

Fleming-Viot Processes in an Environment*

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Abstract

We consider a new type of lookdown processes where spatial motion of each individual is influenced by an individual noise and a common noise, which could be regarded as an environment. Then a class of probability measure-valued processes on real line \mathbb{R} are constructed. The sample path properties are investigated: the values of this new type process are either purely atomic measures or absolutely continuous measures according to the existence of individual noise. When the process is absolutely continuous with respect to Lebesgue measure, we derive a new stochastic partial differential equation for the density process. At last we show that such processes also arise from normalizing a class of measure-valued branching diffusions in a Brownian medium as the classical result that Dawson-Watanabe superprocesses, conditioned to have total mass one, are Fleming-Viot superprocesses.

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1 Introduction

In this work, we construct and study a new class of probability measure-valued Markov processes on the real line \mathbb{R} . Our model arises from a modified stepwise mutation model (see Section 1.1.10 of [7] for classical stepwise mutation model): the mutation process of each individual in the model is influenced by an independent noise and a common noise. More precisely, suppose that $\{W(t, x) : x \in \mathbb{R}, t \geq 0\}$ is space-time white noise based on Lebesgue measure, the common noise, and $\{B_i(t) : t \geq 0, i = 1, 2, \dots\}$ is a family of independent standard Brownian motions, the individual noises, which are independent of $\{W(t, x) : x \in \mathbb{R}\}$. The mutation of an individual in the stepwise mutation system with label i is defined by the stochastic equations

$$dx_i(t) = \epsilon dB_i(t) + \int_{\mathbb{R}} h(y - x_i(t)) W(dt, dy), \quad t \geq 0, \quad i = 1, 2, \dots, \quad (1.1)$$

where $W(dt, dy)$ denotes the time-space stochastic integral relative to $\{W_t(B)\}$ and $\epsilon \geq 0$. Suppose that $h \in C^2(\mathbb{R})$ and h as well as its first two derivatives should be square-integrable. Let $\rho_\epsilon = \epsilon^2 + \rho(0)$ and

$$\rho(x) = \int_{\mathbb{R}} h(y - x) h(y) dy, \quad (1.2)$$

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for $x \in \mathbb{R}$. For each integer $m \geq 1$, $\{(x_1(t), \dots, x_m(t)) : t \geq 0\}$ is an m -dimensional diffusion process which is generated by the differential operator

$$G^m := \frac{\rho_\epsilon}{2} \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i,j=1, i \neq j}^m \rho(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j}. \quad (1.3)$$

In particular, $\{x_i(t) : t \geq 0\}$ is a one-dimensional diffusion process with generator $G := (\rho_\epsilon/2)\Delta$. Because of the exchangeability, a diffusion process generated by G^m can be regarded as an interacting particle system or a measure-valued process. Heuristically, ρ_ϵ represents the speed of the particles and $\rho(\cdot)$ describes the interaction between them.

Our interest comes from recent studies on connections between superprocesses and stochastic flows; see [2], [3], [20] and [23]. In those works, particles undergo random branching and their spatial motions are affected by the presence of stochastic flows. Some new classes of measure-valued processes were constructed from the empirical measure of the particles. Those measure-valued processes are quite different with the classical Dawson-Watanabe processes. There are at least two different ways to look at those processes. One is as a superprocess in random environment and the other as an extension of models of the motion of the mass by stochastic flows; see [16], [17]. In this work we remove the branching structure of particle systems in [23] but add a sampling mechanism. That is whenever a particle's exponential 'sampling clock' rings, it jumps to a position chosen at random from the current empirical distribution of the whole population. Its mutation then continues from its new position.

This work is simulated by classical connections between Dawson-Watanabe processes and Fleming-Viot processes investigated in [8] and [18]. It has been shown that Fleming-Viot superprocesses is the Dawson-Watanabe processes, conditioned to have total mass one. So we want to ask what can we obtain if the measure-valued processes constructed in [2], [3] [20] and [23] are conditioned to have total mass one? The particle picture described in [18] suggests that the branching structure of such conditioned measure-valued branching processes may be changed to sampling mechanism. Thus measure-valued branching processes constructed in [23], conditioned to have total mass one, may have generator as:

$$\mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu), \quad (1.4)$$

where

$$\begin{aligned} \mathcal{A}F(\mu) := & \frac{1}{2} \int_{\mathbb{R}} \rho_\epsilon \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ & + \frac{1}{2} \int_{\mathbb{R}^2} \rho(x-y) \frac{d^2}{dx dy} \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \mu(dx) \mu(dy), \end{aligned} \quad (1.5)$$

$$\mathcal{B}F(\mu) := \frac{\gamma}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} (\mu(dx) \delta_x(dy) - \mu(dx) \mu(dy)), \quad (1.6)$$

for some bounded continuous functions $F(\mu)$ on $P(\mathbb{R})$. The variational derivative is defined by

$$\frac{\delta F(\mu)}{\delta \mu(x)} = \lim_{r \rightarrow 0^+} \frac{1}{r} [F(\mu + r \delta_x) - F(\mu)], \quad x \in \mathbb{R}, \quad (1.7)$$

if the limit exists and $\delta^2 F(\mu)/\delta \mu(x) \delta \mu(y)$ is defined in the same way with F replaced by $(\delta F/\delta \mu(y))$ on the right hand side. If we replace \mathcal{B} in (1.6) by

$$\frac{\gamma}{2} \int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx),$$

then \mathcal{L} is the generator of the measure-valued process constructed in [23], where \mathcal{L} acted on some bounded continuous functions on $M(\mathbb{R})$, space of finite measures on \mathbb{R} ; see (1.8) of [23]. If the second term in \mathcal{A} vanishes, then \mathcal{L} is just the generator of an usual Fleming-Viot process.

The main work in this paper is to solve the martingale problem and analyze the sample path properties of the solution. For $f \in B(\mathbb{R}^m)$, define $F_{m,f}(\mu) = \langle f, \mu^m \rangle$. For $\mu \in P(\mathbb{R})$, we say a $P(\mathbb{R})$ -valued continuous process $\{Z(t) : t \geq 0\}$ is a solution of the (\mathcal{L}, μ) -martingale problem if $Z(0) = \mu$ and

$$F(Z(t)) - F(Z(0)) - \int_0^t \mathcal{L}F(Z(s))ds, \quad t \geq 0, \quad (1.8)$$

is a martingale for each $F \in \mathcal{D}(\mathcal{L}) := \bigcup_{m \geq 1} \{F_{m,f}(\mu), f \in C^2(\mathbb{R}^m)\}$. A simple calculation yields

$$\mathcal{L}F_{m,f}(\mu) = \langle \mu^m, G^m f \rangle + \sum_{1 \leq i < j \leq m} \gamma (\langle \mu^{m-1}, \Psi_{ij} f \rangle - \langle \mu^m, f \rangle), \quad (1.9)$$

where Ψ_{ij} denotes the operator from $B(\mathbb{R}^m)$ to $B(\mathbb{R}^{m-1})$ defined by

$$\Psi_{ij} f(x_1, \dots, x_{m-1}) = f(x_1, \dots, x_{m-1}, \dots, x_{m-1}, \dots, x_{m-2}), \quad (1.10)$$

where x_{m-1} is in the places of the i th and the j th variables of f on the right hand side. We shall show that the (\mathcal{L}, μ) -martingale problem is well-posed and call the solution as Fleming-Viot process in an environment (FVE for short). We will use look-down construction suggested by [4] with some modifications to show the existence of the solution. This look-down construction will help us on analyzing the sample path properties. The uniqueness of the (\mathcal{L}, μ) -martingale problem will be proved by classical duality argument. Since the spatial motions of individuals in the look-down system are not independent with each other, when solving the martingale problem, we need some technical lemmas which will be given in the Appendix.

Our other main results include:

1. State classification: when $\epsilon > 0$, FVE is absolutely continuous respect to dx and we also deduce a new SPDE for the density process; when $\epsilon = 0$ its values are purely atomic;
2. When conditioned to have total mass one, a measure-valued branching process in a Brownian medium constructed in [23] is an FVE.

The remaining of this paper is organized as follows. In Section 2, we solve the (\mathcal{L}, μ) -martingale problem. The state classification of the process will be investigated in Section 3. In the last section, Section 4, we derive the connection between FVE and the process constructed in [23]. Two technical lemmas will be given in the Appendix.

Remark 1.1. *By Theorem 8.2.5 of [9], the closure of $\{(f, G^m f) : f \in C_c^\infty(\mathbb{R}^m)\}$ denoted by \bar{G}^m is single-valued and generates a Feller semigroup $(T_t^m)_{t \geq 0}$ on $\hat{C}(\mathbb{R}^m)$. Note that this semigroup is given by a transition probability function and can therefore be extended to all of $B(\mathbb{R}^m)$.*

Notation: For reader's convenience, we introduce here our main notation. Let $\hat{\mathbb{R}}$ denote the one-point compactification of \mathbb{R} . Given a topological space E , let $M(E)$ ($P(E)$) denote space of finite measures (probability measures) on E . Let $B(E)$ denote the set of bounded measurable functions on E and let $C(E)$ denote its subset comprising of bounded continuous functions. Let $\hat{C}(\mathbb{R}^n)$ be the space of continuous functions on \mathbb{R}^n which vanish at infinity and let $C_c^\infty(\mathbb{R}^n)$ be functions with compact support and bounded continuous derivatives of any order. Let $C^2(\mathbb{R}^n)$ denote the set of functions in $C(\mathbb{R}^n)$ which is twice continuously differential functions with

bounded derivatives up to the second order. Let $\hat{C}^2(\mathbb{R}^n)$ be the subset of $C^2(\mathbb{R}^n)$ of functions that together with their derivatives up to the second order vanish at infinity.

Let

$$C_{\partial}^2(\mathbb{R}^n) = \{f + c : c \in \mathbb{R} \text{ and } f \in \hat{C}^2(\mathbb{R}^n)\}$$

We denote by $C_E[0, \infty)$ the space of continuous paths taking values in E . Let $D_E[0, \infty)$ denote the Skorokhod space of càdlàg paths taking values in E . For $f \in C(\mathbb{R})$ and $\mu \in M(\mathbb{R})$ we shall write $\langle \mu, f \rangle$ for $\int f d\mu$.

2 Construction

2.1 Uniqueness

In this subsection, we define a dual process to show the uniqueness of the (\mathcal{L}, μ) -martingale problem. Let $\{M_t : t \geq 0\}$ be a nonnegative integer-valued càdlàg Markov process. For $i \geq j$, the transition intensities $q_{i,i-1} = \gamma^i(i-1)/2$ and $q_{ij} = 0$ for all other pairs i, j . Let $\tau_0 = 0$ and let $\{\tau_k : 1 \leq k \leq M_0 - 1\}$ be the sequence of jump times of $\{M_t : t \geq 0\}$. That is $\tau_1 = \inf\{t \geq 0 : M_t \neq M_0\}, \dots, \tau_k = \inf\{t > \tau_{k-1} : M_t \neq M_{\tau_{k-1}}\}$.

Let $\{\Gamma_k : 1 \leq k \leq M_0 - 1\}$ be a sequence of random operators which are conditionally independent given $\{M_t : t \geq 0\}$ and satisfy

$$\mathbf{P}\{\Gamma_k = \Psi_{ij} | M(\tau_k-) = l, M(\tau_k) = l - 1\} = \binom{l}{2}^{-1}, \quad 1 \leq i < j \leq l,$$

where Ψ_{ij} are defined by (1.10). Let \mathbf{B} denote the topological union of $\{B(\mathbb{R}^m) : m = 1, 2, \dots\}$ endowed with pointwise convergence on each $B(\mathbb{R}^m)$. Then

$$F_t = T_{t-\tau_k}^{M_{\tau_k}} \Gamma_k T_{\tau_k-\tau_{k-1}}^{M_{\tau_{k-1}}} \Gamma_{k-1} \cdots T_{\tau_2-\tau_1}^{M_{\tau_1}} \Gamma_1 T_{\tau_1}^{M_0} F_0, \quad \tau_k \leq t < \tau_{k+1}, \quad 0 \leq k \leq M_0 - 1, \quad (2.1)$$

defines a Markov process $\{F_t : t \geq 0\}$ taking values from \mathbf{B} . Clearly, $\{(M_t, F_t) : t \geq 0\}$ is also a Markov process. Let $\mathbf{E}_{m,f}$ denote the expectation given $M_0 = m$ and $F_0 = f \in B(\mathbb{R}^m)$.

Theorem 2.1. *Suppose that $\{Z(t) : t \geq 0\}$ is a solution of the (\mathcal{L}, μ) -martingale problem and assume that $\{Z(t) : t \geq 0\}$ and $\{(M_t, F_t) : t \geq 0\}$ are defined on the same probability space and independent of each other, then*

$$\mathbf{E} \langle Z(t)^m, f \rangle = \mathbf{E}_{m,f} [\langle \mu^{M_t}, F_t \rangle] \quad (2.2)$$

for any $t \geq 0$, $f \in C(\mathbb{R}^m)$ and integer $m \geq 1$.

Proof. In this proof we set $F_\mu(m, f) = F_{m,f}(\mu) = \langle \mu^m, f \rangle$. It suffices to prove (2.2) for $f \in C^2(\mathbb{R}^m)$. By the definition of F_t and elementary properties of M_t , we know that $\{(M_t, F_t) : t \geq 0\}$ has weak generator $\mathcal{L}^\#$ given by

$$\mathcal{L}^\# F_\mu(m, f) = F_\mu(m, G^m f) + \sum_{1 \leq i < j \leq m} \gamma (F_\mu(m-1, \Psi_{ij} f) - F_\mu(m, f)) \quad (2.3)$$

with $f \in C^2(\mathbb{R}^m)$. In view of (1.9) we have

$$\mathcal{L}^\# F_\mu(m, f) = \mathcal{L} F_{m,f}(\mu). \quad (2.4)$$

Thus if we can show that for $F_0 \in C^2(\mathbb{R}^m)$, $F_t \in C^2(\mathbb{R}^m)$ for all $t \geq 0$, then dual relationship (2.2) follows from Corollary 4.4.13 of [9]. To this end, it suffices to show that $T_t^m C^2(\mathbb{R}^m) \subset C^2(\mathbb{R}^m)$. When $\epsilon > 0$, G^m is uniform elliptic. The desired result follows from Theorem 0.5 on page 227 of [5]. When $\epsilon = 0$, Lemma B.1 yields the desired conclusion. We are done. \square

2.2 Look Down Processes

Suppose that $x_t = (x_1(t), \dots, x_m(t))$ is a Markov process in \mathbb{R}^m generated by G^m . By Lemma 2.3.2 of [1] we know that $(x_1(t), \dots, x_m(t))$ is an exchangeable Feller process. Let $P_t^{(m)}$ denote its transition semigroup. Then $\{P_t^{(m)}, m \geq 1\}$ is a consistent family of Feller semigroups on $C(\mathbb{R}^m)$, i.e., for all $k \leq m$, any k -component of G^m -diffusion evolve as a G^k -diffusion.

Let $\{B_{ijk}, 1 \leq i < j, 1 \leq k < \infty\}$ and $\{B_{i0}, i \geq 1\}$ be independent Brownian motions, independent of W . Let $\{N_{ij}, 1 \leq i < j\}$ be independent, unit rate Poisson processes, independent of $\{B_{ijk}\}$, W and let τ_{ijk} denote the k th jump time of N_{ij} . Let $\{X_i(0), i \geq 1\}$ be an exchangeable sequence of random variables, independent of $\{U_{ijk}\}$, $\{U_{i0}\}$, W and $\{N_{ij}\}$. Define $\gamma_{ijk} = \min\{\tau_{i'jk'}, i' < j : \tau_{i'jk'} > \tau_{ijk}\}$; that is, γ_{ijk} is the first jump time of $N_j \equiv \sum_{i < j} N_{ij}$ after τ_{ijk} , and define $\gamma_{j0} = \min\{\tau_{ij1} : i < j\}$. Finally, for $0 \leq t < \gamma_{j0}$ define

$$X_j(t) = X_j(0) + \epsilon B_{j0}(t) + \int_0^t \int_{\mathbb{R}} h(y - X_j(s)) W(dy ds) \quad (2.5)$$

and for $\tau_{ijk} \leq t < \gamma_{ijk}$,

$$X_j(t) = X_i(\tau_{ijk}) + \epsilon(B_{ijk}(t) - B_{ijk}(\tau_{ijk})) + \int_{\tau_{ijk}}^t \int_{\mathbb{R}} h(y - X_j(s)) W(dy ds). \quad (2.6)$$

Since G^m -diffusion is an exchangeable consistent family of Feller diffusions, between the jump times of the Poisson processes, the X_j behave as a G^1 -diffusion and any n -component of the particle systems evolve as a G^n -diffusion. At the jump times of N_{ij} , X_j “looks down” at X_i , assumes the value of X_i at the jump time, and then evolves as a G^1 -diffusion and also any n -component of the particle systems evolve as a G^n -diffusion. Then $X = (X_1, X_2, \dots)$ is a Markov process with generator given by

$$\begin{aligned} \mathbb{A}f(x_1, \dots, x_m) &= G^m f(x_1, \dots, x_m) \\ &+ \sum_{1 \leq i < j \leq m} (f(\theta_{ij}(x_1, \dots, x_m)) - f(x_1, \dots, x_m)), \end{aligned} \quad (2.7)$$

where $f \in C^2(\mathbb{R}^m)$ and $\theta_{ij}(x_1, \dots, x_m)$ denote the element of \mathbb{R}^m obtained by replacing x_j by x_i in (x_1, \dots, x_m) .

As in [4], we want to compare the \mathbb{R}^∞ -valued process X to a sequence of modified Moran-type models. Let S_m denote the collection of permutations of $(1, \dots, m)$ which we write as ordered m -tuples $s = (s_1, \dots, s_m)$. Let $\pi_{ij} : S_m \rightarrow S_m$ denote the mapping such that $\pi_{ij}s$ is obtained from s by interchanging s_i and s_j and let $\{M_{ijk} : 1 \leq i \neq j \leq m, k \geq 1\}$ be independent random mappings $M_{ijk} : S_m \rightarrow S_m$ such that $P\{M_{ijk}s = s\} = P\{M_{ijk}s = \pi_{ij}s\} = \frac{1}{2}$. In following we define an S_m -valued process Σ^m and counting processes $\{\tilde{N}_{ij}, 1 \leq i \neq j \leq m\}$ recursively. Let $\Sigma^m(0)$ be uniformly distributed on S_m and independent of all other processes. Let

$$\tilde{N}_{ij}(t) = \sum_{1 \leq k < l \leq m} \int_0^t \mathbf{1}_{\{\Sigma_i^m(r-) = k, \Sigma_j^m(r-) = l\}} dN_{kl}(r) \quad (2.8)$$

and let Σ^m be constant except for discontinuities determined by $\Sigma^m(\tilde{\tau}_{ijk}) = M_{ijk}\Sigma^m(\tilde{\tau}_{ijk}-)$, where $\tilde{\tau}_{ijk}$ is the k -th jump time of \tilde{N}_{ij} , or more precisely, interpreting Σ^m as a \mathbb{Z}^m -valued process,

$$\Sigma^m(t) = \sum_{1 \leq i < j \leq m} \int_0^t \left(M_{ij(\tilde{N}_{ij}(r-)+1)} \Sigma^m(r-) \right) d\tilde{N}_{ij}(r). \quad (2.9)$$

Next, define $\{\hat{N}_{ij}, 1 \leq i \leq m < j\}$ by

$$\hat{N}_{ij}(t) = \sum_{k=1}^m \int_0^t \mathbf{1}_{\{\Sigma_i^m(r-) = k\}} dN_{kj}(r) \quad (2.10)$$

and let $\hat{\tau}_{ijk}$ denote the k -th jump time of \hat{N}_{ij} . Note that for $j > m$,

$$N_j = \sum_{1 \leq i < j} N_{ij} = \sum_{1 \leq i \leq m} \hat{N}_{ij} + \sum_{m < i \leq j} N_{ij}. \quad (2.11)$$

By Lemma 2.1 of [4], $\{\tilde{N}_{ij}\}$ and $\{\hat{N}_{ij}\}$ are Poisson processes with intensities $\frac{1}{2}$ and 1, respectively. And for each $t \geq 0$, $\Sigma^m(t)$ is independent of $\mathcal{G}_t = \sigma(\tilde{N}_{ij}(s), \hat{N}_{kl}(s) : s \leq t, 1 \leq i \neq j \leq m, 1 \leq k \leq m < l)$. Define

$$Y_j^m(t) = X_{\Sigma_j^m(t)}(t), \quad j = 1, \dots, m.$$

Lemma 2.1. $Y^m = (Y_1^m, \dots, Y_m^m)$ is a Markov process with generator given by

$$\begin{aligned} \mathbb{A}_m f(y_1, \dots, y_m) &= G^m f(y_1, \dots, y_m) \\ &+ \frac{1}{2} \sum_{1 \leq i \neq j \leq m} (f(\theta_{ij}(y_1, \dots, y_m)) - f(y_1, \dots, y_m)), \end{aligned} \quad (2.12)$$

where $f \in C^2(\mathbb{R}^m)$ and $\theta_{ij}(x_1, \dots, x_m)$ denote the element of \mathbb{R}^m obtained by replacing x_j by x_i in (x_1, \dots, x_m) .

Proof. The proof is similar to that of part (b) in Lemma 2.1 of [4]. For $1 \leq i, j \leq m$, define

$$\begin{aligned} \tilde{B}_{j0} &= B_{\alpha 0}, & \text{where } \alpha &= \Sigma_j^m(0), \\ \tilde{B}_{ijk} &= B_{\alpha\beta\gamma}, & \text{where } \alpha &= \Sigma_i^m(\tilde{\tau}_{ijk-}), \beta = \Sigma_j^m(\tilde{\tau}_{ijk-}), \\ & & \gamma &= N_{\alpha\beta}(\tilde{\tau}_{ijk-}) \end{aligned} \quad (2.13)$$

Define $\tilde{\gamma}_{ijk} = \min\{\tilde{\tau}_{i'jk'}, i' \neq j : \tilde{\tau}_{i'jk'} > \tilde{\tau}_{ijk}\}$ and let $\tilde{\gamma}_{j0}$ be the first jump time of $\tilde{N}_j \equiv \sum_{i \neq j} \tilde{N}_{ij}$. By Lemma A.1, $Y_j^m(t) = X_{\Sigma_j^m(t)}^m(t)$ yields that for $0 \leq t < \tilde{\gamma}_{j0}$

$$Y_j^m(t) = Y_j^m(0) + \epsilon \tilde{B}_{j0}(t) + \int_0^t \int_{\mathbb{R}} h(y - Y_j^m(s)) W(dy ds) \quad (2.14)$$

and for $\tilde{\tau}_{ijk} \leq t < \tilde{\gamma}_{ijk}$,

$$Y_j^m(t) = Y_i(\tilde{\tau}_{ijk}) + \epsilon(\tilde{B}_{ijk}(t) - \tilde{B}_{ijk}(\tilde{\tau}_{ijk})) + \int_{\tilde{\tau}_{ijk}}^t \int_{\mathbb{R}} h(y - Y_j^m(s)) W(dy ds). \quad (2.15)$$

By Lemmas A5.1 and A5.2 of [4], $\{\tilde{B}_{j0}\}$, $\{\tilde{B}_{ijk}\}$ and $\{Y_j(0)\}$ are independent of $\{\tilde{N}_{ij}\}$ and Σ^m . Furthermore, the \tilde{B}_{j0} and the \tilde{B}_{ijk} are independent Brownian motions and $(Y_1^m(0), \dots, Y_m^m(0))$ has the same distribution as $(X_1(0), \dots, X_m(0))$. Then the desired result follows from (2.14) and (2.15). \square

By (2.14) and (2.15), we see $(Y_1^m(t), \dots, Y_m^m(t))$ is exchangeable and has the same empirical measures as (X_1, \dots, X_m) . From the construction above, $\Sigma^m(t)$ must be independent of $Y^m(t)$. Thus for each $t > 0$, $(X_1(t), X_2(t), \dots)$ is exchangeable. To show the existence of (\mathcal{L}, μ) -martingale problem, we need the following lemma.

Lemma 2.2. (a). Suppose that $Z(t)$ is a $P(\mathbb{R})$ -valued process satisfying the martingale formula (1.8) for every $F \in \mathcal{D}(\mathcal{L})$. Then $\{Z(t) : t \geq 0\}$ has a continuous modification and for $\phi \in C^2(\mathbb{R})$

$$M_t(\phi) := \langle Z(t), \phi \rangle - \langle Z(0), \phi \rangle - \frac{\rho\epsilon}{2} \int_0^t \langle Z(s), \phi'' \rangle ds \quad (2.16)$$

is a martingale with quadratic variation

$$\gamma \int_0^t (\langle Z(s), \phi^2 \rangle - \langle Z(s), \phi \rangle^2) ds + \int_0^t ds \int_{\mathbb{R}} \langle Z(s), h(\cdot - y)\phi' \rangle^2 dy. \quad (2.17)$$

(b). If a continuous $P(\mathbb{R})$ -valued process $Z(t)$ satisfies the martingale problem (2.16) and (2.17), then it is also a solution of (\mathcal{L}, μ) -martingale problem.

Proof. (a). The existence of continuous modification follows from Lemma 2.1 of [10] and the fact that (1.8) is a martingale for each $F \in \mathcal{D}(\mathcal{L})$ which also yields (2.16) and (2.17). The proof for assertion (b) is a classical approximation procedure. We left it to the interested readers. \square

Now, we come to our main result in this section.

Theorem 2.2. Given $\mu \in P(\mathbb{R})$, suppose that $\{X_i(0), i \geq 1\}$ is an exchangeable sequence of random variables such that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \delta_{X_i(0)} = \mu.$$

Let

$$Z_m(t) = \frac{1}{m} \sum_{i=1}^m \delta_{X_i(t)} = \frac{1}{m} \sum_{i=1}^m \delta_{Y_i^m(t)}. \quad (2.18)$$

Then the (\mathcal{L}, μ) -martingale problem has a solution Z such that for each $t > 0$,

$$\lim_{m \rightarrow \infty} \sup_{s \leq t} \rho(Z_m(s), Z(s)) = 0 \quad a.s., \quad (2.19)$$

where ρ denotes the Prohorov metric on $P(\mathbb{R})$.

Proof. With the help of Lemma 2.2 which can be regarded as a version of Lemma 2.3 of [4], the proof is similar to Theorem 2.4 of [4]. We omit it here. \square

3 Sample Path Properties

In this section, we show that when $\epsilon > 0$, $Z(t)$ is absolutely continuous respect to dx for almost all $t \geq 0$ and when $\epsilon = 0$ the values of Z are purely atomic. We first describe the *weak atomic topology* on $M(\mathbb{R})$ introduced by Ethier and Kurtz [11]. Recall that ρ denotes the Prohorov metric on $M(\mathbb{R})$, which induces the topology of the weak convergence. Define the metric ρ_a on $M(\mathbb{R})$ by

$$\rho_a(\mu, \nu) = \rho(\mu, \nu) + \sup_{0 < \epsilon \leq 1} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(|x - y|/\epsilon) \mu(dx) \mu(dy) - \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(|x - y|/\epsilon) \nu(dx) \nu(dy) \right|, \quad (3.1)$$

where $\Phi(\cdot) = (1 - \cdot)_+$. The topology on $M(\mathbb{R})$ induced by ρ_a is called the *weak atomic topology*. For $\mu \in M(\mathbb{R})$, define $\mu^* = \sum \mu(\{x\})^2 \delta_x$. We need the following results of [11].

Lemma 3.1. *Let $\mu_n, \mu \in M(\mathbb{R})$.*

- (a). *Suppose $\rho(\mu_n, \mu) \rightarrow 0$. Then $\rho(\mu_n^*, \mu^*) \rightarrow 0$ if and only if $\mu_n^*(\mathbb{R}) \rightarrow \mu^*(\mathbb{R})$;*
- (b). *$\rho_a(\mu_n, \mu) \rightarrow 0$ if and only if $\rho(\mu_n, \mu) \rightarrow 0$ and $\rho(\mu_n^*, \mu^*) \rightarrow 0$;*
- (c). *Suppose $Z \in C_{(M(\mathbb{R}), \rho)}[0, \infty)$. If $Z^*(\mathbb{R}) \in C_{[0, \infty)}[0, \infty)$, then $Z \in C_{(M(\mathbb{R}), \rho_a)}[0, \infty)$.*

Proof. See Lemmas 2.1, 2.2 and 2.11 of [11] for (a), (b) and (c), respectively. \square

Our first main result in this section is the following theorem.

Theorem 3.1. *Suppose Z is a solution of (\mathcal{L}, μ) -martingale problem. Assume $\epsilon = 0$. Then $\mathbf{P}\{Z(t) \in P_a(\mathbb{R}), t > 0\} = \mathbf{P}\{Z(\cdot) \in C_{(M(\mathbb{R}), \rho_a)}[0, \infty)\} = 1$, where $P_a(\mathbb{R})$ denotes the collection of purely atomic probability measures on \mathbb{R} .*

Proof. According to the look down construction, (2.5) and (2.6), if X_j ‘looks down’ to X_i , and assumes the value X_i at the jump time, then X_j and X_i have the same sample path before the next jump time. Define

$$x_i(t) = X_i(0) + \int_0^t \int_{\mathbb{R}} h(y - x_i(s)) W(dy ds), \quad t \geq 0, \quad i = 1, 2, \dots$$

Therefore, by Lemma 3.1, $Z_m(\cdot) \in D_{(P(\mathbb{R}), \rho_a)}[0, \infty)$ and $Z_m^*(t, \mathbb{R})$ is monotone in $t \geq 0$. According to Proposition 3.3 of [4] and Lemma B.1, almost surely for $t > 0$, there are only finite number paths, denoted by $D(t)$ which is independent of m , alive in the ‘look down system’. Let $t_0 > 0$ be fixed. Note that D is càdlàg on $[t_0, +\infty)$. Typically, $D(t) \leq D(s)$ for $t > s$. Let $\{x_{c_i}(t_0), i = 1, 2, \dots, D(t_0)\}$ be the enumeration of the living paths at t_0 with $x_{c_1}(t_0) < x_{c_2}(t_0) < \dots < x_{c_{D(t_0)}}(t_0)$. Thus for $t > t_0$, we may represent $Z_m(t)$ by

$$Z_m(t) = \sum_{i=1}^{D(t_0)} \frac{b_{i,m}(t)}{m} \delta_{x_{c_i}(t)}, \quad t \geq t_0, \quad (3.2)$$

where $b_{i,m}(t), i = 1, 2, \dots$ are nonnegative integer-valued càdlàg random processes defined on $[t_0, +\infty)$ with $\sum_{i=1}^{D(t)} b_{i,m} = m$. Note that by Lemma B.1, for every $T > t_0$, almost surely,

$$\inf_{i \neq j} \inf_{t_0 \leq t \leq T} |x_{c_i}(t) - x_{c_j}(t)| > 0. \quad (3.3)$$

Therefore, according to (2.19) we may represent $Z(t)$ by

$$Z(t) = \sum_{i=1}^{D(t_0)} b_i(t) \delta_{x_{c_i}(t)}, \quad t \geq t_0, \quad (3.4)$$

where $b_i(t) \geq 0, i = 1, 2, \dots$ are càdlàg random processes defined on $[t_0, +\infty)$ with

$$\sup_{t_0 \leq t \leq T} \sum_{i=1}^{D(t_0)} |b_{i,m}(t)/m - b_i(t)| \rightarrow 0, \quad a.s. \quad \text{as } m \rightarrow \infty. \quad (3.5)$$

Since t_0 is arbitrary, $\mathbf{P}\{Z(t) \in P_a(\mathbb{R}), t > 0\} = 1$. From above and Lemma 3.1, we see $Z(\cdot \vee t_0) \in D_{(P(\mathbb{R}), \rho_a)}[0, \infty)$, *a.s.* Typically,

$$Z_m^*(\cdot \vee t_0, \mathbb{R}) \rightarrow Z^*(\cdot \vee t_0, \mathbb{R}) \quad \text{in } D_{\mathbb{R}}[0, \infty) \quad \text{as } m \rightarrow \infty \quad a.s.$$

On the other hand, according to the ‘look down construction’, if we define

$$J(Z_m^*(t \vee t_0, \mathbb{R})) := \int_0^\infty e^{-u} [1 \wedge \sup_{0 \leq t \leq u} |Z_m^*(t \vee t_0, \mathbb{R}) - Z_m^*((t \vee t_0)^-, \mathbb{R})|] du,$$

then

$$J(Z_m^*(t \vee t_0, \mathbb{R})) \leq \frac{4m+2}{m^2} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

By Theorem 3.10.2 of [9] and Lemma 3.1, $Z(\cdot \vee t_0) \in C_{(P(\mathbb{R}), \rho_a)}[0, \infty)$, *a.s.* Set $D = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and $D_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2, x_3 = x_4\}$. By approximating an indicate function with continuous functions, we see that (2.2) holds for $f = 1_D$ and $g = 1_{D_2}$. Note that $\langle Z(t)^2, f \rangle = Z^*(t, \mathbb{R})$ and $\langle Z(t)^2, f \rangle^2 = \langle Z(t)^4, g \rangle$. Therefore, by (2.2) and the right continuity of (F_t, M_t) ,

$$\lim_{t \downarrow 0} \mathbf{E} |Z^*(t, \mathbb{R}) - \mu^*(\mathbb{R})|^2 = \lim_{t \downarrow 0} \mathbf{E} |\langle Z(t)^2, f \rangle - \langle \mu^2, f \rangle|^2 = 0.$$

By Lemma 3.1 and the monotonicity of $Z_m^*(t, \mathbb{R})$, $\rho_a(Z(t), \mu) \rightarrow 0$ almost surely as $t \rightarrow 0$. Thus $Z(\cdot) \in C_{(P(\mathbb{R}), \rho_a)}[0, \infty)$, *a.s.* \square

In the next theorem, we shall show that when $\epsilon > 0$ $Z(t, dx)$ is absolutely continuous with respect to dx and derive the SPDE for the density.

Theorem 3.2. *Suppose Z is a solution of (\mathcal{L}, μ) -martingale problem. Assume $\epsilon > 0$. Then for $t > 0$, $Z(t, dx)$ is absolutely continuous with respect to dx and the density $Z_t(x)$ satisfies the following SPDE: for $\phi \in \mathcal{S}(\mathbb{R})$,*

$$\begin{aligned} \langle Z_t, \phi \rangle - \langle \mu, \phi \rangle &= \int_0^t \int_{\mathbb{R}} \sqrt{\gamma Z_s(x)} \phi(x) V(dsdx) - \int_0^t \int_{\mathbb{R}} \langle Z_s, \phi \rangle \sqrt{\gamma Z_s(x)} V(dsdx) \\ &\quad + \int_0^t \int_{\mathbb{R}} \langle Z_s, h(x - \cdot) \phi' \rangle W(dsdx) + \frac{\rho \epsilon}{2} \int_0^t \langle Z_s, \phi'' \rangle ds, \end{aligned} \quad (3.6)$$

where V and W are two independent Brownian sheets and $\mathcal{S}(\mathbb{R})$ is the space of rapidly decreasing C^∞ -function defined on \mathbb{R} equipped with the Schwartz topology.

Proof. We borrow the ideas in Theorem 1.7 of [13]. First by dual relationship (2.2), one can derive that for any $\phi, \psi \in C(\mathbb{R})$,

$$\mathbf{E} \langle Z(t), \phi \rangle = \langle \mu, T_t^1 \phi \rangle \quad (3.7)$$

and

$$\mathbf{E} [\langle Z(t), \phi \rangle \langle Z(t), \psi \rangle] = e^{-\gamma t} \langle \mu^2, T_t^2 \phi \psi \rangle + \gamma \int_0^t e^{-\gamma s} \langle \mu T_{t-s}^1, \Psi_{12}(T_s^2 \phi \psi) \rangle ds. \quad (3.8)$$

For $\epsilon > 0$, the semigroup $(T_t^m)_{t>0}$ is uniformly elliptic and has density $q_m(t, x, y)$ satisfying

$$q_m(t, x, y) \leq c \cdot g_m(\epsilon' t, x, y), \quad t > 0, \quad x, y \in \mathbb{R}^m,$$

where c is a constant and $g_m(t, x, y)$ denotes the transition density of the m -dimensional standard Brownian motion; see [5]. Without loss of generality, we assume $\epsilon' = 1$. Note that

$$\begin{aligned} &\int_{\mathbb{R}^2} q_1(u, x, z_1) q_1(u', x, z_2) q_1(t-s, z, y) q_2(s, (y, y), (z_1, z_2)) dz_1 dz_2 \\ &\rightarrow q_1(t-s, z, y) q_2(s, (y, y), (x, x)) \end{aligned}$$

as $u, u' \rightarrow 0$. Meanwhile,

$$\begin{aligned} & \int_{\mathbb{R}^2} q_1(u, x, z_1)q_1(u', x, z_2)q_1(t-s, z, y)q_2(t, (y, y), (z_1, z_2))dz_1dz_2 \\ & \leq c \int_{\mathbb{R}^2} g_1(u, x, z_1)g_1(u', x, z_2)g_1(t-s, z, y)g_2(t, (y, y), (z_1, z_2))dz_1dz_2 \\ & = cg_1(u+s, x, y)g_1(u'+s, x, y)g_1(t-s, z, y). \end{aligned}$$

Take $\phi = \phi_{u,x} = q_1(u, x, \cdot)$ and $\psi = \psi_{u',x} = q_1(u', x, \cdot)$ in (3.8). By dominated convergence theorem, when $u, u' \rightarrow 0$,

$$\begin{aligned} & \int_0^T dt \int dx \int_0^t e^{-\gamma s} \langle \mu T_{t-s}^1, \Psi_{12}(T_s^2 \phi \psi) \rangle ds \\ & \rightarrow \int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^2} e^{-\gamma s} q_1(t-s, z, y)q_2(s, (y, y), (x, x))dy\mu(dz). \end{aligned} \quad (3.9)$$

Similarly, we have

$$\begin{aligned} & \int_0^T dt \int dx e^{-\gamma t} \langle \mu^2, T_t^2 \phi \psi \rangle \\ & \rightarrow \int_0^T dt \int dx \int_{\mathbb{R}^4} e^{-\gamma t} q_2(t, (x_1, x_2), (x, x))\mu(dx_1)\mu(dx_2). \end{aligned} \quad (3.10)$$

Combining (3.9) and (3.10) together yields $\{\langle Z(t), q_1(u, x, \cdot) \rangle, u > 0\}$ is a Cauchy sequence in $L^2(\Omega \times [0, T] \times \mathbb{R})$. This implies the existence of the density $Z_t(x)$ of Z_t in $L^2(\Omega \times [0, T] \times \mathbb{R})$.

Next, we derive the SPDE (3.6). Choose an one dimensional standard Brownian motion \hat{B}_t independent of Z_t . For any fixed $c > \gamma/2$, set $G_t = \exp(\sqrt{\gamma}\hat{B}_t + (c - \gamma/2)t)$. So $G_t > 0$ and $G_t \rightarrow \infty$ as $t \rightarrow \infty$ a.s. It also satisfies

$$dG_t = \sqrt{\gamma}G_t d\hat{B}_t + cG_t dt, \quad G_0 = 1.$$

Define $C_t = \int_0^t G_s ds$. C_t is strictly increasing and $C_t \rightarrow \infty$ as $t \rightarrow \infty$ a.s.. Let C_t^{-1} denote its inverse function on $[0, \infty)$. Define measure-valued process I_t by

$$I_t(dx) = G_{C_t^{-1}} \cdot Z_{C_t^{-1}}(dx).$$

By Ito's formula, (2.16) and (2.17)

$$\begin{aligned} \langle I_t, \phi \rangle &= \langle I_0, \phi \rangle + \int_0^{C_t^{-1}} G_s dM_s(\phi) + \int_0^{C_t^{-1}} \sqrt{\gamma}G_s \langle Z_s, \phi \rangle d\hat{B}_s \\ & \quad + c \int_0^{C_t^{-1}} G_s \langle Z_s, \phi \rangle ds + \frac{\rho\epsilon}{2} \int_0^{C_t^{-1}} G_s \langle Z_s, \phi'' \rangle ds. \end{aligned}$$

Then

$$\tilde{M}_t(\phi) := \int_0^{C_t^{-1}} G_s dM_s(\phi) + \int_0^{C_t^{-1}} G_s \langle Z_s, \phi \rangle d\hat{B}_s, \quad t \geq 0,$$

is a local martingale with quadratic function

$$\langle \tilde{M}(\phi) \rangle_t = \gamma \int_0^t \langle I_s, \phi^2 \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle I_s, h(x - \cdot) \phi' \rangle^2 / \langle I_s, 1 \rangle dx.$$

Clearly, $I_t(dx)$ is also absolutely continuous with respect to dx . Denote the corresponding density by $I_t(x)$. Similar to the martingale representation theorem (see Theorem 3.3.6 of [12] or Theorem III-7 of [6]), there exists two independent $L^2(\mathbb{R})$ -cylindrical Brownian motion \tilde{V} and \tilde{W} (may be on an extension probability space) such that

$$\tilde{M}_t(\phi) = \int_0^t \langle f(s, I_s)^* \phi, d\tilde{V}_s \rangle_{L^2(\mathbb{R})} + \int_0^t \langle g(s, I_s)^* \phi, d\tilde{W}_s \rangle_{L^2(\mathbb{R})},$$

where $f(s, I_s)$ and $g(s, I_s)$ are linear maps from $L^2(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$, the space of Schwartz distributions, such that for $\phi \in \mathcal{S}(\mathbb{R})$,

$$f(s, I_s)^* \phi(x) = \sqrt{\gamma I_s(x)} \phi(x)$$

and

$$g(s, I_s)^* \phi(x) = \int_{\mathbb{R}} h(x-y) \phi'(y) I_s(y) dy / \sqrt{\langle I_s, 1 \rangle}.$$

Thus

$$\begin{aligned} \langle I_t, \phi \rangle &= \int_0^t \langle f(s, I_s)^* \phi, d\tilde{V}_s \rangle_{L^2(\mathbb{R})} + \int_0^t \langle g(s, I_s)^* \phi, d\tilde{W}_s \rangle_{L^2(\mathbb{R})} \\ &\quad + c \int_0^t \langle I_s, \phi \rangle / \langle I_s, 1 \rangle ds + \frac{\rho_\epsilon}{2} \int_0^t \langle I_s, \phi'' \rangle / \langle I_s, 1 \rangle ds. \end{aligned}$$

Define two new $L^2(\mathbb{R})$ -cylindrical Brownian motions \hat{V} and \hat{W} by

$$\langle \hat{V}_t, \phi \rangle = \int_0^{C_t} \frac{1}{\langle I_s, 1 \rangle} \langle d\tilde{V}, \phi \rangle, \quad \langle \hat{W}_t, \phi \rangle = \int_0^{C_t} \frac{1}{\langle I_s, 1 \rangle} \langle d\tilde{W}, \phi \rangle.$$

Since \tilde{V} and \tilde{W} are independent, \hat{V} and \hat{W} are orthogonal (hence they are independent). Then we can find two independent Brownian sheets $V(dtdx)$ and $W(dtdx)$ such that

$$\tilde{V}_t(l) = \int_0^t \int_{\mathbb{R}} l(x) V(dsdx), \quad \tilde{W}_t(l) = \int_0^t \int_{\mathbb{R}} l(x) W(dsdx), \quad \forall l \in L^2(\mathbb{R}).$$

Using Ito's formula and noting that $\langle Z_t, \phi \rangle = \langle I_{C_t}, \phi \rangle / \langle I_{C_t}, 1 \rangle$ yield

$$\begin{aligned} \langle Z_t, \phi \rangle - \langle \mu, \phi \rangle &= \int_0^t \int_{\mathbb{R}} \sqrt{\gamma Z_s(x)} \phi(x) V(dsdx) - \int_0^t \int_{\mathbb{R}} \langle Z_s, \phi \rangle \sqrt{\gamma Z_s(x)} V(dsdx) \\ &\quad + \int_0^t \int_{\mathbb{R}} \langle Z_s, h(x-\cdot) \phi' \rangle W(dsdx) + \frac{\rho_\epsilon}{2} \int_0^t \langle Z_s, \phi'' \rangle ds \end{aligned}$$

for $\phi \in \mathcal{S}(\mathbb{R})$. We have completed the proof. \square

Remark 3.1. By comparing the martingale formula for I_t and \tilde{M}_t with \bar{M}_t in (4.1) and (4.2), one may find that we define I_t so that under the transformation, Z_t becomes a Dawson-Watanabe superprocess in a random medium as shown in Section 4. (In fact, it is not very explicit because of the random medium (common noise)). We refer to [18] for the classical case where the result is explicit ($h = 0$). Thus, techniques from the ‘‘Dawson-Watanabe world’’ can be put to use.

4 Connections to Measure-valued Branching Processes in a Random Medium

It has been shown that there are deep connections between the Dawson-Watanabe and Fleming-Viot superprocesses; see [8, 13, 18]. In this section, we shall show that the Fleming-Viot processes in random environment is a class of measure-valued branching processes in a Brownian medium, conditioned to have total mass one. Such measure-valued branching processes were first constructed and studied by [22] and [23]. The argument in this section is similar to those in [18] with some modifications. Let $\{\omega(t), t \geq 0\}$ and $\{\hat{\omega}(t), t \geq 0\}$ denote the coordinate processes on $C_{P(\mathbb{R})}[0, \infty)$ and $C_{M(\mathbb{R})}[0, \infty)$, respectively. Define $\mathcal{F}_t^0 = \sigma(\omega(s); s \leq t)$, $\hat{\mathcal{F}}_t^0 = \sigma(\hat{\omega}(s); s \leq t)$, $\mathcal{F}_t = \mathcal{F}_{t+}^0$ and $\hat{\mathcal{F}}_t = \hat{\mathcal{F}}_{t+}^0$. Based on the results in [23] and the continuity of $\hat{\omega}$, for each $\mu \in M(\mathbb{R})$, there exists a unique probability measure $\hat{\mathbf{Q}}_\mu$ on $C_{M(\mathbb{R})}[0, \infty)$ such that for $\phi \in C^2(\mathbb{R})$

$$\hat{M}_t(\phi) := \langle \hat{\omega}(t), \phi \rangle - \langle \mu, \phi \rangle - \frac{\rho\epsilon}{2} \int_0^t \langle \hat{\omega}(s), \phi'' \rangle ds, \quad t \geq 0, \quad (4.1)$$

under $\hat{\mathbf{Q}}_\mu$ is a continuous $\hat{\mathcal{F}}_t$ -martingale starting at 0 with quadratic variation

$$\langle \hat{M}(\phi) \rangle_t = \gamma \int_0^t \langle \hat{\omega}(s), \phi^2 \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle \hat{\omega}(s), h(\cdot - y)\phi' \rangle^2 dy. \quad (4.2)$$

Let

$$C_+ = \{f : [0, \infty) \rightarrow [0, \infty) : f \text{ continuous}, \exists t_f \in (0, \infty] \text{ such that} \\ f(t) > 0 \text{ if } t \in [0, t_f) \text{ and } f(t) = 0 \text{ if } t \geq t_f\}$$

with the compact-open topology. Let $L_y \in P(C_+)$ denote the law of the unique solution of

$$\eta_t = y + \int_0^t \sqrt{\gamma\eta_s} dB_s,$$

where B is a standard Brownian motion. Note that

$$\hat{\mathbf{Q}}_\mu(\hat{\omega}(\mathbb{R}) \in \cdot) = L_{\mu(\mathbb{R})}(\cdot). \quad (4.3)$$

For $\mu \in M(\mathbb{R}) - \{0\}$, define $\bar{\mu}(\cdot) = \mu(\cdot)/\mu(\mathbb{R})$. Let $\{\mathbf{Q}_{\bar{\mu}, f}(A) : A \in \mathcal{F}, f \in C_+\}$ be a regular conditional probability for $\bar{\omega}$ given $\hat{\omega}(\mathbb{R}) = f(\cdot)$ under $\hat{\mathbf{Q}}_\mu$, where \mathcal{F} denotes the Borel σ -field on $C_{P(\mathbb{R})}[0, \infty)$. That is

$$\hat{\mathbf{Q}}_\mu(\bar{\omega} \in A | \hat{\omega}(\mathbb{R}) = f(\cdot)) = \mathbf{Q}_{\bar{\mu}, f}(A) \quad \forall A \in \mathcal{F}.$$

Lemma 4.1. *For each $\mu \in M(\mathbb{R}) - \{0\}$, there exists a subset C_μ of C_+ such that $L_{\mu(\mathbb{R})}(C_\mu) = 1$ and for $f \in C_\mu$, under $\mathbf{Q}_{\bar{\mu}, f}$*

$$M_t^f(\phi, \omega) := \langle \omega_t, \phi \rangle - \langle \bar{\mu}, \phi \rangle - \frac{\rho\epsilon}{2} \int_0^t \langle \omega_s, \phi'' \rangle ds, \quad t < t_f, \quad (4.4)$$

is an \mathcal{F}_t -martingale starting at 0 for every $\phi \in C^2(\mathbb{R})$ with

$$\langle M^f(\phi) \rangle_t = \gamma \int_0^t (\langle \omega_s, \phi^2 \rangle - \langle \omega_s, \phi \rangle^2) f(s)^{-1} ds \\ + \int_0^t ds \int_{\mathbb{R}} \langle \omega_s, h(\cdot - y)\phi' \rangle^2 dy \quad \forall t < t_f \quad (4.5)$$

and $\omega_t = \omega_{t_f}$ for all $t \geq t_f$.

Remark 4.1. Note that if $f = 1$, then (4.4) and (4.5) are just (2.16) and (2.17), respectively.

Proof. Define $T_n = \inf\{t : \hat{\omega}_t(\mathbb{R}) \leq 1/n\}$ and for $\phi \in C^2(\mathbb{R})$

$$\bar{M}_t^n(\phi) := \int_0^{t \wedge T_n} \hat{\omega}_s(\mathbb{R})^{-1} d\hat{M}_s(\phi) - \int_0^{t \wedge T_n} \langle \hat{\omega}_s, \phi \rangle \hat{\omega}_s(\mathbb{R})^{-2} d\hat{M}_s(1). \quad (4.6)$$

Thus for fixed t , $\{\bar{M}_t^n(\phi) : n \geq 1\}$ is a martingale in n . By Ito's formula,

$$\langle \bar{\omega}_{t \wedge T_n}, \phi \rangle = \langle \bar{\mu}, \phi \rangle + \frac{\rho_\epsilon}{2} \int_0^{t \wedge T_n} \langle \bar{\omega}_s, \phi'' \rangle ds + \bar{M}_t^n(\phi), \quad (4.7)$$

which implies that

$$\sup_{t \leq K, n \geq 1} |\bar{M}_t^n(\phi)| \leq 2\|\phi\|_\infty + \frac{K\rho_\epsilon}{2}\|\phi''\|_\infty. \quad (4.8)$$

Therefore, according to the Martingale Convergence Theorem and maximal inequality, $\bar{M}_t^n(\phi)$ converges as $n \rightarrow \infty$ uniformly for t in compacts a.s. (by perhaps passing to a subsequence). We denote by $\bar{M}_t(\phi)$ the limit which is a continuous martingale satisfying

$$\bar{M}_t^n(\phi) = \bar{M}_{t \wedge T_n}(\phi), \quad \forall t \geq 0, \quad a.s. \quad (4.9)$$

and

$$\sup_{t \leq K} |\bar{M}_t(\phi)| \leq 2\|\phi\|_\infty + \frac{K\rho_\epsilon}{2}\|\phi''\|_\infty. \quad (4.10)$$

Letting $n \rightarrow \infty$ in (4.7) yields

$$\langle \bar{\omega}_t, \phi \rangle = \langle \bar{\mu}, \phi \rangle + \frac{\rho_\epsilon}{2} \int_0^{t \wedge T_0} \langle \bar{\omega}_s, \phi'' \rangle ds + \bar{M}_t(\phi), \quad \forall t \geq 0 \text{ a.s. } \forall \phi \in C^2(\mathbb{R}), \quad (4.11)$$

where $T_0 = \inf\{t : \hat{\omega}_t(\mathbb{R}) = 0\}$. Note that

$$\bar{M}_{t \wedge T_0}(\phi) = \bar{M}_t(\phi). \quad (4.12)$$

Let $s < t$ and let F be a bounded $\sigma(\hat{\omega}_s(\mathbb{R}))$ -measurable random variable. Since $\{\hat{\omega}_t(\mathbb{R}) : t \geq 0\}$ is a martingale under $\hat{\mathbf{Q}}_\mu$, the martingale representation theorem implies that there exists some $\sigma(\hat{\omega}_s(\mathbb{R}) : s \leq t)$ -predictable function f such that

$$F = \hat{\mathbf{Q}}_\mu(F) + \int_0^\infty f(s, \hat{\omega}) d\hat{\omega}_s(\mathbb{R}). \quad (4.13)$$

According to (4.9) and (4.13),

$$\begin{aligned} & \hat{\mathbf{Q}}_\mu \left((\bar{M}_{t \wedge T_n}(\phi) - \bar{M}_{s \wedge T_n}(\phi)) F | \mathcal{F}_s \right) \\ &= \hat{\mathbf{Q}}_\mu \left((\bar{M}_t^n(\phi) - \bar{M}_s^n(\phi)) \int_0^\infty f(s, \hat{\omega}) d\hat{\omega}_s(\mathbb{R}) | \mathcal{F}_s \right) \\ &= \hat{\mathbf{Q}}_\mu \left(\left(\int_{s \wedge T_n}^{t \wedge T_n} \hat{\omega}_u(\mathbb{R})^{-1} d\hat{M}_u(\phi) - \int_{s \wedge T_n}^{t \wedge T_n} \langle \hat{\omega}_u, \phi \rangle \hat{\omega}_u(\mathbb{R})^{-2} d\hat{M}_u(1) \right) \int_s^t f(s, \hat{\omega}) d\hat{\omega}_u(\mathbb{R}) | \mathcal{F}_s \right) \\ &= \hat{\mathbf{Q}}_\mu \left(\int_{s \wedge T_n}^{t \wedge T_n} \gamma (\langle \hat{\omega}_u, \phi \rangle \hat{\omega}_u(\mathbb{R})^{-1} - \langle \hat{\omega}_u, \phi \rangle \hat{\omega}_u(\mathbb{R})^{-1}) f(u, \hat{\omega}) du | \mathcal{F}_s \right) \\ &= 0 \end{aligned}$$

By letting $n \rightarrow \infty$ in the above, we have

$$\hat{\mathbf{Q}}_\mu((\bar{M}_t(\phi) - \bar{M}_s(\phi))F|\mathcal{F}_s) = 0,$$

which yields for a fixed $\phi \in C^2(\mathbb{R})$, $\{\bar{M}_t(\phi) : t \geq 0\}$ is a martingale with respect to $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\hat{\omega}_s(\mathbb{R}) : s \geq 0)$. On the other hand, by (4.9) and (4.12),

$$\begin{aligned} \langle \bar{M}(\phi) \rangle_t &= \gamma \int_0^{t \wedge T_0} (\langle \bar{\omega}_s, \phi^2 \rangle - \langle \bar{\omega}_s, \phi \rangle^2) \hat{\omega}_s(\mathbb{R})^{-1} ds \\ &\quad + \int_0^{t \wedge T_0} ds \int_{\mathbb{R}} \langle \bar{\omega}_s, h(\cdot - y)\phi' \rangle^2 dy \quad \hat{\mathbf{Q}}_\mu - a.s. \end{aligned} \quad (4.14)$$

Set $M_t^f(\phi, \omega) = M_{t_f^-}^f(\phi)$ for $t \geq t_f$. By (4.11) and (4.12),

$$\bar{M}_t(\phi) = M_t^{\omega(\mathbb{R})}(\phi, \bar{\omega}), \quad \forall t \geq 0 \quad \hat{\mathbf{Q}}_\mu - a.s. \quad \forall \phi \in C^2(\mathbb{R}). \quad (4.15)$$

Then for each $G \in b\mathcal{F}_t^0$ and $s < t$, by the \mathcal{G}_t martingale property of $\bar{M}_t(\phi)$, (4.3) and (4.15),

$$\mathbf{Q}_{\bar{\mu}, f} \left(\left(M_t^f(\phi) - M_s^f(\phi) \right) G \right) = 0 \quad L_{\mu(\mathbb{R})} - a.a.f.$$

By considering rational and the fact that $C_{P(\mathbb{R})}[0, \infty)$ with local uniform topology is a standard measurable space and taking limits in s and G , we could find a $L_{\mu(\mathbb{R})}$ -null set off which the above holds for all $s < t$ and $G \in \mathcal{F}_s$. That is $\{M_t^f(\phi) : t \geq 0\}$ is an \mathcal{F}_t -martingale under $\mathbf{Q}_{\bar{\mu}, f}$ for $L_{\mu(\mathbb{R})} - a.a.f.$ Take $t_n^f = \inf\{u : f(u) \leq 1/n\}$. According to (4.14) and above arguments, we can deduce that for every $n \geq 1$

$$M_{t \wedge t_n^f}^f(\phi)^2 - \gamma \int_0^{t \wedge t_n^f} (\langle \omega_s, \phi^2 \rangle - \langle \omega_s, \phi \rangle^2) f(s)^{-1} ds - \int_0^{t \wedge t_n^f} ds \int_{\mathbb{R}} \langle \omega_s, h(\cdot - y)\phi' \rangle^2 dy, \quad t \geq 0,$$

is an \mathcal{F}_t -martingale under $\mathbf{Q}_{\bar{\mu}, f}$ for $L_{\mu(\mathbb{R})} - a.a.f.$ Now, consider a countable subset of $C^2(\mathbb{R})$, $C_S(\mathbb{R})$, such that we can approximate any function $\phi \in C^2(\mathbb{R})$ by a sequence $\{\phi_k : k \geq 1\} \subset C_S(\mathbb{R})$ in such a way that not only ϕ but all of its derivatives up to the second order are approximated boundedly and pointwise. Taking limits in $M_t^f(\phi)$ and $\langle M^f(\phi) \rangle_t$ yields the desired conclusion. \square

For $T > 0$, define $(\Omega_{T-}, \mathcal{F}_{T-}) = (C_{P(\mathbb{R})}[0, T), \text{Borel sets})$. $(\hat{\Omega}_{T-}, \hat{\mathcal{F}}_{T-})$ denotes the same space with $M(\mathbb{R})$ in place of $P(\mathbb{R})$. If \mathbf{Q} is a probability on $C_{P(\mathbb{R})}[0, \infty)$, then $\mathbf{Q}|_{T-}$ is defined on $(\Omega_{T-}, \mathcal{F}_{T-})$ by $\mathbf{Q}|_{T-}(A) = \mathbf{Q}(\omega|_{[0, T)} \in A)$. Similarly, one defines $(\Omega_T, \mathcal{F}_T)$, $(\hat{\Omega}_T, \hat{\mathcal{F}}_T)$ and $\mathbf{Q}|_T$. Suppose \mathbf{Q}_μ is the unique probability measure on $C_{P(\mathbb{R})}[0, \infty)$ such that $\{\omega(t), t \geq 0\}$ under \mathbf{Q}_μ is a solution of (\mathcal{L}, μ) -martingale problem. Our main result in this subsection is the following theorem which is analogous to Corollary 4 of [18].

Theorem 4.1. *Suppose that $\{\mu_n\} \subset M(\mathbb{R}) - \{0\}$ satisfy $\bar{\mu}_n \rightarrow \mu$ in $P(\mathbb{R})$.*

(a). *If for each n , there exists a function $f_n \in C_{\mu_n}$ such that for some $T > 0$, $\sup_{0 \leq t \leq S} |f_n - 1| \rightarrow 0$ for $S < T$ as $n \rightarrow \infty$, then*

$$\mathbf{Q}_{\bar{\mu}_n, f_n}|_{T-} \rightarrow \mathbf{Q}_\mu|_{T-} \text{ weakly on } (\Omega_{T-}, \mathcal{F}_{T-}). \quad (4.16)$$

(b). Let $\{A_n\}$ be a sequence of Borel subset of C_+ such that $L_{\mu_n(\mathbb{R})}(A_n) > 0$ for every $n \geq 1$. If for some $T > 0$

$$\sup\{|g(t) - 1| : g \in A_n, t \leq S\} \rightarrow 0 \text{ as } n \rightarrow \infty, \forall S < T,$$

then

$$\hat{\mathbf{Q}}_{\mu_n}(\bar{\omega} \in \cdot | \omega(\mathbb{R}) \in A_n) |_{T-} \rightarrow \mathbf{Q}_\mu |_{T-} \text{ weakly on } (\Omega_{T-}, \mathcal{F}_{T-}).$$

Proof. (a). It suffices to prove

$$\mathbf{Q}_{\bar{\mu}_n, f_n} |_S \rightarrow \mathbf{Q}_\mu |_S \text{ weakly on } (\Omega_S, \mathcal{F}_S).$$

Let $\hat{\mathbb{R}} = \mathbb{R} \cup \{\partial\}$ denote the one-point compactification of \mathbb{R} . Since $\sup_{0 \leq t \leq S} |f_n - 1| \rightarrow 0$ for $S < T$ as $n \rightarrow \infty$, $\inf_{t \leq S} f_n \geq 1/2$ for n larger enough and

$$|\langle M^{f_n}(\phi) \rangle_t - \langle M^{f_n}(\phi) \rangle_s| \leq \frac{\gamma}{2} \|\phi\|_\infty^2 |t - s| + \|\rho\|_\infty \|\phi'\|_\infty^2 |t - s|, \quad \forall s, t \leq S, \mathbf{Q}_{\bar{\mu}_n, f_n} - a.s.$$

By Theorem 2.3 of [19], one can check that $\{\mathbf{Q}_{\bar{\mu}_n, f_n} |_S : n \geq 1\}$ is tight in $P(C_{P(\hat{\mathbb{R}})}[0, S])$. Let \mathbf{Q} be a limit point in $P(C_{P(\hat{\mathbb{R}})}[0, S])$. With abuse of notation, we denote by $\{\omega_s : s \leq S\}$ the coordinate processes of $C_{P(\hat{\mathbb{R}})}[0, S]$. One may use Skorohod representation theorem to see that under \mathbf{Q} ,

$$M_t(\phi) := \langle \omega_t, \phi \rangle - \langle \mu, \phi \rangle - \frac{\rho_\epsilon}{2} \int_0^t \langle \omega_s, \phi'' \rangle ds \quad (4.17)$$

is a continuous martingale starting at 0 for $t \leq S$ and $\phi \in C_\delta^2(\mathbb{R})$ with quadratic variation

$$\gamma \int_0^t (\langle \omega_s, \phi^2 \rangle - \langle \omega_s, \phi \rangle^2) ds + \int_0^t ds \int_{\mathbb{R}} \langle \omega_s, h(\cdot - y) \phi' \rangle^2 dy. \quad (4.18)$$

We claim that

$$\mathbf{Q}\{\omega_t(\{\partial\}) = 0 \text{ for all } t \in [0, S]\} = 1.$$

Consequently, \mathbf{Q} is supported by $C_{P(\mathbb{R})}[0, S]$. For $k \geq 1$, let

$$\phi_k(x) = \begin{cases} \exp\{-\frac{1}{|x|^2 - k^2}\}, & \text{if } |x| > k, \\ 0, & \text{if } |x| \leq k. \end{cases}$$

One can check that $\{\phi_k\} \subset C_\delta^2(\mathbb{R})$ such that $\lim_{|x| \rightarrow \infty} \phi_k(x) = 1$, $\lim_{|x| \rightarrow \infty} \phi_k(x)' = 0$ and $\phi_k(\cdot) \rightarrow 1_{\{\partial\}}(\cdot)$ boundedly and pointwise. $\|\phi_k'\| \rightarrow 0$ and $\|\phi_k''\| \rightarrow 0$ as $k \rightarrow \infty$. By martingale inequality, we have

$$\begin{aligned} & \mathbf{Q}\left\{ \sup_{0 \leq t \leq S} |M_t(\phi_k) - M_t(\phi_j)|^2 \right\} \\ & \leq 4\gamma \int_0^S \mathbf{Q}_\mu \{ \langle \omega_s, (\phi_k - \phi_j)^2 \rangle \} ds + 8\gamma \int_0^S \mathbf{Q}_\mu \{ \langle \omega_s, |\phi_k - \phi_j| \rangle \} ds \\ & \quad + 4 \int_0^S ds \int_{\mathbb{R}} \mathbf{Q} \{ \langle \omega_s, h(z - \cdot) (\phi_k' - \phi_j') \rangle^2 \} dz. \end{aligned}$$

By dominated convergence theorem, $\mathbf{Q}\{\sup_{0 \leq t \leq S} |M_t(\phi_k) - M_t(\phi_j)|^2\} \rightarrow 0$ as $k, j \rightarrow \infty$. Therefore, there exists $M^\partial = (M_t^\partial)_{t \leq S}$ such that for every $t \leq S$,

$$\mathbf{Q}\{|M_t(\phi_k) - M_t^\partial|^2\} \rightarrow 0$$

and (by perhaps passing to a subsequence)

$$\sup_{0 \leq s \leq t} |M_s(\phi_k) - M_s^\partial| \rightarrow 0 \quad \mathbf{Q} - a.s.$$

as $k \rightarrow \infty$. We obtain M^∂ is a continuous martingale. It follows from (4.17) that $M_t^\partial = \omega_t(\{\partial\})$ is a continuous martingale with mean zero. Thus $\mathbf{Q}(\omega_t(\{\partial\})) = 0$. Then the claim follows from the continuity of $\{\omega_t(\{\partial\}) : t \geq 0\}$. Extend \mathbf{Q} to $C_P(\mathbb{R})[0, \infty)$ by setting the conditional distribution of $\{\omega_{t+S} : t \geq 0\}$ given \mathcal{F}_S^0 equal to \mathbf{Q}_{ω_S} . Then $\mathbf{Q} = \mathbf{Q}_\mu$ and so $\mathbf{Q}|_S = \mathbf{Q}_\mu|_S$. We complete the proof of (a).

(b). Let $H : \Omega|_{T-} \rightarrow \mathbb{R}$ be bounded and continuous. Then by Lemma 4.1 and (4.16),

$$\begin{aligned} & |\hat{\mathbf{Q}}_{\mu_n}(H(\bar{\omega})|\omega.(\mathbb{R}) \in A_n) - \mathbf{Q}_\mu(H)| \\ &= |\hat{\mathbf{Q}}_{\mu_n}(H(\bar{\omega})|\omega.(\mathbb{R}) \in A_n \cap C_{\mu_n}) - \mathbf{Q}_\mu(H)| \\ &\leq \left| \int_{A_n \cap C_{\mu_n}} \mathbf{Q}_{\bar{\mu}_n, g}(H) - \mathbf{Q}_\mu(H) dL_{\mu_n(\mathbb{R})} L_{\mu_n(\mathbb{R})}(A_n \cap C_{\mu_n})^{-1} \right| \\ &\leq \sup_{g \in A_n \cap C_{\mu_n}} |\mathbf{Q}_{\bar{\mu}_n, g}(H) - \mathbf{Q}_\mu(H)| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We are done. □

Corollary 4.1. *Suppose that $\{\mu_n\} \subset M(\mathbb{R}) - \{0\}$ satisfy $\bar{\mu}_n \rightarrow \mu$ in $P(\mathbb{R})$. For $T > 0$, let $T_n \rightarrow T$ and $\delta_n \rightarrow 0$ and assume $|\mu_n(\mathbb{R}) - 1| < \delta_n$. Then*

$$(a). \quad \hat{\mathbf{Q}}_{\mu_n}(\bar{\omega} \in \cdot | \sup_{t \leq T_n} |\omega_t(\mathbb{R}) - 1| < \delta_n) \xrightarrow{\text{weakly}} \mathbf{Q}_\mu|_{T-} \quad \text{on } (\Omega_{T-}, \mathcal{F}_{T-});$$

$$(b). \quad \hat{\mathbf{Q}}_{\mu_n}(\hat{\omega} \in \cdot | \sup_{t \leq T_n} |\hat{\omega}_t(\mathbb{R}) - 1| < \delta_n) \xrightarrow{\text{weakly}} \mathbf{Q}_\mu|_{T-} \quad \text{on } (\hat{\Omega}_{T-}, \hat{\mathcal{F}}_{T-}).$$

Proof. Setting

$$A_n = \{g \in C_+ : \sup_{t \leq T_n} |g(t) - 1| < \delta_n\}$$

and Theorem 4.1 yield (a). (b) follows from (a) and the fact that for $S < T$ and n large enough,

$$\hat{\mathbf{Q}}_{\mu_n} \left(\sup_{t \leq S} |\hat{\omega}_t(\mathbb{R})^{-1} - 1| < \frac{\delta_n}{1 - \delta_n} \middle| A_n \right) = 1.$$

□

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APPENDIX

A Random selections of stochastic integrals

Lemma A.1. *Let $W(dsdy)$ be a space-time white noise on $[0, \infty) \times \mathbb{R}$ based on Lebesgue measure measure. Let $\{X_i(t), t \geq 0, i = 1, 2, \dots\}$ be a sequence of real valued predictable stochastic processes. Let $h(x, y)$ be a measurable function on $\mathbb{R} \times \mathbb{R}$. Define stochastic integrals*

$$Y_i(t) := \int_0^t \int_{\mathbb{R}} h(X_i(s), y)W(dsdy), \quad t \geq 0, \quad i \geq 1.$$

Suppose π is a random variable taking values in $\{1, 2, \dots\}$, independent of $\{X_i, i = 1, 2, \dots\}$ and W . Then

$$Y_\pi(t) = \int_0^t \int_{\mathbb{R}} h(X_\pi(s), y)W(dsdy), \quad t \geq 0.$$

Proof. If h is a simple function, the desired conclusion is obvious. For general result, one can consider the L^2 approximation and Ito's isometry; see Theorem 2.2.5 of [21]. \square

B Stochastic flow of diffeomorphism

In this part, we consider the following stochastic differential equation

$$\xi_t = x + \int_s^t \int_{\mathbb{R}} h(y - \xi(s))W(dsdy), \quad x \in \mathbb{R}, \quad t \geq s, \quad (2.1)$$

where $W(dsdy)$ is a space-time white noise on $[0, \infty) \times \mathbb{R}$ based on Lebesgue measure measure. The existence and pathwise uniqueness for (2.1) have been proved in [2].

Lemma B.1. *Suppose $h \in C^2(\mathbb{R})$. There is a modification of the solution, denoted by $\xi_{s,t}(x)$, such that almost surely*

- (1) $\xi_{s,t}(x, \omega)$ is continuous in (s, t, x) and satisfies $\lim_{t \downarrow s} \xi_{s,t}(x, \omega) = x$;
- (2) $\xi_{s,t+u}(x, \omega) = \xi_{t,t+u}(\xi_{s,t}(x, \omega), \omega)$ is satisfied for all $s < t$ and $u > 0$;
- (3) the map $\xi_{s,t}(\cdot, \omega) : \mathbb{R} \rightarrow \mathbb{R}$ is an onto homeomorphism for all $s < t$;
- (4) the map $\xi_{s,t}(\cdot, \omega) : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 -diffeomorphism for all $s < t$.

Proof. The argument is exactly similar to that in Chapter 2 of [15]. We omit it here and left it to interested readers. \square

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