Optimal Tracking Portfolio with A Ratcheting Capital Benchmark

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Outline

Portfolio with Benchmark

- Portfolio allocation with benchmark performance has been an active research topic in the asset management.
- Related work in *Browne (2000)*, *Gaivoronski et al. (2005)*, *Yao et al. (2006)* and *Strub and Baumann (2018)*.
- The target benchmark is either a prescribed capital process or a fixed portfolio in the financial market.
- The goal is to choose the portfolio in a passive way to dynamically track the return or the value of the benchmark process.
- In practice, both professional and individual investors may measure their porfolio performance using different benchmarks, such as S&P500 index, inflation and exchange rates.

Formulation of Control Problem

- Some dominating mathematical problems in the existing studies are to minimize the difference between the controlled portfolio and the benchmark
- The resulting problem is a LQ control problem using the mean-variance analysis or a utility maximization problem at the terminal time.
- In our paper, we formulate a different tracking procedure and examining the associated control problem.
- Consider a fund manager who can dynamically inject capital into the portfolio account such that the total capital stays above the benchmark process as an American type floor constraint at each intermediate time.

Ratcheting Capital Benchmark

- Our control problem combines the regular portfolio control and the singular capital injection control.
- The optimality is attained when the cost from the accumulative capital injection is minimized.
- In particular, we are interested in the case when the benchmark process in nondecreasing, representing a ratcheting capital that the fund manager wants to closely follow.
- Our problem in fact focuses on the opposite side of the monotone follower problem studied in *Karatzas and Shreve (1984)* and *Bayraktar and Egami (2008)*.

Outline

Risky Asset Model

- The financial market consists of d risky assets and the price dynamics is described by, for $t \in [0, T]$,

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_t^j, \quad i = 1, \ldots, d.$$

- Denote by θ_t^i the amount of wealth that the fund manager allocates in asset $S^i = (S_t^i)_{t \in [0,T]}$ at time t.
- The wealth process under the control $\theta = (\theta_t^1, \ldots, \theta_t^d)_{t \in [0,T]}^\top$ is

$$V_t^\theta = \mathrm{v} + \int_0^t \theta_s^\top \mu ds + \int_0^t \theta_s^\top \sigma dW_s, \quad t \in [0, T].$$

Ratcheting Capital Benchmark Model

- The benchmark process is defined as a nondecreasing process $A = (A_t)_{t\in[0,T]}$ taking the absolutely continuous form that

$$A_t := a + \int_0^t f(s, Z_s)ds, \quad dZ_t = \mu_Z(Z_t)dt + \sigma_Z(Z_t)dW_t^{\gamma}.$$

- The process $W^{\gamma} = (W_t^{\gamma})_{t\in[0,T]}$ is a linear combination of $(W^1, \ldots, W^d)$ with weights $\gamma = (\gamma_1, \ldots, \gamma_d)^{\top} \in [-1, 1]^d$, which itself is a Brownian motion.
- We impose the following assumption:

($\mathbf{A}_f$) the function $f : [0, T] \times \mathbb{R} \to \mathbb{R}_+$ is continuous and for $t \in [0, T]$, $f(t, \cdot) \in C^2(\mathbb{R})$ with bounded first and second order derivatives.

($\mathbf{A}_Z$) the coefficients $\mu_Z : \mathbb{R} \to \mathbb{R}$ and $\sigma_Z : \mathbb{R} \to \mathbb{R}$ belong to $C^2(\mathbb{R})$ with bounded first and second order derivatives.

Formulation of Optimal Tracking

- The optimal tracking problem is defined to

minimize the expected cost of the discounted accumulative capital injection subjecting to a American-type floor constraint that

$$u(a, \mathrm{v}, z) := \inf_{C,\theta} E\left[C_0 + \int_0^T e^{-\rho t} dC_t\right]$$

subject to $A_t \leq C_t + V_t^\theta$ at each $t \in [0, T]$.

- The constant $\rho \geq 0$ is the discount rate and $C_0 = (a - \mathrm{v})^+$ is the initial injected capital to match with the initial benchmark.
- We refer to θ as the regular control and $C = (C_t)_{t\in[0,T]}$ as the singular control.

Representation of Optimal Singular Control

- Observe that: for a fixed control θ, the (nonnegative) optimal C is always the smallest adapted right-continuous and nondecreasing process that dominates $A - V^{\theta}$.

Lemma (Equivalent Unconstrained Control Problem)

For each fixed regular control θ, the optimal singular control C^* satisfies

$$C_t^* = 0 \vee \sup_{s\in[0,t]} (A_s - V_s^{\theta}), \quad t \in [0, T].$$

The optimal tracking problem admits the equivalent formulation as a unconstrained control problem under a running maximum cost that

$$u(a, \mathrm{v}, z) = (a - \mathrm{v})^{+} + \inf_{\theta\in\mathcal{U}} E\left[\int_0^T e^{-\rho t} d\left(0 \vee \sup_{s\in[0,t]} (A_s - V_s^{\theta})\right)\right].$$

Optimal Singular Control C_t

- Observe that: for a fixed control θ, the (nonnegative) optimal C is always the smallest adapted right-continuous and nondecreasing process that dominates $A - V^{\theta}$.

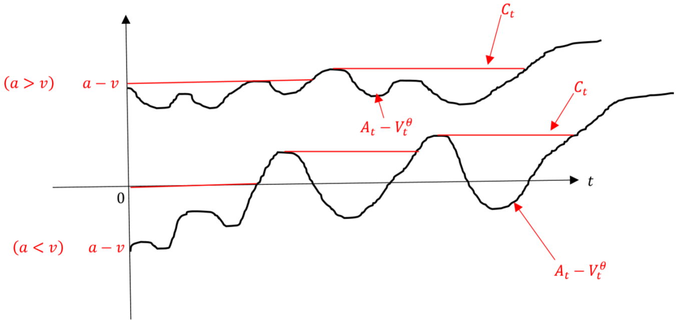

Figure: A sample path of the optimal singular control $t \to C_t$

Outline

Formulation of Auxiliary Control Problem

- Let us first define the difference process $D_t := A_t - V_t^{\theta} + \mathrm{v} - a$ with the initial value $D_0 = 0$.
- For any $x \geq 0$, we then consider its running maximum process:

$$L_t := x \vee \sup_{s \in [0,t]} D_s \geq 0, \quad t \in [0, T].$$

- Then, $(a - \mathrm{v})^+ - u(a, \mathrm{v}, z)$ is equivalent to

Auxiliary Control Problem given by

$$\text{(ACP)} \qquad \sup_{\theta \in \mathcal{U}} E\left[-\int_0^T e^{-\rho s} dL_s \right], \quad L_0 = x = (\mathrm{v} - a)^+.$$

Auxiliary Controlled State Process

- We introduce auxiliary controlled state process $X = (X_t)_{t\in[0,T]}$ for (ACP), which is defined as the reflected process $X_t := L_t - D_t$ that satisfies SDE: $X_0 = x \geq 0$, and

$$X_t = -\int_0^t f(s, Z_s)ds + \int_0^t \theta_s^\top \mu ds + \int_0^t \theta_s^\top \sigma dW_s + L_t.$$

- The running maximum process L_t increases if and only if $X_t = 0$, i.e., $L_t = D_t$.
- In view of "the Skorokhod problem", it satisfies the representation:

$$L_t = x \vee \int_0^t \mathbf{1}_{\{X_s=0\}} dL_s, \quad t \in [0, T].$$

- Hereafter, we shall replace L with L^X so to highlight the dependence of L on X.

Dynamic Version of (ACP)

- For $(t,z,x)\in\mathcal{D}_T:=[0,T]\times\mathbb{R}\times[0,\infty)$, the dynamic version of (ACP) is given by

$$v(t,z,x):=\sup_{\theta\in\mathcal{U}_t}E_{t,z,x}\left[-\int_t^T e^{-\rho s}dL_s^X\right].$$

- The Admissible Control Set (ACS) is defined as:

$\mathcal{U}_t$: the set of the feedback controls $\theta_s=\theta(s,Z_s,X_s)$ for $s\in[t,T]$, where $\theta:\mathcal{D}_T\to\mathbb{R}^n$ is a measurable function s.t. the following reflected SDE has a weak solution: $X_0=x\geq 0$ and

$$X_t=-\int_0^t f(s,Z_s)ds+\int_0^t\theta(s,Z_s,X_s)^\top\mu ds+\int_0^t\theta(s,Z_s,X_s)^\top\sigma dW_s+L_t^X.$$

Properties of Auxiliary Value Function

- It is important to note the equivalence that

$$u(a,\mathrm{v},z)=\begin{cases} a-\mathrm{v}-v(0,z,0), & \text{if } a\geq \mathrm{v},\\ -v(0,z,\mathrm{v}-a), & \text{if } a<\mathrm{v}. \end{cases}$$

- The value function v is Lipschitz in x:

Lemma (Lipschitzity of the value function v)

For $(t,z,x)\in\mathcal{D}_T$, the value function $v(t,z,x)$ of (ACP) is nondecreasing in $x\geq 0$. Moreover, for all $(t,z)\in[0,T]\times\mathbb{R}$, we have

$$|v(t,z,x_1)-v(t,z,x_2)|\leq e^{-\rho t}\left|x_1-x_2\right|,\quad \text{for all } x_1,x_2\geq 0.$$

Prime Hamilton-Jacobi-Bellman Equation I

- By some heuristic arguments of dynamic programming, we can show that $v = v(t, z, x)$ satisfies

HJB equation with Neumann boundary: for $(t, z, x) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+$,

$$\begin{cases} v_t + \sup_{\theta \in \mathbb{R}^n} \left[v_x \theta^\top \mu + \frac{v_{xx}}{2} \theta^\top \sigma \sigma^\top \theta + v_{xz} \sigma_Z(z) \theta^\top \sigma \gamma \right] \\ \qquad + v_z \mu_Z(z) + v_{zz} \frac{\sigma_Z^2(z)}{2} - f(t, z) v_x = \rho v; \\ v(T, z, x) = 0, \quad \forall\ (z, x) \in \mathbb{R} \times [0, \infty); \\ v_x(t, z, 0) = 1, \quad \forall\ (t, z) \in [0, T] \times \mathbb{R}. \end{cases}$$

Prime Hamilton-Jacobi-Bellman Equation II

- Suppose $v_{xx} < 0$ on $[0, T) \times \mathbb{R} \times \mathbb{R}_+$, the feedback optimal control is

$$\theta^*(t, z, x) = -(\sigma\sigma^\top)^{-1} \frac{v_x(t, z, x)\mu + v_{xz}(t, z, x)\sigma_Z(z)\sigma\gamma}{v_{xx}(t, z, x)}.$$

- Plugging θ^* into the HJB equation, we have

For $(t, z, x) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+$,

$$v_t - \rho v - \alpha \frac{v_x^2}{v_{xx}} + \frac{\sigma_Z^2(z)}{2} \left(v_{zz} - \frac{v_{xz}^2}{v_{xx}} \right) - \phi(z) \frac{v_x v_{xz}}{v_{xx}} + \mu_Z(z) v_z$$
$$- f(t, z) v_x = 0,$$

where the coefficients are given by

$$\alpha := \frac{1}{2} \mu^\top (\sigma\sigma^\top)^{-1} \mu, \qquad \phi(z) := \sigma_Z(z) \mu^\top (\sigma\sigma^\top)^{-1} \sigma\gamma, \quad z \in \mathbb{R}.$$

Outline

Legendre-Fenchel Dual Transform of v

- We start by assuming v satisfies $v \in C^{1,2,2}([0,T) \times \mathbb{R} \times [0,\infty)) \cap C(\mathcal{D}_T)$ and $v_{xx} < 0$ on $[0,T) \times \mathbb{R} \times \mathbb{R}_+$, which will be verified later.
- For $(t,z,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_+$, consider the dual transform function:

$$\hat{v}(t,z,y) := \sup_{x>0}\{v(t,z,x) - xy\} \text{ and } x^*(t,z,y) := v_x(t,z,\cdot)^{-1}(y).$$

- For $x^* = x^*(t,z,y)$, it holds that

$$v_x(t,z,x^*) = y, \quad (t,z) \in [0,T] \times \mathbb{R}.$$

- Since $x \to v(t,z,x)$ is Lipschitz, we have $y \in (0,1]$.
- Therefore, for all $(t,z,y) \in [0,T] \times \mathbb{R} \times (0,1)$,

$$\hat{v}(t,z,y) = v(t,z,x^*) - x^*y.$$

Dual HJB Equation Satisfied By $\hat{v}$

- The dual HJB equation satisfied by $\hat{v}$ is given by

$$\hat{v}_t(t,z,y) - \rho\hat{v}(t,z,y) + \rho y\hat{v}_y(t,z,y) + \alpha y^2\hat{v}_{yy}(t,z,y) + \mu_Z(z)\hat{v}_z(t,z,y) + \frac{\sigma_Z^2(z)}{2}\hat{v}_{zz}(t,z,y) - \phi(z)y\hat{v}_{yz}(t,z,y) - f(t,z)y = 0.$$

- By the terminal condition $v(T,z,x)=0$, for $(z,y)\in\mathbb{R}\times(0,1)$,

$$\hat{v}(T,z,y) = \sup_{x>0}\{v(T,z,x)-xy\} = \sup_{x>0}\{-xy\} = 0.$$

- By Neumann boundary condition $v_x(t,z,0)=1$, we have $x^*(t,z,1)=0$, and hence $\hat{v}_y(t,z,1)=-x^*(t,z,1)=0$.

Classical Solutions of Dual HJB Equation I

- For $(t, z, u) \in \mathcal{D}_T$, let us introduce the function:

$$h(t,z,u) := -E\left[\int_t^T e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s^{t,u}} ds\right].$$

- The process $(M_s^{t,z})_{s\in[t,T]}$ with $(t,z) \in [0,T] \times \mathbb{R}$ satisfies SDE:

$$M_s^{t,z} = z + \int_t^s \mu_Z(M_r^{t,z})dr + \varrho \int_t^s \sigma_Z(M_r^{t,z})dB_r^1 + \sqrt{1-\varrho^2}\int_t^s \sigma_Z(M_r^{t,z})dB_r^2.$$

- The process (B^1, B^2) is a two-dimensional Brownian motions with a specific correlation coefficient

$$\varrho := \frac{(\sigma^{-1}\mu)^\top}{|\sigma^{-1}\mu|}\gamma.$$

Classical Solutions of Dual HJB Equation II

- Moreover, the process $(R_s^{t,u})_{s\in[t,T]}$ with $(t,u)\in[0,T]\times[0,\infty)$ is a reflected Brownian motion with drift defined by

$$R_s^{t,u} := u + \sqrt{2\alpha}\int_t^s dB_r^1 + \int_t^s (\alpha-\rho)dr + \int_t^s dL_r^R \geq 0.$$

- The reflected term $t\mapsto L_t^R$ is a continuous and nondecreasing process that increases only on $\{t\in[0,T];\ R_t^{0,u}=0\}$ with $L_0^R=0$.
- By the solution representation of "the Skorokhod problem", we obtain that, for $(s,u)\in[t,T]\times[0,\infty)$,

$$L_s^R = 0 \vee \left\{-u + \max_{r\in[t,s]}\left[-\sqrt{2\alpha}(B_r^1 - B_t^1) - (\alpha-\rho)(r-t)\right]\right\}.$$

Smoothness of h

- The smoothness of the function h is given by

Proposition (Smoothness of h)

Let assumptions $(\mathbf{A}_f)$ *and* $(\mathbf{A}_Z)$ *hold. We have that* $h \in C^{1,2,2}(\mathcal{D}_T)$. *Moreover, for* $(t,z,u) \in \mathcal{D}_T$,

$$h_u(t,z,u) = E\left[\int_t^T e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s^{t,u}} \mathbf{1}_{\left\{\max_{r\in[t,s]}[-\sqrt{2\alpha}B_r^1 - (\alpha-\rho)r] \leq u\right\}} ds\right]$$

$$= E\left[\int_t^{\tau_u^t \wedge T} e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s^{t,u}} ds\right],$$

where $\tau_u^t := \inf\{s \geq t;\ -\sqrt{2\alpha}B_s^1 - (\alpha-\rho)s = u\}$ *(we assume* $\inf\emptyset = +\infty$ *by convention).*

Feynman-Kac Representation I

- The corresponding Feynman-Kac formula is stated as:

Theorem (Feynman-Kac formula)

Suppose that $(\mathbf{A}_f)$ *and* $(\mathbf{A}_Z)$ *hold. Then, h solves the Neumann boundary problem:*

$$\begin{cases} h_t + \alpha h_{uu} + (\alpha - \rho) h_u + \phi(z) h_{uz} + \mu_Z(z) h_z + \dfrac{\sigma_Z^2(z)}{2} h_{zz} \\ \qquad\qquad = f(t,z) e^{-u-\rho t}, \quad (t,z,u) \in [0,T) \times \mathbb{R} \times \mathbb{R}_+; \\ h(T,z,u) = 0, \quad \forall\, (z,u) \in \mathbb{R} \times [0,\infty); \\ h_u(t,z,0) = 0, \quad \forall\, (t,z) \in [0,T] \times \mathbb{R}. \end{cases}$$

On the other hand, if h defined on $\mathcal{D}_T$ *with a polynomial growth is a classical solution of the above Neumann boundary problem, then h has its probabilistic representation.*

Proof of Feynman-Kac Formula I

- **Sketch of Proof:**
- For $(t,u)\in[0,T]\times[0,\infty)$, define

$$B_s^{t,u}:=u+\sqrt{2\alpha}(B_s^1-B_t^1)+(\alpha-\rho)(s-t),\quad s\in[t,T].$$

- By Remark 4.17 of Chapter 5 in Karatzas and Shreve (1991), $(\mathbf{A}_Z)$ gives that the martingale problem on $(M^{t,z},B^{t,u})$ is well posed.
- By applying Theorem 5.4.20 in Karatzas and Shreve (1991), $(M^{t,z},B^{t,u})$ is a strong Markov process.
- For $\varepsilon\in(0,u)$, let us define that

$$\tau_\varepsilon^t:=\inf\left\{s\geq t;\ \left|B_s^{t,u}-u\right|\geq\varepsilon \text{ or } \left|M_s^{t,z}-z\right|\geq\varepsilon\right\}\wedge T.$$

Proof of Feynman-Kac Formula II

- **Sketch of Proof-Continued:**
- Because the paths of $(M^{t,z}, B^{t,u})$ are continuous, we have $\tau_\varepsilon^t > t$, P-a.s.. For any $\hat{t} \in [t, T]$, it holds that

$$B^{t,u}_{\hat{t}\wedge\tau_\varepsilon^t} = R^{t,u}_{\hat{t}\wedge\tau_\varepsilon^t}.$$

- The strong Markov property yields that

$$-E\left[\int_{\hat{t}\wedge\tau_\varepsilon^t}^T f(t, M_s^{t,z}) e^{-R_s^{t,u}-\rho s} ds \middle| \mathcal{F}_{\hat{t}\wedge\tau_\varepsilon^t}\right] = h\left(\hat{t}\wedge\tau_\varepsilon^t, M^{t,z}_{\hat{t}\wedge\tau_\varepsilon^t}, B^{t,u}_{\hat{t}\wedge\tau_\varepsilon^t}\right).$$

- By the probabilistic representation of h, for $(t, y, u) \in \mathcal{D}_T$,

$$h(t, z, u) = E\left[h\left(\hat{t}\wedge\tau_\varepsilon^t, M^{t,z}_{\hat{t}\wedge\tau_\varepsilon^t}, B^{t,u}_{\hat{t}\wedge\tau_\varepsilon^t}\right) - \int_t^{\hat{t}\wedge\tau_\varepsilon^t} e^{-\rho s} f(s, M_s^{t,z}) e^{-B_s^{t,u}-L_s^R} ds\right].$$

Proof of Feynman-Kac Formula III

- **Sketch of Proof-Continued:**
- By above Proposition and Itô's formula, we have

$$\frac{1}{\hat{t}-t}E\left[\int_t^{\hat{t}\wedge\tau_\varepsilon^t} e^{-\rho s}f(s,M_s^{t,z})e^{-B_s^{t,u}-L_s^R}ds\right]\xrightarrow[]{DCT\ \ \hat{t}\downarrow t} f(t,z)e^{-u-\rho t}$$

$$=\frac{1}{\hat{t}-t}E\left[\int_t^{\hat{t}\wedge\tau_\varepsilon^t}(h_t+\mathcal{L}h)\left(s,M_s^{t,z},B_s^{t,u}\right)ds\right]\xrightarrow[]{DCT\ \ \hat{t}\downarrow t}(h_t+\mathcal{L}h)(t,z,u)$$

$$+\frac{1}{\hat{t}-t}E\left[\int_t^{\hat{t}\wedge\tau_\varepsilon^t}h_u\left(s,M_s^{t,z},B_s^{t,u}\right)dL_s^R\right]\xrightarrow[]{\hat{t}\downarrow t}0,\ \text{since } R_s^{t,u}>0 \text{ on } s\in[t,\hat{t}\wedge\tau_\varepsilon^t]$$

- The operator $\mathcal{L}$ acted on $C^2(\mathbb{R}\times[0,\infty))$ is defined as:

$$\mathcal{L}g:=\alpha g_{uu}+(\alpha-\rho)g_u+\phi(y)g_{uz}+\mu_Z(z)g_z+\frac{\sigma_Z^2(z)}{2}g_{zz}.$$

Proof of Feynman-Kac Formula IV

- **Sketch of Proof-Continued:**
- It remains to show the homogeneous Neumann boundary condition.
- In fact, for $s \in [t, T]$ and any positive sequence $(u_n)_{n\geq 1}$ satisfying $u_n \downarrow 0$ as $n \to \infty$, $P(\bigcup_{n\geq 1} A_s^{u_n}){=}1$ where

$$\bigcup_{n\geq 1} A_s^{u_n} {:=} \bigcup_{n\geq 1} \left\{ \max_{r\in[t,s]} \left[-\sqrt{2\alpha} B_r^1 - (\alpha-\rho) r \right] > u_n \right\} \in \mathcal{F}_s.$$

- By the above Proposition and $(\mathbf{A}_Z)$, it follows from DCT that

$$h_u(t,z,0) = \lim_{n\to\infty} \int_t^T E\left[f(s, M_s^{t,z}) e^{-R_s^{t,u} - \rho s} \mathbf{1}_{(A_s^{u_n})^c} \right] ds = 0.$$

Well-posedness of Dual HJB Equation

- The well-posedness of the dual HJB equation with Neumann boundary is given by

Theorem (Well-posedness of the dual HJB equation)

Let assumptions $(\mathbf{A}_f)$ *and* $(\mathbf{A}_Z)$ *hold. The dual HJB equation with Neumann boundary admits a unique classical solution* $\hat{v}$ *such that for* $(t,z,y) \in [0,T] \times \mathbb{R} \times (0,1]$,

$$\left|\hat{v}(t,z,y)\right| \leq C(1+|z|^p+|\ln y|^p), \quad \textit{for some } p>1,$$

and the function

$$h(t,z,u) := e^{-\rho t}\hat{v}(t,z,e^{-u}), \quad (t,z,u) \in \mathcal{D}_T,$$

has its probabilistic representation. Moreover, for each $(t,z) \in [0,T) \times \mathbb{R}$, *the solution* $(0,1] \ni y \mapsto \hat{v}(t,z,y)$ *is strictly convex.*

Outline

Optimal Regular Control and Verification I

- We next recover the solution v of the primal HJB equation via $\hat{v}$ and state the verification theorem.

Theorem (Verification theorem)

Let assumptions $(\mathbf{A}_f)$ *and* $(\mathbf{A}_Z)$ *hold. We have that*

(i) *The HJB PDE has a solution* $v \in C^{1,2,2}([0,T) \times \mathbb{R} \times [0,\infty)) \cap C(\mathcal{D}_T)$. *Moreover, for* $(t,z,x) \in \mathcal{D}_T$,

$$v(t,z,x) = \begin{cases} \inf\limits_{y \in (0,1]} \{\hat{v}(t,z,y) + xy\}, & \text{if } (t,z,x) \in \mathcal{O}_T \text{ or } x = 0, \\ \\ 0, & \text{if } (t,z,x) \in \mathcal{O}_T^c \cap \mathcal{D}_T, \end{cases}$$

where the region $\mathcal{O}_T$ *is given by*

$$\mathcal{O}_T := \{(t,z,x) \in [0,T) \times \mathbb{R} \times \mathbb{R}_+;\ x \in (0, \xi(t,z))\},$$

Optimal Regular Control and Verification II

- The function $\xi(t,z)$ with $(t,z)\in[0,T)\times\mathbb{R}$ is defined by

$$\xi(t,z) := E\left[\int_t^T e^{-\rho(s-t)} f(s, M_s^{t,z}) e^{\sqrt{2\alpha}(B_s^1 - B_t^1)+(\alpha-\rho)(s-t)} ds\right].$$

Theorem (Verification theorem-Continued)

(ii) *Define the feedback control function as, for* $(t,z,x)\in\mathcal{D}_T$,

$$\theta^*(t,z,x) := \begin{cases} -(\sigma\sigma^\top)^{-1}\dfrac{v_x(t,z,x)\mu + v_{xz}(t,z,x)\sigma_Z(z)\sigma\gamma}{v_{xx}(t,z,x)}, & \text{if } (t,z,x)\in\mathcal{O}_T \\ & \text{or } x=0, \\ -\mu(\sigma\sigma^\top)^{-1}\lim\limits_{y\downarrow 0} y\hat{v}_{yy}(t,z,y) & \text{if } (t,z,x)\in\mathcal{O}_T^c\cap\mathcal{D}_T. \\ \quad +(\sigma\sigma^\top)^{-1}\sigma_Z(z)\sigma\gamma\lim\limits_{y\downarrow 0}\hat{v}_{yz}(t,z,y), & \end{cases}$$

Optimal Regular Control and Verification III

- Define $\theta_t^* := \theta^*(t, Z_t, X_t)$ for $t \in [0, T]$. Then $\theta^* = (\theta_t^*)_{t \in [0,T]} \in \mathcal{U}_t$ is an optimal strategy.
- Moreover, for all $\theta \in \mathcal{U}_t$, it holds that $\tilde{J}(\theta; t, z, x) \leq e^{-\rho t} v(t, z, x)$, where $(t, z, x) \in [0, T) \times \mathbb{R} \times [0, \infty)$.

Remark: We explain the role of $\xi(t, z)$ in the above Theorem. In fact, for $(t, z) \in [0, T) \times \mathbb{R}$ and $x \geq \xi(t, z)$, by Theorem-**(i)**, $v(t, z, x) = 0$. Then, by Theorem-**(ii)**, we have that, for the strategy $\theta^* \in \mathcal{U}_t$,

$$E_{t,z,x}\left[-\int_t^T e^{-\rho s} dL_s^{X^*}\right] = 0.$$

Optimal Regular Control and Verification IV

- Remark-Continued:

Remark: This implies from integration by parts that

$$e^{-\rho T} L_T^{X^*} + \rho \int_t^T e^{-\rho s} L_s^{X^*} ds = x, \ \ P\text{-a.s.}$$

and hence $L_T^{X^*} = L_t^{X^*} = x$, P-a.s. because $\xi(t,z) > 0$ for $(t,z) \in [0,T) \times \mathbb{R}$. Therefore, with the strategy $\theta^* \in \mathcal{U}_t$, the (nonnegative) process X^* is given by

$$X_t^* = x - \int_0^t f(s, Z_s) ds + \int_0^t (\theta_s^*)^\top \mu ds + \int_0^t (\theta_s^*)^\top \sigma dW_s.$$

On the other hand, for $0 \leq x < \xi(t,z)$, we have that $v_x(t,z,x) > 0$ and hence $v(t,z,x) < 0$. This implies that, with this initial value x, the reflected term $L_t^{X^*}$ is strictly increasing in $t \in [0,T)$.

Outline

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End

THANK YOU FOR YOUR ATTENTION!