Phase transition in the Ising model on a random 2D lattice

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joint work with Joonas Turunen

Probability Victoria Seminar
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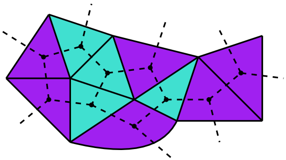
Introduction

The 2D Ising model

Input: • A finite graph G *embedded in* $\mathbb{R}^2$. • A coupling constant $\nu > 0$.

The *nearest-neighbor Ising model* on the *faces* of G is a random assignment of the numbers $+1$ and -1 (spins) to the faces of G according to the probability distribution such that for all $\sigma \equiv (\sigma_x)_{x\in F(G)} \in \{+1,-1\}^{F(G)}$,

$$\mathbb{P}(\sigma) \propto \nu^{\#\{x\sim y:\, \sigma_x=\sigma_y\}} =: \nu^{\mathcal{E}(G,\sigma)}$$



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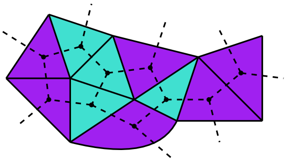

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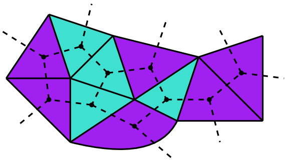

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The usually studied case:

- $\nu > 1$ (ferromagnetic).
- G is a subgraph of a regular 2D lattice ($\triangle$, $\square$ or $\hexagon$).

The 2D Ising model

Initial motivation: • Model of ferromagnet: Competition between

"*weight of*  $>$ *weight of*  " and *Entropy*

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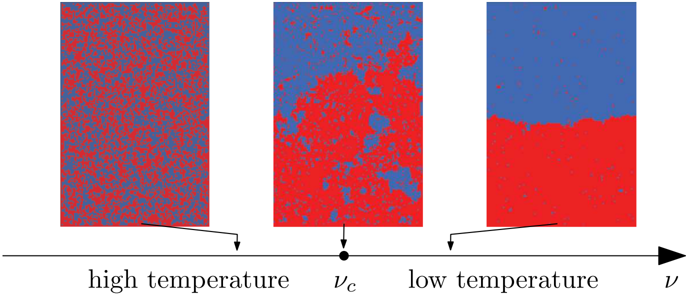

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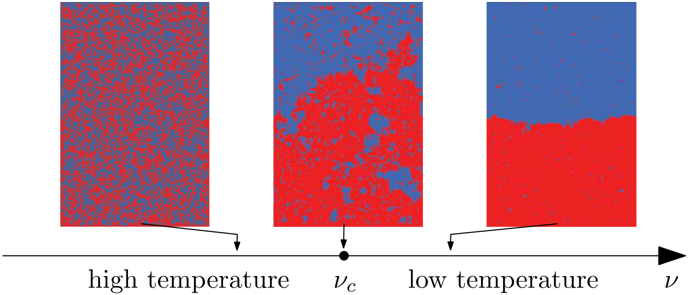

Why would this extremely idealized model be related to the real physics?

$\rightsquigarrow$ *Universality*: the phenomenon that *some macroscopic observables* of the system *at or near criticality* is independent from microscopic details of the system.

$\longrightarrow$ Distinction between *non-universal observables* and *universal observables*.

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- 1980'–1990': development of the CFT and its predictions of the scaling limits of the Ising correlation functions.
- 2000+: proof of the convergence of correlation functions to the predicted limits. prediction and proof of the scaling limits of the interfaces (SLE, CLE)

Random planar maps

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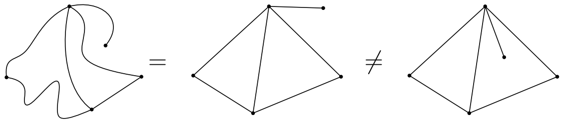

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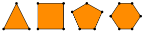
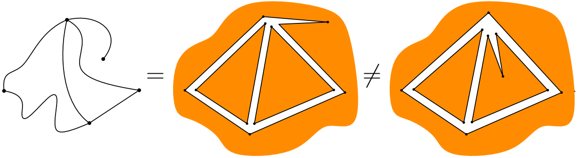
 : faces



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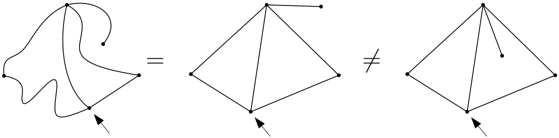

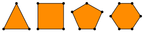

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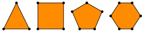
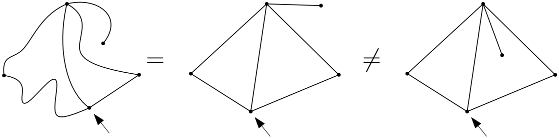
 : faces



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External face: the face containing the root. *Internal faces*: the other faces.
Boundary: the edges/vertices incident to the external face.
Perimeter: the number of the boundary edges.

A (rooted planar) *triangulation* is a map whose all faces are triangles.*
When only the *internal* faces are triangles, we talk about *triangulation with boundary*.

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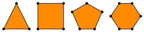

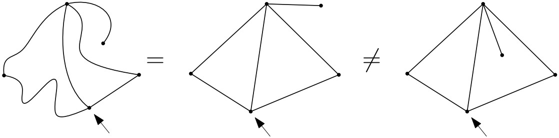

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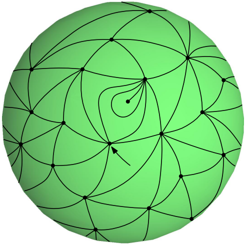
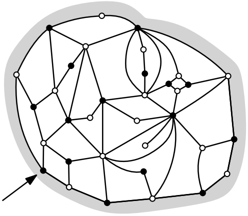


A planar triangulation



A quadrangulation
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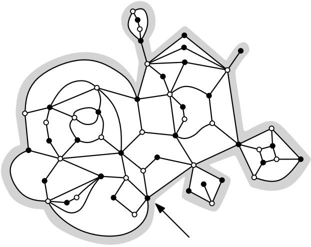

A more general map
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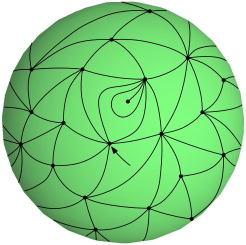
Random planar maps



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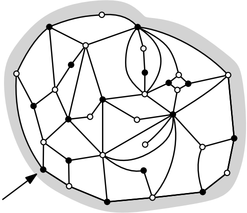

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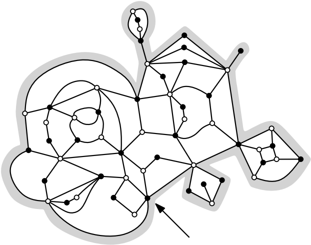

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A *uniform triangulation* of size n is a random triangulation chosen uniformly among the (rooted planar) triangulations with n faces.
A *Boltzmann triangulation* of weight t is a random triangulation chosen among all the triangulations with a probability $\mathbb{P}(\mathfrak{t}) \propto t^{\#faces(\mathfrak{t})}$.

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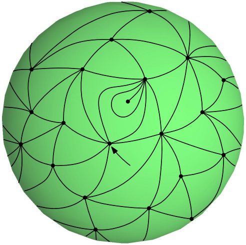

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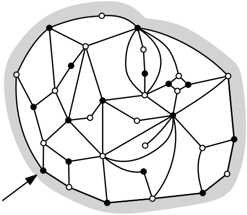

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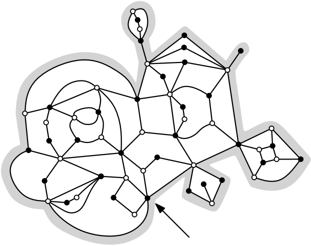

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The partition function: $\mathcal{Z}(t) := \sum_{\mathfrak{t}} t^{\#faces(\mathfrak{t})} = \sum_n T_n \cdot t^n$,
where $T_n := \#\{\text{triangulations with } n \text{ faces}\}$.

Random planar maps

Asymptotic counting formula: $\boxed{T_n \underset{n\to\infty}{\sim} c \cdot t_c^{-n} \cdot n^{-5/2}}$ with $c = \frac{\sqrt{6}}{32\sqrt{\pi}}$ and $t_c = \frac{27}{256}$.

This is the standard asymptotic behavior of the number of planar maps:

- planar maps with n edges: $c = \frac{2}{\sqrt{\pi}}$ and $t_c = \frac{1}{12}$,
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$\rightsquigarrow$ *Two universality classes.* $\dashrightarrow$ Other universal observables ? $\rightsquigarrow$ Yes, many !

Theorem (Aldous 1990, etc.)

Let $\mathfrak{T}_n$ be a random tree of size n chosen uniformly from one of many classes of rooted plane trees,

$$\left(\mathfrak{T}_n, \frac{C}{n^{1/2}} d_{gr}^{\mathfrak{T}_n}\right) \xrightarrow[n\to\infty]{GM} (\mathcal{T}, d)$$

in distribution, where $(\mathcal{T}, d)$ is Adlous' Continuous Random Tree (a compact metric space of Haudorff dimension 2, independent of the choice of the class).

Random planar maps

Theorem (Le Gall & Paulin 2008, Le Gall 2013, Miermont 2013, etc.)

Let $\mathfrak{m}_n$ be a random map of size n chosen uniformly from one of many classes of rooted planar maps,

$$\left(\mathfrak{m}_n, \frac{C}{n^{1/4}} d_{gr}^{\mathfrak{m}_n}\right) \xrightarrow[n\to\infty]{GM} (\mathcal{M}, \mathcal{D})$$

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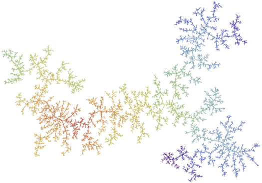
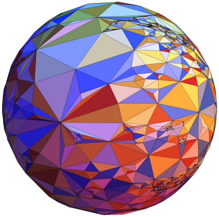
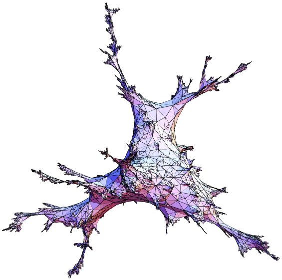
in distribution, where $(\mathcal{M}, \mathcal{D})$ is the Browninan map (a compact metric space of Haudorff dimension 4 and homeomorphic to the sphere, independent of the class).







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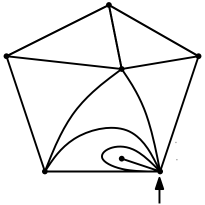
$\sim$2000+ : rigorous and more systematic methods for the enumeration of various classes of maps with additional structures.

The model: Boltzmann Ising-triangulation

Ising-triangulation of the disk

Let $\mathcal{T}_2$ be the set of triangulations with a *simple* boundary, endowed with a partition of its boundary into 2 intervals. Denote by $p(\mathfrak{t})$ and $q(\mathfrak{t})$ the lengths of these intervals.



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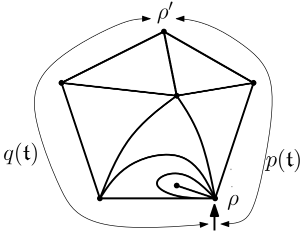

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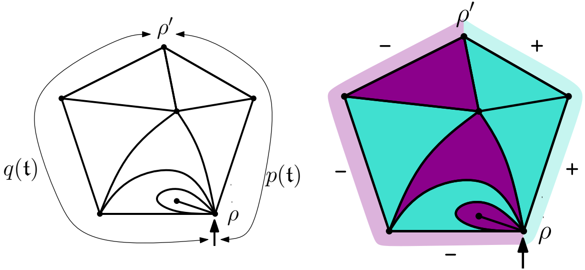

Consider the set of Ising-decorated triangulations (*Ising-triangulation* for short):

$$\mathcal{IT}_{+-} = \{(\mathfrak{t}, \sigma) \mid \mathfrak{t} \in \mathcal{T}_2 \text{ and } \sigma \in \{+,-\}^{faces(\mathfrak{t})}\}$$

The elements of $\mathcal{IT}_{+-}$ are endowed with *Dobrushin boundary condition*: we assign a sequence of $p(\mathfrak{t})$ spins + followed by $q(\mathfrak{t})$ spins – to the *outside of the boundary*.

Let $\mathcal{E}(\mathfrak{t}, \sigma)$ be the number of monochromatic edges in $(\mathfrak{t}, \sigma)$.

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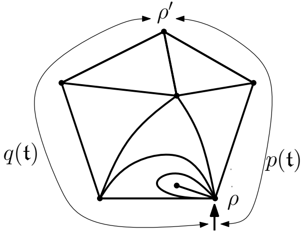

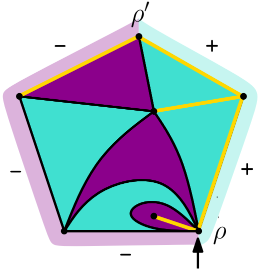

Example: $(\mathfrak{t}, \sigma) \in \mathcal{IT}_{+-}$
$p(\mathfrak{t}) = 2,\ q(\mathfrak{t}) = 3,$
$\#faces(\mathfrak{t}) = 7,\ \mathcal{E}(\mathfrak{t}, \sigma) = 5.$

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Boltzmann-Ising triangulation

Generating functions

$$Z(u,v;t,\nu) := \sum_{(\mathfrak{t},\sigma)\in\mathcal{IT}_{+-}} u^{p(\mathfrak{t})} v^{q(\mathfrak{t})} t^{\#F(\mathfrak{t})} \nu^{\mathcal{E}(\mathfrak{t},\sigma)}$$

$$Z_q(u;t,\nu) := [v^q]Z(u,v;t,\nu)$$

$$z_{p,q}(t,\nu) := [u^p v^q]Z(u,v;t,\nu) = [u^p]Z_q(u;t,\nu)$$

By convention $z_{0,0}(t,\nu) = Z(0,0;t,\nu) = 1$.

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For all $p,q \geq 0$ and $t,\nu > 0$ such that $z_{p,q}(t,\nu) < \infty$, we define a probability measure on the set $\{(\mathfrak{t},\sigma) \in \mathcal{IT}_{+-} \mid p(\mathfrak{t}) = p \text{ and } q(\mathfrak{t}) = q\}$ by

$$\mathbb{P}^{t,\nu}_{p,q}(\mathfrak{t},\sigma) = \frac{t^{\#F(\mathfrak{t})}\nu^{\mathcal{E}(\mathfrak{t},\sigma)}}{z_{p,q}(t,\nu)}.$$

We call a random variable of law $\mathbb{P}^{t,\nu}_{p,q}$ *Boltzmann Ising-triangulation of (p,q)-gon*.

Previous results

Let $t_c(\nu)$ is the radius of convergence of $t \mapsto z_{1,0}(t,\nu)$. Let $\nu_c = 1 + 2\sqrt{7}$.

Theorem (Bernardi-Bousquet-Mélou 11, Albenque-Laurent-Schaeffer 18)

For all $\nu > 1$ and $(p,q) \neq (0,0)$, we have

$$[t^n] z_{p,q}(t,\nu) \underset{n\to\infty}{\sim} \begin{cases} \kappa_{p,q}(\nu) \cdot t_c(\nu)^{-n} \cdot n^{-5/2} & (\nu \neq \nu_c) \\ \kappa_{p,q}(\nu_c) \cdot t_c(\nu_c)^{-n} \cdot n^{-7/3} & (\nu = \nu_c) \end{cases}$$

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- $t_c(\nu)$ is also the radius of convergence of $t \mapsto z_{p,q}(t,\nu)$ and $z_{p,q}(t_c(\nu),\nu) < \infty$, for all $\nu > 1$ and $(p,q) \neq (0,0)$.
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We will focus on the case $t = t_c(\nu)$ ("maximal volume") and $\nu > 1$ (ferromagnetic).

From now on, we assume $t = t_c(\nu)$ and omit the parameter t from the notations.

Main results

Asymptotics of the partition functions $z_{p,q}(\nu)$

Let $u_c(\nu)$ be the radius of convergence of $Z_0(u;\nu)$.

Continuous phase transition in the "surface tension" *

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α_1	0	4/3	5/2
α_2	5/2	11/3	5

$$c(\lambda) = \begin{cases} (1+\lambda)^{-5/2} & (\nu < \nu_c) \\ \frac{4}{3} \int_0^\infty (1+r)^{-\frac{7}{3}} (\lambda + r)^{-\frac{7}{3}} \mathrm{d}r & (\nu = \nu_c) \\ \lambda^{-5/2} & (\nu > \nu_c) \end{cases}$$

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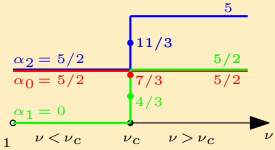

* In both of the two limits above, we have $-\lim \frac{1}{p+q} \log\left(z_{p,q}(\nu)\right) = \log u_c(\nu)$.

Theorem (Near-critical asymptotics)

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$$z_{p,q}(\nu) \sim \tilde{a}_q(\beta) \cdot u_c(\nu)^{-p} \cdot p^{-\alpha_0(\beta)} \qquad \text{and} \qquad a_p(\nu) \sim \tilde{b}(\beta) \cdot u_c(\nu)^{-p} \cdot p^{-\tilde{\alpha}_1(\beta)}$$

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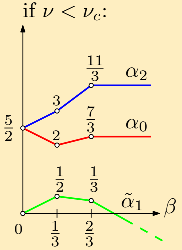

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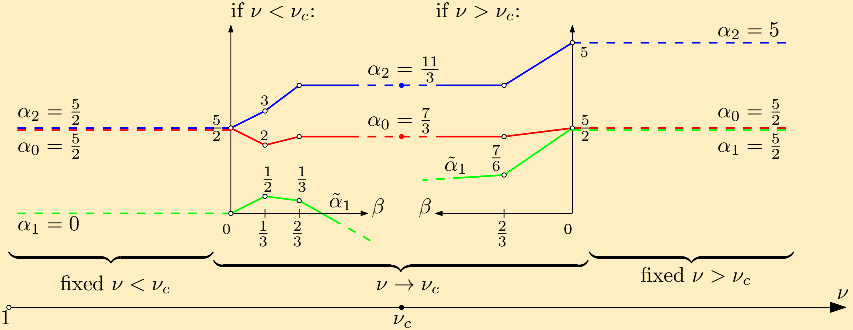

Local weak limit of the distributions $\mathbb{P}^{\nu}_{p,q}$

The *local distance* between two Ising-triangulations $(\mathfrak{t}, \sigma)$ and $(\mathfrak{t}', \sigma')$ is defined by

$$d_{\mathtt{loc}}((\mathfrak{t}, \sigma), (\mathfrak{t}', \sigma')) = 2^{-\sup\{r \in \mathbb{N} :\ B_r(\mathfrak{t}, \sigma) = B_r(\mathfrak{t}', \sigma')\}}$$

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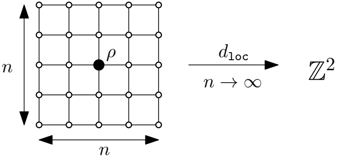

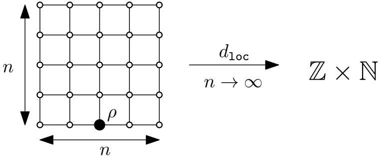

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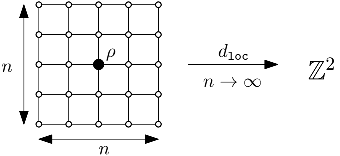
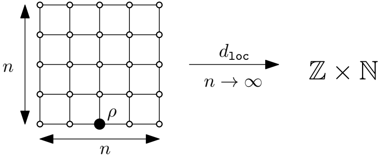
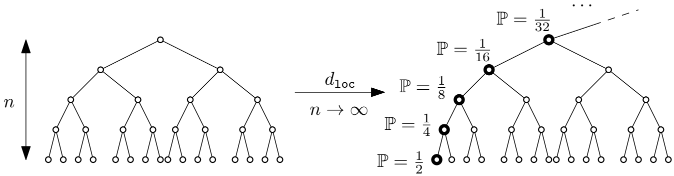




G_n = complete binary tree of height n,
ρ_n = a vertex chosen uniformly in the tree.



Local weak limit of the distributions $\mathbb{P}^{\nu}_{p,q}$

Theorem (Critical and off-critical local limits)

For each $\nu > 1$*, one can construct probability distributions* $(\mathbb{P}^{\nu}_{p})_{p\geq 0}$ *and* $\mathbb{P}^{\nu}_{\infty}$ *such that* $\mathbb{P}^{\nu}_{p,q} \xrightarrow[q\to\infty]{} \mathbb{P}^{\nu}_{p} \xrightarrow[p\to\infty]{} \mathbb{P}^{\nu}_{\infty}$ *weakly with respect to the local distance.*

In the limit $p, q \to \infty$ *and* $q/p \to \lambda \in (0, \infty)$*, the convergence becomes* $\mathbb{P}^{\nu}_{p,q} \to \mathbb{P}^{\nu}_{\infty}$*.*

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Moreover

- *For $\nu > 1$ and $p < \infty$, $\mathbb{P}^{\nu}_{p}$ is supported on the set of one-ended triangulations with one infinite boundary (i.e. triangulations of the half plane).*
- *For $\nu \geq \nu_c$, $\mathbb{P}^{\nu}_{\infty}$ is also supported on the above set.*
- *For $\nu \in (1, \nu_c)$, $\mathbb{P}^{\nu}_{\infty}$ is supported on the set of two-ended triangulations.*

Local weak limit of the distributions $\mathbb{P}^\nu_{p,q}$

Theorem (Critical and off-critical local limits)

For each $\nu > 1$, one can construct probability distributions $(\mathbb{P}^\nu_p)_{p\geq 0}$ and $\mathbb{P}^\nu_\infty$ such that $\mathbb{P}^\nu_{p,q} \xrightarrow[q\to\infty]{} \mathbb{P}^\nu_p \xrightarrow[p\to\infty]{} \mathbb{P}^\nu_\infty$ weakly with respect to the local distance.
In the limit $p, q \to \infty$ and $q/p \to \lambda \in (0,\infty)$, the convergence becomes $\mathbb{P}^\nu_{p,q} \to \mathbb{P}^\nu_\infty$. Moreover

- *For $\nu > 1$ and $p < \infty$, $\mathbb{P}^\nu_p$ is supported on the set of one-ended triangulations with one infinite boundary (i.e. triangulations of the half plane).*
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- *For $\nu \in (1, \nu_c)$, $\mathbb{P}^\nu_\infty$ is supported on the set of two-ended triangulations.*

Theorem (Near-critical local limit)

When $\nu \to \nu_c$ at the same time as $p, q \to \infty$, we have $\mathbb{P}^\nu_{p,q} \xrightarrow[q\to\infty]{} \mathbb{P}^{\nu_c}_p$, $\mathbb{P}^\nu_p \xrightarrow[p\to\infty]{} \mathbb{P}^{\nu_c}_\infty$ and $\mathbb{P}^\nu_{p,q} \xrightarrow[p,q\to\infty]{} \mathbb{P}^{\nu_c}_\infty$ weakly with respect to the local distance.

Scaling limit of the main interface (work in progress)

Let $L^{\nu}_{p,q}$ be the length* of the left-most Ising interface going from ρ to ρ' in a Boltzmann Ising-triangulation of law $\mathbb{P}^{\nu}_{p,q}$.

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Theorem (off-critical and critical limit)

Fix $\nu > 1$ and $\lambda \in (0, \infty)$. In the limit $p, q \to \infty$ and $q/p \to \lambda$, the random variable $L_{p,q}^{\nu}/p$ converges in law to 0 if $\nu > \nu_c$, to a deterministic value $\ell(\lambda; \nu) > 0$ if $\nu < \nu_c$, and to the random variable of density $\frac{1}{\mathcal{Z}}(1 + \mu x)^{-7/3}(\lambda + \mu x)^{-7/3}\mathbb{1}_{\{x>0\}}$ if $\nu = \nu_c$.

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Conjecture (near-critical limit)

Fix $\beta > 0$ and $\lambda \in (0,\infty)$. In the limit $p,q \to \infty$, $q/p \to \lambda$ and $|\nu - \nu_c| = p^{-\beta}$,

1. if $\nu > \nu_c$, or $\nu < \nu_c$ and $\beta > 1/3$, then $L^\nu_{p,q}/p^{\delta(\beta)}$ converges in distribution to a non-trivial random variable on $(0,\infty)$, where $\delta(\beta) = 2\alpha_0(\beta) - \alpha_2(\beta) \in (0,1]$.
2. if $\nu < \nu_c$ and $\beta < \frac{1}{3}$, then $L^\nu_{p,q}/p$ converges to a deterministic value $\ell(\lambda;\beta) > 0$.

Thank you for your attention!