Strong uniqueness for a class of singular SDEs for catalytic branching diffusions [1]

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Abstract

A new result for the strong uniqueness for catalytic branching diffusions is established, which improves the work of Dawson, D.A.; Fleischmann, K.; Xiong, J.[Strong uniqueness for cyclically symbiotic branching diffusions. *Statist. Probab. Lett.* **73**, no. 3, 251–257 (2005)].

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1 Introduction

Stochastic differential equation(SDE) is a very important tool in the theory of diffusion processes. Many investigations were devoted to the problems of existence, uniqueness, and properties of solutions of SDEs. The well-known result of Yamada and Watanabe says that if a solution of a SDE exists and the pathwise uniqueness of solutions holds, then the SDE admits a unique strong solution; see Ikeda and Watanabe (1989, p.163) and Revuz and Yor (1991, p.341). Then the study of pathwise uniqueness is of great interest. For a long time much has been known about uniqueness for one-dimensional stochastic differential equations (SDEs) with singular coefficients. The diffusion coefficient can be non-Lipschitz and degenerate; the drift can be singular and involve local time. See, e.g., Cherny and Engelbert (2005) for a survey. Especially, some results on pathwise uniqueness (strong uniqueness) for SDEs have been obtained for certain Hölder continuous diffusion coefficients; see Revuz and Yor (1991, Chapter IX-3) and Ikeda and Watanabe (1989, p.168). These results are sharp; see Barlow (1982). However, there are much less results on the pathwise uniqueness beyond the Lipschitz (or locally Lipschitz) conditions in the higher-dimensional case. Recent work in this direction includes the papers of Fang and Zhang (2005), Swart (2001, 2002) and DeBlassie (2004).

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In this work, we shall study the pathwise uniqueness for a class of degenerate stochastic differential equations with non-Lipschitz coefficients. Our interest is motivated by models of catalytic branching networks that include *mutually catalytic branching* and *cyclically catalytic branching* diffusions; see Dawson and Fleischmann (2000) for a survey on these systems. For models with mutually catalytic branching and cyclically catalytic branching, the branching rate of one type is allowed to depend on the frequency of the other types. The intuition is that the presence of different types affects the branching of other types. By the interaction over all species, the basic independence assumption in classical branching theory is violated. Uniqueness for those models is usually hard to prove. Recently, Athreya et al (2002), Bass and Perkins (2003) and Dawson and Perkins (2006) studied *weak uniqueness* for

$$dX_t^i = b^i(X_t)dt + \sqrt{2\sigma_i(X_t)X_t^i}dB_t^i, \quad i = 1, 2, \cdots, d \tag{1.1}$$

in $\mathbb{R}_+^d$, where b and σ satisfy non-negative and suitable regularity conditions, and $X_t = (X_t^1, \cdots, X_t^d)$ represents d populations. The branching rate of the ith population of X is a function (σ_i) of the mass of d populations.

Infinite systems of mutually catalytic branching and cyclically catalytic branching diffusions with $d \geq 2$ and a linear interaction between the components have been extensively studied in Dawson and Perkins (1998), Dawson et al (2003) and Fleischmann and Xiong (2001). Uniqueness for these systems follows from Mytnik's self-duality; see Mytnik(1998). But this argument works only for $d = 2$. Swart (2004) described a new way to generalize mutually catalytic branching diffusion to the case $d > 2$, but the set-up there was rather special. Moreover, in all of the work mentioned above only weak uniqueness has been obtained.

Dawson et al (2005) studied the *strong uniqueness* problem for cyclically catalytic branching diffusions in the simplified space-less case. They addressed *pathwise uniqueness* for the SDE

$$dX_t^i = \alpha_i X_t^i dt + \sqrt{\gamma_i X_t^{i-1} X_t^i} dB_t^i, \quad t > 0, \quad i = 1, \cdots, d, \tag{1.2}$$

where $(\gamma_i,\ i = 1, \cdots, d)$ are strictly positive constants. In this note, we study a slightly more general form of (1.2). Fix an integer $d \geq 1$, let $I^d := \{1, 2, \cdots, d\}$ and $\mathbf{f} := (f_1, f_2, \cdots, f_d)$, where for each $i \in I^d$,

$$f_i : \mathbb{R}^d \mapsto [0, \infty)$$

is a continuous function. Consider the following stochastic differential equation

$$\begin{cases} dX_t^i &= \alpha_i X_t^i dt + \sqrt{f_i(X_t) X_t^i} dB_t^i, \quad t > 0, \quad i \in I^d, \\ X_0 &= \mathbf{a} = (a_1, a_2, \cdots, a_d) \in \mathbb{R}_+^d \end{cases} \tag{1.3}$$

for a diffusion process $\mathbf{X} = (X^i)_{i \in I^d}$ in $\mathbb{R}_+^d$. Here $(\alpha_i)_{i \in I^d}$ are real constants and $\mathbf{B} = (B_t^1, B_t^2 \cdots, B_t^d)$ is a $\mathbb{R}^d$-valued standard Brownian motion. Our main purpose is to establish the pathwise uniqueness for equation (1.3). The idea behind this uniqueness is as follows. Indeed, away from the zero boundary, uniqueness holds by an "extended Lipschitz condition" which was suggested by Fang and Zhang (2005). On the other hand, once a component, say X^k, reaches zero, it is trapped there. But after this trapping, the model simplifies drastically. Then we can repeat the previous argument for the simplified model and get the uniqueness result when the cycle is closed.

In section 2, we will describe the main results. The proof of the uniqueness result will be given in section 3. With C we denote a positive constant which might change from line to line. For $x \in \mathbb{R}^d$, let $|x|$ denote the Euclidean norm. For the definitions of weak solution, strong solution, weak uniqueness, pathwise uniqueness, explosion, etc., see Ikeda and Watanabe (1989) for example.

2 Main results

Theorem 2.1 *Let r be a strictly positive C^1-function defined on an interval $(0, c_0]$, satisfying*
(i) $\liminf_{s\to 0} r(s) > 0$,
(ii) $\lim_{s\to 0} \frac{sr'(s)}{r(s)} = 0$,
(iii) $\int_0^a \frac{ds}{sr(s)} = +\infty$ *for any* $a > 0$.
Assume that equation (1.3) *has no explosion and for* $|x-y| \le c_0$,

$$|\boldsymbol{f}(x) - \boldsymbol{f}(y)|^2 \le C|x-y|^2 r(|x-y|^2). \tag{2.1}$$

Then the pathwise uniqueness holds for stochastic differential equation (1.3) *if one of the following conditions holds:*

(I) $f_i(x) > 0$ *for all* $i \in I^d$ *and* $x \in \mathbb{R}^d_+$;

(II) $\{x : f_i(x) = 0\} \subset \partial\mathbb{R}^d_+$ *for all* $i \in I^d$ *and if* $f_i(z) = 0$ *for some* $z = (z_1, \cdots, z_d) \in \partial\mathbb{R}^d_+$ *satisfying* $z_{i_1} = \cdots = z_{i_k} = 0$ *and* $z_j > 0$ *for* $j \notin \{i_1, \cdots, i_k\}$, *then* $f_i(x) = 0$ *for all* $x = (x_1, \cdots, x_d) \in \partial\mathbb{R}^d_+$ *satisfying* $x_{i_1} = \cdots = x_{i_k} = 0$.

Remark 2.1 *By letting* $f_i(x) = \gamma_i x_i$, *we shall see that equation* (1.2) *is a special case of equation* (1.3). *Therefore, the result of uniqueness of Dawson et al (2005) is a consequence of Theorem 2.1.*

Remark 2.2 *Functions* $r(s) = \log 1/s$, $r(s) = \log 1/s \cdot \log\log 1/s, \cdots$ *are typical examples satisfying the conditions (i)-(iii) in Theorem 2.1.*

Remark 2.3 *Define the function* V *on* $\mathbb{R}$ *by the series*

$$V(x) = \sum_{k=1}^{+\infty} \frac{|\sin kx|}{k^2}.$$

For $X = (x_1, x_2) \in \mathbb{R}^2$ *and* $\theta_1 > 0$, $\theta_2 > 0$, *define*

$$f_i(X) = V(x_1) + V(x_2) + \theta_i, \quad i = 1, 2.$$

According to Example 2.8 of Fang and Zhang (2005), the bounded function $\boldsymbol{f} = (f_1, f_2)$ *is of linear growth and satisfies the inequality* (2.1) *with* $r(s) = \log 1/s$ *and the condition (I) in Theorem 2.1.*

Theorem 2.1 is based on the non-explosion assumption of the equation (1.3). The following proposition gives a sufficient condition for non-explosion.

Proposition 2.1 *Let ρ be a strictly positive C^1-function defined on an interval $[K, +\infty)$ satisfying (i) $\lim_{s\to\infty}\rho(s) = +\infty$, (ii) $\lim_{s\to\infty}\frac{s\rho'(s)}{\rho(s)} = 0$ and (iii) $\int_K^{+\infty}\frac{ds}{s\rho(s)+1} = +\infty$. Assume that for $|x| \geq K$,*

$$\sum_{i=1}^{d} f_i^2(x) \leq C[|x|^2\rho(|x|^2) + 1].$$

Then the equation (1.3) has no explosion.

Remark 2.4 *Functions $\rho(s) = \log s$, $\rho(s) = \log s \cdot \log\log s, \cdots$ are typical examples satisfying the conditions (i)-(iii) in Proposition 2.1.*

Proof. By Theorem A of Fang and Zhang (2005), the desired result is obvious. □

3 Proof of Theorem 2.1

According to condition (i) on the function r, we can assume that there exists a constant $C_1 > 0$ such that $r(\zeta) \geq 1/C_1$ for all $\zeta \in (0, c_0]$. Let $\delta > 0$, we define

$$\phi_\delta(\zeta) = \int_0^\zeta \frac{ds}{sr(s) + \delta} \quad \text{and} \quad \Phi_\delta(\zeta) = e^{\phi_\delta(\zeta)}, \quad \zeta \geq 0.$$

By condition (iii) on r, we see that $\Phi_0(\zeta) = +\infty$ for all $\zeta > 0$. We have

$$\Phi'_\delta(\zeta) = \frac{\Phi_\delta(\zeta)}{\zeta r(\zeta) + \delta} \quad \text{and} \quad \Phi''_\delta(\zeta) = \frac{1 - r(\zeta) - \zeta r'(\zeta)}{(\zeta r(\zeta) + \delta)^2}\Phi_\delta(\zeta).$$

By conditions (i) and (ii) on the function r, there exists a constant $C_2 > 0$ such that

$$|1 - r(\zeta) - \zeta r'(\zeta)| \leq C_2 r(\zeta).$$

So that

$$\Phi''_\delta(\zeta) \leq C_2\frac{\Phi_\delta(\zeta)r(\zeta)}{(\zeta r(\zeta) + \delta)^2}. \tag{3.1}$$

Without loss of generality, we may assume that $d \geq 2$. Fix $i \in I^d$ and $\mathbf{a} \in \mathbb{R}^d_+$. Clearly, from Itô's formula, $t \mapsto e^{-\alpha_i t}X^i_t$ is a non-negative martingale, implying that the zero state is a trap for this martingale. Hence, X^i is trapped at 0 once it reaches 0.

Suppose we have two solutions $\mathbf{X} = (x^1_t, x^2_t, \cdots, x^d_t)$ and $\mathbf{Y} = (y^1_t, y^2_t, \cdots, y^d_t)$ to (1.3) with the same $\mathbf{B}$ and satisfying $\mathbf{X}_0 = \mathbf{a} = \mathbf{Y}_0$. Let

$$\eta^i_t := \sqrt{f_i(\mathbf{X}_t)x^i_t} - \sqrt{f_i(\mathbf{Y}_t)y^i_t}, \qquad \xi^i_t := |x^i_t - y^i_t|^2, \quad \text{for} \quad i \in I^d,$$

and

$$\zeta_t := |\mathbf{X}_t - \mathbf{Y}_t|^2.$$

Fix $\mathbf{a} \in \mathbb{R}^d_+$ and $a_i > 0$ for $i \in I_d$. Let $\epsilon > 0$ be such that $\epsilon < a_i < \epsilon^{-1},\ i \in I^d$. Introduce two stopping times

$$\tau^d_\epsilon := \inf\Big\{t > 0 : \exists i \in I^d \text{ with } f_i(\mathbf{X}_t) \wedge f_i(\mathbf{Y}_t) \wedge x^i_t \wedge y^i_t \leq \epsilon \\ \text{or} \quad f_i(\mathbf{X}_t) \vee f_i(\mathbf{Y}_t) \vee x^i_t \vee y^i_t \geq \epsilon^{-1}\Big\}, \tag{3.2}$$

and

$$\tau := \inf\{t > 0 : \zeta_t \geq c_0^2\}.$$

Lemma 3.1 *For any fixed $T > 0$, we have $\boldsymbol{X} = \boldsymbol{Y}$ on $[0, T \wedge \tau \wedge \tau^d_\epsilon]$.*

Proof. From equation (1.3), we have

$$d\xi^i_t = 2\alpha_i \xi^i_t dt + 2(x^i_t - y^i_t)\eta^i_t dB^i_t + (\eta^i_t)^2 dt \tag{3.3}$$

and

$$d\langle \xi^i, \xi^i \rangle_t = 4\xi^i_t (\eta^i_t)^2 dt. \tag{3.4}$$

According to (2.1), for $s \leq \tau \wedge \tau^d_\epsilon$,

$$(\eta^i_s)^2 \leq \frac{1}{2\epsilon^4}(|f_i(\mathbf{X}_s) - f_i(\mathbf{Y})_s|^2 + \zeta_s) \leq \frac{1}{2\epsilon^4}(C\zeta_s r(\zeta_s) + \zeta_s), \tag{3.5}$$

where the first inequality is due to the elementary inequalities

$$|\sqrt{bc} - \sqrt{de}| \leq \frac{1}{2\epsilon^2}(|b - d| + |c - e|) \ \text{ if } \ \epsilon \leq b, c, d, e \leq \epsilon^{-1}$$

and $(g + h)^2 \leq 2(g^2 + h^2)$.
Applying Itô's formula and according to (3.3) and (3.4), we have

$$\begin{aligned} \Phi_\delta(\zeta_{t\wedge\tau\wedge\tau^d_\epsilon}) &= \Phi_\delta(\zeta_0) + \sum_{i=1}^d \int_0^{t\wedge\tau\wedge\tau^d_\epsilon} 2\alpha_i \Phi'_\delta(\zeta_s)\xi^i_s ds \\ &\quad + \sum_{i=1}^d \int_0^{t\wedge\tau\wedge\tau^d_\epsilon} 2\Phi'_\delta(\zeta_s)(x^i_s - y^i_s)\eta^i_s dB^i_s + \sum_{i=1}^d \int_0^{t\wedge\tau\wedge\tau^d_\epsilon} \Phi'_\delta(\zeta_s)(\eta^i_s)^2 ds \\ &\quad + \sum_{i=1}^d \int_0^{t\wedge\tau\wedge\tau^d_\epsilon} 2\Phi''_\delta(\zeta_s)\xi^i_s(\eta^i_s)^2 ds \\ &= \Phi_\delta(\zeta_0) + I_1(t) + I_2(t) + I_3(t) + I_4(t) \end{aligned} \tag{3.6}$$

respectively. For any $s \leq \tau \wedge \tau^d_\epsilon$, by condition (i) on r,

$$\sum_{i=1}^d 2\alpha_i \Phi'_\delta(\zeta_s)\xi^i_s \leq \frac{\alpha\zeta_s\Phi_\delta(\zeta_s)}{\zeta_s r(\zeta_s) + \delta} \leq \alpha C_1 \Phi_\delta(\zeta_s), \tag{3.7}$$

where $\alpha = 2\max_{1\le i\le d}|\alpha_i|$. According to (3.5), we have

$$|\Phi'_\delta(\zeta_s)(x_s^i - y_s^i)\eta_s^i|^2 \le \frac{1}{2\epsilon^4}\cdot\frac{\Phi_\delta^2(\zeta_s)(C\zeta_s^2 r(\zeta_s)+\zeta_s^2)}{(\zeta_s r(\zeta_s)+\delta)^2} \le \frac{(CC_1+C_1^2)}{2\epsilon^4}\sup_{0\le\zeta\le c_0}\Phi_\delta(\zeta)^2 < \infty$$

and

$$\Phi'_\delta(\zeta_s)(\eta_s^i)^2 \le \frac{\Phi_\delta(\zeta_s)}{2\epsilon^4}\cdot\frac{C\zeta_s r(\zeta_s)+\zeta_s}{\zeta_s r(\zeta_s)+\delta} \le \frac{(C+C_1)}{2\epsilon^4}\Phi_\delta(\zeta_s).$$

On the other hand, by (3.1) and (3.5),

$$2\Phi''_\delta(\zeta_s)\xi_s^i(\eta_s^i)^2 \le \frac{C_2}{\epsilon^4}\cdot\frac{\Phi_\delta(C\zeta_s^2 r(\zeta_s)+\zeta_s^2)}{(\zeta_s r(\zeta)+\delta)^2} \le \frac{(CC_1C_2+C_1^2C_2)}{\epsilon^4}\Phi_\delta(\zeta_s).$$

Therefore, $I_2(t)$ is a martingale and $\mathbb{E}(I_2(t)) = 0$. Let

$$K := \alpha C_1 + \frac{d(C+C_1)}{2\epsilon^4} + \frac{dC_1C_2(C+C_1)}{\epsilon^4}.$$

We have

$$\mathbb{E}\left(\Phi_\delta(\zeta_{t\wedge\tau\wedge\tau_\epsilon^d})\right) \le \Phi_\delta(\zeta_0) + K\int_0^t \mathbb{E}\left(\Phi_\delta(\zeta_{s\wedge\tau\wedge\tau_\epsilon^d})\right)ds.$$

Thanks to Gronwall's inequality, we have that, for all $t>0$,

$$\mathbb{E}\left(\Phi_\delta(\zeta_{t\wedge\tau\wedge\tau_\epsilon^d})\right) \le \Phi_\delta(\zeta_0)e^{Kt}.$$

Letting $\delta\downarrow 0$ in the above inequality, we find that

$$\mathbb{E}\left(\Phi_0(\zeta_{t\wedge\tau\wedge\tau_\epsilon^d})\right) \le e^{Kt}.$$

By the continuity of the samples and the fact that $\Phi_0(\zeta) = +\infty$ for $\zeta > 0$, we can get almost surely

$$\zeta_{t\wedge\tau\wedge\tau_\epsilon^d} = 0 \ \text{ for all } t > 0.$$

This yields the desired result. □

As $\epsilon\downarrow 0$, we have the non-decreasing convergence of τ_ϵ^d to some stopping time $\tau^d \le \infty$. On $\{\tau^d = \infty\}$, we clearly have $\zeta_{t\wedge\tau} = 0$, and by the continuity of samples and the definition of τ, we have that almost surely for all $t>0$, $\zeta_t = 0$. Thus, we may assume that $\mathbb{P}(\tau^d < \infty) > 0$. And on $\{\tau^d<\infty\}$, by similar reasoning, $\zeta_{t\wedge\tau\wedge\tau^d} = 0$ implies

$$\zeta_{t\wedge\tau^d} = 0. \tag{3.8}$$

Now, we are in position to complete the proof of Theorem 2.1.

Let $\Omega_0^d = \{\tau^d < \infty\}$. We only need to show that for all $t \ge 0$

$$\zeta_t = 0, \ \text{on } \Omega_0^d.$$

Case I: Since the stochastic differential equation (1.3) has no explosion, for each $\omega \in \Omega_0^d$, there exists a $k \in I^d$ such that $x^k_{\tau^d}(\omega) = 0 = y^k_{\tau^d}(\omega)$. For $k \in I^d$, define

$$\Omega_k^d = \{\omega \in \Omega_0^d \;:\; x^k_{\tau^d} = 0 = y^k_{\tau^d}\}.$$

We shall see from the following argument that there is no loss of generality if we assume that for each $k \in I^d$, $\mathbb{P}(\Omega_k^d) > 0$ and $\Omega_i^d \cap \Omega_j^d = \emptyset$ for $i \neq j$. Note that

$$\Omega_0^d = \cup_{k=1}^d \Omega_k^d.$$

Since 0 is a trap, for $\omega \in \Omega_k^d$, $x_t^k(\omega) = 0 = y_t^k(\omega)$, for any $t > \tau^d(\omega)$. Together with (3.8), we have for any $t \geq 0$, $x_t^k = y_t^k$ on Ω_k^d. To show that $\mathbf{X}$ is pathwise uniquely determined, we only need to show that for each $l \in I^d \backslash \{k\}$,

$$\xi^l_\cdot = 0, \;\text{on } \Omega_k^d.$$

For $\epsilon > 0$, define two new stopping times such that on Ω_k^d

$$\tau_\epsilon^{d-1} := \inf \Big\{ t > 0 : \exists\, i \in I^d \backslash \{k\} \text{ with } f_i(\mathbf{X}_t) \wedge f_i(\mathbf{Y}_t) \wedge x_t^i \wedge y_t^i \leq \epsilon \\ \text{or } \; f_i(\mathbf{X}_t) \vee f_i(\mathbf{Y}_t) \vee x_t^i \vee y_t^i \geq \epsilon^{-1} \Big\} \tag{3.9}$$

and

$$\tau_1^\epsilon := \inf \Big\{ t > 0 : \; x_t^k \geq \epsilon^{-1} \text{ or } f_k(\mathbf{X}_t) \wedge f_k(\mathbf{Y}_t) \leq \epsilon \\ \text{or } f_k(\mathbf{X}_t) \vee f_k(\mathbf{Y}_t) \geq \epsilon^{-1} \Big\} \tag{3.10}$$

and $\tau_1^\epsilon = \tau_\epsilon^{d-1} = \tau_\epsilon^d$ on $\{\tau^d = \infty\}$. Note that $\tau_1^\epsilon \to \infty$ as $\epsilon \to 0$. Let $\gamma^\epsilon = \tau_1^\epsilon \wedge \tau_\epsilon^{d-1}$. For $s \leq \tau \wedge \gamma^\epsilon$,

$$(\eta_s^i)^2 \leq \frac{1}{2\epsilon^4}(|f_i(\mathbf{X}_s) - f_i(\mathbf{Y})_s|^2 + \zeta_s) \leq \frac{1}{2\epsilon^4}(C\zeta_s r(\zeta_s) + \zeta_s), \;\text{on } \Omega \backslash \Omega_i^d$$

and

$$(\eta_s^i)^2 = (x_s^i)^2 (\sqrt{f_i(\mathbf{X}_s)} - \sqrt{f_i(\mathbf{Y})_s})^2 \leq \frac{1}{4\epsilon^6}|f_i(\mathbf{X}_s) - f_i(\mathbf{Y})_s|^2 \leq \frac{1}{4\epsilon^6} C\zeta_s r(\zeta_s), \;\text{on } \Omega_i^d.$$

Thus

$$(\eta_s^i)^2 \leq \left(\frac{1}{2\epsilon^4} \vee \frac{1}{4\epsilon^6} \right) (C\zeta_s r(\zeta_s) + \zeta_s). \tag{3.11}$$

By the same argument as in Lemma 3.1, we have that almost surely

$$\zeta_{t \wedge \tau \wedge \gamma^\epsilon} = 0, \;\text{for all } t \geq 0.$$

And, we also have that there exists a stopping time $\tau^{d-1} \leq \infty$ such that $\tau_\epsilon^{d-1} \to \tau^{d-1}$. Note that $\{\tau^d = \infty\} \subset \{\tau^{d-1} = \infty\}$ and we have for $t \geq 0$,

$$\zeta_t = 0, \;\text{on } \{\tau^{d-1} = \infty\}.$$

Let $\Omega_0^{d-1} = \{\tau^{d-1} < \infty\}$. For each $\omega \in \Omega_0^{d-1} \cap \Omega_k^d$, there exists a $l \in I^d \backslash \{k\}$ such that $x^l_{\tau^{d-1}}(\omega) = 0 = y^l_{\tau^{d-1}}(\omega)$. For $l \in I^d \backslash \{k\}$, define

$$\Omega_{kl}^{d-1} := \{\omega \in \Omega_0^{d-1} \cap \Omega_k^d : \ x^l_{\tau^{d-1}}(\omega) = 0 = y^l_{\tau^{d-1}}(\omega)\}.$$

Note that

$$\bigcup_{k \in I^d} \bigcup_{l \in I^d \backslash \{k\}} \Omega_{kl}^{d-1} = \{\tau^{d-1} < \infty\}.$$

Since 0 is a trap, we see for all $t \geq 0$,

$$x_t^k = y_t^k \text{ and } x_t^l = y_t^l, \text{ on } \Omega_{kl}^{d-1}.$$

Note that if $d = 2$, then we are done. Next, we still assume that for $i \neq j$, $\Omega_{ki}^{d-1} \cap \Omega_{kj}^{d-1} = \emptyset$. For $\epsilon > 0$, define two new stopping times such that on Ω_{kl}^{d-1}

$$\tau_\epsilon^{d-2} := \inf \Big\{ t > 0 : \exists\, i \in I^d \backslash \{k, l\} \text{ with } f_i(\mathbf{X}_t) \wedge f_i(\mathbf{Y}_t) \wedge x_t^i \wedge y_t^i \leq \epsilon \\ \text{or} \quad f_i(\mathbf{X}_t) \vee f_i(\mathbf{Y}_t) \vee x_t^i \vee y_t^i \geq \epsilon^{-1} \Big\} \tag{3.12}$$

and

$$\tau_2^\epsilon := \inf \Big\{ t > 0 : \exists\, i \in \{k, l\} \text{ with } \ x_t^i \geq \epsilon^{-1} \text{ or } f_i(\mathbf{X}_t) \wedge f_i(\mathbf{Y}_t) \leq \epsilon \\ \text{or } f_i(\mathbf{X}_t) \vee f_i(\mathbf{Y}_t) \geq \epsilon^{-1} \Big\} \tag{3.13}$$

and $\tau_2^\epsilon = \tau_\epsilon^{d-2} = \tau_\epsilon^{d-1}$ on $\{\tau^{d-1} = \infty\}$. Repeat the previous argument for τ_ϵ^{d-2} instead of τ_ϵ^{d-1} and τ_2^ϵ instead of τ_1^ϵ. Then we get a new partition on Ω, say $\{\Omega_i\}_{i=0}^n$, and for each Ω_i there exist at least three components of $\mathbf{X}$ such that they are pathwise uniquely determined on Ω_i. By this way the argument can be repeated until the cycle is closed. We conclude that pathwise uniqueness holds for equation (1.3).

Case II: By similar reasoning, for $\omega \in \Omega_0^d$, we have that there exists a $k \in I^d$ such that $x^k_{\tau^d}(\omega) = 0 = y^k_{\tau^d}(\omega)$ and a $l \in I^d$ such that $f_l(\mathbf{X}_{\tau^d}(\omega)) = 0 = f_l(\mathbf{Y}_{\tau^d}(\omega))$ [k may be equal to l]. Using the previous argument, we find that

$$x_t^k = y_t^k, \quad t \geq 0.$$

Since 0 is a trap, and by the condition on f_l, 0 is also a trap for random processes $\{f_l(\mathbf{X}_t) : t \geq 0\}$ and $\{f_l(\mathbf{Y}_t) : t \geq 0\}$. This implies, after the trapping event, $dx_t^l = \alpha_l x_t^l dt$ and $dy_t^l = \alpha_l y_t^l dt$. That is

$$x_t^l = x^l_{\tau^d} e^{\alpha_l t} = y^l_{\tau^d} e^{\alpha_l t} = y_t^l, \quad t \geq \tau^d.$$

Also, for all $t \geq 0$, $f_l(\mathbf{X}_t) = f_l(\mathbf{Y}_t)$ on $\Omega_{lf}^d := \{\omega \in \Omega_0^d : \ f_l(\mathbf{X}_{\tau^d}) = 0 = f_l(\mathbf{Y}_{\tau^d})\}$. Note that $\{x : f_l(x) = 0\} \subset \partial \mathbb{R}_+^d$. Define Ω_k^d as that in previous case. Assume that $\Omega_i^d \cap \Omega_j^d = \emptyset$ and $\Omega_{if}^d \cap \Omega_{jf}^d = \emptyset$ for $i \neq j$. For $\epsilon > 0$, introduce two stopping times such that on $\Omega_k^d \cap \Omega_{lf}^d$

$$\tau_\epsilon^{k,lf} := \inf \Big\{ t > 0 : \exists\, i \in I^d \setminus \{k, l\} \text{ with } f_i(\mathbf{X}_t) \wedge f_i(\mathbf{Y}_t) \wedge x_t^i \wedge y_t^i \leq \epsilon$$

$$\text{or} \quad f_i(\mathbf{X}_t) \vee f_i(\mathbf{Y}_t) \vee x_t^i \vee y_t^i \geq \epsilon^{-1} \Big\}$$

and

$$\tau_{k,lf}^{\epsilon} := \inf \left\{ t > 0 : \ x_t^k \vee x_t^l \vee f_k(\mathbf{X}_t) \vee f_l(\mathbf{X}_t) \geq \epsilon^{-1} \right\}$$

and on $\Omega_k^d \cap (\cup_{l \in I^d} \Omega_{lf}^d)^c$, they are defined by (3.9) and (3.10) respectively and they equal to τ_ϵ^d on $\{\tau^d = \infty\}$. Then the argument would be exactly parallel to that used in Case I. We omit it here and get the pathwise uniqueness of $\mathbf{X}$. This completes the proof of the theorem. □

References

[ABBP02] Athreya, S.R.; Barlow, M.T.; Bass, R.F.; Perkins, E.A. (2002): Degenerate stochastic differential equations and super-Markov chains. *Probab. Theory Related Fields* **123**, no. 4, 484–520.

[Ba82] Barlow, M.T. (1982): One-dimensional stochastic differential equations with no strong solution. *J. London Math. Soc. (2)* **26** , no. 2, 335–347.

[BP03] Bass, R.F.; Perkins, E.A. (2003): Degenerate stochstic differential equations with Hölder continuous coefficients and super-Markov chains. *Trans. Amer. Math. Soc.* **355**, no. 1, 373-405.

[CE04] Cherny, A.S.; Engelbert, H.-J. (2005): *Singular stochastic differential equations.* In: Lecture Notes in Mathematics 1858, Springer-Verlag, Berlin.

[DF00] Dawson, D.A.; Fleischmann, K. (2000): Catalytic and mutually catalytic branching, in *Infinite dimensional stochastic analysis*, eds Ph. Clément, F. den Hollander, J. van Neerven and B. de Pagter, Royal Netherlands Academy, Amsterdam, pp.145-170

[DFX05] Dawson, D.A.; Fleischmann, K.; Xiong, J. (2005): Strong uniqueness for cyclically symbiotic branching diffusions. *Statist. Probab. Lett.* **73**, no. 3, 251–257.

[DFMPX01] Dawson, D.A.; Fleischmann, K.; Mytnik, L.; Perkins, E.A.; Xiong, J. (2003): Mutually catalytic branching in the plane: uniqueness. *Ann. Inst. H. Poincaré Probab. Statist.* **39**, no. 1, 135–191.

[DP98] Dawson, D.A.; Perkins, E.A. (1998): Long-time behavior and coexistence in a mutually catalytic branching model. *Ann. Probab.* **26**, no. 3, 1088–1138.

[DP06] Dawson, D.A.; Perkins, E.A. (2006): On the uniqueness problem for catalytic branching networks and other singular diffusions. *Illinois J. Math.* **50**, no. 1-4, 323–383 (electronic).

[De04] DeBlassie, D. (2004): Uniqueness for diffusions degenerating at the boundary of a smooth bounded set. *Ann. Probab.* **32**, no. 4, 3167–3190.

[FX01] Fleischmann, K.; Xiong, J. (2001): A cyclically catalytic super-Brownian motion. *Ann. Probab.* **29**, no. 2, 820–861.

[FZ05] Fang, S.Z.; Zhang, T.S. (2005): A study of a class of stochastic differential equations with non-Lipschizian coefficients. *Probab.Theory Relat.Fields* **132**, no. 3, 356–390.

[IW89] Ikeda, N.; Watanabe, S. (1989): *Stochastic differential equations and diffusion processes.* North-Holland, Amsterdam.

[My98] Mytnik, L. (1998): Weak uniqueness for the heat equation with noise. *Ann. Probab.* **26**, no. 3, 968–984.

[RY91] Revuz, D.; Yor, M. (1991): *Continuous martingales and Brownian motion.* Grund. der math. Wissenschaften **293**, Springer-Verlag.

[Sw01] Swart, J.M. (2001): A 2-dimensional SDE whose solutions are not unique. *Electron. Comm. Probab.* **6**, 67–71 (electronic).

[Sw02] Swart, J.M. (2002): Pathwise uniqueness for a SDE with non-Lipschitz coefficients. *Stochastic Process. Appl.* **98**, no. 1, 131–149.

[Sw04] Swart, J.M. (2004): Uniqueness for isotropic diffusions with a linear drift. *Probab. Theory Related Fields* **128**, no. 4, 517–524.