Two-curve Green's function of SLE

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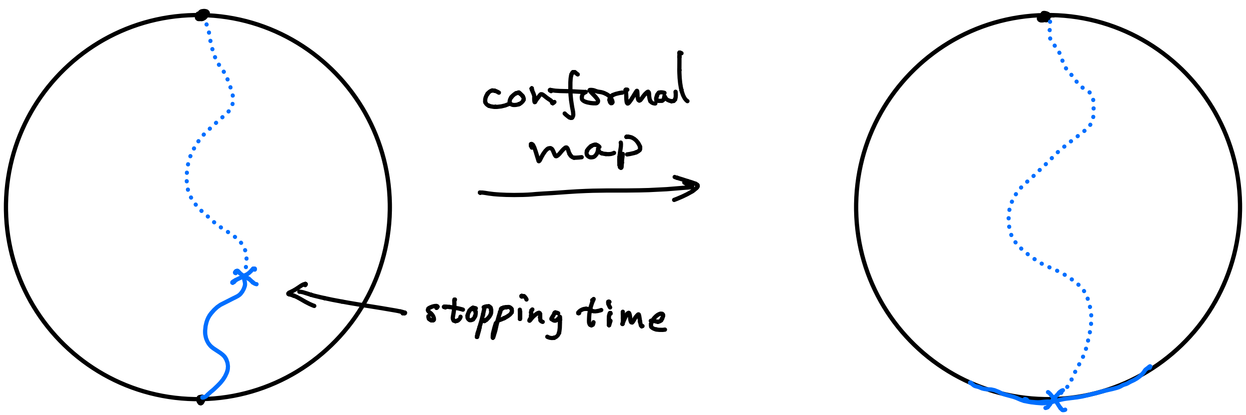
Schramm-Loewner evolution (SLE$_\kappa$) is a one-parameter family of random fractal curves characterized by

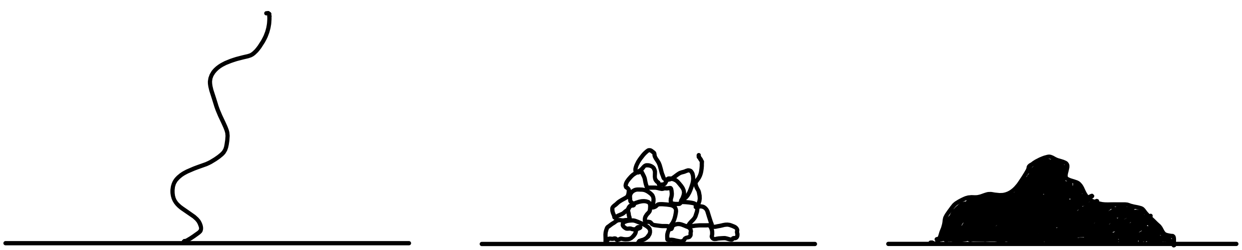
- conformal invariance (CI)
- domain Markov property (DMP)

We focus on *chordal* SLE, which grows in a simply connected domain from one boundary point to another boundary point.



The geometric property of an SLE_κ curve depends on the value of κ:

- simple curve if $\kappa \in (0, 4]$,
- space-filling if $\kappa \in [8, \infty)$,
- neither simple nor space-filling if $\kappa \in (4, 8)$.
- The Hausdorff dimension of the curve is $\min\{2, 1 + \frac{\kappa}{8}\}$.



$0 < \kappa < 4$ $\quad$ $4 < \kappa < 8$ $\quad$ $\kappa \geq 8$

We focus on the range $\kappa \in (0, 8)$. A chordal SLE$_\kappa$ curve γ

- is not space-filling:
- satisfies reversibility,
- may or may not be simple

In fact, for any $z_0 \in D$,

$$\lim_{r \downarrow 0} \mathbb{P}[\text{dist}(z_0, \gamma) < r] = \mathbb{P}[z_0 \in \gamma] = 0.$$

We are interested in the decay rate of $\mathbb{P}[\text{dist}(z_0, \gamma) < r]$ as $r \to 0$.

- The Green's function at z_0 is the limit

$$G(z_0) := \lim_{r \downarrow 0} r^{-\alpha} \mathbb{P}[\mathrm{dist}(z_0, \gamma) < r],$$

  for some suitable exponent $\alpha > 0$.
- We are interested in the value of α, the convergence of the limit, the exact formula of G, and the convergence rate.

The term "*Green's function*" is used for the following reasons. Recall that the Laplacian Green's function $G_D(z,w)$, $z \neq w \in D$, for a planar domain D. It is the function determined by the following properties: for $w \in D$,

- $G_D(\cdot, w)$ is positive and harmonic on $D \setminus \{w\}$.
- As $z \to \partial D$, $G_D(z,w) \to 0$.
- As $z \to w$, $G_D(z,w) = \frac{1}{2\pi} \ln |z-w| + O(1)$.

One important fact is

$$G(z, w) = \frac{1}{2\pi} \lim_{r \downarrow 0} (-\ln r) \cdot \mathbb{P}^z[\mathrm{dist}(w, B[0, \tau_D]) \le r],$$

where B is a planar Brownian motion started from z, and τ_D is the exit time of D. For the proof, one stops the local martingale $G_D(B_t^z, w)$ at $\tau_D \wedge \tau_r^z$ to get a bounded martingale, where τ_r^z is the first time that B gets within distance r from z.

Another important fact is: for any measurable set $U \subset D$,

$$\mathbb{E}[|\{t \in [0, \tau_D) : B_t \in U\}|] = \int_U G(w, z_0) dw.$$

This fact is clear for random walk and discrete Green's function.

About SLE Green's function, the following is a result of [Lawler-Rezaei '15]. For every $\kappa \in (0,8)$, there is a constant $\widehat{c} = \widehat{c}(\kappa) > 0$ such that for a chordal SLE_κ curve in $\mathbb{H}$ from 0 to ∞ and any $z_0 \in \mathbb{H}$,

$$\lim_{r \downarrow 0} r^{-(1-\frac{\kappa}{8})}\mathbb{P}[\mathrm{dist}(\gamma, z) < r] = \widehat{c}(\mathrm{Im}\, z_0)^{\frac{\kappa}{8}+\frac{8}{\kappa}-2}|z_0|^{1-\frac{8}{\kappa}}.$$

- The formula on the RHS was predicted by Rohde and Schramm.
- The exponent $\alpha = 1 - \frac{\kappa}{8}$ is related to the Hausdorff dimension $d = 1 + \frac{\kappa}{8}$ by $\alpha = 2 - d$.
- The constant $\widehat{c}$ is unknown so far.

The same paper derives the existence of two-point Green's function, i.e.,

$$G(z_1, z_2) := \lim_{r_1, r_2 \downarrow 0} r_1^{-\alpha} r_2^{-\alpha} \mathbb{P}[\mathrm{dist}(\gamma, z_j) < r_j, j = 1, 2],$$

and then uses those Green's functions to prove that

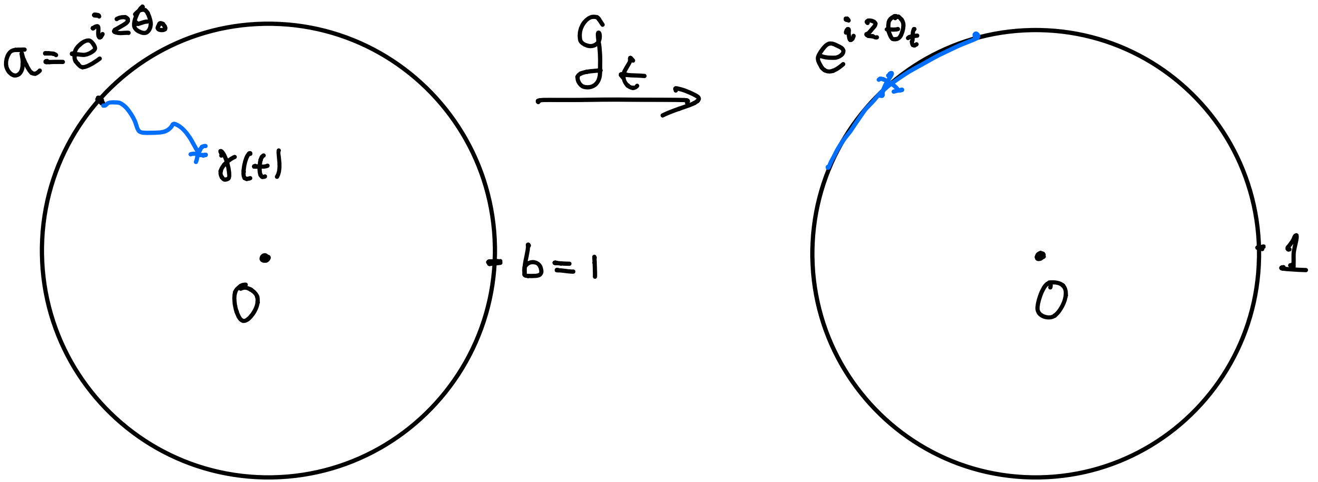
- an SLE_κ curve can be parametrized by its d-dimensional Minkowski content, i.e., for any $t_1 < t_2$, the $(1 + \frac{\kappa}{8})$-dimensional Minkowski content of $\gamma[t_1, t_2]$ is $t_2 - t_1$; and
- under such parametrization, for any measurable set $U \subset D$,

$$\mathbb{E}[|\{t : \gamma(t) \in U\}|] = \int_U G(z) dz.$$

The Minkowski content parametrization agrees with the natural prametrization introduced earlier ([Lawler-Sheffield '11, Lawler-Zhou '13]).

We now briefly review the proof for one-point interior Green's function.

- By conformal covariance, we may assume $D = \mathbb{D} = \{|z| < 1\}$, $a = e^{i2\theta_0}$, $b = 1$, $\theta_0 \in (0, \pi)$, and $z_0 = 0$.
- At each time t before the curve surrounds 0, let D_t denote the connected component of $\mathbb{D} \setminus \gamma[0, t]$ that contains 0.
- Let g_t be the conformal map from D_t onto $\mathbb{D}$, which fixes 0 and 1.
- Let $\theta_t \in (0, \pi)$ be such that $e^{i2\theta_t} = g_t(\gamma(t))$.
- We reparametrize γ such that $|g_t'(0)| = e^t$ for all t.



$a = e^{i2\theta_0}$
$\gamma(t)$
0
$b = 1$
g_t
$e^{i2\theta_t}$
0
1

Using Itô's calculus, we know that $\theta_t \in (0, \pi)$ satisfies SDE:

$$d\theta_t = \frac{\sqrt{\kappa}}{2} dB_t + \frac{\kappa - 4}{4} \cot(\theta_t) dt,$$

where B_t is a standard Brownian motion. After a linear time-change with $ds = \frac{\kappa}{4} dt$, the SDE becomes

$$d\theta_s = d\widehat{B}_s + \frac{\kappa - 4}{\kappa} \cot(\theta_s) ds.$$

After this time-change, $|g_s'(0)| = e^{\frac{4}{\kappa} s}$.

Lawler calls a process $\theta \in (0, \pi)$ that satisfies the SDE

$$d\theta_t = dB_t + \frac{\delta - 1}{2} \cot(\theta_t) dt, \quad 0 \le t < T,$$

a radial Bessel process of dimension δ.

In comparison, a Bessel process X of dimension δ satisfies the SDE:

$$dX_t = dB_t + \frac{\delta - 1}{2X_t} dt, \quad 0 \le t < T.$$

If $\delta \ge 2$, then $T = \infty$ and X stays in $(0, \infty)$; if $\delta < 2$, then $T < \infty$ and $\lim_{t \to T} X_t = 0$.

A radial Bessel process behaves like a Bessel process of the same dimension near 0 and π. If $\delta \ge 2$, $T = \infty$ and θ stays in $(0, \pi)$; if $\delta < 2$, $T < \infty$ and $\lim_{t \to T} \theta(t) \in \{0, \pi\}$.

Another similarity between Bessel processes and radial Bessel processes: when $\delta \in \mathbb{N}$,

- for a Brownian motion B_t in $\mathbb{R}^\delta$ and any $x \in \mathbb{R}^\delta$, $\mathrm{dist}_{\mathbb{R}^\delta}(B_t, x)$ is a Bessel process of dimension δ;
- for a Brownian motion B on the sphere S^δ and any $x \in S^\delta$, $\mathrm{dist}_{S^\delta}(B_t, x)$ is a radial Bessel process of dimension δ;

Recall that our SDE for the Green's function is

$$d\theta_s = d\widehat{B}_s + \frac{\kappa - 4}{\kappa} \cot(\theta_s) ds.$$

So it is a radial Bessel process of dimension $\delta = 3 - \frac{8}{\kappa} < 2$, and has a finite lifetime.

We use a basic tool: Koebe's 1/4 theorem: if f maps a domain D_1 conformally onto a domain D_2, $z_1 \in D_1$ and $z_2 = f(z_1) \in D_2$, then

$$\frac{1}{4}|f'(z_1)| \le \frac{\mathrm{dist}(z_2, \partial D_2)}{\mathrm{dist}(z_1, \partial D_1)} \le 4|f'(z_1)|.$$

Applying Koebe's 1/4 theorem to g_s (from D_s onto $\mathbb{D}$ fixing 0), we get

$$\mathrm{dist}(0, \gamma[0, s]) = \mathrm{dist}(0, \partial D_s) \asymp e^{-\frac{4}{\kappa} s}, \quad 0 \le s < T.$$

Thus, $\mathrm{dist}(0, \gamma) \asymp e^{-\frac{4}{\kappa} T}$. Letting $\alpha = 1 - \frac{\kappa}{8}$, the original limit problem can be converted to the limit

$$\lim_{t \to \infty} e^{(\frac{4}{\kappa} - \frac{1}{2})t} \mathbb{P}[T > t] = \lim_{t \to \infty} e^{\frac{2-\delta}{2} t} \mathbb{P}[T > t].$$

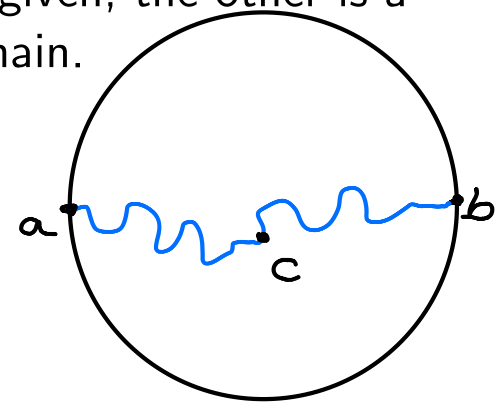
Suppose the limit $\lim_{t\to\infty} e^{\frac{2-\delta}{2}t}\mathbb{P}[T > t]$ exists and equals $G(\theta_0)$. By Markov property of the process θ, we get a martingale

$$N(t) := e^{\frac{2-\delta}{2}t} G(\theta_t).$$

Using Itô's formula, we find that $G(\theta) = c\sin^{2-\delta}\theta$ for some constant $c > 0$. This argument gives the formula of G, but does not prove the convergence of the limit.

For the convergence of the limit, we work in parallel on *two-sided radial SLE_κ curve*. A two-sided radial SLE$_\kappa$ curve

- grows in a simply connected domain D from one marked boundary point a to another marked boundary point b passing through a marked interior point c;
- may be intuitively viewed as a chordal SLE$_\kappa$ curve in D from a to b conditioned on the (singular) event that it passes through c; (The definition resembles the h-processes)
- the two arms of the curve, one from a to c, the other from b to c, satisfy the property that when one curve is given, the other is a chordal SLE$_\kappa$ curve in one complement domain. (This property resembles the multiple SLE.)



Suppose instead of the chordal SLE_κ in $\mathbb{D}$ from $e^{i2\theta_0}$ to 1, we work on the two-sided radial SLE_κ curve in $\mathbb{D}$ from $e^{i2\theta_0}$ to 1 passing through 0. We define D_s, g_s, θ_s, T for the two-sided radial SLE_κ curve up to the time that the curve reaches 0. Then the θ_t obtained is also a radial Bessel process but of dimension $\widetilde{\delta} = 4 - \delta = 1 + \frac{8}{\kappa} > 2$. So its lifetime is ∞.

The law $\mathbb{P}$ of the radial Bessel process of dimension $\delta < 2$ and the law $\widetilde{\mathbb{P}}$ of the radial Bessel process of dimension $\widetilde{\delta} = 4 - \delta > 2$ with the same starting point are related by

$$\frac{d\mathbb{P}|\mathcal{F}_t \cap \{T > t\}}{d\widetilde{\mathbb{P}}|\mathcal{F}_t \cap \{T > t\}} = \frac{N_0}{N_t} = e^{\frac{\delta-2}{2}t}\frac{\sin^{\delta-2}\theta_t}{\sin^{\delta-2}\theta_0}, \quad \forall t \geq 0,$$

where $N_t = e^{\frac{\delta-2}{2}t}\sin^{2-\delta}\theta_t$ is the martingale related to Green's function.

An eigenvalue method is used to prove that a radial Bessel process of dimension $\widetilde{\delta} > 2$ has a transition density $\widetilde{p}_t(x, y)$, which approaches its stationary density $\widetilde{p}_\infty(y) = c \sin^{3-\delta} y$ exponentially fast as $t \to \infty$.

Using the transition density $\widetilde{p}_t(x, y)$ for $\widetilde{\mathbb{P}}$ and the RN derivative connection between $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, we see that if a radial Bessel process of dimension $\delta < 2$ starts from x, then its lifetime T satisfies

$$\mathbb{P}_x[T > t] = e^{\frac{\delta-2}{2}t} \int_0^\pi \frac{\sin^{\delta-2} y}{\sin^{\delta-2} x} \widetilde{p}_t(x, y) dy$$

$$\overset{t\to\infty}{\approx} e^{\frac{\delta-2}{2}t} \int_0^\pi \frac{\sin^{\delta-2} y}{\sin^{\delta-2} x} \widetilde{p}_\infty(y) dy = c(\kappa) e^{\frac{\delta-2}{2}t} \sin^{2-\delta} x.$$

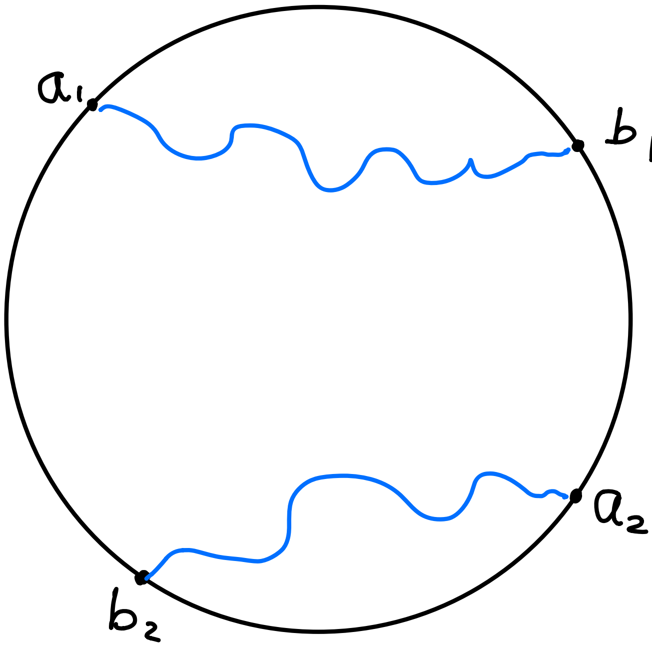
So we get the desired limit $\lim_{t\to\infty} e^{\frac{2-\delta}{2}t} \mathbb{P}[T > t]$.

Below is a list of some existing results on SLE Green's functions.

- 1- and 2-pt Green's function for chordal SLE (Lawler & Rezaei '15)
- 1- and 2-pt boundary Green's function for chordal SLE (Lawler '15)
- Green's function for radial SLE (Alberts, Kozdron & Lawler '12)
- n-point Green's function for chordal SLE (Rezaei & Z. '18)
- One-curve Green's function for 2-SLE (Lenells & Viklund '19).
- Two-curve Green's function for 2-SLE (Z. '20)
- Two-curve boundary Green's function for 2-SLE (Z.)
- Green's function for cut-points of SLE (Z.)

We now study the two-curve Green's function for 2-SLE$_\kappa$ for $0 < \kappa < 8$. Let D be a simply connected domain. Fix four boundary points (a_1, b_1, a_2, b_2) of D ordered clockwise or counterclockwise. A 2-SLE$_\kappa$ configuration in D with link pattern $(a_1 \to b_1; a_2 \to b_2)$ is a pair of random curves (γ_1, γ_2) in $\overline{D}$ satisfying, for $j = 1, 2$,

- γ_j grows from a_j to b_j;
- If γ_{3-j} is given, then γ_j is a chordal SLE$_\kappa$ curve in a complement domain of $D \setminus \gamma_{3-j}$.
- We have existence, uniqueness, and conformal invariance.
- γ_1 and γ_2 are disjoint if $\kappa \le 4$; may intersect if $\kappa \in (4, 8)$.
- The marginal law of each γ_j can be described by a hypergeometric function ([Wu '20]).



a_1
b_1
a_2
b_2

The two-curve Green's function for a 2-SLE$_\kappa$ pair (γ_1, γ_2) at a point $z_0 \in D$ is the limit

$$\lim_{r \downarrow 0} r^{-\alpha} \mathbb{P}[\mathrm{dist}(z_0, \gamma_j) < r, j = 1, 2]$$

for some suitable exponent $\alpha > 0$.

Assuming that $D = \mathbb{D}$ and $z_0 = 0$, we mimic the proof of the one-curve Green's function. Now we have two curves, each has its own parametrization. For each pair of times (t_1, t_2), let D_{t_1,t_2} denote the complement of $\gamma_1[0, t_1] \cup \gamma_2[0, t_2]$ containing 0, and let g_{t_1,t_2} be a conformal map from D_{t_1,t_2} onto $\mathbb{D}$ that fixes 0. By Koebe's 1/4 theorem,

$$|g'_{t_1,t_2}(0)|^{-1} \asymp \mathrm{dist}(0, \partial D_{t_1,t_2}) = \min\{\mathrm{dist}(0, \gamma_j[0, t_j]) : j = 1, 2\}.$$

But we want the decay rate of

$$\mathbb{P}[\max\{\mathrm{dist}(z_0, \gamma_j) : j = 1, 2\} < r], \quad \text{as } r \to 0.$$

To overcome this problem, we use a single time parametrization for both curves. In other words, we grow two curves simultaneously. We want that for any t in the lifespan,

- $\min\{\mathrm{dist}(0, \gamma_j[0, t]) : j = 1, 2\} \asymp e^{-t}$;
- $\mathrm{dist}(0, \gamma_1[0, t]) \asymp \mathrm{dist}(0, \gamma_2[0, t])$.

These two properties together imply that

$$\max\{\mathrm{dist}(0, \gamma_j[0, t]) : j = 1, 2\} \asymp e^{-t}.$$

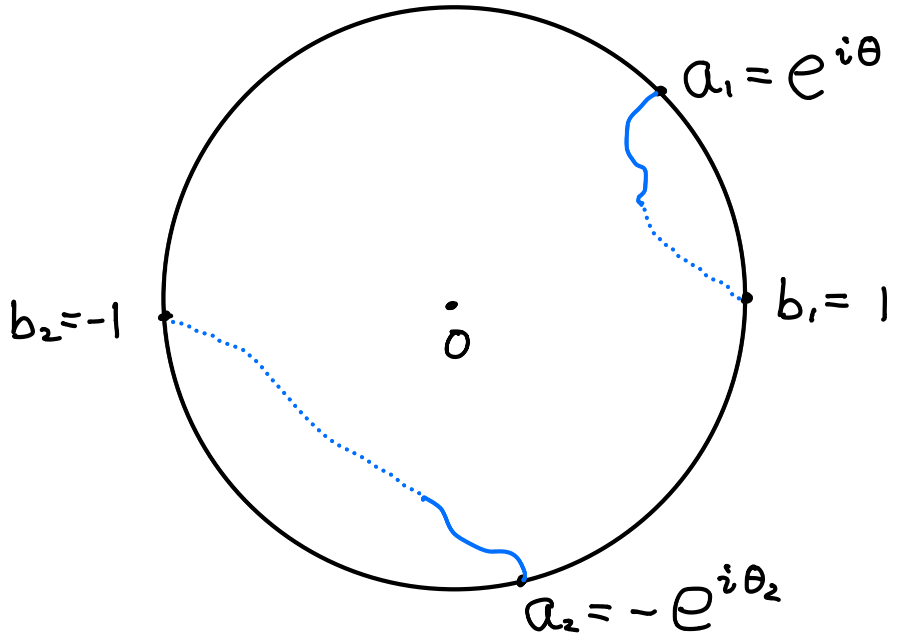
The time parametrization is possible. For simplicity, we suppose $b_1 = 1$, $b_2 = -1$, $a_1 = e^{i\theta_1}$ and $a_2 = -e^{i\theta_2}$ for some $\theta_1, \theta_2 \in (0, \pi)$. At the beginning, b_1 and b_2 equally divide the harmonic measure of $\partial\mathbb{D}$ viewed from 0. Now we grow γ_1 and γ_2 simultaneously such that for any t

- if g_t maps D_t (the connected component of $\mathbb{D} \setminus (\gamma_1[0,t] \cup \gamma_2[0,t])$ that contains 0) conformally onto $\mathbb{D}$, and fixes 0, then $|g_t'(0)| = e^t$.
- $b_1 = 1$ and $b_2 = -1$ equally divide the harmonic measure of ∂D_t viewed from 0.

The first property implies that $\min\{\mathrm{dist}(0, \gamma_j[0,t]) : j = 1, 2\} \asymp e^{-t}$ by Koebe's 1/4 theorem.

if $>>$ $\Rightarrow \notin$ harm measure equally divided

The second property implies that $\mathrm{dist}(0, \gamma_1[0,t]) \asymp \mathrm{dist}(0, \gamma_2[0,t])$ by Beurling's estimate applied to a planar Brownian motion started from 0.



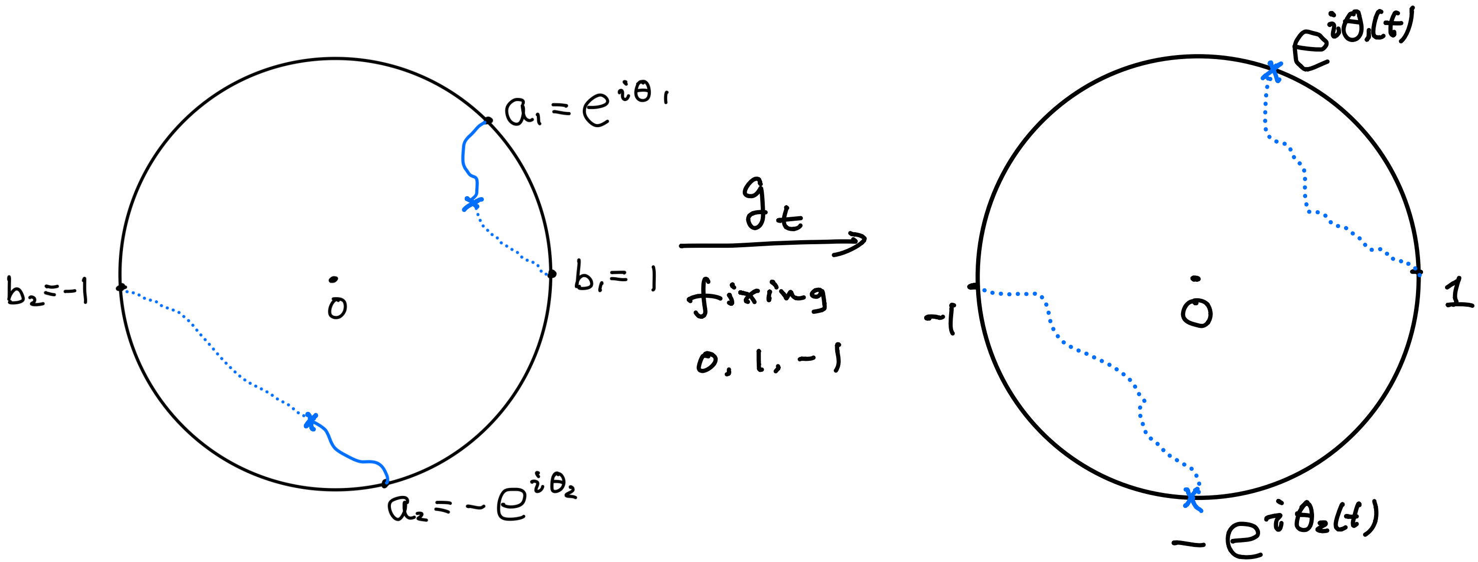
$a_1 = e^{i\theta_1}$
$b_2 = -1$
0
$b_1 = 1$
$a_2 = -e^{i\theta_2}$

The two-curve Green's function is then closely related to the limit

$$\lim_{t\to\infty} e^{\alpha t}\mathbb{P}[T > t],$$

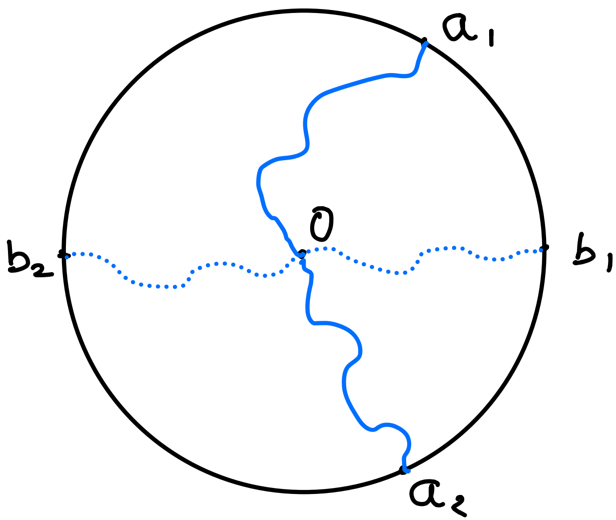
where T is the lifetime of the above growth of two curves.

Since for each $t \in [0, T)$, 1 and -1 equally divide the harmonic measure of ∂D_t viewed from 0, we may assume that g_t maps D_t conformally onto $\mathbb{D}$, and fixes $0, 1, -1$. Then we have processes $\theta_1(t), \theta_2(t) \in (0, \pi)$ such that $e^{i\theta_1(t)} = g_t(\gamma_1(t))$ and $-e^{i\theta_2(t)} = g_t(\gamma_2(t))$. Let $\mathbb{P}$ denote the law of the two-dimensional process $(\theta_1(t), \theta_2(t))_{0\le t<T}$.



$a_1 = e^{i\theta_1}$
$b_1 = 1$
$b_2 = -1$
0
$a_2 = -e^{i\theta_2}$
g_t
fixing
$0, 1, -1$
$e^{i\theta_1(t)}$
-1
0
1
$-e^{i\theta_2(t)}$

We consider another random configuration which can be understood as the 2-SLE$_\kappa$ curves γ_1 and γ_2 conditioned to both pass through 0. This is a four-curve configuration in $\mathbb{D}$ connecting 0 with a_1, b_1, a_2, b_2. They satisfy the property that, when any three curves are given, the last curve is a chordal SLE$_\kappa$ curve in one complement domain. We call it a 4-SLE$_\kappa$.



We now work on this 4-SLE$_\kappa$, and grow γ_1 and γ_2 from a_1 and a_2 towards 0 simultaneously in the same way as before. Then we also get a process $(\theta_1(t), \theta_2(t))$. The lifetime of this new process is ∞. Let $\widetilde{\mathbb{P}}$ denote its law. Then this $\widetilde{\mathbb{P}}$ and the previous $\mathbb{P}$ are related by

$$\frac{d\mathbb{P}|\mathcal{F}_t \cap \{T > t\}}{d\widetilde{\mathbb{P}}|\mathcal{F}_t \cap \{T > t\}} = e^{-\alpha t} \frac{G(\theta_1(0), \theta_2(0))}{G(\theta_1(t), \theta_2(t))}, \quad t \geq 0.$$

where

$$\alpha = \frac{(12-\kappa)(\kappa+4)}{8\kappa},$$

$$G(\theta_1, \theta_2) = (\sin(\frac{\theta_1}{2}) \sin(\frac{\theta_2}{2}))^{\frac{8}{\kappa}-1} \cos(\frac{\theta_1 - \theta_2}{2})^{\frac{4}{\kappa}} F\Big(\frac{\cos(\theta_1/2)\cos(\theta_2/2)}{\cos((\theta_1-\theta_2)/2)}\Big),$$

and F is the hypergeometric function ${}_2F_1(1-\frac{4}{\kappa}, \frac{4}{\kappa}; \frac{8}{\kappa}, \cdot)$.

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n \qquad (a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1),\ n \geq 1$$

Under $\widetilde{\mathbb{P}}$, $(\theta_1(t), \theta_2(t))$ is a diffusion process satisfying the SDE

$$d\theta_j = \sqrt{\frac{\kappa \sin\theta_j}{\sin\theta_1 + \sin\theta_2}}\, dB_j + \frac{4\cos\theta_j}{\sin\theta_1 + \sin\theta_2}\, dt, \quad j = 1, 2,$$

where B_1 and B_2 are independent Brownian motions.

Using an eigenvalue method, we may calculate the transition density $\widetilde{p}_t((x_1, x_2), (y_1, y_2))$ of the process $(\theta_1(t), \theta_2(t))$ under $\widetilde{\mathbb{P}}$. As $t \to \infty$, it converges exponentially fast to the invariant density $\widetilde{p}_\infty(y_1, y_2)$, which is proportional to $(\sin y_1 \sin y_2)^{\frac{8}{\kappa}-1}(\sin y_1 + \sin y_2)$.

Then we calculate

$$\mathbb{P}[T > t] = \int_0^\pi \int_0^\pi e^{-\alpha t} \frac{G(\theta_1, \theta_2)}{G(y_1, y_2)} \widetilde{p}_t((\theta_1, \theta_2), (y_1, y_2)) dy_1 dy_2$$

$$\overset{t \to \infty}{\approx} \int_0^\pi \int_0^\pi e^{-\alpha t} \frac{G(\theta_1, \theta_2)}{G(y_1, y_2)} \widetilde{p}_\infty(y_1, y_2) dy_1 dy_2$$

So we get

$$\lim_{t \to \infty} e^{\alpha t} \mathbb{P}[T > t] = cG(\theta_1, \theta_2).$$

where

$$c = \int_0^\pi \int_0^\pi G(y_1, y_2)^{-1} \widetilde{p}_\infty(y_1, y_2) dy_1 dy_2 \in (0, \infty).$$

For $\kappa \in (4,8)$, the exponent $\alpha = \frac{(12-\kappa)(\kappa+4)}{8\kappa}$ is related to the Hausdorff dimension d of the double points of a single SLE_κ curve ([Miller-Wu '13]) by $\alpha = 2 - d$. A double point of a curve is a point that is visited by the curve for more than once.

Our long term goal is to prove the existence of Minkowski content of double points of SLE_κ, which is related to the Minkowski content of the intersection of the curves of a 2-SLE_κ. For that purpose, we need the two-curve two-point Green's function for 2-SLE_κ, i.e.,

$$\lim_{r_1, r_2 \downarrow 0} r_1^{-\alpha} r_2^{-\alpha} \mathbb{P}[\text{dist}(\gamma_j, z_k) < r_k, j, k \in \{1,2\}].$$

The existence of this limit is currently beyond the reach.

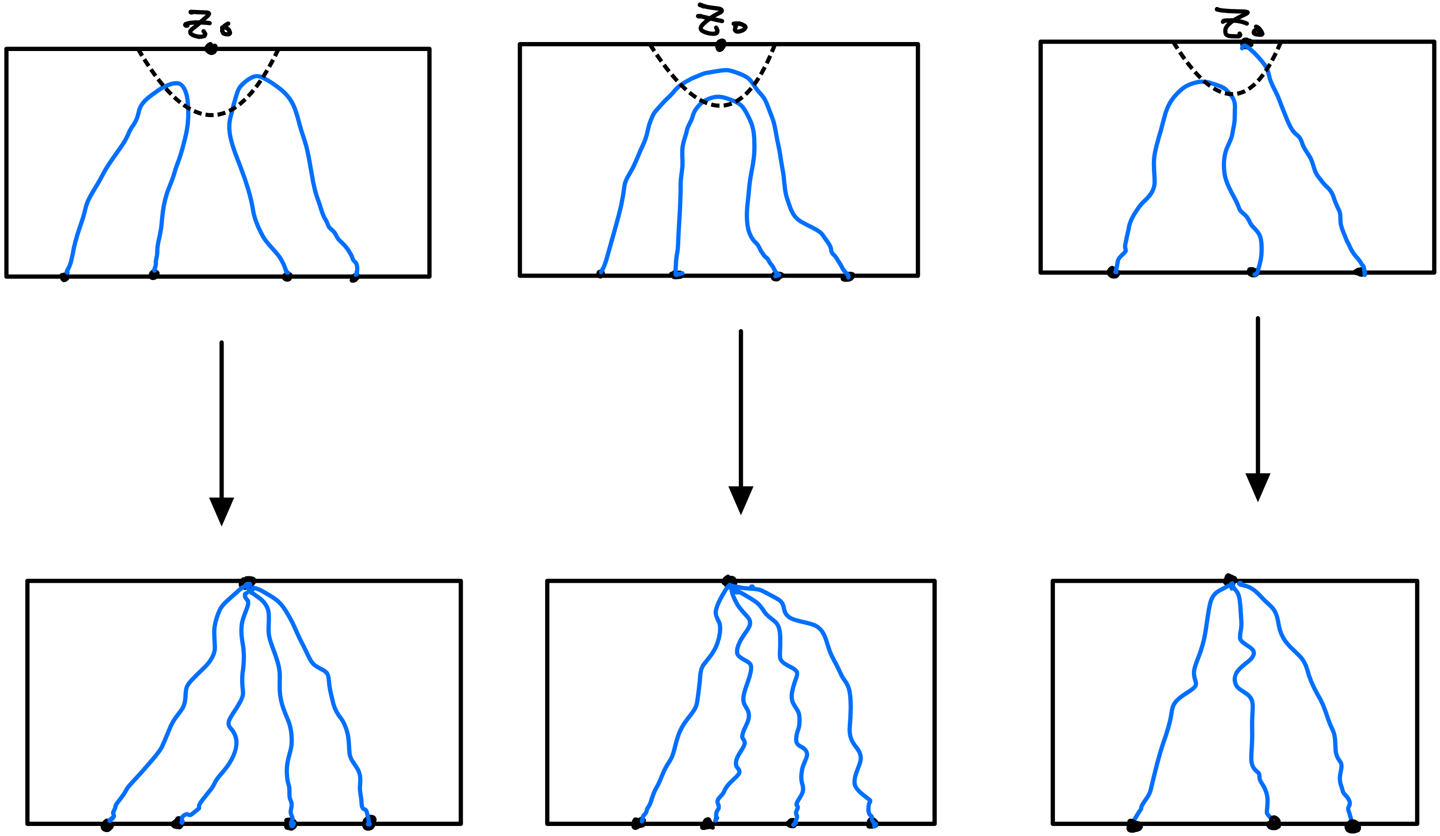
The above technique may be used to study the boundary two-curve Green's function. Let (γ_1, γ_2) be the 2-SLE$_\kappa$ as before. Let $z_0 \in \partial D$ be such that ∂D is analytic near z_0. We are interested in the limit

$$\lim_{r \downarrow 0} r^{-\alpha} \mathbb{P}[\mathrm{dist}(z_0, \gamma_j) < r, j = 1, 2].$$

We may assume that $D = \mathbb{H}$ and $z_0 = \infty$. Then the limit becomes

$$\lim_{R \to \infty} R^{\alpha} \mathbb{P}[\gamma_j \cap \{|z| > R\} \neq \emptyset, j = 1, 2].$$

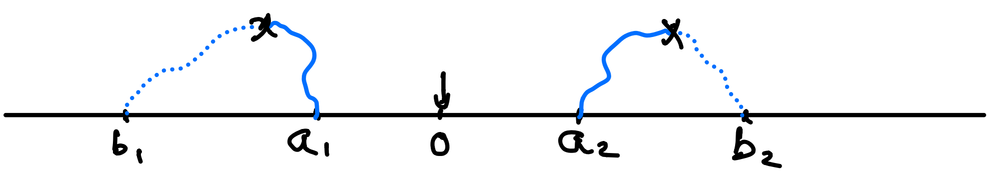
There are three different cases.

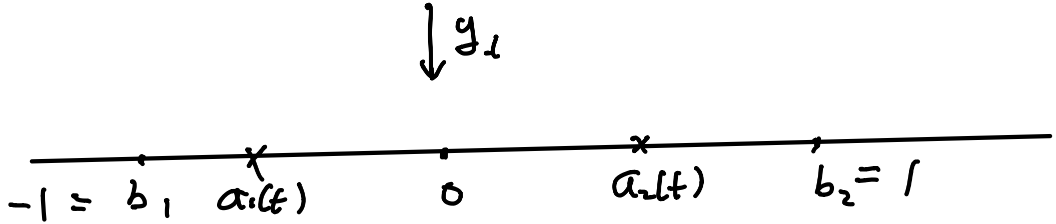

z_0
z_0
z_0

For the first case, we label the end points of the two curves by $b_1 < a_1 < a_2 < b_2$. For simplicity, we assume that $b_1 = -1$, $b_2 = 1$, and $a_1 < 0 < a_2$.

Now we grow γ_1, γ_2 simultaneously from a_1, a_2 such that for every t in the life span $[0, T)$,

- the harmonic measure of $\gamma_1[0,t] \cup \gamma_2[0,t] \cup [b_1, b_2]$ in D_t viewed from ∞ increases exponentially;
- $\gamma_1[0,t] \cup [b_1, 0]$ and $\gamma_2[0,t] \cup [0, b_2]$ have equal harmonic measure in D_t viewed from ∞.





By Koebe's $1/4$ theorem and Beurling's estimate, we then conclude that $1 \vee \mathrm{diam}(\gamma_j[0,t]) \asymp e^t$. So the original limit is closely related to the limit

$$\lim_{t\to\infty} e^{\alpha t}\mathbb{P}[T > t].$$

For each $t \in [0, T)$, suppose g_t maps D_t conformally onto $\mathbb{H}$, and fixes $\infty, 1, -1$. Then we get a two-dimensional process $(a_1(t), a_2(t))$ in $[-1, 0] \times [0, 1]$ by $a_j(t) = g_t(\gamma_j(t))$, $j = 1, 2$.

In comparison, we now work on a 4-SLE$_\kappa$ in $\mathbb{H}$ with link pattern $(a_1 \to \infty, b_1 \to \infty, a_2 \to \infty, b_2 \to \infty)$. We grow the curves from a_1 and a_2 towards ∞ simultaneously with the same property as before, and get a 2-dimensional process $(a_1(t), a_2(t))$, whose lifetime is ∞. The law $\widetilde{\mathbb{P}}$ of $(a_1(t), a_2(t))$ for the 4-SLE$_\kappa$ and the law $\mathbb{P}$ for the 2-SLE$_\kappa$ are related by a Radon-Nikodym derivative process.

Under $\widetilde{\mathbb{P}}$, we have a transition density of the process $(a_1(t), a_2(t))$, which converges to the invariant density as $t \to \infty$.

Using the above facts and the same argument as in the interior case, we conclude that, for $\alpha = 2(\frac{12}{\kappa} - 1)$, the limit $\lim_{t\to\infty} e^{\alpha t}\mathbb{P}[T > t]$ converges to a nontrivial number as $t \to \infty$.

The technique also works in the other two cases, in which we compare 2-SLE_κ respectively with 4-SLE_κ and 3-SLE_κ.

Another application of the two-curve technique is the Green's function for cut points of SLE_κ. For a connected set K, a point z is called a cut point of K if $K \setminus z$ is not connected.

- For $\kappa \in (0, 4]$, every point on an SLE_κ curve is a cut point.
- For $\kappa \geq 8$, an SLE_κ curve has no cut point.
- For $\kappa \in (4, 8)$, the set of cut points of an SLE_κ curve is not empty, and has Hausdorff dimension $3 - \frac{3}{8}\kappa$ ([Miller-Wu '13]).

We now assume that $\kappa \in (4, 8)$.

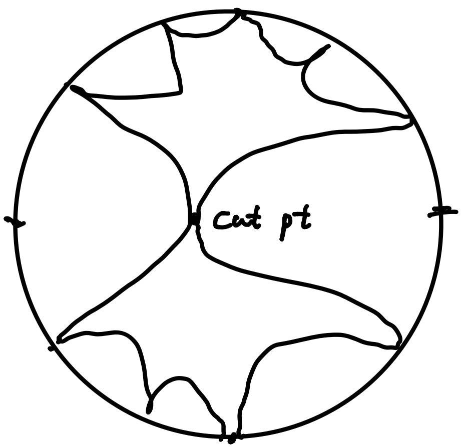
In order to apply the two-curve technique, we attach the SLE curve with two open boundary arcs and consider the cut points of the union.

Setup: Let D be a simply connected domain with four distinct boundary points a_1, a_2, u, v. Suppose u and v divide ∂D into two open boundary arcs: I_1, I_2 such that $a_j \in I_j$, $j = 1, 2$. Let γ be an SLE$_\kappa$ curve in D from a_1 to a_2. Let S_c denote the set of cut points of $\gamma \cup I_1 \cup I_2$. Let $z_0 \in D$. Then we study the limit

$$\lim_{r \downarrow 0} r^{-\alpha} \mathbb{P}[\mathrm{dist}(z_0, S_c) < r].$$

We may assume that $D = \mathbb{D}$, $z_0 = 0$, $u = 1$ and $v = -1$. We simultaneously grow two curves from a_1 and a_2 respectively along γ and its time-reversal in the same way as before, and get a two-dimensional process $(\theta_1(t), \theta_2(t))$ in $(0, \pi)^2$ with finite lifetime T. Then we need to study the limit

$$\lim_{t \to \infty} e^{\alpha t} \mathbb{P}[T > t].$$



In comparison, we work on a random curve that can be understood as the γ conditioned on the event that $0 \in S_c$. This is a curve from a_1 to a_2 passing though 0. For this process we also get a process $(\theta_1(t), \theta_2(t))$ in $(0, \pi)^2$. This process has lifetime ∞. Its law $\widetilde{\mathbb{P}}$ and the law $\mathbb{P}$ of the original $(\theta_1(t), \theta_2(t)$ are related by Radon-Nikodym derivatives.

Under $\widetilde{\mathbb{P}}$, $(\theta_1(t), \theta_2(t))$ has a transition density, which converges to the invariant density as $t \to \infty$.

Using the above facts and the same argument as before, we conclude that, for $\alpha = \frac{3}{8}\kappa - 1$, the limit $\lim_{t\to\infty} e^{\alpha t}\mathbb{P}[T > t]$ converges as $t \to \infty$.

Thank you!