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Rescaled Lotka–Volterra Models Converge to Super-Stable Processes

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Abstract Recently, it has been shown that stochastic spatial Lotka–Volterra models, when suitably rescaled, can converge to a super-Brownian motion. We show that the limit process can be a super-stable process if the kernel of the underlying motion is in the domain of attraction of a stable law. The corresponding results in the Brownian setting were proved by Cox and Perkins (Ann. Probab. 33(3):904–947, 2005; Ann. Appl. Probab. 18(2):747–812, 2008). As applications of the convergence theorems, some new results on the asymptotics of the voter model started from single 1 at the origin are obtained, which improve the results by Bramson and Griffeath (Z. Wahrsch. Verw. Geb. 53:183–196, 1980).

Keywords Super-stable process · Lotka–Volterra · Voter model · Domain of attraction · Stable law · Stable random walk

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1 Introduction

1.1 Motivation

Originally, a super-Brownian motion arises as the limit of branching random walks; see [4, 10, 18]. Recently, it has been shown that many interacting particle systems with very different dynamics, when suitably rescaled, all converge to a super-

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Brownian motion. Such examples include the voter model, the contact process, interacting diffusion process, and the spatial Lotka–Volterra model; see [4, 5, 7, 9, 11]. Donsker's invariance principle is deeply involved in those results; see [22] for an excellent nontechnical introduction. So if we assume that the transition kernel of the underlying motion has finite variance, a super-Brownian motion is obtained as the limit process. On the other hand, the general class of stable distributions was introduced and given this name by the famous French mathematician Paul Lévy. The inspiration for Lévy was the desire to generalize the Central Limit Theorem, which is the foundation of Donsker's principle. Thus we can expect that if we let the transition kernel of the underlying motion be in the domain of attraction of a stable law, the limit process could be a super-stable process.

A motivation for proving those limit theorems is to actually use them in the study of complicated approximating systems. For example, the Lotka–Volterra invariance principle established in [7] was used to study the coexistence and survival problem of the Lotka–Volterra model; see [8]. Cox and Perkins [6] used the voter invariance principle to give a probabilistic proof of the asymptotics for the voter model obtained in [3]. In this paper, we will show that rescaled stochastic spatial Lotka–Volterra models can converge to super-stable processes and also use those limit theorems to get some new results on the asymptotics for the voter model. Coexistence and survival for the Lotka–Volterra model will be discussed in a future work.

1.2 Our Model

A stochastic spatial version of the Lotka–Volterra model was first introduced and studied by Neuhauser and Pacala [17]. In this paper, we follow the construction of the model suggested by [7] but assume that the kernel of the model is in the domain of attraction of a symmetric stable law. We first briefly describe the model. Let $\{p(x,y)\}$ be a random walk kernel on $\mathbb{Z}^d$ (the d-dimensional integer lattice). Suppose that, at each site of $\mathbb{Z}^d$, there is a plant of one of two types. We label the two types 0 and 1. At random times plants die and are replaced by new plants. The times and types depend on the configuration of surrounding plants. We denote by ξ_t, an element of $\{0,1\}^{\mathbb{Z}^d}$, the state of the system at time t, and $\xi_t(x)$ gives the type of the plant at x at time t. To describe the evolution of the system, for $\xi \in \{0,1\}^{\mathbb{Z}^d}$, define

$$f_i(x,\xi) = \sum_{y\in\mathbf{Z}^d} p(x,y)1_{\{\xi(y)=i\}}, \quad i = 0, 1. \tag{1.1}$$

Let α_0, α_1 be nonnegative parameters. Define the Lotka–Volterra *rate function* $c(x,\xi)$ by

$$c(x,\xi) = \begin{cases} f_1(f_0+\alpha_0 f_1) & \text{if } \xi(x)=0, \\ f_0(f_1+\alpha_1 f_0) & \text{if } \xi(x)=1. \end{cases}$$

The Lotka–Volterra process ξ_t is the unique $\xi \in \{0,1\}^{\mathbf{Z}^d}$-valued Feller process with rate function $c(x,\xi)$, meaning that the generator of ξ_t is the closure of the operator

Ω defined by

$$\Omega\phi(\xi) = \sum_x c(x,\xi)\big(\phi(\xi^x) - \phi(\xi)\big)$$

on the set of functions $\phi : \xi \in \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$ depending on only finitely many coordinates, where $\xi^x(y) = \xi(y)$ for $y \neq x$ and $\xi^x(x) = 1 - \xi(x)$.

Note that $f_0 + f_1 = 1$. The dynamics of ξ_t can now be described as follows: at site x in configuration ξ, the coordinate $\xi(x)$ makes transitions

$$0 \to 1 \quad \text{at rate } f_1(f_0 + \alpha_0 f_1) = f_1 + (\alpha_0 - 1) f_1^2,$$
$$1 \to 0 \quad \text{at rate } f_0(f_1 + \alpha_1 f_0) = f_0 + (\alpha_1 - 1) f_0^2.$$

These rates are interpreted in [17] as follows. A plant of type i at site x dies at rate $f_i + \alpha_i f_{1-i}$ and is replaced by a plant of type $\xi(y)$, where y is chosen with probability $p(x,y)$. α_i measures the strength of interspecific competition of type i, and we set the self-competition parameter equal to one.

In [4] an invariance principle was proved for the voter model. That is, appropriately rescaled voter models converge to a super-Brownian motion. Thus we can expect that when the parameters α_i are close to one, a similar result holds for the Lotka–Volterra model. The results in [7] and [9] say that it is true. The intuition of the voter invariance principle is that, when appropriately rescaled, the dependence on the local density of particles gets washed out, and the rescaled voter models should behave like the rescaled branching random walk. The asymptotics behavior of the latter is well known: it approaches a super-Brownian motion. On the other hand, if the kernel of the underlying motion is in the domain of attraction of a stable law, appropriately rescaled branching random walk could approach a super-stable process; see Theorem II.5.1 of [18]. The above reasoning suggests the possibility of that suitably rescaled Lotka–Volterra should approach a super-stable process. Our main results in this paper will show that it is the case.

Let $M(\mathbb{R}^d)$ denote the space of finite measures on $\mathbb{R}^d$, endowed with the topology of weak convergence of measures. Let $\Omega_D = D([0,\infty), M(\mathbb{R}^d))$ be the Skorokhod space of càdlàg paths taking values in $M(\mathbb{R}^d)$. Let Ω_C be the space of continuous $M(\mathbb{R}^d)$-valued paths with the topology of uniform convergence on compact set. We denote by $X_t(\omega) = \omega_t$ the coordinate function. We write $\mu(\phi)$ for $\int \phi\, d\mu$. For $1 \le n \le \infty$, let $C_b^n(\mathbb{R}^d)$ be the space of bounded continuous functions whose partial derivatives of order less than $n+1$ are also bounded and continuous, and let $C_0^n(\mathbb{R}^d)$ be the space of functions from $C_b^n(\mathbb{R}^d)$ with compact support.

An $\mathbb{R}^d$-valued Lévy process Y_t is said to be a symmetric α-stable process with index $\alpha \in (0,2]$ and diffusion speed $\sigma^2 > 0$ if

$$\Psi(\eta) := E\big(e^{i\eta\cdot Y_1}\big) = e^{-\sigma^2|\eta|^\alpha}, \tag{1.2}$$

where $|y|$ is the Euclidean norm of y. The distribution of Y_1 will be called the (σ^2,α)-stable law. When $\alpha = 2$, $Y_t \in \mathbb{R}^d$ is a d-dimensional σ^2-Brownian motion whose generator is $A\phi = \frac{\sigma^2\Delta\phi}{2}$ for $\phi \in C_b^2(\mathbb{R}^d)$. When $0 < \alpha < 2$, the generator of Y_t is given by

$$A\phi(x) = \frac{\sigma^2\Delta^{\alpha/2}\phi(x)}{2} = \sigma^2 \int \left[\phi(x+y) - \phi(x) - \frac{1}{1+|y|^2} \sum_{i=1}^{d} y_j D_j \phi(x) \right] \nu(dy)$$

for $\phi \in C_b^2(\mathbb{R}^d)$ and $D_j = \frac{\partial}{\partial x_j}$, where

$$\nu(dy) = c|y|^{-d-\alpha} 1_{\{|y|\neq 0\}}(dy)$$

for an appropriate $c > 0$; see [20] for details. In both cases, $C_b^\infty(\mathbb{R}^d)$ is a core for A in that the bp-closure of $\{(\phi, A\phi) : \phi \in C_b^\infty\}$ contains $\{(\phi, A\phi) : \phi \in D(A)\}$, where $D(A)$ denotes the domain of the weak generator for the process Y; see [18].

An adapted a.s.-continuous $M(\mathbb{R}^d)$-valued process $\{X_t : t \geq 0\}$ on a complete filtered probability space (Ω, F, F_t, P) is said to be a *super symmetric α-stable process with branching rate $b \geq 0$, drift $\theta \in \mathbb{R}$, and diffusion coefficient $\sigma^2 > 0$ starting at* $X_0 \in M(\mathbb{R}^d)$ if it solves the following martingale problem:

For all $\phi \in C_b^\infty(\mathbb{R}^d)$,

$$M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s\left(\frac{\sigma^2\Delta^{\alpha/2}\phi(x)}{2}\right) ds - \theta \int_0^t X_s(\phi)\, ds \quad (1.3)$$

is a continuous (F_t)-martingale with $M_0(\phi) = 0$ and predictable square function

$$\langle M(\phi)\rangle_t = \int_0^t X_s\big(b\phi^2\big)\, ds. \quad (1.4)$$

The existence and uniqueness in law of a solution to this martingale problem is well known; see Theorem II.5.1 and Remark II.5.13 of [18] and references therein. Let $P_{X_0}^{b,\theta,\sigma^2,\alpha}$ denote the law of the solution on Ω_C. So b and θ can be regarded as branching parameters, and parameters σ and α determine the underlying motion.

Let $\{Z_n : n \geq 1\}$ be a discrete time random walk on $\mathbb{Z}^d$,

$$Z_n = z_0 + \sum_{i=1}^{n} U_i,$$

where $z_0 \in \mathbb{Z}^d$, and the random variables $(U_i : i \geq 1)$ are independent identically distributed on $\mathbb{Z}^d$. Let $\{p(x, y)\}$ be a random walk kernel. In the following of this paper we assume that

(A1) $p(x, y) = p(x - y)$ is an irreducible, symmetric, random walk kernel on $\mathbb{Z}^d$, and $p(0) = 0$. For $\alpha \in (0, 2]$ and $\sigma^2 > 0$, $\{p(x)\}$ is in the domain of attraction of a symmetric (σ^2, α)-stable law, i.e.,

$$P(U_1 = x) = p(x),$$

and there exists a function $L(n)$ of regular variation of index $1/\alpha$ such that

$$L(n)^{-1} \sum_{i=1}^{n} U_i \xrightarrow{(d)} Y_1 \quad \text{as } n \to \infty, \quad (1.5)$$

where Y_1 is determined by (1.2), and the symbol $\xrightarrow{(d)}$ means convergence in distribution.

We will call a random (discrete- or continuous-time) walk with kernel satisfying assumption (A1) a stable random walk. In the following of this paper, we always assume that

$$\text{(A1)} \quad \textit{holds for some } \sigma > 0 \textit{ and } \alpha \in (0, 2].$$

Remark 1.1 Without loss of generality, we may and will assume that function L is continuous and monotonically increasing from $\mathbb{R}^+$ onto $\mathbb{R}^+$ and $L(0) = 0$; see [15] or [13]. We also have that

$$L(x) = x^{1/\alpha} s(x), \quad x > 0,$$

where $s : (0, \infty) \to (0, \infty)$ is a slowly varying function, meaning that for any $c > 0$,

$$\lim_{x \to \infty} \frac{s(cx)}{s(x)} = 1,$$

where the convergence holds uniformly when c varies over the interval $[\epsilon, 1/\epsilon]$ for any $\epsilon > 0$; see Lemma 2 of VIII.8 of [13].

Remark 1.2 According to Proposition 2.5 of [15] and its proof, we have that under (A1), the random walk $\{Z_n\}$ is transient if and only if

$$\sum_{k=1}^{\infty} L(k)^{-d} < \infty.$$

By Lemma 2 in Sect. VIII.8 of [13], the random walk is always transient when $d > \alpha$. Typically, when $d = \alpha = 1$, the random walk is recurrent if and only if

$$\sum_{k=1}^{\infty} \frac{1}{ks(k)} = \infty.$$

Now, we are ready to define our rescaled Lotka–Volterra models. For $N = 1, 2, \ldots$, let

$$\mathbb{S}_N = \mathbb{Z}^d / L(N).$$

Define the kernel p_N on $\mathbb{S}_N$ by

$$p_N(x) = p\big(xL(N)\big), \quad x \in \mathbb{S}_N.$$

For $\xi \in \{0, 1\}^{\mathbb{S}_N}$, define the densities $f_i^N = f_i^N(\xi) = f_i^N(x, \xi)$ by

$$f_i^N(x, \xi) = \sum_{y \in \mathbb{S}_N} p_N(y - x) 1_{\{\xi(y) = i\}}, \quad i = 0, 1.$$

Let $\alpha_i = \alpha_i^N$ depend on N, and let ξ_t^N be the process taking values in $\{0, 1\}^{\mathbb{S}_N}$ determined by the rates: at site x in configuration ξ, the coordinate $\xi(x)$ makes transitions

$$\begin{aligned} 0 \to 1 \quad & \text{at rate } N f_1^N \big(f_0^N + \alpha_0^N f_1^N \big), \\ 1 \to 0 \quad & \text{at rate } N f_0^N \big(f_1^N + \alpha_1^N f_0^N \big). \end{aligned}$$

That is, ξ_t^N is rate-N Lotka–Volterra process determined by the parameters α_i^N and the kernel p_N. More precisely, if we set

$$c_N(x, \xi) = \begin{cases} N f_1^N (f_0^N + \alpha_0^N f_1^N) & \text{if } \xi(x) = 0, \\ N f_0^N (f_1^N + \alpha_1^N f_0^N) & \text{if } \xi(x) = 1, \end{cases}$$

ξ_t^N is the unique Feller process taking values in $\{0, 1\}^{\mathbb{S}^N}$ whose generator is the closure of the operator

$$\Omega_N \phi(\xi) = \sum_{x \in \mathbb{S}^N} c_N(x, \xi) \big(\phi\big(\xi^x\big) - \phi(\xi) \big)$$

on the set of functions $\phi : \xi \in \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R}$ depending on only finitely many coordinates. Here $\xi^x(y) = \xi(y)$ for $y \neq x$ and $\xi^x(x) = 1 - \xi(x)$.

Remark 1.3 If we assume that $\sum_{x \in \mathbb{Z}^d} x^i x^j p(x) = \delta_{ij} \sigma^2 < \infty$, then $p(x)$ is in the domain of attraction of a normal law. That is the case of $\alpha = 2$. So we recover the fixed kernel models in [7]. When the stable random walk is recurrent, since there are significant differences between the case of $d = \alpha = 1$ and the case of $d = \alpha = 2$, we only consider the case of $d = \alpha = 1$. For $d = \alpha = 2$, see the work [9].

Define

$$g(x) = \int_1^x L(s)^{-1} \, ds$$

for $d = \alpha = 1$ and $x \geq 1$. According to Remark 1.2, the one-dimensional random walk Z is recurrent if and only if $\lim_{x \to \infty} g(x) = \infty$.

Set

$$N' = \begin{cases} N & \text{if } d > \alpha, \\ N & \text{if } d = \alpha = 1 \text{ and } \lim_{x \to \infty} g(x) < \infty, \\ N/g(N) & \text{if } d = \alpha = 1 \text{ and } \lim_{x \to \infty} g(x) = \infty. \end{cases}$$

That is, if the stable random walk is transient, then $N' = N$, and $N' = N/g(N)$ if the stable random walk is recurrent.

We define the corresponding measure-valued process X_t^N by

$$X_t^N = \frac{1}{N'} \sum_{x \in \mathbb{S}_N} \xi_t^N(x) \delta_x. \tag{1.6}$$

As in [7] and [9], we make the following assumptions:

$$\begin{aligned} &(1) \quad \sum_{x \in \mathbb{S}_N} \xi_0^N(x) < \infty; \\ &(2) \quad X_0^N \to X_0 \quad \text{in } M\big(\mathbb{R}^d\big) \text{ as } N \to \infty; \\ &(3) \quad \theta_i^N = N'\big(\alpha_i^N - 1\big) \to \theta_i \in \mathbb{R} \quad \text{as } N \to \infty,\ i = 0, 1. \end{aligned} \tag{A2}$$

Now, we are ready to describe our main results.

1.3 Main Results

To describe the limit process, we introduce a coalescing random walk systems $\{\hat{B}_t^x, x \in \mathbb{Z}^d\}$. Each $\hat{B}_t^x$ is a rate 1 random walk on $\mathbb{Z}^d$ with kernel p and $\hat{B}_0^x = x$. The walks move independently until they collide and then move together after that. For finite $A \subset \mathbb{Z}^d$, let

$$\hat{\tau}(A) = \inf\big\{s : \big|\big\{\hat{B}_t^x, x \in A\big\}\big| = 1\big\}$$

be the time at which the particles starting from A coalesce into a single particle, and write $\hat{\tau}(a, b, \ldots)$ when $A = \{a, b, \ldots\}$. Note that when the stable random walk is transient, we can define the "escape" probability by

$$\gamma_e = \sum_{e \in \mathbb{Z}^d} p(e) P\big(\hat{\tau}(0, e) = \infty\big).$$

We also define

$$\beta = \sum_{e, e' \in \mathbb{Z}^d} p(e) p(e') P\big(\hat{\tau}(e, e') < \infty, \hat{\tau}(0, e) = \hat{\tau}(0, e') = \infty\big),$$

$$\delta = \sum_{e, e' \in \mathbb{Z}^d} p(e) p(e') P\big(\hat{\tau}(0, e) = \hat{\tau}(0, e') = \infty\big).$$

Here we are considering a system of three coalescing random walks starting at 0, e, and e', where e and e' are independent with law p. Then β is the probability the walks starting at e and e coalesce, but this coalescing system does not meet the random walk starting at 0, while δ is the strictly larger probability that the coalescing system starting at $\{e, e'\}$ does not meet the random walk starting at 0.

We also need a collection of independent (noncoalescing) rate-1 continuous-time random walks with step function p, which we will denote $\{B_t^x : x \in \mathbb{Z}^d\}$, such that $B_0^x = x$. Define the collision times

$$\tau(x, y) = \inf\big\{t \geq 0 : B_t^x = B_t^y\big\}, \quad x, y \in \mathbb{Z}^d.$$

Let P_N denote the law of $X_\cdot^N$. Our first result is the following:

Theorem 1.1 *Assume* (A1), (A2), *and $d \geq \alpha$. If the stable random walk is transient, then*

$$P_N \xrightarrow{(d)} P_{X_0}^{2\gamma_e,\theta,\sigma^2,\alpha}$$

as $N \to \infty$, where $\theta = \theta_0\beta - \theta_1\delta$.

Remark 1.4 Note that if we assume that $\sum_{x\in\mathbb{Z}^d} x^i x^j p(x) = \delta_{ij}\sigma^2 < \infty$, then $\{p(x)\}$ is in the domain of attraction of a normal law with $L(N) = \sqrt{N}$. So Theorem 1.1 generalizes Theorem 1.2 in [7].

Remark 1.5 The proof of Theorem 1.1 is very similar to that used in [7] and is much easier than that of the recurrent case. To shorten the paper, we omit it in this paper. The key is the following bound, which is exactly the same as the bound in Proposition 3.3 of [7] the proof of which did not use any of the kernel assumptions. For $K, T > 0$, there exists a finite constant $C_1(K, T)$ such that if $\sup_N X_0^N(1) \leq K$, then

$$\sup_N E\Big(\sup_{t\leq T} X_t^N(1)^2\Big) \leq C_1(K, T).$$

This bound allows us to employ the L^2 arguments of [7]. For details of the proof, see Sect. 3 of [14].

Next, we consider the recurrent case. For some technical reasons, we need to assume that the $\{p(x)\}$ is in the *domain of normal attraction* of $(\sigma^2, 1)$-stable law; see Remark 3.10 below. To state our result, we introduce the one-dimensional potential kernel $a(x)$,

$$a(x) = \int_0^\infty \big[P\big(B_t^0 = 0\big) - P\big(B_t^x = 0\big)\big]\,dt. \tag{1.7}$$

We will discuss the existence of $a(x)$ later. Note that $a(x) \geq 0$. Let $\{p_t(x) : t \geq 0, x \in \mathbb{R}\}$ denote the transition density of $\{Y_t\}$. Now we define

$$\gamma^* = \big(p_1(0)\big)^{-1} \int_0^\infty \sum_{x,y,e,e'} p(e)p(e')$$
$$\times P\big(\tau(0,e) \wedge \tau(0,e') > \tau(e,e') \in du,\ B_u^0 = x, B_u^e = y\big)a(y - x). \tag{1.8}$$

Our critical Lotka–Volterra invariance principle is the following:

Theorem 1.2 *Assume that $d = \alpha = 1$. Let* (A1) *hold with $L(t) = t$ and let* (A2) *hold with $N' = N/\log N$. Then*

$$P_N \xrightarrow{(d)} P_{X_0}^{2\hat{p},\theta,\sigma^2,1}$$

as $N \to \infty$, where $\theta = \gamma^*(\theta_0 - \theta_1)$ and $\hat{p} = (p_1(0))^{-1}$.

Remark 1.6 According to Remark 1.2, assumption (A1) with $L(t) = t$ implies that the stable random walk is recurrent.

Now, we consider the applications of the convergence theorems. One can see from the rate function form that if we set $\alpha_0 = \alpha_1 = 1$, ξ_t is just the well-known *voter model*. Identify ξ_t with the set $\{x : \xi_t(x) = 1\}$ and let ξ_t^A denote the voter model starting from 1's exactly on A, $\xi_0^A = A$. Write ξ_t^x for $\xi_t^{\{x\}}$. The usual additive construction of the voter models yields

$$\xi_t^A = \bigcup_{x \in A} \xi_t^x.$$

The fact that $|\xi_t^0| = \sum_x \xi_t^0(x)$ is a martingale tells us that $|\xi_t^0|$ hits 0 eventually with probability 1. Letting $\bar{p}_t = P(|\xi_t^0| > 0)$, it follows that $\bar{p}_t \to 0$ as $t \to \infty$. People always want to determine the rate at which $\bar{p}_t \to 0$. By using a result in [21], Bramson and Griffeath [3] were able to obtain precise asymptotics under the assumption that the underlying motion is a simple random walk. By making the voter model invariance principle, Cox and Perkins [6] reproved the main result in [3] under the weaker assumption that the jump kernel has finite variance. In this paper, as applications of the convergence theorems above, we want to determine the rate at which $\bar{p}_t \to 0$ under assumption (A1). The notation $f(t) \sim g(t)$ as $t \to \infty$ means that $\lim_{n\to\infty} f(t)/g(t) = 1$. Our result is the following theorem.

Theorem 1.3 *Assume that $d \geq \alpha$ and that* (A1) *holds with $L(t) = t^{1/\alpha}$; i.e., $\{p(x)\}$ is in the domain of normal attraction of the (σ, α)-stable law. Let $\gamma_1 = p_1(0)^{-1}$ for $d = \alpha$. Then, as $t \to \infty$,*

$$\bar{p}_t \sim \frac{\log t}{\gamma_1 t}, \quad d = \alpha,$$
$$\sim (\gamma_e t)^{-1}, \quad d > \alpha.$$

Moreover,

$$P\big(\bar{p}_t |\xi_t^0| > u \big| |\xi_t^0| > 0\big) \overset{t\to\infty}{\longrightarrow} e^{-u}, \quad u > 0.$$

The method of this paper starts with the ideas [9], but our arguments are deeply involved. In particular,

1. If $p(\cdot)$ is in the domain of attraction of an α-stable law, we can only consider its α'-order moment for $\alpha' < \alpha < 2$. Thus our techniques are more delicate.
2. We need to deduce several results for a stable random walk such as local limit theorems, hitting time probability estimates, and results on range and growth of stable random walk, which we believe are of independent interest.
3. The proof for uniform convergence of random walk generators to $\Delta^{\alpha/2}$ is new. When $\alpha = 2$, it was proved by using Taylor's formula; see Lemma 2.6 of [4].

At last, we introduce some notation which will plays an important role in our proofs of the main results. First, according to [13], for $0 < \underline{\alpha} < \alpha$, we can define

$$|p|_{\underline{\alpha}} := \sum_{x \in \mathbb{Z}^d} |x|^{\underline{\alpha}} p(x) < \infty.$$

By (A2) we can define

$$\bar{\theta} = 1 \vee \sup_{N,i} N' \left|\alpha_i^N - 1\right| < \infty.$$

For $D \subset \mathbb{R}^d$ and $\phi : D \to \mathbb{R}$, define

$$\|\phi\|_{\mathrm{Lip}} = \|\phi\|_\infty + \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|}.$$

For $0 < \underline{\alpha} \le 1$, let

$$\|\phi\|_{\underline{\alpha}} = \begin{cases} 0, & \phi \equiv c \text{ for some constant } c \in \mathbb{R}, \\ \sup_{x \neq y, |x-y| \le 1} \frac{|\phi(x)-\phi(y)|}{|x-y|^{\underline{\alpha}}} \vee 2\|\phi\|_\infty, & \text{otherwise,} \end{cases}$$

and for $\underline{\alpha} > 1$, let

$$\|\phi\|_{\underline{\alpha}} = 2\|\phi\|_{\mathrm{Lip}}.$$

Note that for $\underline{\alpha} \le 1$,

$$\sup_{x \neq y, |x-y| \le 1} \frac{|\phi(x) - \phi(y)|}{|x - y|^{\underline{\alpha}}} \le \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|}.$$

Thus, for any $\underline{\alpha} > 0$,

$$\|\phi\|_{\underline{\alpha}} \le 2\|\phi\|_{\mathrm{Lip}} \quad \text{and} \quad \left|\phi(x) - \phi(y)\right| \le \|\phi\|_{\underline{\alpha}} |x - y|^{\underline{\alpha}}. \tag{1.9}$$

Remark 1.7 Since $p(\cdot)$ in this paper may not have a bounded first-order moment, we cannot use the Lipschitz norm to do estimates. Thus a “Hölder” norm is introduced to instead.

The remaining of this paper is organized as follows. In Sect. 2, we give some random walk estimates and the uniform convergence of random walk generators to the generator of the symmetric stable process. In Sect. 3, we follow the strategy in [9] to prove Theorem 1.2. Our proofs will be deeply involved due to the lack of high moments. We will carry out in detail only the part that differs. Theorem 1.3 will be proved in Sect. 4.

2 Stable Random Walk

2.1 Random Walk Estimate

Recall that $\{B_t^x, x \in \mathbb{Z}^d\}$ is a collection of rate-one independent stable random walks with $B_0^x = x$. Let $p_t(x, y) = P(B_t^x = y)$ denote the transition function of $\{B_t^x\}$. We denote by l the inverse of L. Define the characteristic function of the step function $p(\cdot)$ by

$$\psi(\eta) = \sum_x p(x) e^{iy \cdot \eta} \quad \text{for } \eta \in T^d := (-\pi, \pi]^d.$$

Since p is symmetric, $\psi(\eta)$ is real. So

$$p_t(0,x) \le p_t(0,0). \tag{2.1}$$

The following proposition is taken from [15].

Proposition 2.1 *The following are equivalent*:

(1) $p(\cdot)$ *is in the domain of attraction of* (σ^2,α)-*stable law.*
(2) $\psi(\eta) = 1 - \frac{\sigma^2}{l(1/|\eta|)} + o(\frac{1}{l(1/|\eta|)})$ *as* $|\eta|$ *tends to* 0.
(3) $\psi(\frac{\eta}{L(n)})^n \overset{n\to\infty}{\longrightarrow} \Psi(\eta)$, $\eta \in \mathbb{R}^d$.

We also have that l is of regular variation of index α and

$$l(x) = x^\alpha t(x),$$

where

$$t(x) = s\big(l(x)\big)^{-\alpha}.$$

By Lemma 2.1 in [15], for any $\epsilon > 0$, we have that there exist two positive constants C_ϵ, C'_ϵ such that, for any $1 \le y \le z$,

$$C_\epsilon y^{\alpha-\epsilon} \le l(y) \le C'_\epsilon y^{\alpha+\epsilon} \quad \text{and} \quad C_\epsilon \left(\frac{z}{y}\right)^{\alpha-\epsilon} \le \frac{l(z)}{l(y)} \le C'_\epsilon \left(\frac{z}{y}\right)^{\alpha+\epsilon}. \tag{2.2}$$

A similar result also holds for L, with α replaced by $1/\alpha$. Since $p(\cdot)$ is symmetric and irreducible, ψ is real, and $\psi(\eta) = 1$ if and only if $\eta = 0$; see [23]. According to Proposition 2.1, we may assume that there exists a constant $C > 0$ such that

$$\frac{C}{l(1/|\eta|)} \le 1 - \psi(\eta) \le 1$$

for every $\eta \in T^d$. (2.2) tells us that, for $L(t) \ge d\pi$ and $0 \le \epsilon \le \alpha$,

$$t\left(1 - \psi\left(\frac{\eta}{L(t)}\right)\right) \ge \frac{Cl(L(t))}{l(L(t)/|\eta|)} \ge (C_\epsilon \vee C'_\epsilon)\big(|\eta|^{\alpha+\epsilon} + |\eta|^{\alpha-\epsilon}\big). \tag{2.3}$$

Recall that $\{p_t(x) : t \ge 0, x \in \mathbb{R}\}$ denotes the transition density of $\{Y_t\}$. The local limit theorem for the stable random walk which plays an important role in our proofs of main results will be given in the following proposition.

Proposition 2.2 *If* (A1) *holds, then*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{Z}^d} \left| L(t)^d p_t(0,x) - p_1\left(\frac{x}{L(t)}\right)\right| = 0, \tag{2.4}$$

and there exists a constant C depending on $p(\cdot)$ such that for all $t \ge 0$ and $x \in \mathbb{R}^d$,

$$p_t(0,x) \le CL(t)^{-d}. \tag{2.5}$$

Moreover, if $L(t)=t$ *and* $d=1$,

$$\sup_{x\in\mathbb{Z}} P\big(B_t^0=x\big)\le C_{2.6}(t+1)^{-1}. \tag{2.6}$$

Proof Since l is a function of regular variation, by Proposition 2.1, for each $|\eta|>0$,

$$\lim_{t\to\infty} t\left(1-\psi\left(\frac{\eta}{L(t)}\right)\right)=\lim_{t\to\infty}\frac{l(L(t))}{l(L(t)/|\eta|)}\big(\sigma^2+o(1)\big)=\sigma^2|\eta|^\alpha. \tag{2.7}$$

Then

$$\begin{aligned}
&\left|L(t)^d p_t(0,x)-p_1\left(\frac{x}{L(t)}\right)\right|\\
&\quad\le (2\pi)^{-d}\left|\int_{L(t)T^d} e^{-ix\cdot(\eta/L(t))}\exp\left\{-t\left(1-\psi\left(\frac{\eta}{L(t)}\right)\right)\right\}d\eta\right.\\
&\qquad\left.-\int_{L(t)T^d} e^{-i(x/L(t))\cdot\eta}\Psi(\eta)\,d\eta\right|\\
&\qquad+(2\pi)^{-d}\int_{\mathbb{R}^d\setminus L(t)T^d}\exp\big\{-\sigma^2|\eta|^\alpha\big\}\,d\eta\\
&\quad\le (2\pi)^{-d}\int_{L(t)T^d}\left|\exp\left\{-t\left(1-\psi\left(\frac{\eta}{L(t)}\right)\right)\right\}-\exp\big\{-\sigma^2|\eta|^\alpha\big\}\right|d\eta\\
&\qquad+(2\pi)^{-d}\int_{\mathbb{R}^d\setminus L(t)T^d}\exp\big\{-\sigma^2|\eta|^\alpha\big\}\,d\eta.
\end{aligned}$$

Then the Dominated Convergence Theorem with (2.3) yields (2.4). For (2.5), when $L(t)\ge d\pi$,

$$\begin{aligned}
p_t(0,x)&=(2\pi)^{-d}\int_{T^d} e^{-ix\cdot\eta}\exp\big\{-t\big(1-\psi(\eta)\big)\big\}\,d\eta\\
&\le (2\pi)^{-d}L(t)^{-d}\int_{L(t)T^d}\exp\left\{-t\left(1-\psi\left(\frac{\eta}{L(t)}\right)\right)\right\}d\eta\\
&\le (2\pi)^{-d}L(t)^{-d}\int_{\mathbb{R}^d}\exp\big\{-(C_\epsilon\vee C'_\epsilon)\big(|\eta|^{\alpha+\epsilon}+|\eta|^{\alpha-\epsilon}\big)\big\}\,d\eta\\
&\le CL(t)^{-d},
\end{aligned}$$

where the second inequality follows from (2.3). Then (2.5) holds for every $t\ge 0$. We complete the proof. □

The following two propositions consider the growth of the stable random walk.

Proposition 2.3

(a) *If* $z_T \in \mathbb{Z}^d$ *and* $t_T > 0$ *satisfy*

$$\lim_{T\to\infty} \frac{z_T}{L(T)} = z \quad \text{and} \quad \lim_{T\to\infty} \frac{t_T}{T} = s > 0, \tag{2.8}$$

then

$$\lim_{T\to\infty} L(T)^d P\left(B^0_{t_T} = z_T\right) = \frac{p_1(z/s)}{s^d}. \tag{2.9}$$

(b) *For each* $K > 0$, *there is a constant* $C_{2.10}(K) > 0$ *such that*

$$\liminf_{T\to\infty} \inf_{|x|\le KL(T)} L(T)^d P\left(B^0_T = x\right) \ge C_{2.10}(K). \tag{2.10}$$

Proof By (2.8) and Remark 1.1, we have $\lim_{T\to\infty} \frac{L(t_T)}{L(T)} = s$. Then (2.9) follows from (2.4). For (b), when $\alpha = 2$, by (2.4), the desired result is immediate. When $0 < \alpha < 2$, recall that $\{p_t(x) : t \ge 0, x \in \mathbb{R}^d\}$ is the transition density of a symmetric α-stable process. By the arguments after Remark 5.3 of [1], there exists two positive constants c_1 and c_2 such that

$$c_1\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \le p_t(x) \le c_2\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right). \tag{2.11}$$

By the bounds above and (2.4),

$$\begin{aligned}\liminf_{T\to\infty} \inf_{|x|\le KL(T)} L(T)^d P\left(B^0_T = x\right) &= \liminf_{T\to\infty} \inf_{|x|\le KL(T)} p_1\left(x/L(T)\right)\\ &\ge c\left(1 \wedge K^{d+\alpha}\right).\end{aligned}$$

The desired result follows readily. □

Proposition 2.4 *Assume that* $d = 1$. *If* g_1 *and* g_2 *are two positive functions on* $\mathbb{R}^+$ *such that* $g_1(x) \to +\infty$ *and* $g_2(x) \to +\infty$ *as* $x \to +\infty$, *then there is exists a constant* $C_{2.12}$ *which only depends on* p *such that*

$$P\left(\left|B^0_{g_1(N)}\right| \ge g_2(N)\right) \le \frac{C_{2.12}\, g_1(N)}{l(g_2(N))}. \tag{2.12}$$

Proof First,

$$P\left(\left|B^0_{g_1(N)}\right| \ge g_2(N)\right) \le P\left(\max_{u\le g_1(N)} \left|B^0_u\right| \ge g_2(N)\right).$$

Note that $\{B^0_u : u \ge 0\}$ is a compound Poisson process whose Lévy measure is given by

$$\nu_0(dz) := \sum_{y\in\mathbb{Z}^d} p(y)\delta_y(dz),$$

which is a symmetric measure. According to the arguments in Sect. 3 of [19],

$$P\Big(\max_{u\le g_1(N)}\big|B_u^0\big|\ge g_2(N)\Big)$$
$$\le Cg_1(N)\bigg(\nu_0\big(z:|z|>g_2(N)\big)+g_2(N)^{-2}\int_{|z|\le g_2(N)}z^2\nu_0(dz)\bigg),$$

where C is a positive constant; see (3.2) of [19]. Since $p(\cdot)$ is in the domain of attraction of (σ,α)-stable law, we have

$$\frac{x^2[\nu_0(z:|z|>x)]}{\int_{|z|\le x}z^2\nu_0(dz)}\longrightarrow\frac{2-\alpha}{\alpha}\tag{2.13}$$

and

$$\frac{x\int_{|z|\le L(x)}z^2\nu_0(dz)}{L(x)^2}\longrightarrow C_0\tag{2.14}$$

as $x\to\infty$ for some constant $C_0>0$; see (5.16) and (5.23) in Chap. XVII of [13]. By (2.13) there exists a constant C_1 independent of N such that

$$\nu_0\big(z:|z|>g_2(N)\big)\le C_1g_2(N)^{-2}\int_{|z|\le g_2(N)}z^2\nu_0(dz).$$

According to (2.14), there exists another constant C_2 independent of N such that

$$g_2(N)^{-2}\int_{|z|\le g_2(N)}z^2\nu_0(dz)\le\frac{C_2}{l(g_2(N))}.$$

(Recall that l is the inverse function of L.) Thus,

$$P\Big(\max_{u\le g_1(N)}\big|B_u^0\big|\ge g_2(N)\Big)\le CC_2(C_1+1)\frac{g_1(N)}{l(g_2(N))},$$

which yields the desired result. □

At last we give the asymptotics of the range of the stable random walk. Define the mean range of the stable random walk B_t^0 by

$$R(t)=E\bigg(\sum_x 1_{\{B_s^0=x\text{ for some }s\le t\}}\bigg).$$

By the results for the range of the discrete-time stable random walk in [15], we see that

$$\begin{aligned}\lim_{t\to\infty}\frac{R(t)}{t}\int_1^t L(s)^{-d}\,ds=p_1(0)^{-1}\quad&\text{if } B_t \text{ is recurrent},\\ \lim_{t\to\infty}\frac{R(t)}{t}=\gamma_e\quad&\text{if } B_t\text{ is transient.}\end{aligned}\tag{2.15}$$

2.2 Hitting Time for One-dimensional Walk

In this subsection, we assume that $d = \alpha = 1$. Recall that

$$g(x) = \int_1^x L(s)^{-1}\, ds$$

for $x \geq 1$. Assume that $\lim_{x\to\infty} g(x) = \infty$. This means that the one-dimensional random walk is recurrent. Let $\tau_x = \inf\{t \geq 0 : B_t^0 = x\}$, and write P^x to indicate the law of the walk $B_\cdot^x$. Let $\tilde{P}(\cdot) = \sum_e p(e) P^e(\cdot)$ and define

$$H(t) = \tilde{P}(\tau_0 > t). \tag{2.16}$$

Recall the definition of $a(x)$ in (1.7).

Proposition 2.5

$$\lim_{t\to\infty} H(t) g(t) = p_1(0)^{-1}; \tag{2.17}$$

$$\frac{P^x(\tau_0 > t)}{H(t)} \leq 2a(x) \quad \textit{for all } x \in \mathbb{Z},\ t > 0; \tag{2.18}$$

$$\lim_{t\to\infty} \frac{P^x(\tau_0 > t)}{H(t)} = a(x) \quad \textit{for all } x \in \mathbb{Z}; \tag{2.19}$$

$$a(x)/|x|,\ x \neq 0 \textit{ is bounded on } \mathbb{Z}. \tag{2.20}$$

Proof For (2.17), let $G(t) = \int_0^t p_s(0,0)\, ds$. Proposition 2.2 implies that $G(t) \sim p_1(0) g(t)$ as $t \to \infty$ in $d = 1$. Then one can follow the arguments in the proof of Lemma A.3 in [4] by using the last exit time decomposition of Lemma A.2 there and with (A.7) replaced by (2.5) to obtain that $G(t)H(t) \to 1$ as $t \to \infty$; see the arguments after (A.8) of [4]. Then (2.17) holds.

Recall that $\{Z_n : n = 0, 1, 2, \ldots\}$ denotes the discrete-time stable random walk defined in Sect. 1.2. With abuse of notation, let P^x denote the law of the walk starting at $Z_0 = x$. Let $\sigma_x = \inf\{n \geq 1 : Z_n = x\}$. By T29.1 of [23],

$$a(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \left[P^0(Z_k = 0) - P^0(Z_k = x)\right] < \infty \quad \text{exists for all } x \text{ in } \mathbb{Z}.$$

Note that P11.1, P11.2, and P11.3 in Chap. III of [23] are available for one-dimensional recurrent random walk; see arguments before P28.1 of [23]. Meanwhile, according to T29.1 and P30.1 of [23], (i)$'$ and (ii)$'$ on p. 116 in Chap. III of [23] also hold for a one-dimensional random walk. Then we can check that both P11.4 and P11.5 in Chap. III of [23] are also available. Thus we have

$$P^0(\sigma_x < \sigma_0) = 1/2a(x).$$

Since the sequence of states visited by the walk B_t^0 is equal in law to the sequence visited by the walk Y_n (with $Y_0 = 0$), we have $\tilde{P}(\tau_x < \tau_0) = 1/2a(x)$. The strong

Markov property implies that

$$H(t) \geq \sum_e p(e) P^e(\tau_x < \tau_0, \tau_0 > t) \geq \sum_e P^e(\tau_x < \tau_0) P^x(\tau_0 > t),$$

and then (2.18) follows.

For (2.19), by T32.1 of [23],

$$\lim_{n\to\infty} \frac{P^x(\sigma_0 > n)}{P^0(\sigma_0 > n)} = a(x). \tag{2.21}$$

Define

$$h(n) = \sum_{0\leq k\leq n} P^0(Y_k = 0).$$

Then

$$h(n) \sim p_1(0) \sum_{k=1}^n \frac{1}{L(k)} \quad \text{as } n \to \infty; \tag{2.22}$$

see p. 696 of [15]. We also have that

$$P^0(\sigma_0 > n) = \frac{1}{h(n)} + o\left(\frac{1}{h(n)^2 s(n)}\right);$$

see the proof of Theorem 6.9 of [15]. Thus,

$$P^0(\sigma_0 > n) g(n) \to p_1(0)^{-1}. \tag{2.23}$$

According to a standard large-deviations estimate for a rate-1 Poisson process, say $S(t)$, $e^{Ct} P(S(t) \notin [t/2, 2t]) \to 0$ as $n \to \infty$ for some constant $C > 0$. Then the fact that $Y_{S(\cdot)}$ is a realization of $B^0_{\cdot}$ yields

$$\left(1 - o\left(e^{-Ct}\right)\right) P^x(\sigma_0 > 2t) \leq P^x(\tau_0 > t) \leq o\left(e^{-Ct}\right) + P^x(\sigma_0 > t/2).$$

The inequalities above, together with (2.21) and (2.23), imply

$$\lim_{t\to\infty} \frac{P^x(\tau_0 > t)}{P^x(\sigma_0 > t)} = 1. \tag{2.24}$$

By (2.17) we see $H(t)/P^0(\sigma_0 > t) \to 1$ as $t \to \infty$. Then (2.21) and (2.24) tell us that (2.19) holds readily. Finally, (2.20) follows from the fact that

$$\lim_{|x|\to\infty} \frac{a(x)}{|x|} = 0;$$

see P29.3 of [23] and elsewhere. We have completed the proof. □

2.3 Convergence of Generators

For

$$\phi = \phi_s(x), \qquad \dot{\phi}_s(x) \equiv \frac{\partial}{\partial s}\phi(s,x) \in C_b\big([0,T]\times \mathbb{S}_N\big),$$

and $s \le T$, define

$$\mathsf{A}_N(\phi_s)(x) = \sum_{y\in\mathbb{S}_N} N p_N(y-x)\big(\phi_s(y)-\phi_s(x)\big). \tag{2.25}$$

In this subsection we consider the uniform convergence of A_N. Recall the definition of generators of symmetric stable processes and the stable random walk Z_n defined in Sect. 1.2. For each $N > 1$, let $\{P_t^{(N)} : t \ge 0\}$ be a rate-N Poisson process which is independent of $\{U_i : i \ge 1\}$. Then

$$\hat{Z}_t^N = L(N)^{-1}\sum_{i=1}^{P_t^{(N)}} U_i$$

is a compound Poisson process on $\mathbb{R}^d$ whose Lévy measure is given by

$$\nu_N^0(dy) := \sum_{z\in\mathbb{S}_N} N p_N(z)\delta_z(dy);$$

see [20]. Note that both the law of $\hat{Z}_1^N$ and the (σ^2,α)-stable law are infinitely divisible distributions. We also have that

$$\mathbf{E}\big(e^{i\hat{Z}_1^N\cdot\eta}\big) = \exp\left\{-N\left(\psi\left(\frac{\eta}{L(N)}\right)-1\right)\right\}.$$

By (2.7),

$$\hat{Z}_1^N \xrightarrow{(d)} Y_1 \quad \text{as } N\to\infty.$$

According to Theorem 8.7 of [20] and its proof, we see that

$$\rho_N(dy) := \frac{|y|^2}{1+|y|^2}\nu_N^0(dy) \to \rho(dy) := \frac{\sigma^2|y|^2}{1+|y|^2}\nu(dy) \quad \text{in } M\big(\mathbb{R}^d\big).$$

For $f \in C_b(\mathbb{R}^d)$, define

$$\|f\|_{BL} = \sup_x\big|f(x)\big| \vee \sup_{x\ne y}\frac{|f(x)-f(y)|}{|x-y|}.$$

Let P, Q be two probability measures on $\mathbb{R}^d$. Set

$$\|\mathsf{P}-\mathsf{Q}\|_{BL} := \sup_{\|f\|_{BL}=1}\left|\int f\,d\mathsf{P} - \int f\,d\mathsf{Q}\right|.$$

It is easy to see that

$$\|P - Q\|_{BL} = \sup_{\|f\|_{BL}<\infty} \frac{|\int f\,dP - \int f\,dQ|}{\|f\|_{BL}}. \tag{2.26}$$

By Problem 3.11.2 of [12],

$$\|P - Q\|_{BL} \le 3\,M(P, Q), \tag{2.27}$$

where M denotes the Prokhorov metric; see Chap. 3 of [12].

Lemma 2.1 *For $\phi \in C_b^{1,3}([0, T] \times \mathbb{R}^d)$,*

$$\lim_{N\to\infty} \sup_{s\le T} \left\| A_N\phi_s - \frac{\sigma^2 \Delta^{\alpha/2}\phi_s}{2} \right\|_\infty = 0.$$

Moreover, for each $R < \infty$, the rate of convergence is uniform on

$$H_R := \Big\{\phi \in C_b^{1,3}\big([0, T] \times \mathbb{R}^d\big) : \sup_{s,i,j,k} \big(\|\phi_s\|_\infty + \|(\phi_s)_i\|_\infty + \|(\phi_s)_{ij}\|_\infty + \|(\phi_s)_{ijk}\|_\infty\big) < R\Big\},$$

where the subscripts i, j, k indicate partial derivatives with respect to the spatial variable.

Proof Recall that $D_j = \frac{\partial}{\partial x_j}$. Define

$$g_s(x, y) = \left[\phi_s(x + y) - \phi_s(x) - \frac{1}{1 + |y|^2} \sum_{i=1}^{d} y_j D_j\phi_s(x)\right] \cdot \frac{1 + |y|^2}{|y|^2}.$$

Since p_N is symmetric, we may rewrite

$$A_N\phi_s(x) = \int g_s(x, y)\rho_N(dy),$$

and we also have that

$$\frac{\sigma^2 \Delta^{\alpha/2}\phi_s(x)}{2} = \int g_s(x, y)\rho(dy).$$

Let $h : \mathbb{R}^d \to [0, 1]$ be a C_b^∞ function such that

$$B(0, 1) \subset \big\{x : h(x) = 0\big\} \subset \big\{x : h(x) < 1\big\} \subset B(0, 2)$$

and

$$B(0, 2)^c \subset \big\{x : h(x) = 1\big\}.$$

Define $h_k(x) = h(kx)$ for $k \geq 1$. Let

$$g_k(s, x, y) := h_k(y) g_s(x, y).$$

Then $g_k(s, x, y) = g_s(x, y)$ for $|y| > 2/k$. One can check that

$$\sup_k \sup_{\phi \in H_R} \sup_s \sup_x \left(\left\| g_k(s, x, \cdot) \right\|_\infty + \left\| g_s(x, \cdot) \right\|_\infty \right) < C_d R$$

and for each $k \geq 1$,

$$\sup_{\phi \in H_R} \sup_s \sup_x \left\| \sum_{j=1}^d \left| \frac{\partial g_k(s, x, y)}{\partial y_j} \right| \right\|_\infty < k C_d R,$$

where C_d is a constant depending only on d. Typically, for each $k \geq 1$,

$$\sup_{\phi \in H_R} \sup_s \sup_x \left\| g_k(s, x, \cdot) \right\|_{BL} < (k+1) C_d R.$$

By (2.26) and (2.27), we obtain

$$\begin{aligned}
&\sup_{\phi \in H_R} \sup_{s \leq T} \sup_x \left| \frac{\int g_k(s, x, y) \rho_N(dy)}{\rho_N(\mathbb{R}^d)} - \frac{\int g_k(s, x, y) \rho(dy)}{\rho(\mathbb{R}^d)} \right| \\
&\quad \leq (k+1) C_d R \cdot 3 \mathsf{M} \left(\frac{\rho_N}{\rho_N(\mathbb{R}^d)}, \frac{\rho}{\rho(\mathbb{R}^d)} \right) \\
&\quad \to 0 \quad \text{as } N \to \infty.
\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}
&\sup_{\phi \in H_R} \sup_{s \leq T} \sup_x \left| \int g_k(s, x, y) \rho_N(dy) - \int g_k(s, x, y) \rho(dy) \right| \\
&\leq C_d R \left| \rho_N\left(\mathbb{R}^d\right) - \rho\left(\mathbb{R}^d\right) \right| \\
&\quad + \rho\left(\mathbb{R}^d\right) \sup_{\phi \in H_R} \sup_{s \leq T} \sup_x \left| \frac{\int g_k(s, x, y) \rho_N(dy)}{\rho_N(\mathbb{R}^d)} - \frac{\int g_k(s, x, y) \rho(dy)}{\rho(\mathbb{R}^d)} \right| \\
&\to 0 \quad \text{as } N \to \infty.
\end{aligned}$$

Using the triangle inequality again, we have

$$\begin{aligned}
&\sup_{\phi \in H_R} \sup_{s \leq T} \left\| \mathsf{A}_N \phi_s - \frac{\sigma^2 \Delta^{\alpha/2} \phi_s}{2} \right\|_\infty \\
&\quad \leq \sup_{\phi \in H_R} \sup_{s \leq T} \sup_x \left| \int g_s(x, y) \rho_N(dy) - \int g_k(s, x, y) \rho_N(dy) \right| \\
&\qquad + \sup_{\phi \in H_R} \sup_{s \leq T} \sup_x \left| \int g_k(s, x, y) \rho_N(dy) - \int g_k(s, x, y) \rho(dy) \right|
\end{aligned}$$

$$+\sup_{\phi\in H_R}\sup_{s\le T}\sup_{x}\left|\int g_k(s,x,y)\rho(dy)-\int g_s(x,y)\rho(dy)\right|$$
$$\le C_d R\rho_N\big(\{y:|y|\le 2/k\}\big)+C_d R\rho\big(\{y:|y|\le 2/k\}\big)$$
$$+\sup_{\phi\in H_R}\sup_{s\le T}\sup_{x}\left|\int g_k(s,x,y)\rho_N(dy)-\int g_k(s,x,y)\rho(dy)\right|.$$

Note that $\rho(dy)$ is absolutely continuous with respect to the Lebesgue measure. Letting N go to infinity above yields

$$\lim_{N\to\infty}\sup_{\phi\in H_R}\sup_{s\le T}\left\|\mathsf{A}_N\phi_s-\frac{\sigma^2\Delta^{\alpha/2}\phi_s}{2}\right\|_\infty\le 2C_d R\rho\big(\{y:|y|\le 2/k\}\big).$$

Then since $\rho(\{0\})=0$, the desired result follows readily if we let $k\to\infty$. □

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. In Sects. 3.2–3.5, we assume that

$$d=\alpha=1\quad\text{and}\quad L(t)=t,$$

and by $\underline{\alpha}$ we always mean a constant which is strictly less than 1. We can adopt some of the arguments of [9] to prove some analogous results to those in [9] without using the fact that $p(\cdot)$ is in the domain of attraction of a stable law. We will refer the reader to these results as we use them.

3.1 Semimartingale Decompositions

Some results in this subsection are exactly the same as those in Sect. 3 of [9]. For completeness, we list them here. Let ξ_t^N be the rescaled Lotka–Volterra model we have constructed in Sect. 1.2. As in [9], we introduce the following notation. For

$$\phi=\phi_s(x),\qquad \dot{\phi}_s(x)\equiv\frac{\partial}{\partial s}\phi(s,x)\in C_b\big([0,T]\times\mathbb{S}_N\big),$$

and $s\le T$, define

$$D_t^{N,1}(\phi)=\int_0^t X_s^N(\mathsf{A}_N\phi_s+\dot{\phi}_s)\,ds,\tag{3.1}$$

$$D_t^{N,2}(\phi)=\frac{N(\alpha_0^N-1)}{N'}\int_0^t\sum_{x\in\mathbb{S}_N}\phi_s(x)1_{\{\xi_s^N(x)=0\}}\big(f_1^N\big(x,\xi_s^N\big)\big)^2\,ds,\tag{3.2}$$

$$D_t^{N,3}(\phi)=\frac{N(\alpha_1^N-1)}{N'}\int_0^t\sum_{x\in\mathbb{S}_N}\phi_s(x)1_{\{\xi_s^N(x)=1\}}\big(f_0^N\big(x,\xi_s^N\big)\big)^2\,ds,\tag{3.3}$$

$$\langle M^N(\phi)\rangle_{1,t} = \frac{N}{(N')^2}\int_0^t \sum_{x\in\mathbb{S}_N}\phi_s^2(x)\sum_{y\in\mathbb{S}_N} p_N(y-x)\big(\xi_s^N(y)-\xi_s^N(x)\big)^2\,ds, \tag{3.4}$$

$$\langle M^N(\phi)\rangle_{2,t} = \frac{1}{(N')^2}\int_0^t \sum_{x\in\mathbb{S}_N}\phi_s^2(x)\Big[\big(\alpha_0^N-1\big)1_{\{\xi_s^N(x)=0\}}\big(f_1^N\big(x,\xi_s^N\big)\big)^2 + \big(\alpha_1^N-1\big)1_{\{\xi_s^N(x)=1\}}\big(f_0^N\big(x,\xi_s^N\big)\big)^2\Big]\,ds. \tag{3.5}$$

If $X.$ is a process, let $(\mathcal{F}_t^X, t\ge 0)$ be the right-continuous filtration generated by $X.$. The following proposition is a version of Proposition 3.1 of [9]. For its proof, see Sect. 2 of [7].

Proposition 3.1 *For $\phi, \dot{\phi}\in C_b([0,T]\times\mathbb{S}_N)$ and $t\in[0,T]$,*

$$X_t^N(\phi_t) = X_0^N(\phi_0) + D_t^N(\phi) + M_t^N(\phi), \tag{3.6}$$

where

$$D_t^N(\phi) = D_t^{N,1}(\phi) + D_t^{N,2}(\phi) - D_t^{N,3}(\phi), \tag{3.7}$$

and $M_t^N(\phi)$ is an $\mathcal{F}_t^{X^N}$-square-integrable martingale with predictable square function

$$\langle M^N(\phi)\rangle_t = \langle M^N(\phi)\rangle_{1,t} + \langle M^N(\phi)\rangle_{2,t}. \tag{3.8}$$

The following lemma is a generalization of Lemma 3.5 of [7] and Lemma 4.8 of [9].

Lemma 3.1 *There is a constant C such that if $\phi:[0,T]\times\mathbb{S}_N\to\mathbb{R}$ is a bounded measurable function, then*

(a) $\langle M^N(\phi)\rangle_{2,t} = \int_0^t m_{2,s}^N(\phi)\,ds$, *where*

$$\big|m_{2,s}^N(\phi)\big| \le C\frac{\|\phi_s\|_\infty^2}{(N')^2}X_s^N(1). \tag{3.9}$$

(b) *For $\underline{\alpha} < 1\wedge\alpha$,*

$$\langle M^N(\phi)\rangle_{1,t} = 2\int_0^t X_s^N\big((N/N')\phi_s^2 f_0^N\big(\xi_s^N\big)\big)\,ds + \int_0^t m_{1,s}^N(\phi_s)\,ds, \tag{3.10}$$

where

$$\big|m_{1,s}^N(\phi)\big| \le \bigg[X_s^N(1)\frac{2N\|\phi\|_{\underline{\alpha}}^2|p|_{\underline{\alpha}}}{N'L(N)^{\underline{\alpha}}}\bigg]\wedge\bigg[\frac{2N\|\phi\|_\infty^2 X_s^N(1)}{N'}\bigg]. \tag{3.11}$$

(c) *For $i=2,3$, $D_t^{N,i}(\phi) = \int_0^t d_s^{N,i}(\phi)\,ds$ for $t\le T$, where for all N and $s\le T$,*

$$\big|d_s^{N,i}(\phi)\big| \le C\|\phi_s\|_\infty X_s^N\big((N/N')f_0^N\big(\xi_s^N\big)\big).$$

Remark 3.1 Note that whenever $N' = N$, then, since $f_0^N \le 1$, we have

$$\left|d_s^{N,i}(\phi)\right| \le C\|\phi_s\|_\infty X_s^N(1), \quad i = 2, 3.$$

Proof (a) In the following of this proof, by C we denote a positive constant which may change from line to line. Since $f_0^N \le 1$, $f_1^N \le 1$, and $1_{\{\xi_s^N(x)=1\}} = \xi_s^N(x)$, the definition of $\langle M^N(\phi)\rangle_{2,t}$ and the fact that $f_0^N + f_1^N = 1$ imply

$$\begin{aligned}
\left|m_{2,s}^N(\phi)\right| &\le \frac{\|\phi\|_\infty^2 \sup_N N'|\alpha_0^N - 1|}{(N')^3} \sum_{x\in\mathbb{S}_N} \left(f_1^N\left(x, \xi_s^N\right)\right) 1_{\{\xi_s^N(x)=0\}} \\
&\quad + \frac{\|\phi\|_\infty^2 \sup_N N'|\alpha_1^N - 1|}{(N')^2} X_s^N(1) \\
&\le \frac{C\|\phi\|_\infty^2}{(N')^3} \sum_{x,y} p_N(x-y)(1 - 1_{\{\xi_s^N(x)=1\}}) 1_{\{\xi_s^N(y)=1\}} + \frac{C\|\phi\|_\infty^2}{(N')^2} X_s^N(1) \\
&\le \frac{C\|\phi\|_\infty^2}{(N')^2} X_s^N(1),
\end{aligned}$$

where the second inequality follows from (A2). For (b), note that

$$\begin{aligned}
\left\langle M^N(\phi)\right\rangle_{2,t} &= \frac{1}{(N')^2} \int_0^t \sum_{x\in\mathbb{S}_N} \phi_s^2(x) \sum_{y\in\mathbb{S}_N} N p_N(y-x)\left(\xi_s^N(y) - \xi_s^N(x)\right)^2 ds \\
&= \frac{1}{(N')^2} \int_0^t \sum_{x\in\mathbb{S}_N} \phi_s^2(x) \sum_{y\in\mathbb{S}_N} N p_N(y-x)\left(2\xi_s^N(x)\left(1 - \xi_s^N(y)\right)\right) ds \\
&\quad + \frac{1}{(N')^2} \int_0^t \sum_{x\in\mathbb{S}_N} \phi_s^2(x) \sum_{y\in\mathbb{S}_N} N p_N(y-x)\left(\xi_s^N(y) - \xi_s^N(x)\right) ds.
\end{aligned}$$

Thus, (3.10) holds with

$$\begin{aligned}
m_{1,s}^N(\phi) &= \frac{N}{(N')^2} \sum_{x\in\mathbb{S}_N} \phi_s^2(x) \sum_{y\in\mathbb{S}_N} p_N(y-x)\left(\xi_s^N(y) - \xi_s^N(x)\right) \\
&= \frac{N}{(N')^2} \sum_{x\in\mathbb{S}_N} \phi_s^2(x) \sum_{y\in\mathbb{S}_N} p_N(y-x)\left(\xi_s^N(y) 1_{\{\xi_s^N(x)=0\}} - \xi_s^N(x) 1_{\{\xi_s^N(y)=0\}}\right) \\
&= \frac{N}{(N')^2} \sum_{x,y\in\mathbb{S}_N} p_N(y-x)\left(\phi_s^2(x) - \phi_s^2(y)\right) \xi_s^N(y)\left(1 - \xi_s^N(x)\right) \\
&\le \frac{2N\|\phi\|_\infty^2 X_s^N(1)}{N'}.
\end{aligned}$$

On the other hand,

$$\left|\phi_s^2(x) - \phi_s^2(y)\right| \le 2\|\phi\|_{\underline{\alpha}}^2 |x-y|^{\underline{\alpha}}$$

for $\underline{\alpha} < 1 \wedge \alpha$. Thus,

$$\begin{aligned} m_{1,s}^N(\phi) &\le 2(N/N')\|\phi\|_{\underline{\alpha}}^2 \frac{1}{N'} \sum_y \xi_s^N(y) \sum_x |y-x|^{\underline{\alpha}} p_N(y-x) \\ &\le X_s^N(1) \frac{2N\|\phi\|_{\underline{\alpha}}^2 |p|_{\underline{\alpha}}}{N' L(N)^{\underline{\alpha}}}. \end{aligned}$$

This completes the proof of (b). For (c), according to (A2), the fact that both f_0^N and f_1^N are less than 1 yields

$$\begin{aligned} \left|d_s^{N,i}(\phi)\right| &\le \frac{N \sup_N N'|\alpha_{i-2}^N - 1|}{N'} \|\phi_s\|_\infty \frac{1}{N'} \sum_x \sum_y p_N(y-x)\xi_s^N(x)\big(1-\xi_s^N(y)\big) \\ &\le C\|\phi_s\|_\infty X_s^N\big((N/N') f_0^N\big(\xi_s^N\big)\big). \end{aligned}$$

We are done. □

3.2 Voter and Biased Voter Estimates

In this subsection, we consider voter and biased voter bounds. We follow the arguments in Sect. 5 of [9] step by step. For $b, \nu \ge 0$, the 1-biased voter model $\bar{\xi}_t$ is the Feller process taking values in $\{0,1\}^{\mathbb{Z}}$ with rate function

$$\bar{c}(x,\xi) = \begin{cases} (\nu+b) f_1(x,\xi) & \text{if } \xi(x) = 0, \\ \nu f_0(x,\xi) & \text{if } \xi(x) = 1, \end{cases} \tag{3.12}$$

where $f_i(x,\xi)$ is as in (1.1). The 0-biased voter model is the Feller process $\underline{\xi}_t$ taking values in $\{0,1\}^{\mathbb{Z}}$ with rate function

$$\underline{c}(x,\xi) = \begin{cases} \nu f_1(x,\xi) & \text{if } \xi(x) = 0, \\ (\nu+b) f_0(x,\xi) & \text{if } \xi(x) = 1. \end{cases} \tag{3.13}$$

The voter model $\hat{\xi}_t$ is the 1-biased voter model with bias $b = 0$. Then by Theorem III.1.5 of [16], assuming that $\underline{\xi}_0 = \hat{\xi}_0 = \bar{\xi}_0$, we may define $\underline{\xi}_t$, $\hat{\xi}_t$, and $\bar{\xi}_t$ on a common probability space so that

$$\underline{\xi}_t \le \hat{\xi}_t \le \bar{\xi}_t \quad \text{for all } t \ge 0. \tag{3.14}$$

For $\xi, \zeta \in \{0,1\}^{\mathbb{Z}}$, $\xi \le \zeta$ means that $\xi(x) \le \zeta(x)$ for all $x \in \mathbb{Z}$.

Let us recall the voter model duality; see [16]. Recall also the coalescing random walk system $\{\hat{B}_t^x : x \in \mathbb{Z}\}$ defined in Sect. 1.3. The duality equation for the rate-1 $(\nu = 1)$ voter model is: for finite $A \subset \mathbb{Z}$,

$$P\big(\hat{\xi}_t(x) = 1\ \forall x \in A\big) = P\big(\hat{\xi}_0\big(\hat{B}_t^x\big) = 1\ \forall x \in A\big). \tag{3.15}$$

First, we consider the voter estimates. Let $P_t,\ t \geq 0$, be the semigroup of a rate-1 random walk with step distribution $p(\cdot)$. Recall the definition of $|p|_{\underline{\alpha}}$ in Sect. 3. For $\phi : \mathbb{Z} \to \mathbb{R}$ and $\xi \in \{0,1\}^{\mathbb{Z}}$, let

$$\xi(\phi) = \sum_x \phi(x)\xi(x).$$

Lemma 3.2 *Let $\hat{\xi}_t$ denote the rate-ν voter model. Then for all bounded $\phi : \mathbb{Z} \to \mathbb{R}^+$, $0 < \underline{\alpha} < 1$, and $t \geq 0$,*

$$E\left(\hat{\xi}\left(\phi f_0(\hat{\xi}_t)\right)\right) \leq \left(\nu t |p|_{\underline{\alpha}} H(2\nu t)\right)^{1/2} \|\phi\|_{\underline{\alpha}/2} |\bar{\xi}_0| + H(2\nu t)\hat{\xi}_0(\phi). \tag{3.16}$$

Remark 3.2 Equation (3.16) is just a version of (5.8) in Lemma 5.1 of [9]. We slightly abuse our notation, and we can prove that the other statements in Lemma 5.1 of [9] ((5.6), (5.7), and (5.9) there) hold without modifying any arguments of their proofs.

Remark 3.3 Recall the definition of $\|\phi\|_{\underline{\alpha}}$ in Sect. 3. We see that for $\phi = 1$, the right side of (3.16) is just $H(2\nu t)|\hat{\xi}_0|$.

Proof It suffices to consider $\nu = 1$. Using the voter duality equation (3.15) and following the arguments in the proof of (5.8) of [9], we have

$$E\left(\hat{\xi}\left(\phi f_0(\hat{\xi}_t)\right)\right) \leq \sum_{e,z} \hat{\xi}_0(z) p(e) E\left(\phi\left(z + B_t^0\right) 1_{\{\tau(0,e)>t\}}\right).$$

For any z and $0 < \underline{\alpha} < 1$,

$$\begin{aligned}
&\sum_e p(e) E\left(\phi\left(z + B_t^0\right) 1_{\{\tau(0,e)>t\}}\right) \\
&\quad \leq \sum_e p(e) E\left(\left(\|\phi\|_{\underline{\alpha}/2} |B_t^0|^{\underline{\alpha}/2} + \phi(z)\right) 1_{\{\tau(0,e)>t\}}\right) \\
&\quad \leq \|\phi\|_{\underline{\alpha}/2} \left(E\left(|B_t^0|^{\underline{\alpha}}\right) \sum_e p(e) P\left(\tau(0,e) > t\right)\right)^{1/2} \\
&\qquad + \phi(z) \sum_e p(e) P\left(\tau(0,e) > t\right).
\end{aligned}$$

Since $E(|B_t^0|^{\underline{\alpha}}) \leq t|p|_{\underline{\alpha}}$, this proves (3.16). □

Next, we give some biased voter model bounds. Let $\bar{\xi}_t$ be the 1-biased voter model with rate function (3.12). By the same arguments as in Sect. 4 of [7], we can prove the following inequalities without using any of kernel assumptions:

$$E\left(|\bar{\xi}_t|\right) \leq e^{bt} |\bar{\xi}_0|, \tag{3.17}$$

$$E\left(|\bar{\xi}_t|^2\right) \leq e^{2bt} \left(|\bar{\xi}_0|^2 + \frac{2\nu + b}{b}\left(1 - e^{-bt}\right)|\bar{\xi}_0| \right) \tag{3.18}$$

$$\leq e^{2bt}\left(|\bar{\xi}_0|^2 + (2\nu + b)t|\bar{\xi}_0|\right). \tag{3.19}$$

In Sect. 4.3 below, we will compare the Lotka–Volterra model ξ_t^N with the biased voter models $\underline{\xi}_t^N, \bar{\xi}_t^N$ on $\mathbb{S}_N$. In order to construct coupling $\underline{\xi}_t^N \le \xi_t^N \le \bar{\xi}_t^N$, we assume that the voting and bias rates ν_N and b_N are

$$\nu = \nu_N = N - \bar{\theta} \log N \quad \text{and} \quad b = b_N = 2\bar{\theta} \log N. \tag{3.20}$$

As in [9], we need improved versions of (3.17) and (3.18). For $p \ge 2$ and $0 < \underline{\alpha} < 1$, define

$$\begin{aligned}
\kappa_p &= \kappa_p(b,\nu) = 3\big(bH\big(2\nu/b^p\big) + e^2\big) \quad \text{and} \quad \kappa = \kappa_3, \\
A &= A(b,\nu) = bR\big(2\nu/b^3\big) + 3e^2(1 + 2\nu/b), \\
B_p &= B_p(b,\nu,\underline{\alpha}) = \big(|p|_{\underline{\alpha}} \nu b^{2-p} H\big(2\nu/b^p\big)\big)^{1/2} + bH\big(2\nu/b^p\big)\big(|p|_{\underline{\alpha}}\big(\nu/b^p + 1\big)\big)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
h_1(b,\nu)(t) &= e^2 t^{-1/3} + 2\kappa e^{2+2\kappa t}, \\
h_2(b,\nu)(t) &= e^2 t^{-1/3}(1 + 2\nu/b) + 5\kappa A e^{1+3\kappa t}.
\end{aligned}$$

Put $P\phi(x) = \sum_y p(y-x)\phi(y)$, define the operators

$$\bar{A}\phi = \nu(P\phi - \phi) \quad \text{and} \quad A^* = (1 + b/\nu)\bar{A}, \tag{3.21}$$

and denote the associated semigroups by $\bar{P}_t$ and P_t^*, respectively.

Remark 3.4 Comparing the constants and functions defined above with those defined in (5.16) and (5.17) of [9], we see that only B_p is different. We replaced $2\sigma^2$ by $|p|_{\underline{\alpha}}$.

Remark 3.5 For the parameters $\nu = \nu_N$ and $b = b_N$ in (3.20), (2.17) and (2.15) imply that $\kappa_p = O(1)$, $A = O(N/\log N)$, and $B_p = O(N^{1/2}(\log N)^{(1-p)/2})$ as $N \to \infty$.

The following proposition is a version of Proposition 5.4 of [9].

Proposition 3.2 *Assume that* $b \ge 1$ *and* $p \ge 2$. *For all* $t \ge 0$,

$$E\big(|\bar{\xi}_t|\big) \le e^{b^{1-p} + \kappa_p t}|\bar{\xi}_0|, \tag{3.22}$$

$$E\big(|\bar{\xi}_t|^2\big) \le e^{2+2\kappa t}|\bar{\xi}_0|^2 + 4Ae^{1+3\kappa t}|\bar{\xi}_0|, \tag{3.23}$$

$$bE\big(\bar{\xi}_t\big(f_0(\bar{\xi}_t)\big)\big) \le h_1(t)|\bar{\xi}_0|, \tag{3.24}$$

$$bE\big(|\bar{\xi}_t|\bar{\xi}_t\big(f_0(\bar{\xi}_t)\big)\big) \le h_1(t)|\bar{\xi}_0|^2 + h_2(t)|\bar{\xi}_0|. \tag{3.25}$$

For all bounded $\phi : \mathbb{Z} \to [0,\infty)$, $p \ge 3$, *and* $0 < \underline{\alpha} < 1$,

$$E\big(\bar{\xi}_t(\phi)\big) \le e^{b^{1-p} + (1+\kappa_p)t}\big(\bar{\xi}_0\big(P_t^*(\phi)\big) + \big[\kappa_p b^{2-p}\|\phi\|_\infty + B_p\|\phi\|_{\underline{\alpha}/2}\big]|\bar{\xi}_0|\big). \tag{3.26}$$

Remark 3.6 Proposition 5.4 of [9] was proved with the help of Lemmas 5.1, 5.5, and 5.6 there. We can adopt the arguments in [9] to obtain similar results in Lemmas 5.5 and 5.6 of [9]. With abuse of notation, in the following we assume that those two lemmas are available for us.

Remark 3.7 The only difference between Proposition 3.2 and Proposition 5.4 of [9] is that inequality (3.26) is different from inequality (5.23) there. In fact, the key reason is that, when proving inequality (3.26), we will use estimate (3.16) in Lemma 3.2 of this paper instead of estimate (5.8) of Lemma 5.1 of [9].

Proof According to Remark 3.2, Remark 3.6, and the coupling (3.14), we can follow the arguments in [9] to obtain that (5.36), (5.37), and (5.38) there are available, which will be used in the following proof. Put $\epsilon = b^{-p}$ and assume that $\phi \geq 0$. We also have that

$$\begin{aligned} &E\left(\left|\bar{\xi}_\epsilon\left(b\phi f_0(\bar{\xi}_\epsilon)\right) - \hat{\xi}_\epsilon\left(b\phi f_0(\hat{\xi}_\epsilon)\right)\right|\right) \\ &\quad \leq 2b\|\phi\|_\infty E\left(|\bar{\xi}_\epsilon| - |\hat{\xi}_\epsilon|\right) \leq 2b\left(e^{b\epsilon} - 1\right)\|\phi\|_\infty|\bar{\xi}_0|, \end{aligned} \tag{3.27}$$

which is just a version of (5.39) of [9] (In fact, they are the same). The voter model estimate (3.16) tells us

$$\begin{aligned} E\left(\bar{\xi}_\epsilon\left(b\phi f_0(\bar{\xi}_\epsilon)\right)\right) &\leq 2eb^2\epsilon\|\phi\|_\infty|\bar{\xi}_0| \\ &\quad + b\left(|p|_{\underline{\alpha}}\nu\epsilon H(2\nu\epsilon)\right)^{1/2}\|\phi\|_{\underline{\alpha}/2}|\bar{\xi}_0| + bH(2\nu\epsilon)\bar{\xi}_0(\phi). \end{aligned} \tag{3.28}$$

By using Markov property, we see that for $s \geq \epsilon$,

$$\begin{aligned} &E\left(\bar{\xi}_s\left(b\phi f_0(\bar{\xi}_s)\right)|\mathcal{F}_{s-\epsilon}\right) \\ &\quad \leq \left(2eb^2\epsilon\|\phi\|_\infty + b\left(|p|_{\underline{\alpha}}\nu\epsilon H(2\nu\epsilon)\right)^{1/2}\|\phi\|_{\underline{\alpha}/2}\right)|\bar{\xi}_{s-\epsilon}| \\ &\qquad + bH(2\nu\epsilon)\bar{\xi}_{s-\epsilon}(\phi). \end{aligned} \tag{3.29}$$

Take expectations in (3.29) for $\phi = 1$ and recall the definition $\|\phi\|_{\underline{\alpha}}$ in Sect. 3. We have that for $s \geq \epsilon$,

$$E\left(\bar{\xi}_s\left(b\phi f_0(\bar{\xi}_s)\right)\right) \leq \kappa_p E\left(|\bar{\xi}_{s-\epsilon}|\right). \tag{3.30}$$

Using this inequality in (5.36) of [9] yields for $s \geq \epsilon$,

$$E\left(|\bar{\xi}_t|\right) \leq E\left(|\bar{\xi}_\epsilon|\right) + \kappa_p \int_\epsilon^t E\left(|\bar{\xi}_{s-\epsilon}|\right) ds \leq e^{b\epsilon} + \kappa_p \int_0^t E\left(|\bar{\xi}_s|\right) ds,$$

where the second inequality follows from (5.38) of [9]. This bound also holds for $t \leq \epsilon$. Then Gronwall's inequality implies that (3.22) holds.

Again using (5.38) of [9] gives that for $\psi : \mathbb{Z} \to \mathbb{R}^+$,

$$\left|E\left(\bar{\xi}_\epsilon(\psi)\right) - \bar{\xi}_0(\psi)\right| \leq \left(e^{b\epsilon} - 1\right)\bar{\xi}_0\left(P_\epsilon^*\psi\right) + \left|\bar{\xi}_0\left(P_\epsilon^*\right) - \bar{\xi}_0(\psi)\right|.$$

Note that

$$\left|P_\epsilon^*\psi(x)-\psi(x)\right|\le\|\psi\|_{\underline{\alpha}/2}E\left(\left|B^0_{\nu\epsilon(1+b/\nu)}\right|^{\underline{\alpha}/2}\right)\le\|\psi\|_{\underline{\alpha}/2}\left(\epsilon(\nu+b)|p|_{\underline{\alpha}}\right)^{1/2}.$$

Thus,

$$\left|E\left(\bar{\xi}_\epsilon(\psi)\right)-\bar{\xi}_0(\psi)\right|\le\left(eb\epsilon\|\psi\|_\infty+\|\psi\|_{\underline{\alpha}/2}\left(\epsilon(\nu+b)|p|_{\underline{\alpha}}\right)^{1/2}\right)|\bar{\xi}_0|.$$

Then by using Markov property, for $s\ge\epsilon$,

$$E\left(\bar{\xi}_{s-\epsilon}(\psi)\right)\le E\left(\bar{\xi}_s(\psi)\right)+\left(eb\epsilon\|\psi\|_\infty+\|\psi\|_{\underline{\alpha}/2}\left(\epsilon(\nu+b)|p|_{\underline{\alpha}}\right)^{1/2}\right)E\left(|\bar{\xi}_{s-\epsilon}|\right).$$

Since $\|P^*_{t-s}\phi\|_{\underline{\alpha}/2}\le\|\phi\|_{\underline{\alpha}/2}$, using the above inequality in (3.29) with $\psi=P^*_{t-s}\phi$ replacing ϕ, we have for $s\ge\epsilon$,

$$\begin{aligned}E\left(\bar{\xi}_s\left(bP^*_{t-s}\phi f_0(\bar{\xi}_s)\right)\right)\le&\left(\kappa_p b^2\epsilon\|\phi\|_\infty+B_p\|\phi\|_{\underline{\alpha}/2}\right)E\left(|\bar{\xi}_{s-\epsilon}|\right)\\&+\kappa_p E\left(\bar{\xi}_s\left(P^*_{t-s}\phi\right)\right),\end{aligned}\tag{3.31}$$

which is a version of (5.43) of [9]. Then the following arguments for proving (3.26) are very similar to those after (5.43) in [9]. We have proved (3.22) and (3.26). The other statements in the proposition can be proved in a similar way to that used to prove their counterparts in [9] (recall Remarks 3.2 and 3.6). We omit it here. □

Remark 3.8 We have followed the arguments in Sect. 5 of [9] to obtain some voter and biased voter estimates. In fact, we only replaced (5.8) and (5.23) in Sect. 5 of [9] by (3.16) and (3.26), respectively, and modified the arguments in the proof of (5.19) and (5.23) of [9]; please compare (3.28)–(3.31) with their counterparts (5.40)–(5.43) in Sect. 5 of [9]. We can also adopt the arguments there to obtain similar results to all other statements in Sect. 5 of [9] without using the fact the $p(\cdot)$ is in the domain of attraction of a stable law. In the next subsection, we will directly refer to them.

3.3 Four Key Results

In this subsection, we first list four key results and will give their proofs later. Let

$$g_0(s)=C_{3.32}s^{-1/3}e^{C_{3.32}s},\tag{3.32}$$

where $C_{3.32}$ will be chosen later.

Proposition 3.3

(a) *For $T>0$, there is a constant $C_{3.33}(T)$ such that for all $N\in\mathbb{N}$,*

$$\sup_{t\le T}E\left(X_t^N(1)\right)\le C_{3.33}(T)X_0^N(1),\tag{3.33}$$

$$E\left(\sup_{t\le T}X_t^N(1)^2\right)\le C_{3.33}(T)\left(X_0^N(1)^2+X_0^N(1)\right).\tag{3.34}$$

(b) *For all* $s > 0$ *and* $N \in \mathbb{N}$,

$$(\log N)E\big(X_s^N\big(f_0^N\big(\cdot,\xi_s^N\big)\big)\big) \le g_0(s)X_0^N(1), \tag{3.35}$$

$$(\log N)E\big(X_s^N(1)X_s^N\big(f_0^N\big(\cdot,\xi_s^N\big)\big)\big) \le g_0(s)\big(X_0^N(1)^2 + X_s^N(1)\big). \tag{3.36}$$

Let $A_N^*(\psi) = \frac{1}{N}(N + \bar{\theta}\log N)A_N(\psi)$ with semigroup $P_t^{N,*}$.

Proposition 3.4 *For* $p \ge 3$, *there is a constant* $C_{3.37}(p)$ *such that for any* $t \ge 0$ *and* $\phi : \mathbb{R} \to \mathbb{R}^+$,

$$E\big(X_t^N(\phi)\big) \le e^{(\log N)^{1-p}} e^{C_{3.37}t} X_0^N\big(P_t^{N,*}\phi\big) + C_{3.37}e^{C_{3.37}t}\|\phi\|_{1/2}(\log N)^{(1-p)/2}X_0^N(1). \tag{3.37}$$

Proposition 3.5 *For* $p \ge 3$, *there is a constant* $C_{3.38}(p)$ *such that for all* $\phi : \mathbb{R} \to \mathbb{R}^+$, *if* $\epsilon = (\log N)^{-p}$, *then*

$$E\big(X_\epsilon^N\big(\log N\phi f_0^N\big(\cdot,\xi_\epsilon^N\big)\big)\big) \le C_{3.38}X_0^N(1)\|\phi\|_{1/2}(\log N)^{(1-p)/2} + C_{3.38}X_0^N(\phi). \tag{3.38}$$

Let $\sup_{K,T}$ indicate the supremum over all $X_0^N \in M(\mathbb{S}_N)$, $\phi : \mathbb{R} \to \mathbb{R}$, and $t \ge 0$ satisfying $X_0^N(1) \le K$, $\|\phi\|_{\text{Lip}} \le K$, and $t \le T$.

Remark 3.9 Note that if $\|\phi\|_{\text{Lip}} \le K$, then $\|\phi\|_{\underline{\alpha}} \le 2K$ for any $0 < \underline{\alpha} < 1$.

Proposition 3.6 *For all* $K, T > 0$ *and* $0 < p < 2$,

$$\lim_{N\to\infty}\sup_{K,T} E\left(\left|\int_0^t X_s^N\big(\log N\phi^2 f_0^N\big(\cdot,\xi_s^N\big)\big) - p_1(0)^{-1}X_s^N\big(\phi^2\big)\right|^p\right) = 0, \tag{3.39}$$

and for $i = 2, 3$,

$$\lim_{N\to\infty}\sup_{K,T} E\left(\left|D_t^{N,i} - \int_0^t \theta_{i-2}\gamma^* X_s^N(\phi)\,ds\right|^p\right) = 0. \tag{3.40}$$

Recall the rescaled Lotka–Volterra models in Sect. 1.2 and assume that (A2) holds. Also recall the 1-biased voter model and 0-biased voter model with rates $\nu = \nu_N$ and $b = b_N$ defined in the last subsection. Set $\bar{\xi}_t^N(x) = \bar{\xi}_t(Nx)$ and $\underline{\xi}_t^N(x) = \underline{\xi}_t(Nx)$ for $x \in \mathbb{S}_N$. Thus the rate function of $\bar{\xi}_t^N$ is given by

$$\bar{c}(x,\xi) = \begin{cases} (\nu_N + b_N)f_1^N(x,\xi) & \text{if } \xi(x) = 0, \\ \nu_N f_0^N(x,\xi) & \text{if } \xi(x) = 1, \end{cases}$$

and the rate function of $\underline{\xi}_t^N(x)$ is given by

$$\underline{c}(x,\xi) = \begin{cases} \nu_N f_1^N(x,\xi) & \text{if } \xi(x) = 0, \\ (\nu_N + b_N)f_0^N(x,\xi) & \text{if } \xi(x) = 1. \end{cases}$$

Assume that N is large enough ($N \geq N_0$) so that $\nu_N > 0$ and $b_N > 1$. As in the last subsection, we may construct the three processes on one probability space so that $\underline{\xi}_0^N = \hat{\xi}_0^N = \bar{\xi}_0^N$ and

$$\underline{\xi}_t^N \leq \hat{\xi}_t^N \leq \bar{\xi}_t^N \quad \text{for all } t \geq 0. \tag{3.41}$$

Define

$$\bar{X}_t^N = \frac{1}{N'} \sum_{x \in \mathbb{S}_N} \bar{\xi}_t^N(x)\delta_x \quad \text{and} \quad \underline{X}_t^N = \frac{1}{N'} \sum_{x \in \mathbb{S}_N} \underline{\xi}_t^N(x)\delta_x.$$

It follows that

$$\underline{X}_t^N \leq X_t^N \leq \bar{X}_t^N \quad \text{for all } t \geq 0. \tag{3.42}$$

Keep Remark 3.5 in mind. Applying Proposition 3.2 gives that there are constants $C_{3.43}$ and $C_{3.32}$ such that for all $N \geq N_0$ and $t \geq 0$,

$$E\big(\bar{X}_t^N(1)\big) \leq C_{3.43} e^{C_{3.43} t} \bar{X}_0^N(1), \tag{3.43}$$

$$E\big(\bar{X}_t^N(1)^2\big) \leq C_{3.43} e^{C_{3.43} t} \big(\bar{X}_0^N(1)^2 + \bar{X}_0^N(1)\big), \tag{3.44}$$

and if g_0 is as in (3.32), then

$$(\log N) E\big(\bar{X}_t^N\big(f_0^N\big(\cdot, \bar{\xi}_t^N\big)\big)\big) \leq g_0(s) X_0^N(1), \tag{3.45}$$

$$(\log N) E\big(\bar{X}_t^N(1)\bar{X}_t^N\big(f_0^N\big(\cdot, \bar{\xi}_t^N\big)\big)\big) \leq g_0(s)\big(X_0^N(1)^2 + X_s^N(1)\big). \tag{3.46}$$

Typically, we have that there exists a constant $C_{3.47}$ such that

$$E\big(\bar{X}_t^N(1)\big) - E\big(\underline{X}_t^N(1)\big) \leq C_{3.47}\big[(\log N)^{-2} + t\big] X_0^N(1), \quad 0 \leq t \leq 1, \tag{3.47}$$

whose counterpart in [9] is (6.7). We first prove Proposition 3.3. In fact, we only give an outline.

Proof of Proposition 3.3 With inequalities (3.43) and (3.44) and the coupling (3.42) in hand, part (a) follows from the strong L^2 inequality for nonnegative submartingales and the fact that $\bar{X}_t^N(1)^2$ is a submartingale; see Remark 3.8 and (5.29) of [9]. For part (b), if we have similar results to those in Proposition 6.1 of [9], then part (b) follows from Remark 3.5. But the proof of Proposition 6.1 of [9] works here if we replace (5.40) there by (3.28) in the last subsection; see Remark 3.8. □

Proof of Proposition 3.4 Recall that $\bar{\xi}_t$ is the biased voter model with rates $\nu = N - \bar{\theta} \log N$ and $b = 2\bar{\theta} \log N$, and $\bar{\xi}_t^N(x) = \bar{\xi}_t(Nx)$, $x \in \mathbb{S}_N$. For $\psi : \mathbb{R} \to \mathbb{R}^+$, define $\phi : \mathbb{Z} \to \mathbb{R}^+$ by $\phi(x) = \psi(x/N)$. Then $\|\phi\|_\infty = \|\psi\|_\infty$, and for $0 < \underline{\alpha} < 1$,

$$\sup_{x \neq y, |x-y| \leq 1} \frac{|\phi(x) - \phi(y)|}{|x-y|^{\underline{\alpha}/2}} \leq \sup_{x \neq y, |x-y| \leq 1} \frac{|\phi(x) - \phi(y)|}{|x-y|^{1/2}}$$

$$\leq N^{-1/2} \sup_{x \neq y, |x-y| \leq 1/N} \frac{|\psi(x) - \psi(y)|}{|x-y|^{1/2}}.$$

Thus $\|\phi\|_{\alpha/2} \le N^{-1/2}\|\psi\|_{1/2}$. Note that $A_N^*\psi(x) = (N + \bar{\theta}\log N)\sum_{y\in\mathbb{S}_N} p_N(y - x)\psi(y)$ with semigroup $P_t^{N,*}$ and $A^*\phi(x) = (N + \bar{\theta}\log N)\sum_y p(y - x)\phi(y)$ with semigroup P_t^*; see (3.21) for the definition of A^*. We have that $P_t^*\phi(x) = P_t^{N,*}\psi(x/N)$ and $\bar{\xi}_t^N(\psi) = \bar{\xi}_t(\phi)$. According to (3.26), we obtain

$$E\big(\bar{\xi}_t^N(\psi)\big) \le e^{b^{1-p}+(1+\kappa_p)t}\big(\bar{\xi}_0^N\big(P_t^{N,*}(\psi)\big) + \big[\kappa_p b^{2-p}\|\psi\|_\infty + B_p N^{-1/2}\|\psi\|_{1/2}\big]\big|\bar{\xi}_0^N\big|\big). \tag{3.48}$$

Since $p \ge 3$, Remark 3.5 implies $\kappa_p b^{2-p} + B_p N^{-1/2} = O((\log N)^{(1-p)/2})$ as $N \to \infty$. Then the fact that $\bar{\theta} \ge 1$ implies that $b \ge \log N$ and the coupling (3.42) yield the desired inequality (3.37). □

Remark 3.10 One can see that the estimates in Remark 3.5 play important roles in the proof of Proposition 3.4. That is why we are forced to assume that $\{p(x)\}$ is in the domain of normal attraction of a stable law; or we will get that $B_p = O(N^{1/2}(\int_1^N L(s)^{-1}\,ds)^{(1-p)/2})$ and for $\psi : \mathbb{R} \to \mathbb{R}^+$ in this proof, $\phi : \mathbb{Z} \to \mathbb{R}^+$ will be defined by $\phi(x) = \psi(x/L(N))$. Then the term $B_p N^{-1/2}$ in (3.48) would be replaced by $B_p L(N)^{-1/2}$, which equals $s(N)^{-1/2}O((\int_1^N L(s)^{-1}\,ds)^{(1-p)/2})$. But the behaviors of $s(N)$ and $\int_1^N L(s)^{-1}\,ds$ as $N \to \infty$ are unknown, and we are not able to control $E(X_t^N(\phi))$ as (3.37).

Proof of Proposition 3.5 Let $\epsilon = b^{-p}$. According to Remark 3.8, we may use (5.32) of [9] to obtain that

$$E\big(X_\epsilon^N\big(b\phi f_0\big(\xi_\epsilon^N\big)\big)\big) \le E\big(\bar{X}_\epsilon^N\big(b\phi f_0\big(\bar{\xi}_\epsilon^N\big)\big)\big) + 2b\|\phi\|_\infty\big(E\big(\bar{X}_\epsilon^N(1) - X_\epsilon^N(1)\big)\big).$$

Applying (5.62) of [9] and (3.28) gives

$$E\big(X_\epsilon^N\big(b\phi f_0\big(\xi_\epsilon^N\big)\big)\big) \le \big(6eb^{2-p}\|\phi\|_\infty + B_p N^{-1/2}\|\phi\|_{1/2}\big)X_0^N(1) + \kappa_p X_0^N(\phi).$$

Then Remark 3.5 yields (3.38). □

We will give the proof of Proposition 3.6 in Sect. 3.5. In the next subsection with the help of the four propositions in this subsection we prove Theorem 1.2.

3.4 Proof of Theorem 1.2

First, we check the compact containment condition.

Proposition 3.7 *For all $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that*

$$\sup_N P\Big(\sup_{t\le\epsilon^{-1}} X_t^N\big(B(0,n)^c\big) > \epsilon\Big) < \epsilon.$$

Proof Recall that $L(N)=N$. Let $h_n:\mathbb{R}^d\to[0,1]$ be a C^∞ function such that

$$1_{\{|x|>n+1\}}\le h_n(x)\le 1_{\{|x|>n\}}$$

and

$$\sup_n\sum_{i,j,k\le d}\|(h_n)_i\|_\infty+\|(h_n)_{ij}\|_\infty+\|(h_n)_{ijk}\|_\infty\equiv C_h<\infty.$$

By the semimartingale decomposition,

$$\sup_{t\le T}X_t^N(h_n)\le X_0^N(h_n)+\sum_{i=1}^3\sup_{t\le T}\big|D_t^{N,i}(h_n)\big|+\sup_{t\le T}\big|M_t^N(h_n)\big|.$$

We need to check that the right-hand side tends to zero as $N,n\to\infty$. Let

$$\eta_N:=\sup_n\left\|A_N(h_n)-\frac{\sigma^2\Delta^{1/2}h_n}{2}\right\|_\infty.$$

Then $\lim_{N\to\infty}\eta_N=0$ by Lemma 2.1. Note that

$$\begin{aligned}&\frac{1}{N'}\sum_{x,y}\big|h_n(x)-h_n(y)\big|p_N(x-y)\xi_s^N(y)\\&\quad\le\frac{\|h_n\|_{\underline{\alpha}}}{N'}\sum_y\sum_x|x-y|^{\underline{\alpha}}p_N(x-y)\xi_s^N(y)\\&\quad\le\frac{C_h|p|_{\underline{\alpha}}}{N^{\underline{\alpha}}}X_s^N(1).\end{aligned}$$

Set $\eta'_N(T)=C_{3.33}(T)(\eta_N+\bar{\theta}C_h\log N|p|_{\underline{\alpha}}/N^{\underline{\alpha}})T)$. We have, as in the deviation of (4.17) in [9],

$$\begin{aligned}&E\Big(\sup_{t\le T}X_t^N(h_n)\Big)\\&\quad\le X_0^N(h_n)+2\big(\langle M^N(h_n)\rangle_T\big)^{1/2}+\eta'_NX_0^N(1)\\&\qquad+C_h\int_0^T E\big(X_s^N(h_{n-1})\big)\,ds+2\bar{\theta}\int_0^T E\big(X_s^N\big(h_n\log Nf_0^N\big(\xi_s^N\big)\big)\big)\,ds.\end{aligned}\tag{3.49}$$

Applying Proposition 3.5 and (3.33), we obtain the last integral above is bounded by

$$\eta''_N(T)X_0^N(1)+C_{3.38}\int_0^T E\big(X_s^N(h_n)\big)\,ds,\tag{3.50}$$

where $\eta''_N(T)=C_{3.33}(T)[(\log N)^{-2}+C_{3.38}C_hT/\log N]$. By Lemma 3.1 and (3.33) there is a constant $C_{3.51}(T)$ such that if $\phi_s=\psi$, then for any $\underline{\alpha}<1$ and $0\le s\le T$,

$$E\big(\big|m_{1,s}^N\big|+\big|m_{2,s}^N\big|\big)\le C_{3.51}(T)\|\phi\|_{\underline{\alpha}}^2\big(\log N/N^{\underline{\alpha}}\big)X_0^N(1).\tag{3.51}$$

Then the above inequality, (3.50), and Lemma 3.1 give (recall $N/N' = \log N$)

$$E\big(\langle M^N(h_n)\rangle_T\big) \le \eta_N'''(T)X_0^N(1) + 2C_{3.38}\int_0^T E\big(X_s^N(h_n)\big)\,ds, \tag{3.52}$$

where $\eta_N'''(T) = 2\eta_N''(T) + C_{3.51}(T)TC_h^2 \log N/N^{\frac{\alpha}{-}}$. Finally, let $B_t^{N,*}$ be the continuous random walk with semigroup $P_t^{N,*}$ defined before Proposition 3.4 and $B_0^{N,*} = 0$. Note that

$$P\bigg(\big|B_s^{N,*}\big| \ge \frac{n-1}{2}\bigg) = P\bigg(\big|B^0_{(N+\bar\theta\log N)s}\big| \ge \frac{N(n-1)}{2}\bigg).$$

Since $L(t) = l(t) = t$, Proposition 2.4 yields that the left-hand side above goes to 0 uniformly in $N \in \mathbb{N}$ and $0 \le s \le T$ as $n \to \infty$. Thus with the help of Proposition 3.4 and inequalities (3.49), (3.50), and (3.52) we can conclude: for any $T, \epsilon > 0$, there is an N_0 such that

$$\text{for } N \ge N_0, n \ge N_0, \quad E\Big(\sup_{t\le T} X_t^N(h_n)\Big) < \epsilon.$$

The desired result is immediate. □

Proof of Theorem 1.2 In fact, we have already completed all tasks. First, with (3.35) and (3.36) in hand, by the same arguments as those in the proof of Lemma 4.10 of [9], we have that there exists a constant $C_{3.53}(T)$ such that for all $0 \le s \le t \le T$,

$$E\bigg(\bigg[\int_s^t X_r^N\big(\log N f_0^N(\xi_r^N)\big)\,dr\bigg]^2\bigg) \\ \le C_{3.53}(T)(t-s)^{4/3}\big(X_0^N(1)^2 + X_0^N(1)\big). \tag{3.53}$$

Now, recall the decomposition of $X_t^N(\phi_t)$ in Sect. 3.1. With the help of Lemma 3.1 and (3.53), by the same arguments as those in the proof of Proposition 4.11 of [9], for each $\phi \in C_b^{1,3}(\mathbb{R}_+ \times \mathbb{R})$, each of the families $\{X_\cdot^N(\phi), N \in \mathbb{N}\}$, $\{D_\cdot^{N,i}, N \in \mathbb{N}\}$, $i = 1, 2, 3$, $\{\langle M^N(\phi)\rangle_\cdot, N \in \mathbb{N}\}$, and $\{M_\cdot^N(\phi), N \in \mathbb{N}\}$ is C-tight in $D([0,\infty), \mathbb{R})$. The C-tightness of $\{P_N, N \in \mathbb{N}\}$ is now immediate from Proposition 3.7 and Theorem II.4.1 of [18]. Then to check that any limit point of $\{P_N\}$ is the law claimed in the Theorem, one can follow the same arguments as those in the proof of Proposition 4.2 of [9], using Proposition 3.6 above. □

3.5 Proof of Proposition 3.6

Recall the definitions of $\hat\tau$ and τ in Sect. 1.3. For $e, e' \in \mathbb{Z}$, define the event $\varGamma_T(e,e') = \{\hat\tau(e,e') < T, \hat\tau(0,e) \wedge \hat\tau(0,e') > T\}$, and let

$$q_T = \sum_{e,e'} p(e)p(e')P\big(\varGamma_T(e,e')\big). \tag{3.54}$$

We have the following characterization of γ^*.

Proposition 3.8

$$\gamma * = \lim_{T\to\infty} (\log T) q_T < \infty. \tag{3.55}$$

With Proposition (2.5) in hand, the proof of Proposition 3.8 is similar to that of Proposition 2.1 in Sect. 2 of [9]. We omit it here.

For N fixed, let $\hat{\xi}_t$ be the rate $\nu_N = N - \bar{\theta}\log N$ voter model on $\mathbb{Z}$ with rate as in (3.12) for $b=0$ and $\nu = \nu_N$. Define $\hat{\xi}_t^N(x) = \hat{\xi}_t(xN)$, $x \in \mathbb{S}_N$, the rate ν_N voter model on $\mathbb{S}_N$. Recall the independent and coalescing random walks systems $\{B_t^x\}$ and $\{\hat{B}_t^x\}$ defined in Sect. 1.3. We need to introduce their rescaled versions as follows: for $x, y \in \mathbb{S}_N$,

$$B_t^{N,x} = B_{\nu_N t}^{xN}/N, \qquad \hat{B}_t^{N,x} = B_{\nu_N t}^{xN}/N, \tag{3.56}$$

and

$$\tau^N(x,y) = \tau(Nx,Ny)/\nu_N, \qquad \hat{\tau}^N(x,y) = \hat{\tau}(Nx,Ny)/\nu_N.$$

Define

$$\varepsilon(t) = \sup_{x\in\mathbb{Z}} \big| t p_t(0,x) - p_1(x/t)\big| \vee \big(1/t^2\big).$$

By Proposition 2.2, $\varepsilon(t) \to 0$ as $t\to\infty$. Then for each $k \in \mathbb{Z}^+$, there exists a $t(k)$ such that for $t > t(k)$, $\varepsilon(t) \le 1/k$. Define

$$\varepsilon'(t) = \begin{cases} 1, & 0 \le t \le t(1), \\ 1/k, & t(k) < t \le t(k+1). \end{cases} \tag{3.57}$$

Then $\varepsilon'(t) \downarrow 0$ as $t\to\infty$ and $\varepsilon'(t) \ge \varepsilon(t)$ for $t > t(1)$. Let $\hat{\eta}_N = e^{-\sqrt{\log N}}$, $a_N = \nu_N(2-\hat{\eta}_N)/\log N$, and

$$\epsilon'_N = (\log\log N)^{-1} \vee \sqrt{\varepsilon'(a_N/\log\log N)}.$$

Then

$$\begin{aligned} ez_N &:= \left(\varepsilon(a_N\epsilon'_N)/\epsilon'_N + \frac{\log\log N}{\log N}\right) \\ &\le \varepsilon'(a_N\epsilon'_N)\big(\sqrt{\varepsilon'(a_N/\log\log N)}\big)^{-1} + \frac{\log\log N}{\log N} \\ &\le \sqrt{\varepsilon'(a_N/\log\log N)} + \frac{\log\log N}{\log N} \to 0 \end{aligned}$$

as $N\to\infty$. Define the sequences

$$t_N = \frac{\epsilon'_N}{\log N}, \qquad K_N = (\log N)^{1/2}, \qquad \delta_N = K_N t_N. \tag{3.58}$$

We assume that N is large enough so that $\epsilon'_N \vee t_N \vee \delta_N \le 1$ and $\delta_N/\epsilon'_N \to 0$ as $N\to\infty$.

Lemma 3.3 *There is a constant $C_{3.59}$ such that*

$$\frac{\log N}{N'}\sum_{x,e} p_N(e)P\big(\hat{\xi}_0^N\big(B_{t_N}^{N,x}\big)=\hat{\xi}_0^N\big(B_{t_N}^{N,x+e}\big)=1,\ \tau^N(x,x+e)>t_N\big)$$
$$\le C_{3.59}(\epsilon_N')^{-1}\iint_{|w-z|\le\delta_N} d\hat{X}_0^N(w)\,d\hat{X}_0^N(z)+C_{3.59}\epsilon_N\hat{X}_0^N(1)^2. \tag{3.59}$$

Proof By translation invariance and symmetry, the left side of (3.59) is

$$(N')^{-2}\sum_{w,z}\hat{\xi}_0^N(w)\hat{\xi}_0^N(z)\sum_e p_N(e)$$
$$\times\Big[\sum_x NP\big(B_{t_N}^{N,0}=w-x,\,B_{t_N}^{N,e}=z-x,\,\tau^N(0,e)>t_N\big)\Big]$$
$$=(N')^{-2}\sum_{w,z}\hat{\xi}_0^N(w)\hat{\xi}_0^N(z)\sum_e p_N(e)NP\big(B_{2t_N}^{N,e}=z-w,\,\tau_0^{N,e}>2t_N\big)$$
$$\equiv\Sigma_d^N+\Sigma_c^N, \tag{3.60}$$

where $\tau_0^{N,e}=\inf\{s: B_s^{N,e}=0\}$, and Σ_d^N, respectively, Σ_c^N, denotes the contribution to (3.60) from w,z satisfying $|w-z|\le K_N t_N$, respectively, $|w-z|>K_N t_N$. Let

$$\tilde{P}\big(\big(B_\cdot^N,\tau_0^N\big)\in\cdot\big)=\sum_e p_N(e)P\big(\big(B_\cdot^{N,e},\tau_0^{N,e}\big)\in\cdot\big).$$

For Σ_d^N, use (2.5) and the Markov property at time t_N to see that

$$N\tilde{P}^N\big(B_{2t_N}^N=z-w,\,\tau_0^N>2t_N\big)$$
$$\le N\tilde{E}\big(P\big(B_{t_N}^{N,0}=z-w-B_{t_N}^N(w)\big);\,\tau_0^N>t_N\big)$$
$$\le CN\tilde{P}\big(\tau_0^N>t_N\big)(\nu_N t_N)^{-1}$$
$$\le C\frac{NH(\nu_N t_N)}{\nu_N t_N}.$$

By (2.17), there is a constant $C_{3.61}$ such that

$$\Sigma_d^N\le C_{3.61}(\epsilon_N')^{-1}\iint_{|w-z|\le K_N t_N} d\hat{X}_0^N(w)\,d\hat{X}_0^N(z). \tag{3.61}$$

It is more complicated to bound Σ_c^N. Using the Markov property at time $\hat{\eta}_N t_N$ gives

$$\tilde{P}^N\big(B_{2t_N}^N=w-z,\,\tau_0^N>2t_N\big)$$
$$\le\tilde{P}\bigg(\tau_0^N>\hat{\eta}_N t_N,\,\big|B_{\hat{\eta}_N t_N}^N\big|>\frac{K_N t_N}{2}\bigg)\sup_{x'}P\big(B_{(2-\hat{\eta}_N)t_N}^{N,0}=x'\big)$$

$$+\tilde{P}\Big(P\big(B^{N,0}_{(2-\hat{\eta}_N)t_N}=w-z-B^N_{\hat{\eta}_N t_N}\big);\tau^N_0>\hat{\eta}_N t_N,\big|B^N_{\hat{\eta}_N t_N}\big|\le\frac{K_N t_N}{2}\Big)$$
$$=\Sigma^N_{1c}+\Sigma^N_{2c}.$$

Note that

$$\tilde{P}\left(\big|B^N_{\hat{\eta}_N t_N}\big|>\frac{K_N t_N}{2}\right)=\sum_e p_N(e)P\left(\big|B^0_{N\hat{\eta}_N t_N}+e\big|>\frac{NK_N t_N}{2}\right),$$

which is bounded by

$$\frac{2|p|_{1/2}}{(NK_N t_N)^{1/2}}+P\left(\big|B^0_{N\hat{\eta}_N t_N}\big|>\frac{NK_N t_N}{4}\right).$$

By Proposition 2.4,

$$P\left(\big|B^0_{N\hat{\eta}_N t_N}\big|>\frac{NK_N t_N}{4}\right)\le\frac{4C_{2.12}N\hat{\eta}_N t_N}{NK_N t_N}=4C_{2.12}\hat{\eta}_N/K_N.$$

(Note that $l(t)=L(t)=t$.) Thus, by (2.6),

$$\Sigma^N_{1c}\le\frac{C(\hat{\eta}_N/K_N+1/(NK_N t_N)^{1/2})}{\nu_N(2-\hat{\eta_N})t_N}. \tag{3.62}$$

Let us consider Σ^N_{2c}. By the definition of $\varepsilon(t)$ and (2.11) (recall that $d=\alpha=1$),

$$\begin{aligned}p_t(0,x)&\le\frac{\varepsilon(t)}{t}+\frac{p_1(x/t)}{t}\\&\le\frac{1}{t}\left(\varepsilon(t)+c_2\left(1\wedge\left|\frac{t}{x}\right|^2\right)\right).\end{aligned} \tag{3.63}$$

Note that for $|w-z|>K_N t_N$, on $\{|B^N_{\hat{\eta}_N t_N}|\le\frac{K_N t_N}{2}\}$,

$$\big|w-z-B^N_{\hat{\eta}_N t_N}\big|^{-1}\le\frac{2}{K_N t_N}.$$

Thus, by inequality (3.63), Σ^N_{2c} is less than

$$\left(\varepsilon\big(\nu_N(2-\hat{\eta}_N)t_N\big)+c_2\left(1\wedge\left(\frac{2\nu_N(2-\hat{\eta}_N)}{NK_N}\right)^2\right)\right)\frac{H(\nu_N\hat{\eta}_N t_N)}{\nu_N(2-\hat{\eta}_N)t_N}.$$

Thus, by $a_N\epsilon'_N=\nu_N(2-\hat{\eta}_N)t_N$ and (2.17),

$$\begin{aligned}\Sigma^N_{2c}&\le C\big(\varepsilon(a_N\epsilon'_N)+1/K_N^2\big)\frac{\log N}{\nu_N\epsilon'_N\log(\nu_N\hat{\eta}_N t_N)}\\&\le C\big(\varepsilon(a_N\epsilon'_N)/(N\epsilon'_N)+(N\log N\epsilon'_N)^{-1}\big)\end{aligned}$$

$$\leq C\left(\varepsilon(a_N\epsilon'_N)/(N\epsilon'_N)+\frac{\log\log N}{N\log N}\right)$$
$$= C\epsilon_N/N, \tag{3.64}$$

where C may change its values from line to line, and the second inequality follows from

$$\log(\nu_n\hat{\eta}_N t_N)=\log(\epsilon'_N)+\log(\nu_N)-\log\log N-\sqrt{\log N}$$

and $\lim_{N\to\infty}\frac{N}{\nu_N}=1$. With (3.61), (3.62), and (3.64) in hand, (3.60) yields the desired result, (3.59). □

For $\phi:\mathbb{R}^2\to\mathbb{R}$, $\zeta\in\{0,1\}^{\mathbb{S}_N}$, and $X(\phi)=(1/N')\sum_x\phi(x)\zeta(x)$, define

$$\Delta_1^{N,+}(\phi,\zeta)=X\big(\log N\phi^2 f_0^N(\cdot,\zeta)\big),$$
$$\Delta_2^{N,+}(\phi,\zeta)=\frac{1}{N'}\sum_x\big(1-\zeta(x)\big)\phi(x)\log Nf_1^N(x,\zeta)^2,$$
$$\Delta_3^{N,+}(\phi,\zeta)=X\big(\log N\phi f_0^N(\cdot,\zeta)^2\big),$$

and

$$\Delta_j^N(\phi,\zeta)=\Delta_j^{N,+}(\phi,\zeta)\gamma_jX(\phi),\quad j=1,2,3,$$

where $\gamma_1=p_1(0)^{-1}$ and $\gamma_2=\gamma_3=\gamma^*$. Define

$$m(1)=2\quad\text{and}\quad m(2)=m(3)=1.$$

Proposition 3.9 *There are a constant $C_{3.65}$ and a sequence $\eta_{3.65}(N)\downarrow 0$ such that for $j=1,2,3$, if $\phi:\mathbb{R}^2\to\mathbb{R}$, then for any $0<\underline{\alpha}<1$,*

$$\left|E\big(\Delta_j^N(\phi,\hat{\xi}_{t_N}^N)\big)\right|\leq\eta_{3.65}(N)\big(\hat{X}_0^N(1)+\hat{X}_0^N(1)^2\big)\|\phi\|_{\underline{\alpha}}^{m(j)}$$
$$+\frac{C_{3.65}\|\phi\|_\infty^{m(j)}}{\epsilon'_N}\iint_{|w-z|\leq\epsilon_N}d\hat{X}_0^N(w)\,d\hat{X}_0^N(z). \tag{3.65}$$

Proof To prove the proposition, we can define $\Sigma_j^{i,N}$, $i=1,2$ for $j=1$, and $i=1,2,3$ for $j=2,3$, as in (7.20), (7.21), and (7.22) of [9] and decompose each $E(\Delta_j^{N,+})$ into a sum of those terms. We omit the definitions and decompositions here, since they are the same. By Lemma 3.3, we can show that

$$\Sigma_j^{2,N}\leq C_{3.59}\|\phi\|_\infty^{m(j)}\left[(\epsilon'_N)^{-1}\iint_{|w-z|\leq\delta_N}d\hat{X}_0^N(w)\,d\hat{X}_0^N(z)+\epsilon_N\hat{X}_0^N(1)^2\right]. \tag{3.66}$$

For $\Sigma_j^{3,N}$, $j=2,3$, with Proposition 2.5 in hand, one can check that a similar conclusion to that in Lemma 2.5 of [9] is available. Following the proof of Proposition 7.5 of [9], we have that there exists a constant $C_{3.67}$, depending on $p(\cdot)$, such that

$$\Sigma_2^{3,N}+\Sigma_3^{3,N}\leq C_{3.67}\|\phi\|_\infty\hat{X}_0^N(1)(\log N)^{-1/2}. \tag{3.67}$$

Now, we need to establish that there is a sequence $\eta(N) \to 0$ such that for $j = 1, 2, 3$,

$$\left|\Sigma_j^{1,N} - \gamma_j \hat{X}_0^N(\phi)\right| \le \eta(N)\|\phi\|_{\underline{\alpha}}^{m(j)} \hat{X}_0^N(1). \tag{3.68}$$

Let e denote independent random variable with law $p(\cdot)$. First,

$$P\left(B_{t_N}^{N,e} > \sqrt{\epsilon_N'}\right) = P\left(\left|B_{\nu_N t_N}^0 + e\right| > N\sqrt{\epsilon_N'}\right).$$

We also have

$$\begin{aligned} P\left(\left|B_{\nu_N t_N}^0 + e\right| > N\sqrt{\epsilon_N'}\right) &\le \frac{2|p|_{\underline{\alpha}}}{(N\sqrt{\epsilon_N'})^{\underline{\alpha}}} + P\left(\left|B_{\nu_N t_N}^0\right| > N\sqrt{\epsilon_N'}/2\right) \\ &\le \frac{2|p|_{\underline{\alpha}}}{(N\sqrt{\epsilon_N'})^{\underline{\alpha}}} + \frac{C_{2.12}\nu_N t_N}{N\sqrt{\epsilon_N'}}, \end{aligned} \tag{3.69}$$

where the second inequality follows from Proposition 2.4. Typically, we have

$$P\left(B_{t_N}^{N,0} > \sqrt{\epsilon_N'}\right) \le \frac{C_{2.12}\nu_N t_N}{N\sqrt{\epsilon_N'}} = \frac{C_{2.12}\nu_N\sqrt{\epsilon_N'}}{N\log N}. \tag{3.70}$$

Now, we consider the case of $j = 2$. By the same arguments as in [9], we can show that

$$\begin{aligned} &\left|\Sigma_2^{1,N} - \gamma^* \hat{X}_0^N(\phi)\right| \\ &\quad\le \frac{1}{N'}\sum_w \hat{\xi}_0^N(w)\log N E\left(\left|\phi\left(w - \hat{B}_{t_N}^{N,e}\right) - \phi(w)\right|; \hat{\tau}^N(0,e) \wedge \hat{\tau}^N(0,f) > t_N,\right. \\ &\qquad \left.\hat{\tau}^N(e,f) \le t_N\right) + \left|\frac{1}{N'}\sum_w \hat{\xi}_0^N(w)\phi(w)\left(q_{\nu_N t_N}\log N - \gamma^*\right)\right| \\ &\quad\le \|\phi\|_{\underline{\alpha}}\hat{X}_0^N(1)\log N\left(\sqrt{\epsilon_N'}\right)^{\underline{\alpha}} q_{\nu_N t_N} \\ &\qquad + 2\|\phi\|_\infty \hat{X}_0^N(1)\log N P\left(\left|B_{t_N}^{N,e}\right| > \sqrt{\epsilon_N'}\right)^{1/2} q_{\nu_N t_N}^{1/2} \\ &\qquad + \|\phi\|_\infty \hat{X}_0^N(1)\left|\log N q_{\nu_N t_N} - \gamma^*\right|, \end{aligned}$$

where the second inequality follows from the Cauchy–Schwarz inequality and considering the cases $|B_{t_N}^{N,e}| > \sqrt{\epsilon_N'}$ and $|B_{t_N}^{N,e}| \le \sqrt{\epsilon_N'}$. Thus by (2.17), (3.69), and Proposition 3.8, there exists a sequence $\eta_{3.71}(N)$ which goes to 0 as $N \to \infty$ such that

$$\left|\Sigma_2^{1,N} - \gamma^* \hat{X}_0^N(\phi)\right| \le \eta_{3.71}(N)\|\phi\|_{\underline{\alpha}}\hat{X}_0^N(1). \tag{3.71}$$

By replacing $\hat{B}^{N,e}_{t_N}$, $B^{N,e}_{t_N}$ with $\hat{B}^{N,0}_{t_N}$, $B^{N,0}_{t_N}$, respectively, the same argument as above gives the same bound for $|\Sigma^{1,N}_3 - \gamma^* \hat{X}^N_0(\phi)|$. Typically, inequality (3.69) could be simplified. Next, we turn to $\Sigma^{1,N}_2$. Following the strategy of the proof for term on $\Sigma^{1,N}_2$, we have that

$$\begin{aligned}
&\bigl|\Sigma^{1,N}_1 - p_1(0)^{-1}\hat{X}^N_0(\phi^2)\bigr| \\
&\quad = \left|\frac{1}{N'}\sum_w \hat{\xi}^N_0(w)\bigl[\log N E\bigl(\phi^2\bigl(w - B^{N,0}_{t_N}\bigr); \tau^N(0,e) > t_N\bigr) - p_1(0)^{-1}\phi^2(w)\bigr]\right| \\
&\quad \le \frac{1}{N'}\sum_w \hat{\xi}^N_0(w)\bigl[\log N E\bigl(\bigl|\phi^2\bigl(w - B^{N,0}_{t_N}\bigr) - \phi^2(w)\bigr|; \tau^N(0,e) > t_N\bigr)\bigr] \\
&\qquad + \frac{1}{N'}\sum_w \hat{\xi}^N_0(w)\phi^2(w)\bigl|\log N P\bigl(\tau^N(0,e) > t_N\bigr) - p_1(0)^{-1}\bigr| \\
&\quad \le \Bigl(2\|\phi\|_{\underline{\alpha}} \log N\Bigl(\sqrt{\epsilon'_N}\Bigr)^{\underline{\alpha}} H(\nu_N t_N) \\
&\qquad + 2\|\phi\|_\infty \log N P\Bigl(\bigl|B^{N,0}_{t_N}\bigr| > \sqrt{\epsilon'_N}\Bigr)^{1/2} H(\nu_N t_N)^{1/2} \\
&\qquad + \|\phi\|_\infty \bigl|\log N H(\nu_N t_N) - p_1(0)^{-1}\bigr|\Bigr)\|\phi\|_\infty \hat{X}^N_0(1).
\end{aligned}$$

According to (3.69) and (2.17), we can conclude that

$$\bigl|\Sigma^{1,N}_1 - p_1(0)^{-1}\hat{X}^N_0(\phi^2)\bigr| \le \eta_{3.72}\hat{X}^N_0(1)\|\phi\|^2_{\underline{\alpha}}, \tag{3.72}$$

where $\eta_{3.72} \to 0$ as $N \to \infty$. Thus we get (3.68). By the decompositions in (7.18) of [9], we obtain the desired result. □

With Proposition 3.9 in hand, Proposition 3.6 follows from the following two propositions, which are analogous to Propositions 7.1 and 7.2 in [9] and a similar argument to that in Sect. 8 of [9].

Proposition 3.10 *There are a constant $C_{3.73}(K)$ and sequence $\eta_{3.73}(N) \downarrow 0$ such that for all $\phi : \mathbb{R} \to [0,\infty)$ satisfying $\|\phi\|_{\mathrm{Lip}} \vee X^N_0(1) \le K$ and $j = 1, 2, 3$,*

$$\begin{aligned}
\bigl|E\bigl(\Delta^N_j\bigl(\phi, \xi^N_{t_N}\bigr)\bigr)\bigr| \le C_{3.73}(K)\biggl(&\eta_{3.73}(N)\bigl(X^N_0(1) + X^N_0(1)^2\bigr) \\
&+ (\epsilon'_N)^{-1}\iint_{|w-z|\le\delta_N} dX^N_0(w)\, dX^N_0(z)\biggr).
\end{aligned} \tag{3.73}$$

Proof First, we can follow the strategy in the proof of Lemma 7.8 in [9] to obtain an analogous result to that in Lemma 7.8 of [9]. Then with our coupling, (3.47), and Proposition 3.9 in hand, following the argument in [9], one can get the desired result. □

Proposition 3.11 *There is a constant $C_{3.74}$ such that for all* $0 \le t \le T$,

$$E\left(\iint_{|w-z|\le\delta_N} dX_0^N(w)\,dX_0^N(z)\right)$$
$$\le C_{3.74}e^{C_{3.74}T}\left(X_0^N(1)+X_0^N(1)^2\right)$$
$$\times\left[\frac{\delta_N}{\delta_N+t}\left(1+t^{2/3}\right)+\delta_N t^{-1/3}\log\left(1+\frac{t}{\delta_N}\right)\right]. \tag{3.74}$$

The proof of Proposition 3.11 is also exactly the same as that of Proposition 7.2 of [9]. In fact, we only need to prove the following random walk estimate which is a version of Corollary 7.9 of [9] and can be deduced directly from (2.6) and Proposition 2.3. Let $B_t^{N,*}$ be the random walk with semigroup $(P_t^{N,*}, t\ge 0)$ from Proposition 3.4; at rate $\nu_N + b_N = N + \bar\theta\log N$, $B_\cdot^{N,*}$ takes steps with $p_N(\cdot)$ and $B_0^{N,*}=0$.

Corollary 3.1

(a) *For all* $x\in\mathbb{S}_N$ *and* $t\ge 0$,

$$P\left(B_t^{N,*}=x\right)\le\frac{C_{2.6}}{1+Nt}. \tag{3.75}$$

(b) *Assume that* $\delta_N'\downarrow 0$ *and* $N\delta_N'\to\infty$. *For each* $K>0$, *there is a constant* $C_{3.76}(K)>0$ *such that*

$$\inf_{N\ge 1, w\in\mathbb{S}_N, |w|\le K\delta_N'} N\delta_N' P\left(B_{2\delta_N'}^{N,*}=w\right)\ge C_{3.76}(K)>0. \tag{3.76}$$

Now, one follows the argument in [9] to get Proposition 3.11. To obtain Proposition 3.6, the following arguments are similar to those in Sect. 8 of [9]. We omit them here.

4 Voter Model's Asymptotics

In this section, we will prove Theorem 1.3 and assume that assumption (A1) holds with $L(t)=t^{1/\alpha}$. Recall that $\bar p_t = P(|\xi_t^0|>0)$. Our first object is to prove that

$$\begin{aligned}\bar p_t &= O\left(\frac{\log t}{t}\right)\quad \text{as } t\to\infty \text{ for } d=\alpha,\\ &= O\left(t^{-1}\right)\quad \text{as } t\to\infty \text{ for } d>\alpha.\end{aligned} \tag{4.1}$$

The asymptotics above are similar to the results in Theorem 1 of [3]. Note that Theorem 1 of [3] could be proved under the assumption that the underlying motion has finite variance and one only need to modify the proof of Lemma 5 of [3]; see Lemma 2 of [2]. For our purpose, we also need to generalize the asymptotic results in (14) of [3].

Recall that $\{B_t^x, x \in \mathbb{Z}^d\}$ is a collection of rate-one independent stable random walks with $B_0^x = x$. Let $p_t(x, y) = P(B_t^x = y)$ denote the transition function of $\{B_t^x\}$.

With (2.15) in hand, one can generalize the asymptotics results in (14) of [3]. Now, to prove (4.1) we only need to prove some analogous results to those in Lemma 5 of [3]. Set $G_t(x) = \int_0^t p_s(0, x)\, ds$, let $\tau(x) = \inf\{t \geq 0 : B_t^x = 0\}$, and define $\bar{H}_t(x) = P(\tau(x) \leq t)$.

Lemma 4.1 *If* $x \in \mathbb{Z}^d$ *with* $|x| = r$, *then there is a constant* $C_{d,\alpha} > 0$ *such that*

$$\begin{aligned}\bar{H}_{r^\alpha}(x) &\geq C_{d,\alpha}/\log r \quad \text{for } d = \alpha,\\ &\geq C_{d,\alpha} r^{\alpha-d} \quad \text{for } d > \alpha.\end{aligned}$$

Proof We first consider the asymptotics for the Green's function. According to (2.4) and (2.11), for r large enough,

$$G_{r^\alpha}(x) = \int_0^{r^\alpha} p_s(0, x)\, ds \geq c_1 \int_{r^\alpha/2}^{r^\alpha} \frac{s}{r^{d+\alpha}} ds - \int_{r^\alpha/2}^{r^\alpha} s^{-d/\alpha}\, ds.$$

A bit of calculation shows that there exist constants $\bar{C}_{d,\alpha} > 0$ such that

$$\begin{aligned}G_{r^\alpha}(x) &\geq \bar{C}_{d,\alpha} r^{\alpha-d} \quad \text{for } d > \alpha,\\ &\geq \bar{C}_{d,\alpha} \quad \text{for } d = \alpha.\end{aligned}$$

By (2.5), we see that there exist constants $\underline{C}_{d,\alpha} > 0$ such that

$$\begin{aligned}G_{r^\alpha}(0) &\leq \underline{C}_{d,\alpha} \quad \text{for } d > \alpha,\\ &\leq \underline{C}_{d,\alpha} \log r \quad \text{for } d = \alpha.\end{aligned}$$

Then the desired result follows from inequality $\bar{H}_t(x) \geq G_t(x)/G_t(0)$, whose discrete-time version can be easily deduced from (2.k) on p. 662 of [15]. □

Now, one can follow the arguments in Sect. 3 of [3] to obtain (4.1). (Note that when proving a result analogous to that in Lemma 4 of [3], one may need to set $s_t = d[(2\bar{p}_t^{-1})^{1/d}]^\alpha$.) With (4.1), Theorem 1.1, and Theorem 1.2 in hand, the following proof for Theorem 1.3 is exactly the same as that in [6]. We leave it to the interested readers. The intuition is that the underlying motion has nothing to do with the total mass process.

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