Spatial ergodicity and central limit theorem for stochastic heat equation

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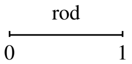
Outline

- Introduction
  - Stochastic Heat Equation (SHE)
  - Poincaré inequality
- Spatial ergodicity for SHE ($\Rightarrow$ law of large numbers)
- Spatial central limit theorem for SHE

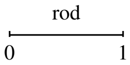
Stochastic heat equation

Stochastic heat equation



- At time $t = 0$, heat the rod with heat density $\sin(\pi x)$, $x \in [0, 1]$.

Stochastic heat equation



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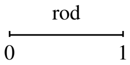

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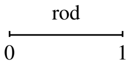

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- Add random external heat density $W(t,x)$ at (t,x),

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\partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) + W(t,x). \qquad \text{(additive)}
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- Space-time white noise: $W(t,x), t > 0, x \in [0,1]$, independent centered Gaussian random variables.

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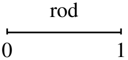

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- Interaction.

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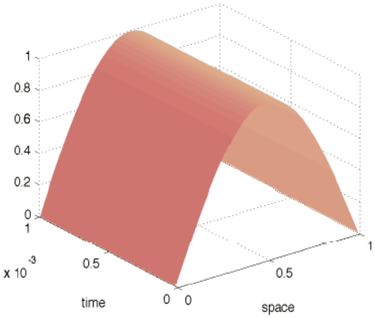

HE

Pictures courtesy by Khoshnevisan

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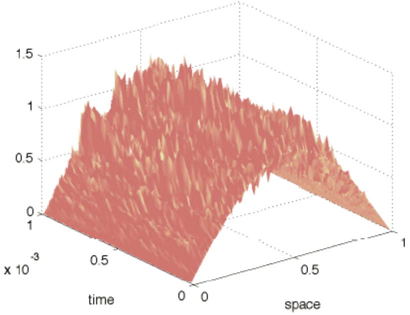

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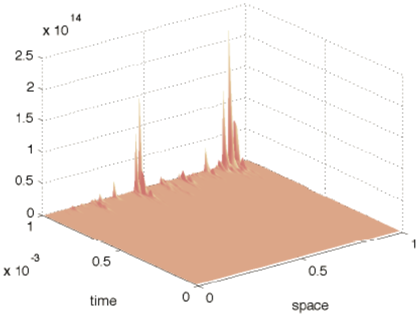

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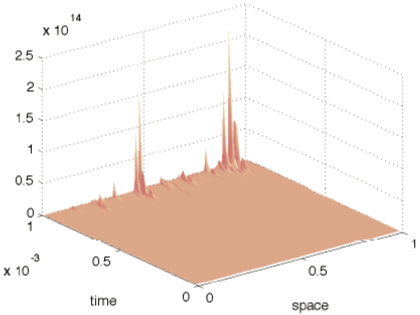

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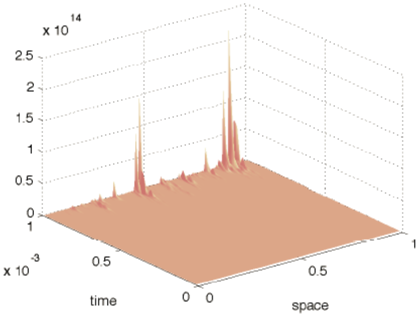

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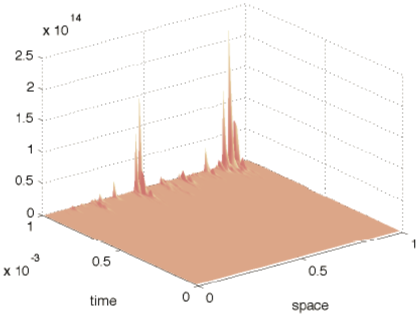

SHE: intermittency

- High peaks; others are small. What is average? Law of large numbers.
- Normalization of average. Central limit theorem.
- Number of peaks higher than a certain level. Poisson limit theorem.

Stochastic heat equation

- Stochastic heat equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma(u(t,x))W(t,x), & t>0, x\in\mathbb{R}^d, \\ u(0)\equiv 1 \end{cases} \tag{SHE}$$

- $\sigma : \mathbb{R} \mapsto \mathbb{R}$ Lipschitz continuous.

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- $\sigma : \mathbb{R} \mapsto \mathbb{R}$ Lipschitz continuous.
- White noise $\{W(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$: Gaussian process with covariance

$$\mathrm{E}[W(t,x)W(s,y)] = \delta_0(t-s)f(x-y),$$

  where f is a nonnegative-definite measure on $\mathbb{R}^d$.
- Space-time white noise: $f = \delta_0$.

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- Mild solution.

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} \boldsymbol{p}_{t-s}(x-y)\sigma(u(s,y))W(s,y)\mathrm{d}s\mathrm{d}y,$$

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- $W(\mathrm{d}s\,\mathrm{d}y)$ denotes Walsh / Itô integral (Walsh 86).

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$$\int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}x)}{1+\|x\|^2} < \infty, \qquad \text{(Dalang's condition 99)}$$

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  - Cole-Hopf solution to KPZ equation: $H(t,x) := \log u(t,x)$, then formally

$$\partial_t H(t,x) = \frac{1}{2}\partial_x^2 H(t,x) + \frac{1}{2}(\partial_x H(t,x))^2 + W(t,x). \qquad \text{(KPZ)}$$

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  - Malliavin-Stein method.

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  - Huang, Nualart, Viitasaari 18: $d = 1, f = \delta_0$, $g(u) = u$.
  - Huang, Nualart, Viitasaari, Zheng 19: $f(\mathrm{d}x) = \|x\|^{-\beta}\mathrm{d}x$, $g(u) = u$.
  - Malliavin-Stein method.

> How does f determine the spatial ergodicity and central limit theorem?

Poincaré inequality

- $\mathrm{d}\gamma(x) = (2\pi)^{-n/2} \mathrm{e}^{-\frac{\|x\|^2}{2}} \mathrm{d}x$ Gaussian probability measure on $\mathbb{R}^n$. $F : \mathbb{R}^n \mapsto \mathbb{R}$ smooth enough,

$$\mathrm{Var}_\gamma(F) := \int_{\mathbb{R}^n} F^2 \mathrm{d}\gamma - \left(\int_{\mathbb{R}^n} F \mathrm{d}\gamma \right)^2 \leq \int_{\mathbb{R}^n} |\nabla F|^2 \mathrm{d}\gamma.$$

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Ergodicity

Theorem 1 (Chen, Khoshnevisan, Nualart and P., 2019)

Assume $u(0) \equiv 1$ and f satisfies Dalang's condition. Then the following are valid:

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Central limit theorem

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Fix $t > 0$ and g Lipschitz continuous, and suppose $f(\mathbb{R}^d) < \infty$. Then,

$$\frac{1}{N^{d/2}} \int_{[0,N]^d} \left\{ g(u(t\,,x)) - \mathrm{E}[g(u(t\,,0))] \right\} \mathrm{d}x \xrightarrow{\mathrm{d}} \mathrm{N}(0, \sigma_{g,t}^2) \qquad \text{as } N \to \infty, \tag{CLT}$$

where $\sigma_{g,t}^2 = \int_{\mathbb{R}^d} \mathrm{Cov}\left[g(u(t\,,x))\,, g(u(t\,,0)) \right] \mathrm{d}x$. (CLT) *holds iff* $f(\mathbb{R}^d) < \infty$ *when* σ *is nonzero constant.*

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Central limit theorem

Theorem 2 (Chen, Khoshnevisan, Nualart, P. 19)

Fix $t > 0$ and g Lipschitz continuous, and suppose $f(\mathbb{R}^d) < \infty$. Then,

$$\frac{1}{N^{d/2}} \int_{[0,N]^d} \left\{ g(u(t\,,x)) - \mathrm{E}[g(u(t\,,0))] \right\} \mathrm{d}x \xrightarrow{\mathrm{d}} \mathrm{N}(0, \sigma^2_{g,t}) \qquad \text{as } N \to \infty, \tag{CLT}$$

where $\sigma^2_{g,t} = \int_{\mathbb{R}^d} \mathrm{Cov}\left[g(u(t\,,x))\,, g(u(t\,,0))\right] \mathrm{d}x$. (CLT) *holds iff* $f(\mathbb{R}^d) < \infty$ *when* σ *is nonzero constant. In the case* $d = 1$, $f = \delta_0$ *and* $\sigma(u) = u$,

$$\frac{1}{\sqrt{N}} \int_{[0,N]} \left\{ \log u(t\,,x) - \mathrm{E}[\log u(t\,,0)] \right\} \mathrm{d}x \xrightarrow{\mathrm{d}} \mathrm{N}(0, \sigma^2_t) \qquad \text{as } N \to \infty.$$

where $\sigma^2_t = \int_{\mathbb{R}} \mathrm{Cov}\left[\log u(t\,,x)\,, \log u(t\,,0)\right] \mathrm{d}x \in (0, \infty)$.

- Approaches to CLT (Poincaré inequality):
  - Blocking argument. (Bradley: *Introduction to Strong Mixing Conditions.* 2007)
  - Compactness argument and Paul Lévy's characterization of Brownian motion.
  - Malliavin-Stein method (Nourdin, Peccati):

    If $F = \delta(v)$ and $\mathrm{E}[F^2] = 1$,

    $$d_{\mathrm{TV}}(F\,, \mathrm{N}(0\,,1)) \le 2\sqrt{\mathrm{Var}\left(\langle DF\,, v\rangle_{\mathcal{H}}\right)}.$$

    Total variation distance: $d_{\mathrm{TV}}(F, G) := \sup_{A\in\mathcal{B}(\mathbb{R})} |\mathrm{P}(F \in A) - \mathrm{P}(G \in A)|$

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- Define
$$X_N(r) := \frac{\int_{[0,rN]\times[0,N]^{d-1}} \left(g(u(t,x)) - \mathrm{E}[g(u(t,0))]\right) \mathrm{d}x}{N^{d/2}}, \quad r \in [0,1].$$
- Objective: $\{X_N(r) : r \in [0,1]\} \xrightarrow{\text{f.d.d}} X = \{\sigma_{g,t} B(r) : r \in [0,1]\}$, as $N \to \infty$.

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Choose $a, b \in \mathbb{R}$,
$$\left|\mathrm{E}\left[\mathrm{e}^{\mathrm{i}(a(X_N(r_2)-X_N(r_1))+b(X_N(r_4)-X_N(r_2)))}\right] - \mathrm{E}\left[\mathrm{e}^{\mathrm{i}a(X_N(r_2)-X_N(r_1))}\right]\mathrm{E}\left[\mathrm{e}^{\mathrm{i}b(X_N(r_4)-X_N(r_3))}\right]\right| \to 0,$$

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$$= \left|\mathrm{Cov}\left(e^{ia(X_N(r_2)-X_N(r_1))}, e^{-ib(X_N(r_4)-X_N(r_3))}\right)\right| \qquad \text{(Poincaré inequatliy)}$$

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$$= \left|\mathrm{Cov}\left(\mathrm{e}^{\mathrm{i}a(X_N(r_2)-X_N(r_1))}, \mathrm{e}^{-\mathrm{i}b(X_N(r_4)-X_N(r_3))}\right)\right| \qquad \text{(Poincaré inequatliy)}$$

$$\le |ab|\int_0^t \mathrm{d}s\int_{\mathbb{R}^{2d}}\mathrm{d}z\mathrm{d}w\,\|D_{s,z}\left(X_N(r_2)-X_N(r_1)\right)\|_2\, f(z-w)\,\|D_{s,w}\left(X_N(r_4)-X_N(r_3)\right)\|_2$$

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&=\left|\mathrm{Cov}\left(e^{\mathrm{i}a(X_N(r_2)-X_N(r_1))},\ e^{-\mathrm{i}b(X_N(r_4)-X_N(r_3))}\right)\right| \qquad \text{(Poincaré inequatliy)}\\
&\le |ab|\int_0^t \mathrm{d}s\int_{\mathbb{R}^{2d}}\mathrm{d}z\mathrm{d}w\,\|D_{s,z}\left(X_N(r_2)-X_N(r_1)\right)\|_2\, f(z-w)\,\|D_{s,w}\left(X_N(r_4)-X_N(r_3)\right)\|_2\\
&\to 0,\quad \text{as } N\to\infty. \qquad \left(\text{using } \|D_{s,z}u(t,x)\|_2\lesssim \boldsymbol{p}_{t-s}(x-z) \text{ and } f(\mathbb{R}^d)<\infty\right)
\end{aligned}$$

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