# Distances associated with Liouville quantum gravity

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Based on joint works with
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Oct 8, 2020

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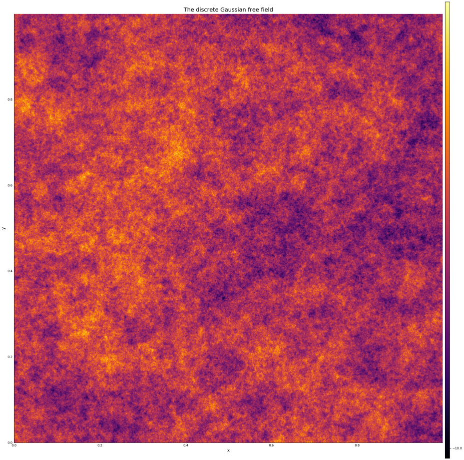

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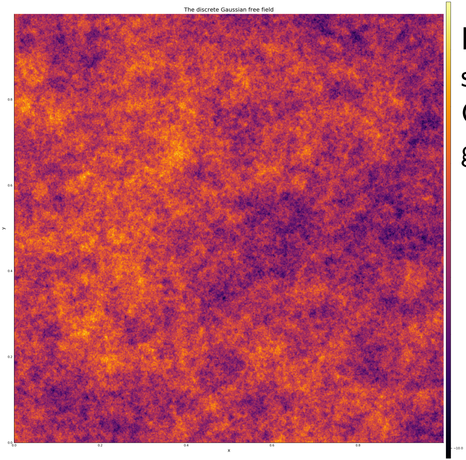


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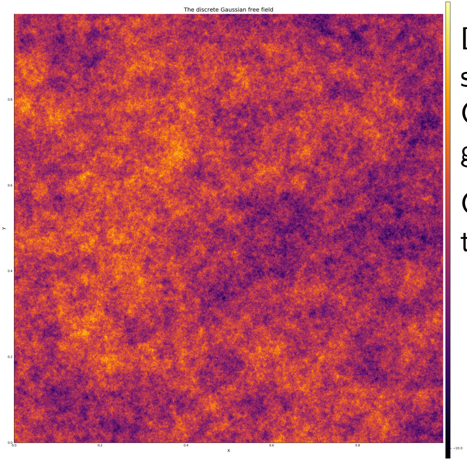


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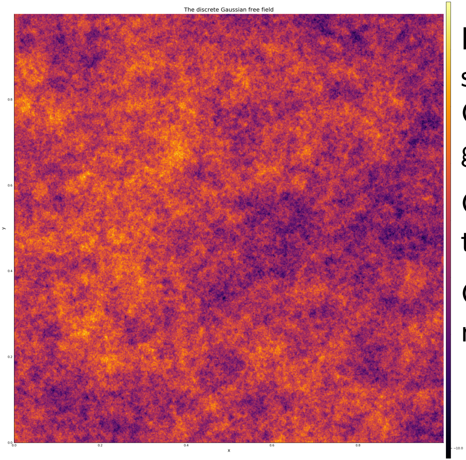


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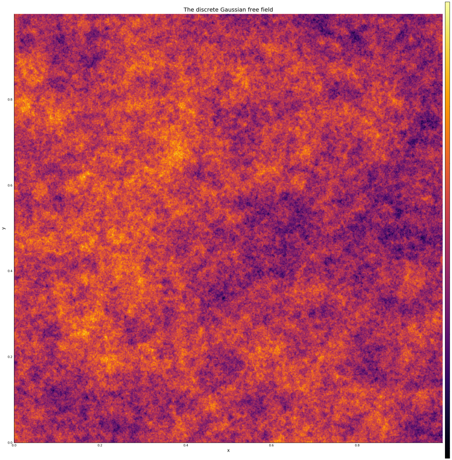


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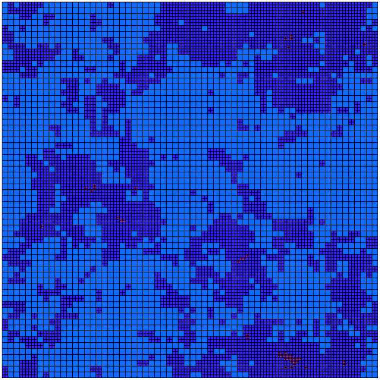

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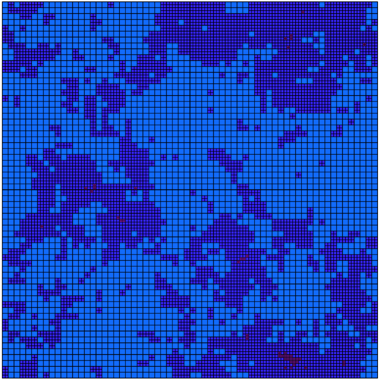

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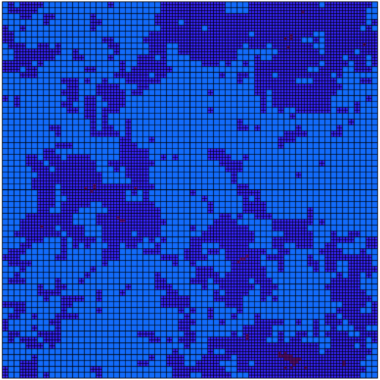

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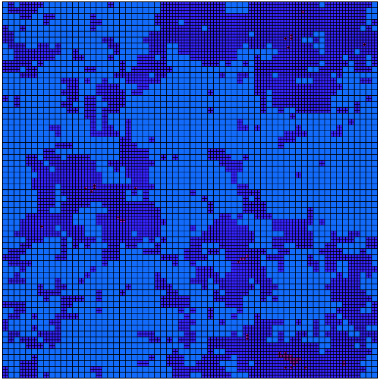

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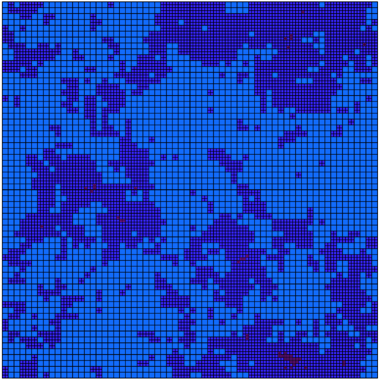

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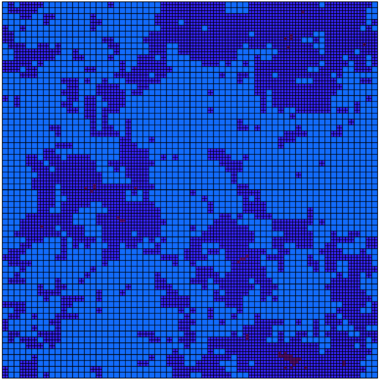

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Distance on LQG: today's focus.

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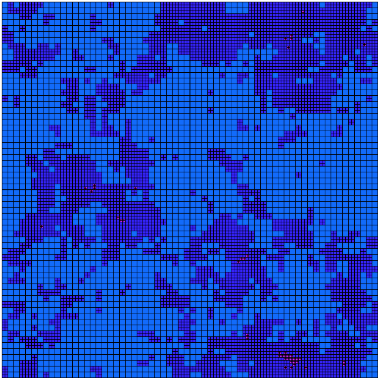

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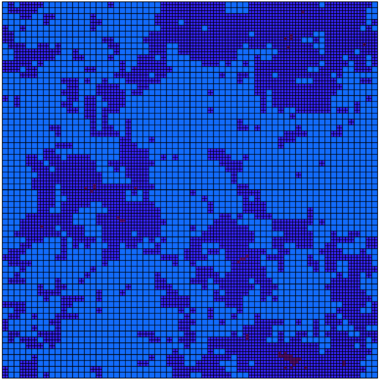

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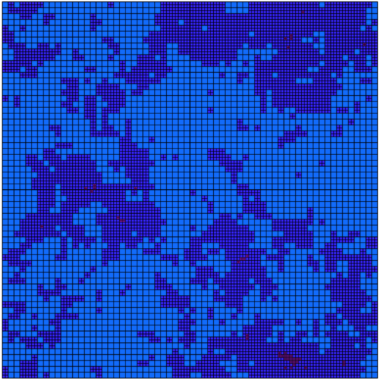

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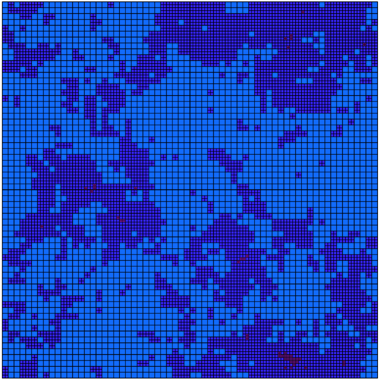

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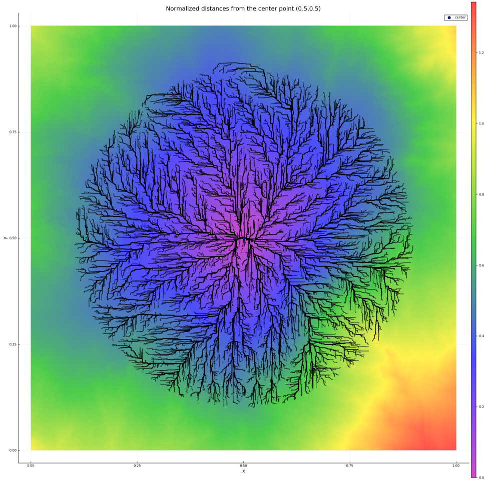
Heat kernel of LBM in short time is another way to make sense of the LQG distance. (Varadhan 67: short time heat kernel of BM relates to geodesic distance, for uniformly elliptic generator.)

# Liouville first passage percolation

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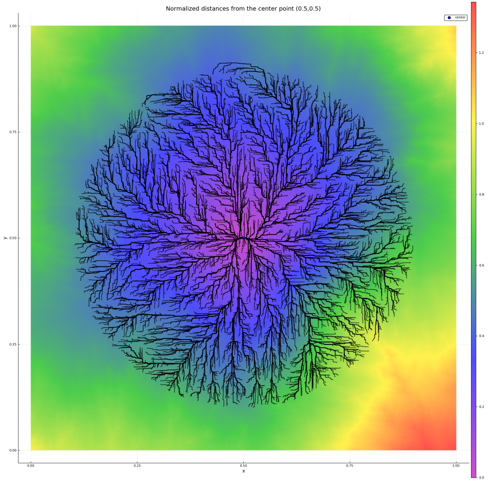


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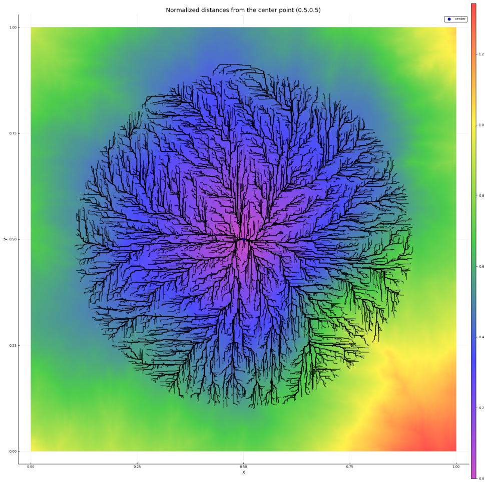


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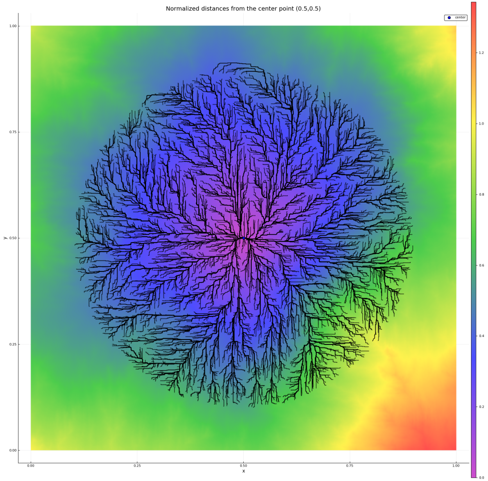


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LFPP v.s. FPP: strong correlation and hierarchical structure of the random media makes a drastic difference.

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May become universal if we pose regularity assumptions on covariance kernel, but most analysis of log-correlated fields do not depend on such regularity assumption.

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**Theorem.** (D.–Gwynne 18, Gwynne–Holden–Sun 17) For certain random planar maps (such as UIPT, mated-CRT maps,...), $d_\gamma$ also describes the exponent for the graph distance of the planar map.

Remark: By known estimates for the UIPT (Angel, 2003) $d_{\sqrt{8/3}} = 4$. New for LQG distance.

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**Theorem.** (D.–Zhang 16) (discrete case) For small but fixed $\gamma > 0$, the box-counting dimension of the geodesic is strictly larger than 1.

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• Gwynne 19 computed the exact dimension for the boundary of the LQG metric ball in terms of $d_\gamma$.

**Open:** relate the dimension of geodesic to $d_\gamma$?

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**D.–Dunlap–Falconet–Dubédat 19, D.–Dunlap 18**
The normalized distance $\tilde{D}_{\gamma,\delta}(\cdot,\cdot)$ is tight with respect to uniform topology of continuous functions from $[0,1]^2 \times [0,1]^2$ to $\mathbb{R}^+$. In addition, all possible (conjecturally unique) scaling limits are bi-Hölder-continuous with respect to the Euclidean distance.

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Remark: the LQG distance constructed by Miller–Sheffield 16 at $\gamma = \sqrt{8/3}$ also satisfies the aforementioned axioms and thus the same as the scaling limit of Liouville FPP; in addition by Miller–Sheffield, this is equivalent to the distance of Brownian map.

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Open: uniqueness for Liouville graph distance; universality for all limits of reasonable discrete approximations of LQG distances?

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