Branching Brownian motion conditioned on small maximum

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Joint work with Xinxin Chen (Lyon 1) and Bastien Mallein (Paris 13)

THU-PKU-BNU probability webinar

Content

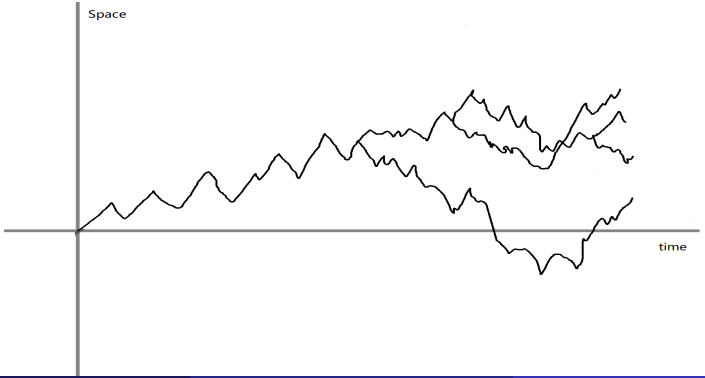
- Branching Brownian motion
- Deviation probabilities
- Idea to the proof

Content

- **Branching Brownian motion**
- Deviation probabilities
- Idea to the proof

Branching Brownian motion (BBM)

- Each particle moves as a BM .
- life time $\sim Exp.(1)$ with two offsprings.
- Each offspring performs the same behaviors (independently).
- M_t: maximal position of BBM at time t.



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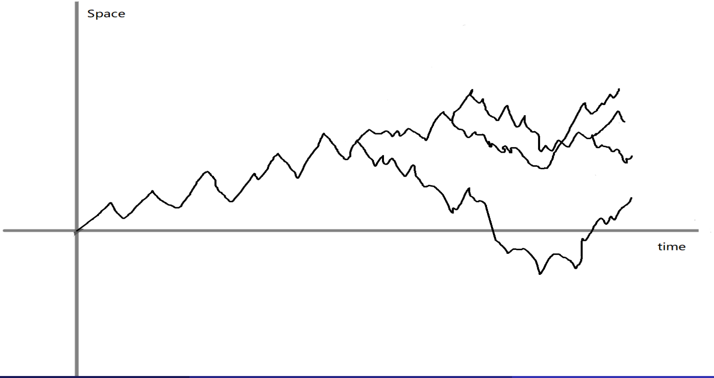

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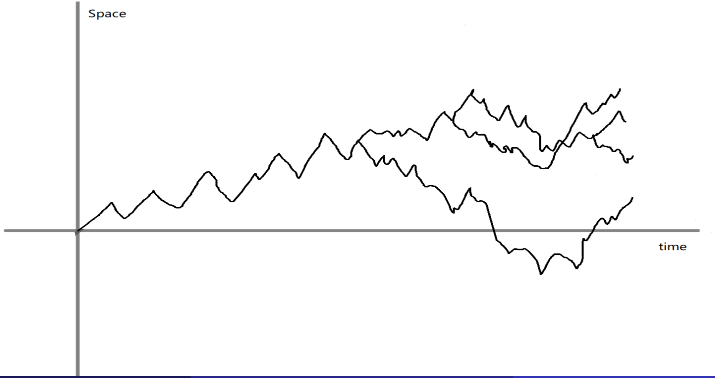

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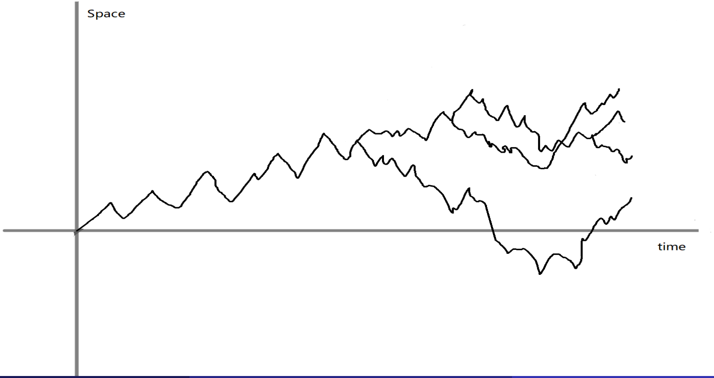

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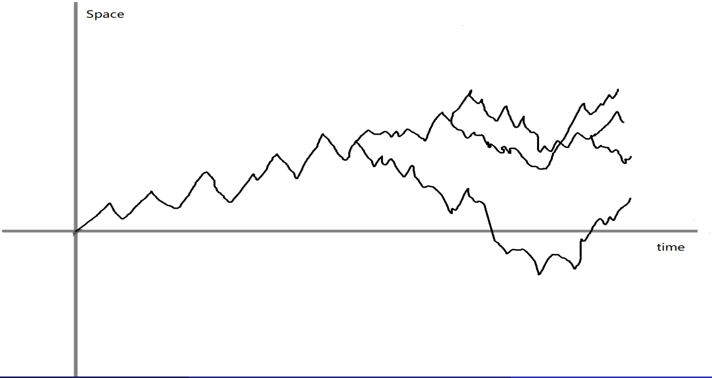

BBM with pic. from J. Berestycki

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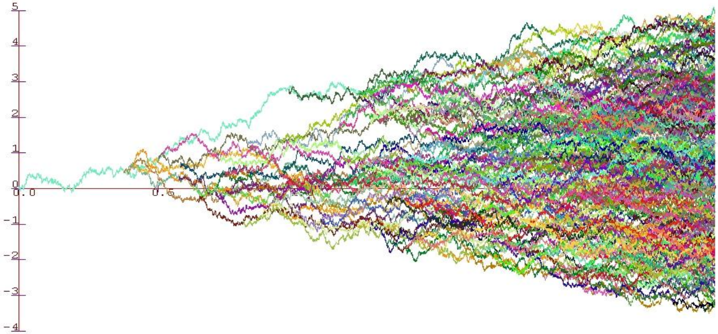

Maximum of BBM and F-KPP equation

- M_t: maximal position of BBM at time t.
- $U(t,x) = P(M_t \le x)$.
- McKean (1976): U satisfies F-KPP equation.

$$\frac{\partial}{\partial t}U(t,x) = \frac{1}{2}\Delta U(t,x) + U^2 - U, \quad U(0,x) = \mathbf{1}_{\{x>0\}}.$$

- Fisher (1937), Kolmogorov, Petrovsky and Piskunov (1937): F-KPP equation.
- Ikeda, Nagasawa and Watanabe (1968a, 1968b, 1969): BBM and F-KPP equation.

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F-KPP equation and the travelling wave solution

- Recall M_t and $U(t,x) = P(M_t \leq x)$.
- Kolmogorov, Petrovsky and Piskunov (1937): exists m_t s.t.

$$U(t, m_t + x) \stackrel{t\to\infty}{\longrightarrow} \omega(x), \quad \frac{1}{2}\omega'' + \sqrt{2}\omega' + \omega(\omega - 1) = 0.$$

- Bramson (1978): $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1)$.
- Bramson (1983): weak convergence of centered maximum

$$U(t, m_t + x) = P\left(M_t \leq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + x\right) \stackrel{t\to\infty}{\longrightarrow} \omega(x).$$

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Spin glasses and random energy model

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- Generalized REM: $g(\sigma,\sigma') = A\left(\frac{\sigma\wedge\sigma'}{N}\right)$ for some function A; Gardner and Derrida (1986, J. Phy. C), Bovier and Kurkova (2004).
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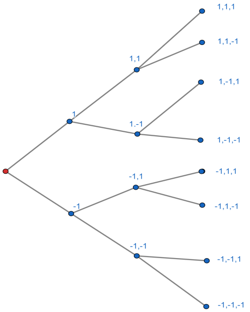

$\{1,-1,1\}\wedge\{1,1,1\} = 1$

$\{-1,1,1\}\wedge\{-1,1,-1\} = 2$

Maximum of BBM: backgrounds

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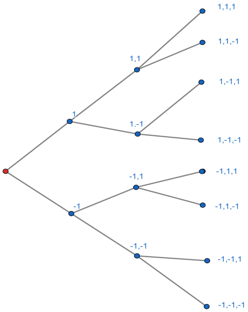

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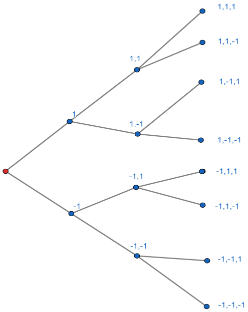

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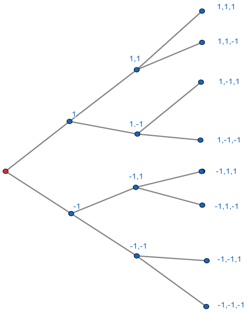

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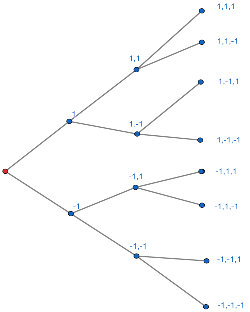

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- Variable speed BBM (BRW): Derrida and Spohn (1988), Fang and Zeitouni (2012), Bovier and Hartung (2014).

Maximum of BBM: backgrounds

Log-correlated Gaussian fields [correlations decay logarithmically with the "distance"; Duplantier, Rhodes, Sheffield and Vargas (2017)]

- 2D discrete Gaussian free field: Bramson, Ding and Zeitouni (2016).
- Random Unitary Matrix; Randomized Riemann-Zeta function; 2D cover times...

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- M_t: Maximum of BBM.
- $m_t := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$.
- Bramson (1983): $P(M_t \le m_t + x) \to \omega(x)$.

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- Deviation probabilities: for $a_t \to +\infty$,
  - Upper deviation: $P(M_t \ge m_t + a_t)$
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Upper deviation probability

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- Chauvin and Rouault (1988): upper deviation probability
  - For $\alpha > 1$,

$$P\left(M_t \geq \sqrt{2}\alpha t\right) \overset{t\to\infty}{\sim} \frac{\text{Const.}}{\sqrt{t}} e^{-(\alpha^2-1)t}.$$

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Rough idea for upper deviation probability

- Recall $P\left(M_t \geq \sqrt{2}\alpha t\right) \overset{t\to\infty}{\sim} \frac{\text{Const.}}{\sqrt{t}} e^{-(\alpha^2-1)t}, \quad \alpha > 1.$
- $\mathcal{N}(t) :=$ number of particles at time t with $E[\mathcal{N}(t)] = e^t$.
- Each particle moves as a BM.

$$P\left(M_t \geq \sqrt{2}\alpha t\right)$$
$$= P(\text{there exists one particle which is higher than } \sqrt{2}\alpha t)$$
$$\leq E[\mathcal{N}(t)]P(B_t \geq \sqrt{2}\alpha t)$$
$$= e^t \frac{1}{\sqrt{2\pi t}} \int_{\sqrt{2}\alpha t}^{\infty} e^{-\frac{x^2}{2t}} dx \sim e^t \frac{1}{\sqrt{2\pi t}} e^{-\frac{2\alpha^2 t^2}{2t}} = \frac{1}{\sqrt{2\pi}\sqrt{t}} e^{-(\alpha^2-1)t}.$$

- Lower bound: second moment method.

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- $\mathcal{N}(t) :=$ number of particles at time t with $E[\mathcal{N}(t)] = e^t$.
- Each particle moves as a BM.

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\begin{aligned}
&P\left(M_t \geq \sqrt{2}\alpha t\right)\\
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Lower deviation probability

- Derrida and Shi (2017, JSP): lower deviation probability.
- For $\alpha < 1$, as $t \to \infty$,

$$\log P(M_t \leq \sqrt{2}\alpha t) \sim -\Phi(\alpha)t,$$

$$\Phi(\alpha) = \begin{cases} 2(\sqrt{2}-1)(1-\alpha), & \alpha \in [1-\sqrt{2}, 1) \\ (1+\alpha^2), & \alpha \in (-\infty, 1-\sqrt{2}). \end{cases}$$

- Question: more explicit decay rate?

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Phase transition at $1-\sqrt{2}$

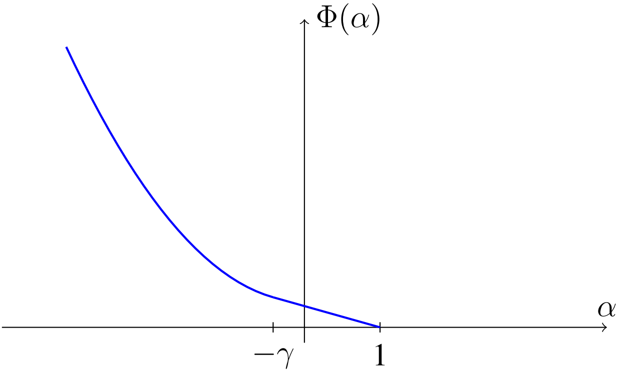
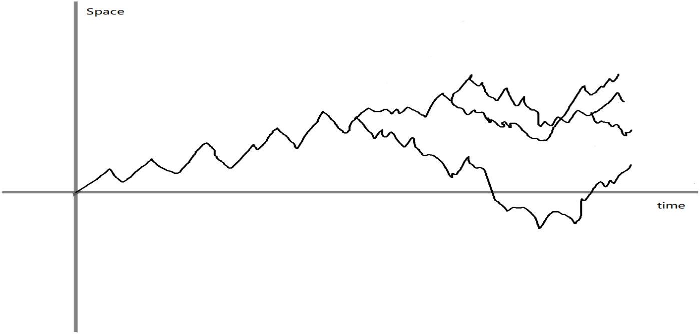


Figure: $-\gamma = 1-\sqrt{2}$.

Phase transition at $1 - \sqrt{2}$: intuition

- First branching happens before time t or not.



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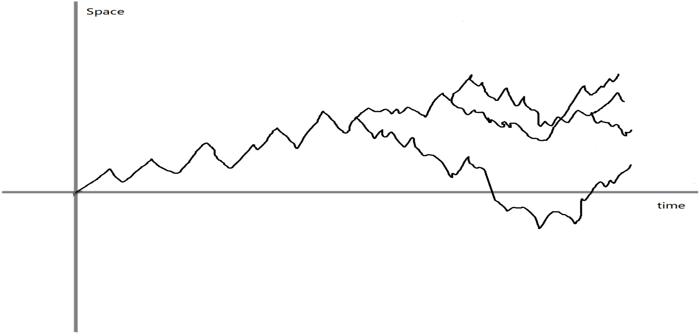

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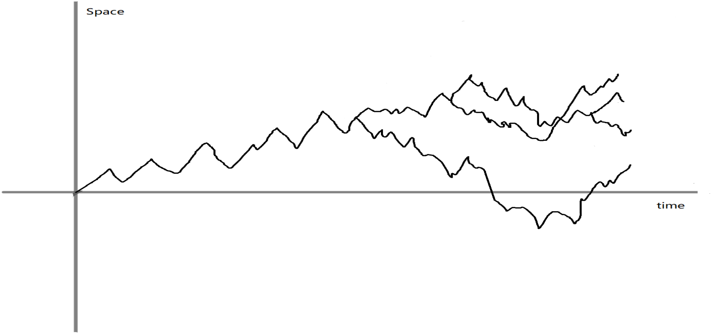

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- Recall $P(M_t \le \sqrt{2}\alpha t) \sim e^{-(1+\alpha^2+o(1))t}$ if $\alpha \le 1-\sqrt{2}$.
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- If $\alpha > 1-\sqrt{2}$, then $\tau = \Theta(1)t$ and $B_\tau \approx y$ so that $m_{t-\tau} \approx \sqrt{2}\alpha t - y$.

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Lower deviation probability

Recall $\gamma = \sqrt{2} - 1$.

Theorem (Chen, H. and Mallein (2020+))

As $t \to \infty$,

$$P(M_t \leq \sqrt{2}\alpha t) \sim \begin{cases} C_1 t^{\frac{3\gamma}{2}} e^{-2\gamma(1-\alpha)t}, & \alpha \in (-\gamma, 1); \\ C_2 t^{\frac{3\gamma}{4}} e^{-(1+\gamma^2)t}, & \alpha = -\gamma; \\ C_3 t^{-\frac{1}{2}} e^{-(1+\alpha^2)t}, & \alpha \in (-\infty, -\gamma). \end{cases}$$

- C_1, C_2, C_3 are explicit constants.
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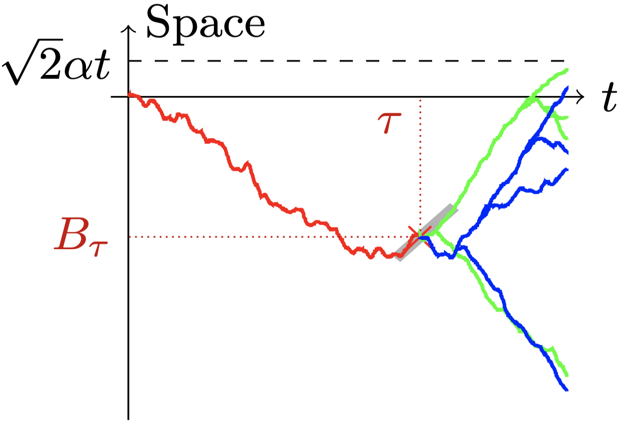
Content

- Branching Brownian motion
- Deviation probabilities
- **Idea to the proof**

Idea to the proof

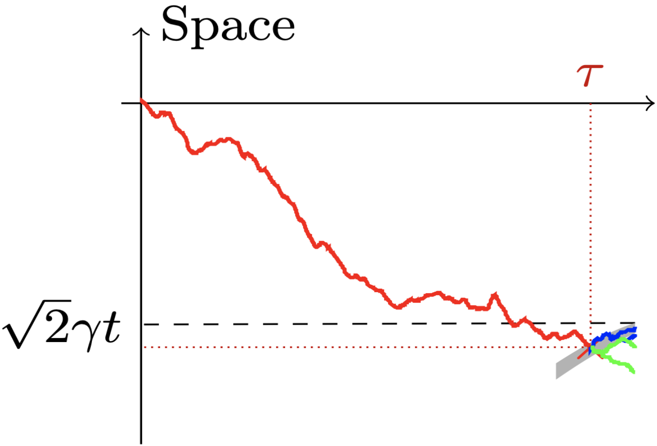
1. First branching event.
2. F-KPP equation.
3. Laplace method.

First branching event: $\alpha > -\gamma$



$$\tau \approx \frac{1-\alpha}{\sqrt{2}} t \pm \Theta(1)\sqrt{t},$$
$$B_\tau \approx \sqrt{2}\alpha t - m_{t-\tau} + O(1).$$

First branching event: $\alpha = -\gamma$



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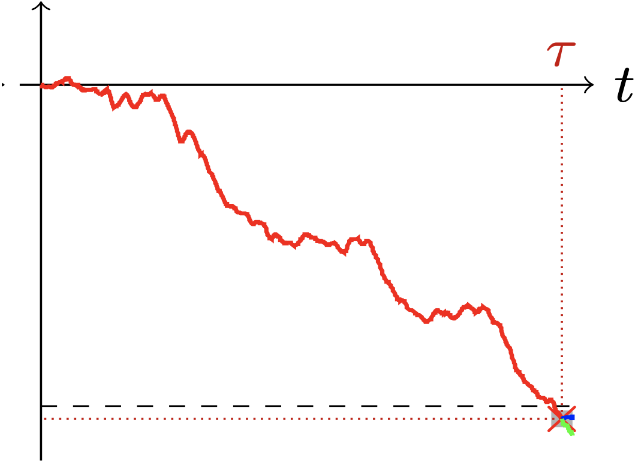
First branching event: $\alpha < -\gamma$



$$\tau = t - O(1), B_\tau \approx \sqrt{2}\alpha t$$

First branching time

Theorem (Chen, H. and Mallein (2020+))

Conditioned on $\{M_t \leq \sqrt{2}\alpha t\}$, *as* $t \to \infty$,

$$\frac{\tau - \frac{(1-\alpha)}{\sqrt{2}} t}{\sqrt{t \frac{(1-\alpha)}{4\sqrt{2}}}} \to N(0,1), \quad \alpha > -\gamma;$$

$$(t-\tau)/\sqrt{t} \xrightarrow{d} \xi, \quad \alpha = -\gamma;$$

where $P(\xi \in du) = 2^{-3(\sqrt{2}+1)/4} \Gamma((3\sqrt{2}-1)/4) u^{3\gamma/2} e^{-2u^2} du, \quad u > 0.$

$$t - t \wedge \tau \to \xi_\alpha, \quad \alpha < -\gamma,$$

where $\xi_\alpha \in [0, \infty)$.

Other related models

- Branching random walk
  - Aidekon (2013): weak convergence of centered maximum.
  - Gantert and Höfelsauer (2018), Chen and H. (2020+): upper/lower deviation probability.
  - Madaule (2017), Mallein (2018): extremal processes.
- 2d discrete Gaussian Free Field
  - Bramson, Ding and Zeitouni (2016): weak convergence of centered maximum.
  - Ding (2013), Ding and Zeitouni (2015): upper/lower deviation probability.
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