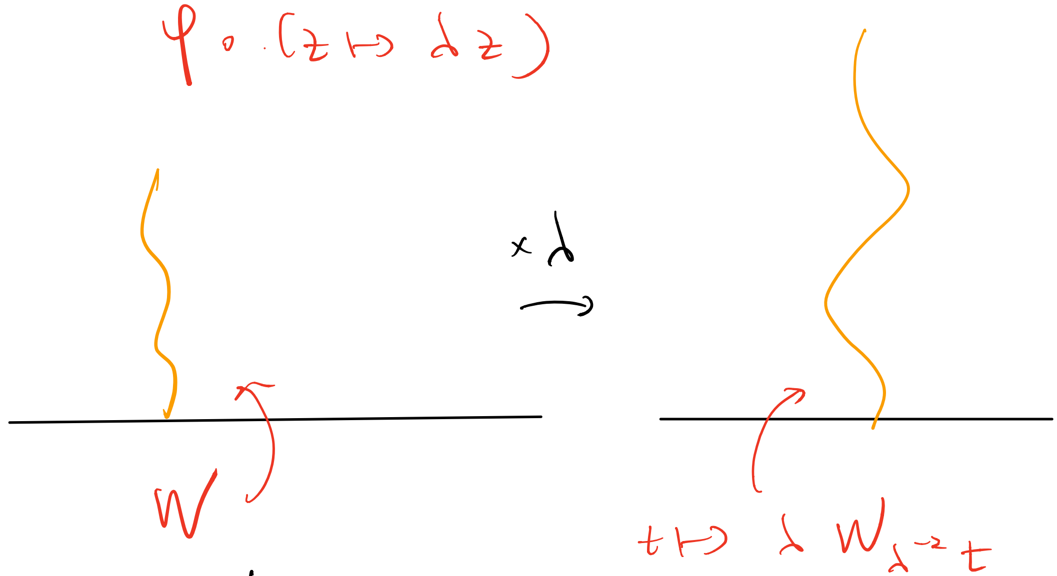
SLE$_\kappa$ in (D, x, y) is defined as the image of SLE$_\kappa$ in $(\mathbb{H}; 0, \infty)$ by $\varphi : \mathbb{H} \to D$

($\exists$ PSL$(2, \mathbb{R})$)

$\varphi(0) = x$, $\varphi(\infty) = y$

Other deg of freedom

$\varphi \circ (z \mapsto \lambda z)$



If $W = \sqrt{\kappa} B$ the law is invariant under this transformation

$\Rightarrow$ SLE$_\kappa$ welldefined in $(D; x, y)$

$\to$ SLE_0 in $(D; x, y)$

$= \varphi(i\mathbb{R})$

$=$ "hyperbolic geodesic in D from x to y"

Next

- LDP of chordal (SLE_κ) $\varepsilon = \kappa$

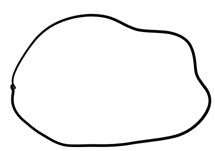
  SLE_{0+}

  $\rightsquigarrow$ Loewner energy

  $\rightsquigarrow$ Reversibility

  $\rightsquigarrow$ generalize to Jordan curves

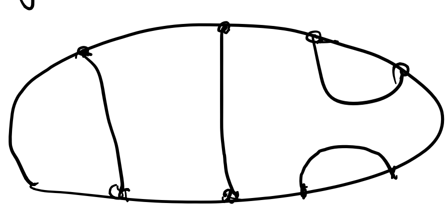
Welding · Cutting flow-lines.



Finite energy Jordan curves

$\Leftrightarrow$ Weil-Petersson (26)

$\rightsquigarrow$ generalize to multi chord



$\Rightarrow$ Multiple SLE_0

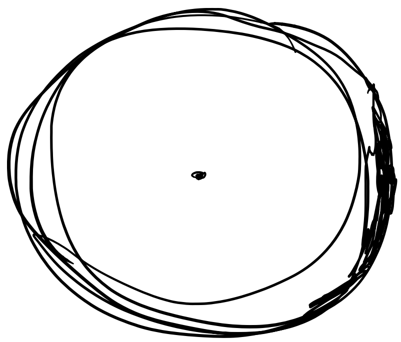
Rational functions

$\searrow$ Semi classical limit of BPZ equations

$\rightsquigarrow$ $\kappa \to \infty$ LDP $\qquad \varepsilon = \frac{1}{\kappa}$

$\longrightarrow$ Radial SLE

$\longrightarrow$ Loewner-Kufarev energy (rate function)



$\rightsquigarrow$ $\kappa \rightsquigarrow 16/\kappa$

SLE_{0+} $\qquad$ SLE_{∞}

$\nwarrow$ $\qquad$ $\nearrow$

Loewner energy $\Longleftrightarrow$ L-K energy