A Gaussian process for particle masses in the near-critical Ising model

Jianping Jiang

NYU Shanghai

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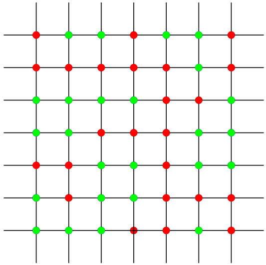
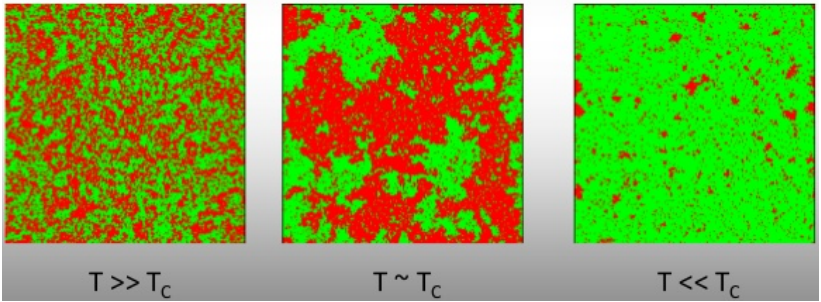
Ising model on finite domains

Let $\Lambda_L := [-L, L]^2 \cap \mathbb{Z}^2$. The classical Ising model at temperature T on Λ_L with free boundary conditions and with external field H is the probability measure $P_{\Lambda_L,f,H}$ on $\{-1,+1\}^{\Lambda_L}$, such that for any $\sigma \in \{-1,+1\}^{\Lambda_L}$,



$$P_{\Lambda_L,f,H}(\sigma) = \frac{1}{Z_{L,H}} e^{(1/T)\sum_{\{u,v\}} \sigma_u \sigma_v + H \sum_{u \in \Lambda_L} \sigma_u}$$

Phase transition ($H = 0$; on $a\mathbb{Z}^2$)



Picture from https://www.zybuluo.com/lostpg/note/625388

Some history for critical Ising ($H = 0$; on $\mathbb{Z}^2$)

- Peierls 1936 proved the existence of phase transition.
- Onsager 1944 computed the **free energy**

$$f_\beta := -\beta^{-1} \lim_{L \to \infty} \frac{\ln Z_L}{(2L+1)^2}$$

The specific heat, i.e., $-k_0 \beta^2 \frac{\partial^2 (\beta f_\beta)}{\partial \beta^2}$, has singularity at $\beta_c = \ln(1+\sqrt{2})/2$

- Yang 1952 proved for each $\beta > \beta_c$,

$$\langle \sigma_0 \rangle^+_{\beta,0} = (1 - \sinh(\beta)^{-4})^{1/8}$$

- Wu 1966, Chelkak, Hongler and Izyurov 2015 proved

$$\langle \sigma_x \sigma_y \rangle_{\beta_c,0} \sim C|x-y|^{-1/4}$$

Some history for near-critical Ising ($H > 0$; on $\mathbb{Z}^2$)

- Camia, Garban and Newman 2014, Camia, J. and Newman 2017 proved
$$\langle \sigma_0 \rangle_{\beta_c,H} \sim H^{1/15}$$
- Camia, J. and Newman 2017 proved
$$C_1(H) e^{-C_2 H^{8/15}|x-y|} \leq \langle \sigma_x ; \sigma_y \rangle_{\beta_c,H} \leq C_3(H) e^{-C_4 H^{8/15}|x-y|}$$

Near-critical scaling limit (for general $d \geq 2$)

We are interested in the $a \downarrow 0$ behavior on $a\mathbb{Z}^d$ with $T = T_c$ and $H = a^{(d+2-\eta)/2} h$ (for $h = 0$ and $h > 0$). Φ^h is generalized random field: for test fcn. f on $\mathbb{R}^d$

$$\Phi^h(f) := \lim_{a \downarrow 0} \Phi^{a,h}(f) = \lim_{a \downarrow 0} a^{(d+2-\eta)/2} \sum_{x \in a\mathbb{Z}^d} \sigma_x f(x).$$

Remark 1

The exponent in H follows from

$$\langle \sigma_{\vec{0}} \sigma_{\vec{x}} \rangle_{\beta_c, 0} \approx |\vec{x}|^{-d+2-\eta} \text{ for } \vec{0}, \vec{x} \in \mathbb{Z}^d.$$

Some known results about Φ^h

- $d = 2$ and $h = 0$, Φ^h is non-Gaussian.
  Aizenman 1982, Camia, Garban and Newman 2015
- $d > 4$ and $h = 0$, Φ^h is Gaussian.
  Aizenman 1982, Fröhlich 1982
- $d = 4$ and $h = 0$, Φ^h is Gaussian.
  Aizenman and Duminil-Copin 2019
- $d = 2$ and $h > 0$, Φ^h is non-Gaussian.
  Camia, Garban and Newman 2016

Why is Φ^h of interest? $(d = 2)$

Zamolodchikov ('89) conjecture: related quantum field has 8 particles with masses $m_1 < m_2 < \ldots < m_8$ related to Lie Algebra E_8 and

$$m_2/m_1 = 2\cos(\pi/5),$$

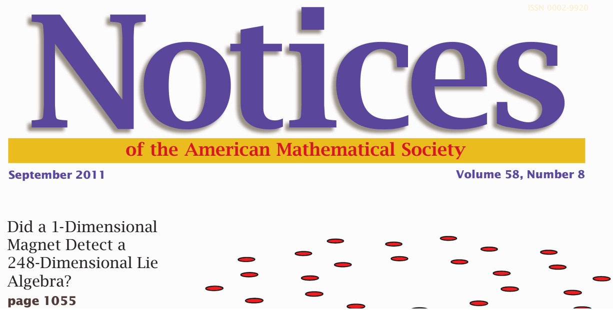
$$\vdots$$

$$m_4/m_1 = 4\cos(\pi/5)\cos(7\pi/30),$$

$$\vdots$$

$$m_8/m_1 = 8\big(\cos(\pi/5)\big)^2\cos(2\pi/15).$$

Why is Φ^h of interest?



REPORT

Quantum Criticality in an Ising Chain: Experimental Evidence for Emergent E_8 Symmetry

R. Coldea[1,*], D. A. Tennant[2], E. M. Wheeler[1,†], E. Wawrzynska[3], D. Prabhakaran[1], M. Telling[4], K. Habicht[2], P. Smeibidl[2], K. Kiefer[2]
+ See all authors and affiliations

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Main results

Masses are related to exponential decay rates of covariances.

Theorem 1 (Camia, J., Newman, 2017)

For $0 \leq f, g \in C_0^\infty(\mathbb{R}^2)$,

$$\left|\text{Cov}\left(\Phi^h(f), \Phi^h(g)\right)\right| \leq C_0 \int\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{f(x)g(y)}{|x-y|^{1/4}} e^{-Ch^{8/15}|x-y|} dxdy.$$

This proves (roughly) $\mathbf{m_1 > 0}$.

Covariance function

Masses are related to exponential decay rates of covariances. Let $H(t, y)$ be the covariance function of Φ^h. Loosely speaking,

$$H(t, y) = \text{Cov}\left(\Phi^h(t_0, y_0), \Phi^h(t_0 + t, y_0 + y)\right) \ \forall (t_0, y_0) \in \mathbb{R}^2.$$

Note that H is a function only of the radial variable $\sqrt{t^2 + y^2}$.

$$H(\sqrt{t^2 + y^2}) = H(t, y).$$

A Gaussian process

We define a mean zero stationary **Gaussian process** $\{X_s : s \in \mathbb{R}\}$ by

$$\text{Cov}(X(s), X(t)) = K(t-s) := \int_{-\infty}^{\infty} H(t-s, y)dy \ \forall s, t \in \mathbb{R}.$$

We can prove

Theorem 2 (Camia, J., Newman, 2019)

$$K(t) = \int_{m_1}^{\infty} e^{-m|t|} d\rho(m),$$

where $\rho(m)$ is a mass spectral measure of the relativistic quantum field theory obtained from Φ^h via the Osterwalder-Schrader reconstruction theorem.

An example-Gaussian free field

For the massive **Gaussian free field** on $\mathbb{R}^d$ with $d \geq 2$, the covariance function is

$$\tilde{H}(\vec{z}) = C \int_{\mathbb{R}^d} e^{i\vec{\xi}\cdot\vec{z}} \frac{1}{|\vec{\xi}|^2 + m^2} d\vec{\xi}, \quad z \in \mathbb{R}^d, m > 0.$$

An explicit computation gives

$$\tilde{K}(t) = C \int_{\mathbb{R}^{d-1}} \tilde{H}(t, \vec{y}) d\vec{y} = Ce^{-m|t|}.$$

Therefore, $\{\tilde{X}_s : s \in \mathbb{R}\}$ is an **Ornstein-Uhlenbeck** process.

Main results

Theorem 3 (Camia, J., Newman, 2019)

$$\lim_{\lambda \downarrow 0} \lambda^{1/4} H(\lambda y) = H^0(y) = C_1 |y|^{-1/4}, \; y \in \mathbb{R} \setminus \{0\}.$$

Moreover,

$$\lim_{\epsilon \downarrow 0} \frac{K(0) - K(\epsilon)}{\epsilon^{3/4}} = 2 \int_0^\infty \left[H^0(y) - H^0(\sqrt{1+y^2}) \right] dy.$$

The main ingredient is the scaling relation for Φ^h:

$$\lambda^{1/8} \Phi^h(\lambda x) \stackrel{d}{=} \Phi^{\lambda^{15/8} h}(x) \; \forall h > 0, \lambda > 0.$$

One remark

Remark 2

$$K(0) - K(\epsilon) \sim \epsilon^{1-\eta} \text{ where } \eta = 1/4.$$

So $X(t)$ has continuous sample paths. Loosely speaking, the sample path of $X(t)$ behaves locally like $t^{3/8}$, which is rougher than a 1D Brownian motion.

Why is $X(s)$ of interest?

We conjecture

1. $d = 2$, for large $|t|$
$$K(t) = B_1 e^{-m_1|t|} + B_2 e^{-m_2|t|} + B_3 e^{-m_3|t|} + O(e^{-2m_1|t|}).$$
2. $d = 3$
$$K(0) - K(\epsilon) \sim \epsilon^{1-\eta} \text{ where } \eta > 0.$$
3. $d \geq 5$
$$K(t) = Ce^{-m_1 t}.$$
4. $d = 4$, there is the possibility of log correction.

Construct $X(s)$ from Φ^h

We define a family of stochastic processes $\{X_M(s) : s \geq 0\}$:

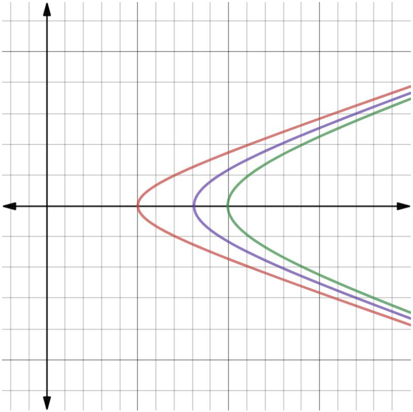
$$X_M(s) := \frac{\Phi^h\left(1_{[-M,M]}(y)\delta_s(t)\right) - \mathbb{E}\Phi^h\left(1_{[-M,M]}(y)\delta_s(t)\right)}{\sqrt{2M}}.$$

Theorem 4 (Camia, J., Newman, 2019)

For any $n \in \mathbb{N}$ and distinct $s_1, \ldots, s_n \in \mathbb{R}$, we have

$$(X_M(s_1), \ldots, X_M(s_n)) \Rightarrow (X(s_1), \ldots, X(s_n)) \text{ as } M \to \infty,$$

Some intuition



Mass hyperbola $E^2 - p^2 = m^2$

Construct $X(s)$ from the near-critical Ising model

We define another family of stochastic processes $\{X_L(s) : s \geq 0\}$:

$$X_L(s) := \frac{a^{7/8} \sum_{k \in a\mathbb{Z} \cap [-L,L]} \left[\sigma_{(s_a,k)} - \langle \sigma_{(s_a,k)} \rangle\right]}{\sqrt{2L}}.$$

Theorem 5 (Camia, J., Newman, 2020+)

Suppose $L(a) > 0$ is a function of a satisfying $L(a) \to \infty$ as $a \downarrow 0$. Then for any $n \in \mathbb{N}$ and distinct $s_1, \ldots, s_n \in \mathbb{R}$, we have

$$\left(X_{L(a)}(s_1), \ldots, X_{L(a)}(s_n)\right) \Rightarrow \left(X(s_1), \ldots, X(s_n)\right) \text{ as } a \downarrow 0.$$

Key ingredients for the proof of Theorems 5

Proposition 1

For fixed $L \in (0,\infty)$ *and* $s,t \in \mathbb{R}$, *we have*

$$\lim_{a\downarrow 0} a^{3/4} \sum_{k\in a\mathbb{Z}\cap[-L,L]} \langle \sigma_{(s_a,0)}; \sigma_{(t_a,k)} \rangle = \int_{-L}^{L} H(t-s,y)dy,$$

$$\lim_{a\downarrow 0} a^{1/4} \langle \sigma_{z_a}; \sigma_{w_a} \rangle = H(|z-w|), \text{ for all } z \neq w \in \mathbb{R}^2.$$

Remark 3

The second limit generalizes the classical Wu result, which corresponds to $h=0$.

Key ingredients for the proof of Theorems 5

An inequality for FKG systems:
Suppose $U_1, \ldots, U_m$ have finite variance and satisfy the FKG inequalities; then for any $r_1, \ldots, r_m$,

$$\left| \left\langle \exp\left(i \sum_{l=1}^{m} r_l U_l \right) \right\rangle - \prod_{l=1}^{m} \langle \exp\left(i r_l U_l \right) \rangle \right| \leq \frac{1}{2} \sum\sum_{l \neq n} |r_l r_n| \mathrm{Cov}(U_l, U_n)$$

Thanks!