Discontinuous Superprocesses with Dependent Spatial Motion [1]

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Abstract

We construct a class of discontinuous superprocesses with dependent spatial motion and general branching mechanism. The process arises as the weak limit of critical interacting-branching particle systems where the spatial motions of the particles are not independent. The main work is to solve the martingale problem. When we turn to the uniqueness of the process, we generalize the localization method introduced by [D.W. Stroock, Diffusion processes associated with Lévy generators, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 32(1975) 209–244] to the measure-valued context. As for existence, we use particle system approximation and a perturbation method. This work generalizes the model introduced in [D.A. Dawson, Z. Li, H. Wang, Superprocesses with dependent spatial motion and general branching densities, Electron. J. Probab. 6(2001), no.25, 33 pp. (electronic)] where quadratic branching mechanism was considered. We also investigate some properties of the process.

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Abbreviated Title: Discontinuous superprocesses

1 Introduction

Notation: For reader's convenience, we introduce here our main notation. Let $\hat{\mathbb{R}}$ denote the one-point compactification of $\mathbb{R}$. Let $\hat{\mathbb{R}}^n$ denote the n-fold Cartesian product of $\hat{\mathbb{R}}$. Let $M(\mathbb{R})$ denote the space of finite measure endowed with topological of weak convergence. We denote by λ^n the Lebesgue measure on $\mathbb{R}^n$. Given a topological space E, let $\mathfrak{B}(E)$ denote borel σ-algebra on E. Let $B(E)$ denote the set of bounded measurable functions on E and let $C(E)$ denote its subset comprising of bounded continuous functions. Let $\hat{C}(\mathbb{R}^n)$ be the space of continuous functions on $\mathbb{R}^n$ which vanish at infinity and let $C_c^\infty(\mathbb{R}^n)$ be functions with compact support and bounded continuous derivatives of any order. Let $C^2(\mathbb{R}^n)$ denote the set of functions in $C(\mathbb{R}^n)$ which is twice continuously differential functions with bounded derivatives up to the second order. Let $C_c^2(\mathbb{R}^n)$ denote the set of functions in $C^2(\mathbb{R}^n)$ with compact support. Let $\hat{C}^2(\mathbb{R}^n)$ be the subset of $C^2(\mathbb{R}^n)$ of functions that together with their derivatives up to the second order

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vanish at infinity.
Let

$$C^2_\partial(\mathbb{R}^n) = \{f + c : c \in \mathbb{R} \text{ and } f \in \hat{C}^2(\mathbb{R}^n)\}$$

and

$$C^2_0(\mathbb{R}^n) = \{f : f \in C^2_\partial(\mathbb{R}^n) \text{ and } (1+|x|^2)D^\alpha f(x) \in \hat{C}(\mathbb{R}^n),\ \alpha = 1, 2\},$$

where $D^1 f = \sum_{i=1}^n |\partial f/\partial x_i|$ and $D^2 f = \sum_{i,j=1}^n |\partial^2 f/\partial x_i \partial x_j|$. We use the superscript "+" to denote the subsets of non-negative elements of the function spaces, and "++" is used to denote the subsets of non-negative elements bounded away from zero, e.g., $B(\mathbb{R}^n)^+$, $C(\mathbb{R}^n)^{++}$. Let f^i denote the first order partial differential derivatives of the function $f(x_1, \cdots, x_n)$ with respect to x_i and let f^{ij} denote the second order partial differential derivatives of the function $f(x_1, \cdots, x_n)$ with respect to x_i and x_j. We denote by $C([0,\infty), E)$ the space of continuous paths taking values in E. Let $D([0,\infty), E)$ denote the Skorokhod space of càdlàg paths taking values in E. For $f \in C(\mathbb{R})$ and $\mu \in M(\mathbb{R})$ we shall write $\langle f, \mu\rangle$ for $\int f d\mu$.

A class of *superprocesses with dependent spatial motion* (SDSM) over the real line $\mathbb{R}$ were introduced and constructed in [18, 19]. A generalization of the model was then given in [4]. We first briefly describe the model constructed in [4]. Suppose that $c \in C^2(\mathbb{R})$ and $h \in C(\mathbb{R})$ is square-integrable. Let

$$\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \tag{1.1}$$

and $a(x) = c(x)^2 + \rho(0)$ for $x \in \mathbb{R}$. We assume in addition that $\rho \in C^2(\mathbb{R})$ and $|c|$ is bounded away from zero. Let σ be a nonnegative function in $C^2(\mathbb{R})$ and can be extended continuously to $\hat{\mathbb{R}}$. Given a finite measure μ on $\mathbb{R}$, the SDSM with parameters (a, ρ, σ) and initial state μ is the unique solution of the $(\mathcal{L}, \mu)$-martingale problem, where

$$\mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu), \tag{1.2}$$

$$\begin{aligned}\mathcal{A}F(\mu) := &\frac{1}{2}\int_{\mathbb{R}} a(x)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \\ &+\frac{1}{2}\int_{\mathbb{R}^2} \rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy),\end{aligned} \tag{1.3}$$

$$\mathcal{B}F(\mu) := \frac{1}{2}\int_{\mathbb{R}} \sigma(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx), \tag{1.4}$$

for some bounded continuous functions $F(\mu)$ on $M(\mathbb{R})$. The variational derivative is defined by

$$\frac{\delta F(\mu)}{\delta\mu(x)} = \lim_{r\to 0^+}\frac{1}{r}[F(\mu + r\delta_x) - F(\mu)], \quad x \in \mathbb{R}, \tag{1.5}$$

if the limit exists and $\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y)$ is defined in the same way with F replaced by $(\delta F/\delta\mu(y))$ on the right hand side. Clearly, the SDSM reduces to a usual critical Dawson-Watanabe superprocess if $h(\cdot) \equiv 0$ (see [2]). A general SDSM arises as the weak limit of critical interacting-branching particle systems. In contrast to the usual branching particle system, the spatial motions of the particles in the interacting-branching particle system are *not* independent. The spatial motions of the particles can be described as follows. Suppose that $\{W(t,x) : x \in \mathbb{R}, t \geq 0\}$ is space-time white noise based on Lebesgue measure, the common noise, and $\{B_i(t) : t \geq 0, i = 1, 2, \cdots\}$ is a family of independent standard Brownian motions, the individual noises, which are independent of $\{W(t,x) : x \in \mathbb{R}\}$. The migration of a particle in the approximating system with label i is defined by the stochastic equations

$$dx_i(t) = c(x_i(t))dB_i(t) + \int_{\mathbb{R}} h(y - x_i(t))W(dt, dy), \quad t \geq 0, \quad i = 1, 2, \cdots, \tag{1.6}$$

where $W(dt, dy)$ denotes the time-space stochastic integral relative to $\{W_t(B)\}$. For each integer $m \geq 1$, $\{(x_1(t), \cdots, x_m(t)) : t \geq 0\}$ is an m-dimensional diffusion process which is generated by the differential operator

$$G^m := \frac{1}{2}\sum_{i=1}^{m} a(x_i)\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i,j=1, i\neq j}^{m} \rho(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{1.7}$$

In particular, $\{x_i(t) : t \geq 0\}$ is a one-dimensional diffusion process with generator $G := (a(x)/2)\Delta$. Because of the exchangeability, a diffusion process generated by G^m can be regarded as an interacting particle system or a measure-valued process. Heuristically, $a(\cdot)$ represents the speed of the particles and $\rho(\cdot)$ describes the interaction between them. The diffusion process generated by $\mathcal{A}$ arises as the high density limit of a sequence of interacting particle systems described by (1.6); see Wang [18, 19] and Dawson *et al* [4]. There are at least two different ways to look at the SDSM. One is as a superprocess in random environment and the other as an extension of models of the motion of the mass by stochastic flows (see [13]). Some other related models were introduced and studied in Skoulakis and Adler [15]. The SDSM possesses properties very different from those of the usual Dawson-Watanabe superprocess. For example, a Dawson-Watanabe superprocess in $M(\mathbb{R})$ is usually absolutely continuous whereas the SDSM with $c(\cdot) \equiv 0$ is purely atomic; see Konno and Shiga [10] and [3, 20], respectively.

To best of our knowledge, in all of the work which considered the SDSM and related models only continuous processes have been introduced and studied. In this paper, we construct a class of discontinuous superprocesses with dependent spatial motion. A modification of the above martingale problem is to replace operator $\mathcal{B}$ in (1.2) by

$$\begin{aligned}\mathcal{B}F(\mu) = {} & \frac{1}{2}\int_{\mathbb{R}} \sigma(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx) \\ & + \int_{\mathbb{R}} \mu(dx)\int_0^\infty \left(F(\mu + \xi\delta_x) - F(\mu) - \frac{\delta F(\mu)}{\delta\mu(x)}\xi\right)\gamma(x, d\xi), \end{aligned}\tag{1.8}$$

whose coefficients satisfy:

(i) $\sigma \in C^2_\partial(\mathbb{R})^+$,

(ii) $\gamma(x, d\xi)$ is a kernel from $\mathbb{R}$ to $(0, +\infty)$ such that $\sup_x[\int_0^{+\infty} \xi \wedge \xi^2\gamma(x, d\xi)] < +\infty$,

(iii) $\int_\Gamma \xi \wedge \xi^2\gamma(x, d\xi) \in C^2_\partial(\mathbb{R})$ for each $\Gamma \in \mathfrak{B}((0, \infty))$.

A Markov process generated by $\mathcal{L}$ is a measure-valued branching process with branching mechanism given by

$$\Psi(x, z) := \frac{1}{2}\sigma(x)z^2 + \int_0^\infty (e^{-z\xi} - 1 + z\xi)\gamma(x, d\xi).$$

This process is naturally called a *superprocess with dependent spatial motion (SDSM)* with parameters (a, ρ, Ψ). This modification is related to the recent work of Dawson *et al* [4], where it was assumed that $\gamma(x, d\xi) = 0$. Though our model is an extension of the model introduced in Wang [18, 19] and Dawson *et al* [4], the construction of our model differ from theirs. We describe our approach to the construction of our model in the following.

The main work of this paper is to solve the $(\mathcal{L}, \mu)$-martingale problem. As for uniqueness, following the idea of Stroock [16] a localization procedure is developed. Therefore, we do not consider the $(\mathcal{L}, \mu)$-martingale problem directly. Instead, we will first solve the $(\mathcal{L}', \mu)$-martingale problem, where

$$\mathcal{L}'F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}'F(\mu), \tag{1.9}$$

$$\begin{aligned}\mathcal{B}' :=& \frac{1}{2}\int_{\mathbb{R}} \sigma(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx) - \int_{\mathbb{R}} \mu(dx)\int_l^\infty \frac{\delta F(\mu)}{\delta\mu(x)}\xi\gamma(x, d\xi) \\ &+ \int_{\mathbb{R}} \mu(dx)\int_0^l \left(F(\mu + \xi\delta_x) - F(\mu) - \frac{\delta F(\mu)}{\delta\mu(x)}\xi\right)\gamma(x, d\xi).\end{aligned} \tag{1.10}$$

and we make the convention that

$$\int_0^l = \int_{(0,l)} \qquad \text{and} \qquad \int_l^\infty = \int_{[l,\infty)}$$

for $0 < l < \infty$. We regard the $(\mathcal{L}', \mu)$-martingale problem as the 'killed' martingale problem. We shall see that the Markov process associated with the 'killed' martingale problem also arises as high density limit of a sequence of interacting-branching particle system and it is an SDSM with branching mechanism given by

$$\Psi_0(x, z) := \frac{1}{2}\sigma(x)z^2 + \int_l^\infty \xi\gamma(x, d\xi)z + \int_0^l (e^{-z\xi} - 1 + z\xi)\gamma(x, d\xi).$$

It is easy to see from the branching mechanism that the process is a subcritical branching process with all 'big' jumps such that the jump size is larger than l been 'killed'. We will use duality method to show the uniqueness of the 'killed' martingale problem. We shall construct a dual process and show its connection with the solutions of the 'killed' martingale problem which gives the uniqueness. When we establish the dual relationship, we point out that there exists a gap in the proof of establishing the dual relationship in [4]; see Remark 2.2 in Section 2 of this paper for details. Then a localization argument is developed to show that if the $(\mathcal{L}', \mu)$ martingale problem is well-posed then uniqueness holds for the $(\mathcal{L}, \mu)$-martingale problem. The argument consists of three parts.

In the first part, we show that each solution of the $(\mathcal{L}, \mu)$-martingale problem , say X, behaves the same as the solution of the killed martingale problem until it has a 'big jump' whose jump size is larger than l. Intuitively, one can think of the branching particle system as follows. In the branching particle system corresponding to the $(\mathcal{L}, \mu)$-martingale problem, if a particle dies and it leaves behind a large number of offsprings, say more than 500, which always be regarded as a 'big jump' event, we kill all its offsprings. Then we get a new branching particle system and before the jump event happens the two systems are the same. The evolution of the new particle system represents the behavior of the solution to the 'killed' martingale problem. It is clear that if the original branching particle system is a critical system, the new particle system is a subcritical branching system. Since the 'killed' martingale problem is well-posed, X is uniquely determined before it has a 'big jump'. Next, we show that when a 'big jump' event happens, the jump size is uniquely determined. This conclusion is not surprising either. Given a branching mechanism, in a branching particle system, when a particle dies, the distribution of its offspring number is uniquely determined by the position of the particle itself (we assume that the branching mechanism is independent of time). Thus we can find a predictable representation for the jump size. According to the argument in the first part, we see the jump size is uniquely determined. At last, we can prove by induction that the distribution of X is uniquely determined, since after the first 'big jump' event happens, X also behaves the same as the solution of the 'killed' martingale problem until the second 'big jump' event happens. Before we use the localization procedure, we follow an argument taken from El-Karoui and Roelly-Coppoletta [7] to decompose each solution of the $(\mathcal{L}, \mu)$-martingale problem into a continuous part and a purely discontinuous part. We will use this argument again when we show the existence of solutions to the $(\mathcal{L}, \mu)$-martingale problem; see next two paragraphs.

When we turn to the existence we also first consider the existence of the 'killed' martingale problem. Although the solution of the 'killed' martingale problem is also an SDSM which arises as high density limit of a sequence of interacting-branching particle systems, in order to deduce the martingale formula the techniques developed in Wang [18, 19] and Dawson *et al* [4] can not be used directly because of the third item in the branching mechanism Ψ_0. We will use the martingale decomposition and special semi-martingale's representation to get the desired result. Our approach is stimulated by El-Karoui and Roelly-Coppoletta [7], who considered the martingale problem of the usual Dawson-Watanabe superprocess. We briefly describe the main idea in next paragraph.

First, a sequence of subcritical branching particle systems is constructed. Let $X = (X_t)_{t\geq 0}$ denote a limit of the particle systems. Then we derive the special semi-martingale property of $\{\exp\{-\langle \phi, X_t\rangle\} : t \geq 0\}$ with ϕ bounded away from zero by using particle system approximation, and obtain a representation for this semi-martingale. This approach is *different* from that of [7], where log-laplace equation was used to deduce the semi-martingale property. Next, we consider an integer-valued random measure $N(ds, d\nu) = \sum_{s>0} 1_{\{\Delta X_s \neq 0\}} \delta_{(s,\Delta X_s)}(ds, d\nu)$ and by an approximation procedure we can show

$$M_t(\phi) := \langle \phi, X_t\rangle - \langle \phi, X_0\rangle - \frac{1}{2}\int_0^t \langle a\phi'', X_s\rangle ds + \int_0^t ds \langle \int_l^\infty \xi\gamma(\cdot, d\xi)\phi, X_s\rangle \tag{1.11}$$

is square-integrable martingale which can be decompose into a continuous martingale $\{M_t^c(\phi) : t \geq 0\}$ and a purely discontinuous martingale $\{M_t^d(\phi) : t \geq 0\}$. We have

$$\langle \phi, X_t\rangle = \langle \phi, X_0\rangle + \frac{1}{2}\int_0^t \langle a\phi'', X_s\rangle ds + M_t^c(\phi) + M_t^d(\phi) - \int_0^t ds \langle \int_l^\infty \xi\gamma(\cdot, d\xi)\phi, X_s\rangle, \tag{1.12}$$

and $M^d(\phi)$ can be represented as a stochastic integral with respect to the corresponding martingale measure of $N(ds, d\nu)$. This argument is also different from the argument of [7], where according to the semi-martingale property of $\{\exp\{-\langle \phi, X_t\rangle\} : t \geq 0\}$ only semi-martingale property of $\{\langle \phi, X_t\rangle : t \geq 0\}$ with ϕ bounded away from zero was derived. By the martingale decomposition (1.12) we can obtain another representation for semi-martingale $\{\exp\{-\langle \phi, X_t\rangle\} : t \geq 0\}$. By identifying two representations for $\{\exp\{-\langle \phi, X_t\rangle\} : t \geq 0\}$ mentioned above, we know the explicit form of the quadratic variation process of $\{M_t^c(\phi) : t \geq 0\}$ and the compensator of the random measure $N(ds, d\nu)$. Then we can deduce X satisfies the martingale formula for the $(\mathcal{L}', \mu)$-martingale problem. At last by a perturbation method we show the existence of the $(\mathcal{L}, \mu)$-martingale problem.

The remainder of the paper is organized as follows. In Section 2, we first introduce the 'killed' martingale problem and define a dual process and investigate its connection to the solutions of the 'killed' martingale problem which gives the uniqueness of the 'killed' martingale problem. Then we deduce that the uniqueness holds for the $(\mathcal{L},\mu)$-martingale problem. In Section 3, we first give a formulation of the system of branching particles with dependent spatial motion and obtain the existence of the solution of the 'killed' martingale problem by taking high density limit of particle systems. Then a perturbation argument is used to show the existence of the $(\mathcal{L},\mu)$-martingale problem. We compute the first and second order moment formulas of the process in Section 4.

Remark 1.1 *By Theorem 8.2.5 of [6], the closure of* $\{(f, G^m f) : f \in C_c^\infty(\mathbb{R}^m)\}$ *which we still denote by* G^m *is single-valued and generates a Feller semigroup* $(P_t^m)_{t\geq 0}$ *on* $\hat{C}(\mathbb{R}^m)$. *Note that this semigroup is given by a transition function and can therefore be extended to all of* $B(\mathbb{R}^m)$. *We also have that* $(1,0)$ *is in the bp-closure of* G^m.

2 Uniqueness

2.1 Killed martingale problem

In this section, we first introduce the *killed martingale problem* for the SDSM and show the uniqueness holds for the killed martingale problem.

Definition 2.1 *Let* $\mathcal{D}(\mathcal{L}) = \bigcup_{m=0}^\infty \{F(\mu) = f(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle),\ f\in C_0^2(\mathbb{R}^m),\ \{\phi_i\}\subset C_c^2(\mathbb{R})^+\}$. *For* $\mu \in M(\mathbb{R})$ *and an* $M(\mathbb{R})$*-valued càdlàg process* $\{X_t : t \geq 0\}$, *we say* X *is a solution of the* $(\mathcal{L},\mu)$*-martingale problem if* $X_0 = \mu$ *and*

$$F(X_t) - F(X_0) - \int_0^t \mathcal{L}F(X_s)ds, \quad t \geq 0, \tag{2.1}$$

is a local martingale for each $F \in \mathcal{D}(\mathcal{L})$ *and for* $l > 1$, *we say* X *is a solution of the* $(\mathcal{L}',\mu)$*-martingale problem if* $X_0 = \mu$ *and*

$$F(X_t) - F(X_0) - \int_0^t \mathcal{L}'F(X_s)ds, \quad t \geq 0, \tag{2.2}$$

is a local martingale for each $F \in \mathcal{D}(\mathcal{L})$.

Let $\mathcal{D}_0(\mathcal{L}) = \bigcup_{m=0}^\infty \{f(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle),\ f\in C_0^2(\mathbb{R}^m),\ \{\phi_i\}\subset C^2(\mathbb{R})^{++}\}$. Note that for $F(\mu) \in \mathcal{D}_0(\mathcal{L}) \cup \mathcal{D}(\mathcal{L})$,

$$\begin{aligned}\mathcal{A}F(\mu) &= \frac{1}{2}\sum_{j=1}^m f^i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)\langle a\phi_i'',\mu\rangle \\ &\quad + \frac{1}{2}\sum_{i,j=1}^m f^{ij}(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)\int_{\mathbb{R}^2}\rho(x-y)\phi_i'(x)\phi_j'(y)\mu^2(dxdy),\end{aligned} \tag{2.3}$$

$$\mathcal{B}F(\mu) = \frac{1}{2}\sum_{i,j=1}^m f^{ij}(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)\langle\sigma\phi_i\phi_j,\mu\rangle$$

$$+\int_{\mathbb{R}}\mu(dx)\int_0^\infty\{f(\langle\phi_1,\mu\rangle+\xi\phi_1(x),\cdots,\langle\phi_m,\mu\rangle+\xi\phi_m(x))$$
$$-f(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)-\xi\sum_{i=1}^m f^i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)\phi_i(x)\}\gamma(x,d\xi) \quad (2.4)$$

and

$$\begin{aligned}\mathcal{B}'F(\mu) &= \frac{1}{2}\sum_{i,j=1}^m f^{ij}(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)\langle\sigma\phi_i\phi_j,\mu\rangle\\ &\quad-\int_{\mathbb{R}}\mu(dx)\int_l^\infty\xi\gamma(x,d\xi)\sum_{i=1}^m f^i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)\phi_i(x)\\ &\quad+\int_{\mathbb{R}}\mu(dx)\int_0^l\{f(\langle\phi_1,\mu\rangle+\xi\phi_1(x),\cdots,\langle\phi_m,\mu\rangle+\xi\phi_m(x))\\ &\qquad-f(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)-\xi\sum_{i=1}^m f^i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)\phi_i(x)\}\gamma(x,d\xi).\end{aligned} \quad (2.5)$$

Thus for every $F\in\mathcal{D}_0(\mathcal{L})$, both $\mathcal{L}F$ and $\mathcal{L}'F$ are bounded functions on $M(\mathbb{R})$.

Remark 2.1 *Let $h\in C_c^2(\mathbb{R}^m)$ satisfy $1_{B(0,1)}\le h\le 1_{B(0,2)}$ and $h_k(x)=h(x/k)\in C_c^2(\mathbb{R}^m)$. Then for each $\phi\in C^2(\mathbb{R})^{++}$, it can be approximated by $\{\phi h_k\}\subset C_c^2(\mathbb{R})^+$ in such a way that not only ϕ but its derivatives up to second order are approximated boundedly and pointwise. Therefore when X is a solution of $(\mathcal{L},\mu)$-martingale problem (or $(\mathcal{L}',\mu)$-martingale problem), (2.1) (or (2.2)) is a martingale for $F\in\mathcal{D}_0(\mathcal{L})$. On the other hand, for every $\phi\in C_c^2(\mathbb{R})^+$, we can approximate ϕ by $\{\phi+1/n\}\subset C_\partial^2(\mathbb{R})^{++}\subset C^2(\mathbb{R})^{++}$ in the same way. Thus if (2.1) (or (2.2)) is a martingale for every $F\in\mathcal{D}_0(\mathcal{L})$, it is a local martingale for every $F\in\mathcal{D}(\mathcal{L})$. We shall see that any solution of the $(\mathcal{L}',\mu)$-martingale problem has bounded moment of any order. Thus if X is a solution of the $(\mathcal{L}',\mu)$-martingale problem, (2.2) is a martingale for every $F\in\mathcal{D}_0(\mathcal{L})\cup\mathcal{D}(\mathcal{L})$.*

We shall see that the Markov process associated with $(\mathcal{L}',\mu)$-martingale problem is a subcritical measure-valued branching process with branching mechanism given by

$$\Psi_0(x,z):=\frac{1}{2}\sigma(x)z^2+\int_l^\infty\xi\gamma(x,d\xi)z+\int_0^l(e^{-z\xi}-1+z\xi)\gamma(x,d\xi).$$

For $i\ge 2$, let $\sigma_i:=\sup_x[\int_0^l\xi^i\gamma(x,d\xi)]$. We first show that each solution of the $(\mathcal{L}',\mu)$-martingale problem has bounded moment of any order.

Lemma 2.1 *Suppose that $\mathbf{Q}'_\mu$ is a probability measure on $D([0,+\infty),M(\mathbb{R}))$ such that under $\mathbf{Q}'_\mu$ $\omega_0=\mu$ a.s. and $\{\omega_t:t\ge 0\}$ is a solution of the $(\mathcal{L}',\mu)$-martingale problem. Then for $n\ge 1,t\ge 0$ we have*

$$\begin{aligned}\mathbf{Q}'_\mu\{\langle 1,\omega_t\rangle^n\}&\le\sigma_2t/2+\langle 1,\mu\rangle^n+C_1(n,\gamma)\int_0^t\mathbf{Q}'_\mu\{\langle 1,\omega_s\rangle\}ds\\ &\quad+C_2(n,\sigma,\gamma)\int_0^t\mathbf{Q}'_\mu\{\langle 1,\omega_s\rangle^{n-1}\}ds+C_3(n,\gamma)\int_0^t\mathbf{Q}'_\mu\{\langle 1,\omega_s\rangle^n\}ds,\end{aligned} \quad (2.6)$$

where $C_1(n,\gamma)$, $C_2(n,\sigma,\gamma)$ and $C_3(n,\gamma)$ are constants which depend on n, σ and γ.

Proof. Let $n \geq 1$ be fixed. For any $k \geq 1$, take $f_k \in C_0^2(\mathbb{R})$ such that $f_k(z) = z^n$ for $0 \leq z \leq k$ and $|f_k'(z)| \leq nz^{n-1}$, $f_k''(z) \leq n^2 z^{n-2}$ for all $z > k$. Let $F_k(\mu) = f_k(\langle 1, \mu\rangle)$. Then $\mathcal{A}F_k(\mu) = 0$ and

$$\begin{aligned}
\mathcal{B}'F_k(\mu) &= \frac{1}{2}f_k''(\langle 1,\mu\rangle)\langle\sigma,\mu\rangle - \int_{\mathbb{R}}\mu(dx)\int_l^\infty \xi f_k'(\langle 1,\mu\rangle)\gamma(x,d\xi) \\
&\quad + \int_{\mathbb{R}}\mu(dx)\int_0^l \{f_k(\langle 1,\mu\rangle+\xi) - f_n(\langle 1,\mu\rangle) - \xi f_k'(\langle 1,\mu\rangle)\}\gamma(x,d\xi) \\
&\leq \frac{1}{2}n^2||\sigma||\langle 1,\mu\rangle^{n-1} + \sup_x[\int_1^\infty \xi\gamma(x,d\xi)]n\langle 1,\mu\rangle^n \\
&\quad + \int_{\mathbb{R}}\mu(dx)\int_0^l \frac{1}{2}n^2(\langle 1,\mu\rangle+\xi)^{n-2}\xi^2\gamma(x,d\xi).
\end{aligned}$$

Then we deduce that

$$\mathcal{B}'F_k(\mu) \leq C_1(n,\gamma)\langle 1,\mu\rangle + C_2(n,\sigma,\gamma)\langle 1,\mu\rangle^{n-1} + n\sup_x[\int_1^\infty \xi\gamma(x,d\xi)]\langle 1,\mu\rangle^n,$$

where $C_2(n,\sigma,\gamma) = n^2||\sigma||/2 + \frac{1}{2}\sigma_2 n^2 2^{(n-3)\vee 0}$ and

$$C_1(n,\gamma) = \begin{cases} n^2 2^{(n-3)\vee 0}\sigma_n/2, & n \geq 2, \\ 0, & n = 1. \end{cases}$$

We have used the Taylor's expansion and elementary inequality

$$(c+d)^\beta \leq 2^{(\beta-1)\vee 0}(c^\beta + d^\beta), \text{ for all } \beta, c, d \geq 0.$$

Note that $F_k \in \mathcal{D}_0(\mathcal{L})$. Thus

$$F_k(\omega_t) - F_k(\omega_0) - \int_0^t \mathcal{L}'F_k(\omega_s)ds, \ t \geq 0,$$

is a martingale. We get

$$\begin{aligned}
\mathbf{Q}_\mu' f_k(\langle 1,\omega_t\rangle) &\leq f_k(\langle 1,\mu\rangle) + C_1(n,\gamma)\int_0^t \mathbf{Q}_\mu'(\langle 1,\omega_s\rangle)ds \\
&\quad + C_2(n,\sigma,\gamma)\int_0^t \mathbf{Q}_\mu'(\langle 1,\omega_s\rangle^{n-1})ds + C_3(n,\gamma)\int_0^t \mathbf{Q}_\mu'(\langle 1,\omega_s\rangle^n)ds,
\end{aligned}$$

where $C_3(n,\gamma) = n\sup_x[\int_1^\infty \xi\gamma(x,d\xi)]$. Now inequality (2.6) follows from Fatou's Lemma. □

Observe that, if $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ for $f \in C^2(\mathbb{R}^m)$, then

$$\begin{aligned}
\mathcal{A}F_{m,f}(\mu) &= \frac{1}{2}\int_{\mathbb{R}^m}\sum_{i=1}^m a(x_i)f^{ii}(x_1,\cdots,x_m)\mu^m(dx_1,\cdots,dx_m) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^m}\sum_{i,j=1,i\neq j}^m \rho(x_i-x_j)f^{ij}(x_1,\cdots,x_m)\mu^m(dx_1,\cdots,dx_m) \\
&= F_{m,G^m f}(\mu),
\end{aligned} \tag{2.7}$$

and

$$\mathcal{B}'F_{m,f}(\mu) = \frac{1}{2}\sum_{i,j=1,i\neq j}^m \int_{\mathbb{R}^{m-1}}\Psi_{ij}f(x_1,\cdots,x_{m-1})\mu^{m-1}(dx_1,\cdots,dx_{m-1})$$

$$+\sum_{a=2}^{m}\int_{\mathbb{R}^{m-a+1}}\sum_{\{a\}}\Phi_{i_1,\cdots,i_a}f(x_1,\cdots,x_{m-a+1})\mu^{m-a+1}(dx_1,\cdots,dx_{m-a+1})$$
$$-\sum_{i=1}^{m}\int_{\mathbb{R}^m}\int_l^\infty \xi\gamma(x_i,d\xi)f(x_1,\cdots,x_m)\mu^m(dx_1,\cdots,dx_m), \tag{2.8}$$

where $\{a\}=\{1\le i_1<i_2<\cdots<i_a\le m\}$. Ψ_{ij} denotes the operator from $B(\mathbb{R}^m)$ to $B(\mathbb{R}^{m-1})$ defined by

$$\Psi_{ij}f(x_1,\cdots,x_{m-1})=\sigma(x_{m-1})f(x_1,\cdots,x_{m-1},\cdots,x_{m-1},\cdots,x_{m-2}), \tag{2.9}$$

where x_{m-1} is in the places of the ith and the jth variables of f on the right hand side and $\Phi_{i_1,\cdots,i_a}$ denotes the operator from $B(\mathbb{R}^m)$ to $B(\mathbb{R}^{m-a+1})$ defined by

$$\Phi_{i_i,\cdots,i_a}f(x_1,\cdots,x_{m-a+1})=f(x_1,\cdots,x_{m-a+1},\cdots,x_{m-a+1},\cdots,x_{m-a})\int_0^l\xi^a\gamma(x_{m-a+1},d\xi), \tag{2.10}$$

where x_{m-a+1} is in the places of the i_1th, i_2th, $\cdots$, i_ath variables of f on the right hand side. For $x=(x_1,\cdots,x_m)\in\mathbb{R}^m$, let $b(x)=\sum_{i=1}^m\int_l^\infty\xi\gamma(x_i,d\xi)$. It follows that

$$\begin{aligned}\mathcal{L}'F_{m,f}(\mu) &= F_{m,G^mf}(\mu)-F_{m,bf}(\mu)\\ &\quad+\frac{1}{2}\sum_{i,j=1,i\neq j}^{m}F_{m-1,\Psi_{ij}f}(\mu)+\sum_{a=2}^{m}\sum_{\{a\}}F_{m-a+1,\Phi_{i_1,\cdots,i_a}f}(\mu).\end{aligned} \tag{2.11}$$

Lemma 2.2 *Suppose that $\mathbf{Q}'$ is a probability measure on $D([0,+\infty),M(\mathbb{R}))$ such that under $\mathbf{Q}'$ $\{\omega_t:t\ge0\}$ is a solution of the $(\mathcal{L}',\mu)$-martingale problem. Then*

$$F(\omega_t)-F(\omega_0)-\int_0^t\mathcal{L}'F(\omega_s)ds,\ t\ge0, \tag{2.12}$$

under $\mathbf{Q}'$ is a martingale for each $F(\mu)=F_{m,f}(\mu)=\langle f,\mu^m\rangle$ with $f\in C^2(\mathbb{R}^m)$.

Proof. For any $k\ge1$, take $f_k\in C_0^2(\mathbb{R}^m)$ such that for $0\le x_i^2\le k,\ 1\le i\le m$,

$$f_k(x_1,\cdots,x_m)=\prod_{i=1}^m x_i.$$

For $\{\phi_i\}\subset C^2(\mathbb{R})^{++}$, let $F_k(\mu)=f_k(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)$. Then $\lim_{k\to\infty}F_k(\mu)=F_{m,f}(\mu)$ for all $\mu\in M(\mathbb{R})$ and if for every $1\le i\le m$, $0\le\langle\phi_i,\mu\rangle^2+l^2||\phi_i||^2\le k$, we have

$$\mathcal{L}'F_k(\mu)=\mathcal{L}'F_{m,f}(\mu).$$

Introduce a sequence stopping times

$$\tau_k:=\inf\{t\ge0,\ \text{there exists } i\in\{1,\cdots m\}\text{ such that }\langle\phi_i,\omega_t\rangle^2+l^2||\phi_i||^2\ge k\}\wedge k.$$

Then $\tau_k\to\infty$ as $k\to\infty$. Suppose that $\{H_i\}_{i=1}^n\subset C(M(\mathbb{R}))$ and $0\le t_1<\cdots<t_n<t_{n+1}$. By Lemma 2.1 and the dominated convergence theorem we deduce that

$$\mathbf{Q}'\Big\{\big[F_{m,f}(\omega_{t_{n+1}})-F_{m,f}(\omega_{t_n})-\int_{t_n}^{t_{n+1}}\mathcal{L}'F_{m,f}(\omega_s)ds\big]\prod_{i=1}^n H(\omega_{t_i})\Big\}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \mathbf{Q}'\Big\{\big[F_k(\omega_{t_{n+1}}) - F_k(\omega_{t_n}) - \int_{t_n}^{t_{n+1}} \mathcal{L}'F_k(\omega_s)ds\big] \prod_{i=1}^n H(\omega_{t_i})\Big\} \\
&\quad + \lim_{k\to\infty} \mathbf{Q}'\Big\{\big[\int_{t_n}^{t_{n+1}} \mathcal{L}'F_k(\omega_s)1_{\{\tau_k\le s\}}ds - \int_{t_n}^{t_{n+1}} \mathcal{L}'F_{m,f}(\omega_s)1_{\{\tau_k\le s\}}ds\big] \prod_{i=1}^n H(\omega_{t_i})\Big\} \\
&= 0.
\end{aligned}
$$

That is under $\mathbf{Q}'$

$$F_{m,f}(\omega_t) - F_{m,f}(\omega_0) - \int_0^t \mathcal{L}'F_{m,f}(\omega_s)ds,\ t \ge 0,$$

is a martingale for $f = \prod_{i=1}^m \phi_i$ with $\{\phi_i\} \subset C^2(\mathbb{R})^{++}$(and therefore $\{\phi_i\} \subset C^2(\mathbb{R})$). Since $f \in C^2(\mathbb{R})$ can be approximated by polynomials in such a way that not only f but its derivatives up to second order are approximated uniformly on compact sets, by an approximating procedure (2.12) is a martingale for $F(\mu) = \langle f, \mu^m\rangle$ with $f \in C_c^2(\mathbb{R}^m)$ (see [6], p.501). By Remark 1.1, (1, 0) is in the bp-closure of G^m. In fact, let $h \in C_c^2(\mathbb{R}^m)$ satisfy $1_{B(0,1)} \le h \le 1_{B(0,2)}$ and $h_k(x) = h(x/k) \in C_c^2(\mathbb{R}^m)$. Then for $f \in C^2(\mathbb{R}^m)$, we can approximate $(f, G^m f)$ by $\{fh_k, G^m fh_k\}$. According to (2.11) and Lemma 2.1, we see the desired result follows by another approximating procedure. □

Let $G_b^m := G^m - b$. By Theorem 5.11 of [5], there exists a diffusion process on $\hat{C}(\mathbb{R}^m)$ generated by $G_b^m \mid_{C_c^2(\mathbb{R}^m)}$ (and therefore $G_b^m \mid_{\hat{C}^2(\mathbb{R}^m)}$). Its transition density $q^m(t,x,y)$ is the fundamental solution of the equation

$$\frac{\partial u}{\partial t} = G_b^m u. \tag{2.13}$$

The semigroup corresponding to the operator G_b^m is defined by

$$T_t^m f(x) = \int q^m(t,x,y)f(y)dy \tag{2.14}$$

for $f \in \hat{C}(\mathbb{R}^m)$ and can therefore be extended to all of $B(\mathbb{R}^m)$. According to $0.24.A_2$ of [5], for $f \in C(\mathbb{R}^m)$

$$\lim_{t\to 0} \int q^m(t,x,y)f(y)dy = f(x) \quad (x \in \mathbb{R}^m),$$

where the convergence is uniform on every bounded subset. On the other hand, $(T_t^m)_{t\ge 0}$ is strong Feller, i.e., for $f \in B(\mathbb{R}^m)$ and $t > 0$, $T_t^m f \in C(\mathbb{R}^m)$. In fact, according to 1° of the proof of Theorem 5.11 of [5], $T_t^m f \in C^2(\mathbb{R}^m)$ satisfies equation (2.13). Hence for $f \in C^2(\mathbb{R}^m)$

$$\frac{T_t^m f(x) - f(x)}{t} = \frac{u_t(x) - f(x)}{t} = \frac{1}{t}\int_0^t G_b^m u_s(x)ds.$$

Therefore

$$\lim_{t\to 0} \frac{T_t^m f(x) - f(x)}{t} = G_b^m f(x),$$

where the convergence is bounded and pointwise. Let $\tilde{G}_b^m$ denote the weak generator of $(T_t^m)_{t\ge 0}$. Thus $C^2(\mathbb{R}^m)$ belong to the domain of $\tilde{G}_b^m$ and $T_t^m C^2(\mathbb{R}^m) \subset C^2(\mathbb{R}^m)$. Also, $\tilde{G}_b^m|_{C^2(\mathbb{R}^m)} = G_b^m|_{C^2(\mathbb{R}^m)}$. Let $p^m(t,x,y)$ denote the transition density corresponding to the semigroup $(P_t^m)_{t\ge 0}$. According to 6° of the proof of Theorem 5.11 of [5], we see for all $t > 0$, $x \in \mathbb{R}^m$, $A \in \mathfrak{B}(\mathbb{R}^m)$,

$$\int_A p^m(t,x,y)dy \ge \int_A q^m(t,x,y)dy.$$

Therefore, for $f \in B(\mathbb{R}^m)^+$,

$$P_t^m f(x) \geq T_t^m f(x).$$

Next, we define a dual process and reveal its connection to the solutions of the $(\mathcal{L}', \mu)$-martingale problem.

Let $\{M_t : t \geq 0\}$ be a nonnegative integer-valued càdlàg Markov process. For $i \geq j$, the transition intensities $\{q_{ij}\}$ defined by

$$q_{ij} = \begin{cases} \sum_{i \neq j} -q_{ij} & if \quad j = i \\ \frac{1}{2} i(i-1) + \begin{pmatrix} i \\ 2 \end{pmatrix} & if \quad j = i-1 \\ \begin{pmatrix} i \\ j-1 \end{pmatrix} & if \quad 1 \leq j \leq i-2 \end{cases}$$

and $q_{ij} = 0$ for $i < j$. Let $\tau_0 = 0$ and $\tau_{M_0} = \infty$, and let $\{\tau_k : 1 \leq k \leq M_0 - 1\}$ be the sequence of jump times of $\{M_t : t \geq 0\}$. That is $\tau_1 = \inf\{t \geq 0 : M_t \neq M_0\}, \cdots, \tau_k = \inf\{t > \tau_{k-1} : M_t \neq M_{\tau_{k-1}}\}$.
Let $\{\Gamma_k : 1 \leq k \leq M_0 - 1\}$ be a sequence of random operators which are conditionally independent given $\{M_t : t \geq 0\}$ and satisfy

$$\mathbf{P}\{\Gamma_k = \Psi_{ij} | M(\tau_k -) = l, M(\tau_k) = l-1\} = \frac{1}{2l(l-1)}, \quad 1 \leq i \neq j \leq l,$$

$$\mathbf{P}\{\Gamma_k = \Phi_{i_1,i_2} | M(\tau_k -) = l, M(\tau_k) = l-1\} = \frac{1}{l(l-1)}, \quad 1 \leq i_1 < i_2 \leq l,$$

and for $a \geq 3$,

$$\mathbf{P}\{\Gamma_k = \Phi_{i_1,\cdots,i_a} | M(\tau_k -) = l, M(\tau_k) = l-a+1\} = \frac{1}{\begin{pmatrix} l \\ a \end{pmatrix}}, \quad 1 \leq i_1 < \cdots < i_a \leq l,$$

where Ψ_{ij} and $\Phi_{i_1,\cdots,i_a}$ are defined by (2.9) and (2.10) respectively. Let $\mathbf{B}$ denote the topological union of $\{B(\mathbb{R}^m) : m = 1, 2, \cdots\}$ endowed with pointwise convergence on each $B(\mathbb{R}^m)$. Then

$$Y_t = T_{t-\tau_k}^{M_{\tau_k}} \Gamma_k T_{\tau_k - \tau_{k-1}}^{M_{\tau_{k-1}}} \Gamma_{k-1} \cdots T_{\tau_2 - \tau_1}^{M_{\tau_1}} \Gamma_1 T_{\tau_1}^{M_0} Y_0, \quad \tau_k \leq t < \tau_{k+1}, \quad 0 \leq k \leq M_0 - 1, \tag{2.15}$$

defines a Markov process $\{Y_t : t \geq 0\}$ taking values from $\mathbf{B}$. Clearly, $\{(M_t, Y_t) : t \geq 0\}$ is also a Markov process. Let $\mathbf{E}_{m,f}^{\sigma,\gamma}$ denote the expectation given $M_0 = m$ and $Y_0 = f \in B(\mathbb{R}^m)$.

Theorem 2.1 *Suppose that $\{X_t : t \geq 0\}$ is a càdlàg $M(\mathbb{R})$-valued process. If $\{X_t : t \geq 0\}$ is a solution of the $(\mathcal{L}', \mu)$-martingale problem and assume that $\{X_t : t \geq 0\}$ and $\{(M_t, Y_t) : t \geq 0\}$ are defined on the same probability space and independent of each other, then*

$$\boldsymbol{E}\langle f, X_t^m \rangle = \mathbf{E}_{m,f}^{\sigma,\gamma} \big[\langle Y_t, \mu^{M_t} \rangle \exp\big\{ \int_0^t (2^{M_s} + \frac{M_s(M_s-1)}{2} - M_s - 1) ds \big\} \big] \tag{2.16}$$

for any $t \geq 0$, $f \in B(\mathbb{R}^m)$ and integer $m \geq 1$.

Proof. In this proof we set $F_\mu(m, f) = F_{m,f}(\mu) = \langle f, \mu^m \rangle$. By Lemma 2.1, we have that for each $m \geq 1$, $\mathbf{E}[\langle 1, X_t \rangle^m]$ is a locally bounded function of $t \geq 0$. Then by martingale inequality we have that $\mathbf{E}[\sup_{0 \leq s \leq t} \langle 1, X_s \rangle^m]$ is a locally bounded function of $t \geq 0$.

By the definition of Y and elementary properties of M, we know that $\{(M_t, Y_t) : t \geq 0\}$ has weak generator $\mathcal{L}^*$ given by

$$\begin{aligned}\mathcal{L}^* F_\mu(m,f) &= F_\mu(m, G_b^m f) + \frac{1}{2}\sum_{i,j=1, i\neq j}^{m} [F_\mu(m-1, \Psi_{ij} f) - F_\mu(m,f)] \\ &\quad + \sum_{k=2}^{m}\left(\sum_{\{1\leq i_1<\cdots<i_k\leq m\}} [F_\mu(m-k+1, \Phi_{i_1,\cdots,_{1_k}} f) - F_\mu(m,f)]\right) \qquad (2.17)\end{aligned}$$

with $f \in C^2(\mathbb{R}^m)$. In view of (2.11) we have

$$\mathcal{L}^* F_\mu(m,f) = \mathcal{L}' F_{m,f}(\mu) - (2^m + \frac{1}{2}m(m-1) - m - 1)F_\mu(m,f). \qquad (2.18)$$

Then it is easy to verify that the inequalities in Theorem 4.4.11 of [6] are satisfied. Then the desired conclusion follows from Corollary 4.4.13 of [6]. □

Remark 2.2 *We point out that there exists a gap in the proof of establishing the dual relationship of [4]. There it was assumed σ is a bounded measurable function and $\gamma = 0$. When they established the dual relationship, they used a relationship which is similar to (2.18). However, note that (2.18) makes sense if $f \in \mathcal{D}(G_b^m)$ and Y_t need not always take values in $\mathcal{D}(G_b^m)$ if we only assume that σ is a bounded measurable function and G^m is elliptic. If we assume that $\sigma \in C_\partial^2(\mathbb{R})$ and G^m is uniformly elliptic, then the argument there can be applied to establish the dual relationship there. If $c = 0$, G^m need not always be uniformly elliptic. Our methods cannot be applied to obtain the uniqueness of the corresponding martingale problem. Dawson and Li [3] constructed SDSM from one-dimensional excursion when $c = 0$ and $\gamma(x, d\xi) = 0$. From the construction there, an important property of the SDSM was revealed. That is when $c = 0$, the process always lives in the space of purely atomic measures. We can also follow the idea there to construct discontinuous SDSM.*

Theorem 2.2 *Suppose that for each $\mu \in M(\mathbb{R})$ there is a probability measure $\mathbf{Q}'_\mu$ on $D([0,\infty), M(\mathbb{R}))$ such that $\mathbf{Q}'_\mu\{\langle 1, \omega_t\rangle^m\}$ is locally bounded in $t \geq 0$ for every $m \geq 1$ and such that $\{\omega_t : t \geq 0\}$ under $\mathbf{Q}'_\mu$ is a solution of the $(\mathcal{L}', \mu)$-martingale problem. Then $\mathbf{Q}' := \{\mathbf{Q}'_\mu : \mu \in M(\mathbb{R})\}$ defines a Markov process with transition semigroup $(Q'_t)_{t\geq 0}$ given by*

$$\int_{M(\mathbb{R})} \langle f, \nu^m\rangle\, Q'_t(\mu, d\nu) = \mathbf{E}^{\sigma,\gamma}_{m,f}\left[\langle Y_t, \mu^{M_t}\rangle \exp\left\{\int_0^t \left(2^{M_s} + \frac{M_s(M_s-1)}{2} - M_s - 1\right) ds\right\}\right] \qquad (2.19)$$

for $f \in B(\mathbb{R}^m)$.

Proof. Let $Q'_t(\mu, \cdot)$ denote the distribution of ω_t under $\mathbf{Q}'_\mu$. By Theorem 2.1, we obtain (2.19). We first consider the case that $\sigma(x) \equiv \sigma_0$ for a constant σ_0 and $\gamma(x, d\xi) \equiv \hat{\gamma}(d\xi)$ such that $\int_l^\infty \hat{\gamma}(d\xi) = 0$. In this case, $\{\langle 1, \omega_t\rangle : t \geq 0\}$ is a critical continuous state branching process with generator $\mathfrak{L}$ given by

$$\mathfrak{L} f(x) = \frac{1}{2}\sigma_0 x f''(x) + x\int_0^l \left(f(x+\xi) - f(x) - \xi f'(x)\right)\hat{\gamma}(d\xi) \qquad (2.20)$$

for $f \in C^2(\mathbb{R})$. By Kawazu and Watanabe [11] we deduce that

$$\int_{M(\mathbb{R})} e^{\lambda\langle 1,\nu\rangle} Q_t(\mu, d\nu) = e^{\langle 1,\mu\rangle \varphi(t,\lambda)}, \quad t \geq 0,\ \lambda \geq 0,$$

where $\varphi(t,\lambda)$ is the solution of

$$\begin{cases}\frac{\partial\varphi}{\partial t}(t,\lambda)=R(\varphi(t,\lambda)),\\ \varphi(0,\lambda)=\lambda,\end{cases}$$

and $R(\lambda)$ is given as follows:

$$R(\lambda)=-\frac{1}{2}\sigma_0\lambda^2-\int_0^l(e^{-\lambda\xi}-1+\lambda\xi)\hat{\gamma}(d\xi).$$

Then for each $f\in B(\mathbb{R})^+$ the power series

$$\sum_{m=0}^{\infty}\frac{1}{m!}\int_{M(\mathbb{R})}\langle f,\nu\rangle^m Q_t'(\mu,d\nu)\lambda^m \tag{2.21}$$

has a positive radius of convergence. By this and Theorem 30.1 of [1], it is easy to show that $Q_t'(\nu,\cdot)$ is the unique probability measure on $M(\mathbb{R})$ satisfying (2.19). Now the result follows from Theorem 4.4.2 of [6]. For general case, let $\sigma_0=||\sigma||$ and $f^{\otimes m}(x_1,\cdots,x_m)=f(x_1)\cdots f(x_m)$. We can find a measure $\hat{\gamma}(d\xi)$ on $(0,+\infty)$ such that for every $k\geq 2$

$$C_\gamma:=\sup_x[\int_0^1\xi^2\gamma(x,d\xi)+\int_1^l\xi\gamma(x,d\xi)]\leq\int_0^l\xi^k\hat{\gamma}(d\xi)<\infty$$

and $\int_l^\infty\hat{\gamma}(d\xi)=0$. In fact, since $l>1$, we can let $\hat{\gamma}(d\xi)=(k_l+1)C_\gamma 1_{(0,l)}(\xi)d\xi$, where $d\xi$ denotes the Lebesgue measure and $k_l=\min\{k\geq 2: l^k/(k+1)>1\}$. We obtain that for each $k\geq 2$

$$\sup_x[\int_0^l\xi^k\gamma(x,d\xi)]\leq l^k\int_0^l\xi^k\hat{\gamma}(d\xi).$$

By (2.19) and (2.15) we have

$$\int_{M(\mathbb{R})}\langle f,\nu\rangle^m Q_t'(\mu,d\nu)\leq\mathbf{E}^{\sigma_0,\hat{\gamma}}_{m,l^m f^{\otimes m}}\left[\langle Y_t,\mu^{M_t}\rangle\exp\left\{\int_0^t\left(2^{M_s}+\frac{M_s(M_s-1)}{2}-M_s-1\right)ds\right\}\right]$$

for $f\in B(\mathbb{R})^+$. Then the power series (2.21) also has a positive radius of convergence and the desired result follows as in previous case. □

Remark 2.3 *From (2.11), we may regard the Markov process associated with $(\mathcal{L}',\mu)$-martingale problem as a measure-valued branching process with branching mechanism given by*

$$\Psi_1(x,z):=\frac{1}{2}\sigma(x)z^2+\int_0^l(e^{-z\xi}-1+z\xi)\gamma(x,d\xi).$$

and its spatial motion is a diffusion process generated by

$$\frac{1}{2}\sum_{i=1}^m a(x_i)\frac{\partial^2}{\partial x_i^2}+\frac{1}{2}\sum_{i,j=1,i\neq j}^m\rho(x_i-x_j)\frac{\partial^2}{\partial x_i\partial x_j}-\sum_{i=1}^m\int_l^\infty\xi\gamma(x_i,d\xi)$$

which represents 'G^m-diffusion killed at a rate $\sum_{i=1}^m\int_l^\infty\xi\gamma(x_i,d\xi)$'; see Rogers and Williams [14] and references therein for more details of 'Markov process with killing'.

2.2 Uniqueness for $(\mathcal{L},\mu)$-martingale problem

In this section, we will consider a localization procedure suggested by Stroock [16] to show that the uniqueness for the $(\mathcal{L},\mu)$-martingale problem follows from the uniqueness of the $(\mathcal{L}',\mu)$-martingale problem. Although some arguments in this subsection are similar to those of [7] and [16], we shall give the details for the convenience of the reader. We assume that the for each $\mu \in M(\mathbb{R})$, $(\mathcal{L}',\mu)$-martingale problem is well-posed. The existence for the $(\mathcal{L}',\mu)$-martingale problem will be revealed in Section 3. Let $\mathbf{Q}'$ denote the Markovian system defined in Theorem 2.2. Let $\mathbf{Q}'_{s,\mu} = \mathbf{Q}'(\cdot|\omega_s = \mu)$. Then $\mathbf{Q}'_{s,\mu}$ is also a Markovian system starting from (s,μ) whose transition semigroup is the same with $\mathbf{Q}'$.

Let $\{\omega_t : t \geq 0\}$ denote the coordinate process of $D([0,\infty), M(\mathbb{R}))$. Let $\Omega = D([0,\infty), M(\mathbb{R}))$. Set $\mathcal{F}_t = \sigma\{\omega_s : 0 \leq s \leq t\}$, and take $\mathcal{F}^t = \sigma\{\omega_s : t \leq s\}$.

Definition 2.2 *For $\mu \in M(\mathbb{R})$, we say a probability measure $\mathbf{Q}_{s,\mu}$ on $(\Omega, \mathcal{F}^s)$ is a solution of the $(\mathcal{L},\mu)$-martingale problem if $\mathbf{Q}_{s,\mu}(\omega_s = \mu) = 1$ and*

$$F(\omega_t) - F(\mu) - \int_s^t \mathcal{L}F(\omega_u)du, \quad t \geq s, \tag{2.22}$$

is a local martingale for each $F \in \mathcal{D}(\mathcal{L})$.

In the following we will write $\mathbf{Q}_\mu$ instead of $\mathbf{Q}_{0,\mu}$ and write $\mathcal{F}$ instead of $\mathcal{F}^0$. Let $S(\mathbb{R})$ denote the space of finite signed Borel measures on $\mathbb{R}$ endowed with the σ-algebra generated by the mappings $\mu \mapsto \langle f,\mu\rangle$ for all $f \in C(\mathbb{R})$. Let $S(\mathbb{R})^\circ = S(\mathbb{R}) \setminus \{0\}$ and $M(\mathbb{R})^\circ = M(\mathbb{R}) \setminus \{0\}$. The following theorem is analogous to Théorèm 7 of [7].

Theorem 2.3 *Suppose that a probability measure $\mathbf{Q}_\mu$ on $(\Omega,\mathcal{F})$ is a solution of the $(\mathcal{L},\mu)$-martingale problem. Define an optional random measure $N(ds,d\nu)$ on $[0,\infty) \times S(\mathbb{R})^\circ$ by*

$$N(ds,d\nu) = \sum_{s>0} 1_{\{\Delta\omega_s \neq 0\}} \delta_{(s,\Delta\omega_s)}(ds,d\nu),$$

where $\Delta\omega_s = \omega_s - \omega_{s-} \in S(\mathbb{R})$. Let $\hat{N}(ds,d\nu)$ denote the predictable compensator of $N(ds,d\nu)$ and let $\tilde{N}(ds,d\nu)$ denote the corresponding martingale measure under $\mathbf{Q}_\mu$. Then $\hat{N}(ds,d\nu) = dsK(\omega_s,d\nu)$ with $K(\mu,d\nu)$ given by

$$\int_{M(\mathbb{R})^\circ} F(\nu)K(\mu,d\nu) = \int_{\mathbb{R}} \mu(dx) \int_0^\infty F(\xi\delta_x)\gamma(x,d\xi),$$

and for $\phi \in C^2(\mathbb{R})^+$,

$$M_t(\phi) := \langle \phi,\omega_t\rangle - \langle\phi,\mu\rangle - \frac{1}{2}\int_0^t \langle a\phi'',\omega_s\rangle ds, \quad t \geq 0, \tag{2.23}$$

is a martingale and we also have that

$$M_t(\phi) = M_t^c(\phi) + M_t^d(\phi),$$

where $M_t^c(\phi)$ under $\mathbf{Q}_\mu$ is a continuous martingale with quadratic variation process given by

$$\langle M^c(\phi)\rangle_t = \int_0^t \langle\sigma\phi^2,\omega_s\rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z-\cdot)\phi',\omega_s\rangle^2 dz, \tag{2.24}$$

and

$$M_t^d(\phi) = \int_0^{t+} \int_{M(\mathbb{R})^\circ} \langle \phi, \nu \rangle \tilde{N}(ds, d\nu) \tag{2.25}$$

is a purely discontinuous martingale under $\mathbf{Q}_\mu$*.*

Proof. Some arguments in the proof of this theorem are similar to those of Theorem 6.1.3 of [2]. The proof will be divided into 4 steps.

Step 1. Since $e^{-\langle \phi, \nu \rangle} \in \mathcal{D}_0(\mathcal{L})$ for $\phi \in C^2(\mathbb{R})^{++}$,

$$W_t(\phi) := e^{-\langle \phi, \omega_t \rangle} - \int_0^t e^{-\langle \phi, \omega_s \rangle}[-\frac{1}{2}\langle a\phi'', \omega_s \rangle + \frac{1}{2}\int_{\mathbb{R}} \langle h(z-\cdot)\phi', \omega_s \rangle^2 dz + \langle \Psi(\phi), \omega_s \rangle] ds, \ t \geq 0, \tag{2.26}$$

is a $\mathbf{Q}_\mu$-martingale with $\phi \in C^2(\mathbb{R})^{++}$, where $\Psi(\phi) := \Psi(x, \phi(x))$. Therefore, $\{W_t(\phi)\}$ is a local martingale for $\phi \in C^2(\mathbb{R})^+$. Let

$$Z_t(\phi) := \exp\{-\langle \phi, \omega_t \rangle\},$$

$$H_t(\phi) := \exp\Big\{ -\langle \phi, \omega_t \rangle + \int_0^t \big[\frac{1}{2}\langle a\phi'', \omega_s \rangle - \frac{1}{2}\int_{\mathbb{R}} \langle h(z-\cdot)\phi', \omega_s \rangle^2 dz - \langle \Psi(\phi), \omega_s \rangle\big] ds \Big\}$$

and

$$Y_t(\phi) := \exp\Big\{ \int_0^t \big[\frac{1}{2}\langle a\phi'', \omega_s \rangle - \frac{1}{2}\int_{\mathbb{R}} \langle h(z-\cdot)\phi', \omega_s \rangle^2 dz - \langle \Psi(\phi), \omega_s \rangle\big] ds \Big\}.$$

By integration by parts,

$$\begin{aligned} & \int_0^t Y_s(\phi) dW_s(\phi) \\ = & \int_0^t Y_s(\phi) dZ_s(\phi) \\ & - \int_0^t Y_s(\phi) e^{-\langle \phi, \omega_s \rangle}\big[-\frac{1}{2}\langle a\phi'', \omega_s \rangle + \frac{1}{2}\int_{\mathbb{R}} \langle h(z-\cdot)\phi', \omega_s \rangle^2 dz + \langle \Psi(\phi), \omega_s \rangle\big] ds \\ = & H_t(\phi) - Z_0(\phi) \end{aligned}$$

is a $\mathbf{Q}_\mu$-local martingale. We also have

$$Z_t(\phi) = Y_t^{-1}(\phi) H_t(\phi),$$

and, again by integration by parts,

$$\begin{aligned} dZ_t(\phi) &= Y_t^{-1}(\phi) dH_t(\phi) + H_{t-}(\phi) dY_t^{-1}(\phi) \\ &= Y_t^{-1}(\phi) dH_t(\phi) \\ &\quad + Z_{t-}(\phi)\big[-\frac{1}{2}\langle a\phi'', \omega_{t-} \rangle + \frac{1}{2}\int_{\mathbb{R}} \langle h(z-\cdot)\phi', \omega_{t-} \rangle^2 dz + \langle \Psi(\phi), \omega_{t-} \rangle\big] dt. \end{aligned} \tag{2.27}$$

Then $\{Z_t(\phi) : t \geq 0\}$ is a special semi-martingale with $\phi \in C^2(\mathbb{R})^+$ (see Definitions 1.4.21 of [9]).

Step 2. By the same argument as in the proof of Lemma 2.1, we have that

$$\mathbf{Q}_\mu[\omega_t(1)] \leq \langle 1, \mu \rangle + C_1(\sigma, \gamma) \int_0^t \mathbf{Q}_\mu[\omega_s(1)] ds,$$

where $C_1(\sigma,\gamma) := ||\sigma|| + 2\sup_x \int_1^\infty \xi\gamma(x,d\xi) + \sup_x \int_0^1 \xi^2\gamma(x,d\xi)$. By Gronwall's inequality

$$\mathbf{Q}_\mu[\omega_t(1)] \leq \langle 1,\mu\rangle e^{C_1(\sigma,\gamma)t}. \tag{2.28}$$

For any $k \geq 1$, take $f_k \in C_0^2(\mathbb{R})$ such that $f_k(x) = x$ for $|x| \leq k$ and $|f_k'(x)| \leq 1$ for all $x \in \mathbb{R}$. We see for each $\phi \in C^2(\mathbb{R})^{++}$,

$$\lim_{k\to\infty} f_k(\langle\phi,\mu\rangle) = \langle\phi,\mu\rangle \quad \text{and} \quad \lim_{k\to\infty} \mathcal{L}f_k(\langle\phi,\mu\rangle) = \frac{1}{2}\langle a\phi'',\mu\rangle.$$

Since $f_k(\langle\phi,\mu\rangle) \in \mathcal{D}_0(\mathcal{L})$, by (2.28) and dominated convergence theorem an approximation argument shows that for $\phi \in C^2(\mathbb{R})^{++}$

$$\langle\phi,\omega_t\rangle = \langle\phi,\mu\rangle + \frac{1}{2}\int_0^t \langle a\phi'',\omega_s\rangle ds + M_t(\phi),$$

where $\{M_t(\phi) : t \geq 0\}$ is a martingale. For $\phi \in C^2(\mathbb{R})^+$, we have $\{M_t(\phi+\varepsilon)\}$ are martingales for $\varepsilon > 0$. By letting $\varepsilon \to 0$, (2.28) ensures that

$$M_t(\phi) = \langle\phi,\omega_t\rangle - \langle\phi,\mu\rangle - \frac{1}{2}\int_0^t \langle a\phi'',\omega_s\rangle ds, \quad t \geq 0,$$

is a martingale for $\phi \in C^2(\mathbb{R})^+$. By Corollary 2.2.38 of [9], $\{M_t(\phi)\}$ admits a unique representation

$$M_t(\phi) = M_t^c(\phi) + M_t^d(\phi),$$

where $\{M_t^c(\phi)\}$ is a continuous local martingale with quadratic variation process $\{C_t(\phi)\}$ and

$$M_t^d(\phi) = \int_0^{t+}\int_{S(\mathbb{R})^\circ} \langle\phi,\nu\rangle \tilde{N}(ds,d\nu) \tag{2.29}$$

is a purely discontinuous local martingale. Moreover, $\{\langle\phi,\omega_t\rangle\}$ is a semimartingale. An application of Itô's formula for semimartingale (see Theorem 1.4.57 of [9]) yields

$$\begin{aligned} dZ_t(\phi) \quad &= \quad Z_{t-}(\phi)[-dU_t(\phi) + \frac{1}{2}dC_t(\phi) + \int_{S(\mathbb{R})^\circ} (e^{-\langle\phi,\nu\rangle} - 1 + \langle\phi,\nu\rangle)N(dt,d\nu)] \\ &\quad\; +d(loc.mart.), \end{aligned} \tag{2.30}$$

where $U_t(\phi) = \frac{1}{2}\int_0^t \langle a\phi'',\omega_s\rangle ds$ is of locally bounded variation. Note that

$$0 \leq Z_{s-}(\phi)(e^{-\langle\phi,\nu\rangle} - 1 + \langle\phi,\nu\rangle) \leq C(|\langle\phi,\nu\rangle| \wedge |\langle\phi,\nu\rangle^2|)$$

for some constant $C \geq 0$. According to Theorem 1.4.47 of [9], $\sum_{s\leq t}(\langle\phi,\Delta\omega_s\rangle)^2 < \infty$. Thus the first term in (2.30) has finite variation over each finite interval $[0,t]$. Since $\{Z_t(\phi)\}$ is a special semimartingale, Proposition 1.4.23 of [9] implies that

$$\int_0^{t+}\int_{S(\mathbb{R})^\circ} Z_{s-}(\phi)(e^{-\langle\phi,\nu\rangle} - 1 + \langle\phi,\nu\rangle)N(ds,d\nu)$$

is of locally integrable variation. Thus it is locally integrable. According to Proposition 2.1.28 of [9],

$$\int_0^{t+}\int_{S(\mathbb{R})^\circ} Z_{s-}(\phi)(e^{-\langle\phi,\nu\rangle} - 1 + \langle\phi,\nu\rangle)\tilde{N}(ds,d\nu)$$

$$= \int_0^{t+}\int_{S(\mathbb{R})^\circ} Z_{s-}(\phi)(e^{-\langle\phi,\nu\rangle}-1+\langle\phi,\nu\rangle)N(ds,d\nu)$$
$$-\int_0^{t+}\int_{S(\mathbb{R})^\circ} Z_{s-}(\phi)(e^{-\langle\phi,\nu\rangle}-1+\langle\phi,\nu\rangle)\hat{N}(ds,d\nu)$$

is a purely discontinuous local martingale. Therefore,

$$\begin{aligned} dZ_t(\phi) &= Z_{t-}(\phi)[-dU_t(\phi)+\frac{1}{2}dC_t(\phi)+\int_{S(\mathbb{R})^\circ}(e^{-\langle\phi,\nu\rangle}-1+\langle\phi,\nu\rangle)\hat{N}(dt,d\nu)] \\ &\quad +d(loc.mart.). \end{aligned} \tag{2.31}$$

Step 3. Since $Z_t(\phi)$ is a special semimartingale we can identify the predictable components of locally integrable variation in the two decompositions (2.27) and (2.31) to get that

$$\begin{aligned} & Z_{t-}(\phi)[-\frac{1}{2}\langle a\phi'',\omega_{t-}\rangle+\frac{1}{2}\int_{\mathbb{R}}\langle h(z-\cdot)\phi',\omega_{t-}\rangle^2dz+\langle\Psi(\phi),\omega_{t-}\rangle]dt \\ =\; & Z_{t-}(\phi)[-dU_t(\phi)+\frac{1}{2}dC_t(\phi)+\int_{S(\mathbb{R})^\circ}(e^{-\langle\phi,\nu\rangle}-1+\langle\phi,\nu\rangle)\hat{N}(dt,d\nu)]. \end{aligned}$$

Then

$$\begin{aligned} & \int_0^t[-\frac{1}{2}\langle a\phi'',\omega_s\rangle+\frac{1}{2}\int_{\mathbb{R}}\langle h(z-\cdot)\phi',\omega_s\rangle^2dz+\langle\Psi(\phi),\omega_s\rangle]ds \\ =\; & -U_t(\phi)+\frac{1}{2}C_t(\phi)+\int_0^t\int_{S(\mathbb{R})^\circ}(e^{-\langle\phi,\nu\rangle}-1+\langle\phi,\nu\rangle)\hat{N}(ds,d\nu). \end{aligned} \tag{2.32}$$

According to (2.28) and (2.29), we can deduce that $C_t(\theta\phi)=\theta^2C_t(\phi)$ with $\theta>0$. Replacing ϕ by $\theta\phi$ with $\theta>0$ in (2.32), we have

$$\begin{aligned} & -\theta\int_0^t\frac{1}{2}\langle a\phi'',\omega_s\rangle ds+\frac{\theta^2}{2}\int_0^t\int_{\mathbb{R}}\langle h(z-\cdot)\phi',\omega_s\rangle^2dzds+\frac{\theta^2}{2}\int_0^t\langle\sigma\phi^2,\omega_s\rangle ds \\ & +\int_0^t ds\int_{\mathbb{R}}\omega_s(dx)\int_0^\infty\gamma(x,d\xi)(e^{-\theta\xi\phi(x)}-1+\theta\xi\phi(x)) \\ =\; & -\theta U_t(\phi)+\frac{\theta^2}{2}C_t(\phi)+\int_0^t\int_{S(\mathbb{R})^\circ}(e^{-\theta\langle\phi,\nu\rangle}-1+\theta\langle\phi,\nu\rangle)\hat{N}(ds,d\nu). \end{aligned} \tag{2.33}$$

We conclude that

$$C_t(\phi)=\int_0^t ds\int_{\mathbb{R}}\langle h(z-\cdot)\phi',\omega_s\rangle^2dz+\int_0^t\langle\sigma\phi^2,\omega_s\rangle ds \tag{2.34}$$

and

$$\begin{aligned} & \int_0^t\int_{S(\mathbb{R})^\circ}(e^{-\theta\langle\phi,\nu\rangle}-1+\theta\langle\phi,\nu\rangle)\hat{N}(ds,d\nu) \\ =\; & \int_0^t ds\int_{\mathbb{R}}\omega_s(dx)\int_0^\infty\gamma(x,d\xi)(e^{-\xi\langle\delta_x,\theta\phi\rangle}-1+\xi\langle\delta_x,\theta\phi\rangle), \end{aligned}$$

where $\theta>0$ and $\phi\in C^2(\mathbb{R})^+$. That is, under $\mathbf{Q}_\mu$ the jump measure N has compensator

$$\hat{N}(ds,d\nu)=ds\omega_s(dx)\gamma(x,d\xi)\cdot\delta_{\xi\delta_x}(d\nu),\quad \nu\in M(\mathbb{R}). \tag{2.35}$$

In particular this implies that the jumps of ω are $\mathbf{Q}_\mu$-a.s. in $M(\mathbb{R})$, i.e. positive measures. Observe that for $\{\phi_i\}_{i=1}^2 \subset C^2(\mathbb{R})^+$, $M_t^c(\phi_1+\phi_2) = M_t^c(\phi_1) + M_t^c(\phi_2)$. According to (2.34),

$$\begin{aligned}\langle M^c(\phi_1), M^c(\phi_2)\rangle_t &= \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \rho(x-y)\phi_1'(x)\phi_2'(y)\omega_s(dx)\omega_s(dy)ds \\ &+\frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \rho(x-y)\phi_2'(x)\phi_1'(y)\omega_s(dx)\omega_s(dy)ds \\ &+\int_0^t \langle \sigma\phi_1\phi_2, \omega_s\rangle ds. \end{aligned} \tag{2.36}$$

Step 4. Let $J_1(\phi,\nu) = \langle\phi,\nu\rangle 1_{\{\langle 1,\nu\rangle\geq 1\}}$ and $J_2(\phi,\nu) = \langle\phi,\nu\rangle 1_{\{\langle 1,\nu\rangle< 1\}}$. First, one can check that

$$\mathbf{Q}_\mu\Big[\int_0^t\int J_1(\phi,\nu)\hat{N}(ds,d\nu)\Big] < \infty \quad \text{and} \quad \mathbf{Q}_\mu\Big[\int_0^t\int J_2(\phi,\nu)^2\hat{N}(ds,d\nu)\Big] < \infty$$

for $\phi \in C^2(\mathbb{R})^+$. Then following the argument in Section 2.3 of [12] we obtain the martingale property of $M^d(\phi)$. By Proposition 2.1.28 and Theorem 2.1.33 of [9] we can deduce that

$$\begin{aligned}\int_0^{t+}\int_{M(\mathbb{R})^\circ} J_1(\phi,\nu)\tilde{N}(ds,d\nu) &= \int_0^{t+}\int_{M(\mathbb{R})^\circ} J_1(\phi,\nu)N(ds,d\nu) \\ &-\int_0^{t}\int_{M(\mathbb{R})^\circ} J_1(\phi,\nu)\hat{N}(ds,d\nu), \quad t\geq 0,\end{aligned}$$

is a martingale and

$$\int_0^{t+}\int_{M(\mathbb{R})^\circ} J_2(\phi,\nu)\tilde{N}(ds,d\nu), \quad t\geq 0,$$

is a square-integrable martingale with quadratic variation process given by

$$\langle\int_0^{\cdot+}\int_{M(\mathbb{R})^\circ} J_2(\phi,\nu)\tilde{N}(ds,d\nu)\rangle_t = \int_0^{t}\int_{M(\mathbb{R})^\circ} J_2(\phi,\nu)^2\hat{N}(ds,d\nu).$$

Recall that

$$M_t^c(\phi) = M_t(\phi) - M_t^d(\phi).$$

The fact that both $M^d(\phi)$ and $M(\phi)$ above are martingales yields the martingale property of $M^c(\phi)$. We are done. □

Lemma 2.3 *Let $\mathbf{Q}_\mu$ be a probability measure on $(\Omega,\mathcal{F})$ such that it is a solution of the $(\mathcal{L},\mu)$-martingale problem. Then*

$$\mathbf{Q}_\mu[\sup_{0\leq s\leq t}\langle 1,\omega_s\rangle] < \infty.$$

Proof. According to Theorem 2.3 and *Step 4* in its proof, we have

$$\langle 1,\omega_t\rangle = \langle 1,\mu\rangle + M_t^c(1) + \int_0^t\int_{M(\mathbb{R})^\circ}\langle 1,\nu\rangle\tilde{N}(ds,d\nu)$$

is a martingale and we obtain

$$\mathbf{Q}_\mu\Big[\sup_{0\leq s\leq t}\langle 1,\omega_s\rangle\Big] \leq \langle 1,\mu\rangle + \mathbf{Q}_\mu\Big[\sup_{0\leq s\leq t}|M_s^c(1)|\Big] + \mathbf{Q}_\mu\Big[\sup_{0\leq s\leq t}|\int_0^s\int J_2(1,\nu)\tilde{N}(ds,d\nu)|\Big]$$

$$
\begin{aligned}
&+\mathbf{Q}_\mu\Big[\sup_{0\le s\le t}\int_0^s\int J_1(1,\nu)N(ds,d\nu)\Big]\\
&+\mathbf{Q}_\mu\Big[\sup_{0\le s\le t}\int_0^s\int J_1(1,\nu)\hat{N}(ds,d\nu)\Big]\\
\le{}& \langle 1,\mu\rangle+4\mathbf{Q}_\mu[C_t(1)]+2+\mathbf{Q}_\mu\Big[\sup_{0\le s\le t}[\int_0^s\int J_2(1,\nu)\tilde{N}(ds,d\nu)]^2\Big]\\
&+2\sup_x\int_1^\infty \xi\gamma(x,d\xi)\int_0^t \mathbf{Q}_\mu[\langle 1,\omega_s\rangle]ds\\
\le{}& \langle 1,\mu\rangle+2+4||\sigma||\int_0^t \mathbf{Q}_\mu[\langle 1,\omega_s\rangle]ds+4\sup_x\int_0^1\xi^2\gamma(x,d\xi)\int_0^t \mathbf{Q}_\mu[\langle 1,\omega_s\rangle]ds\\
&+2\sup_x\int_1^\infty \xi\gamma(x,d\xi)\int_0^t \mathbf{Q}_\mu[\langle 1,\omega_s\rangle]ds\\
\le{}& \langle 1,\mu\rangle+2+C_2(\sigma,\gamma)\langle 1,\mu\rangle t,
\end{aligned}
$$

where $C_2(\sigma,\gamma) := 4||\sigma|| + 2\sup_x\int_1^\infty \xi\gamma(x,d\xi) + 4\sup_x\int_0^1 \xi^2\gamma(x,d\xi)$ and the second and the third inequalities follow from Doob's inequality and the elementary inequality $|x|\le x^2+1$. We complete the proof. □

In accordance with the notation used in Theorem 2.3, set

$$X_t^L := \omega_t - \int_0^{t+}\int_{M(\mathbb{R})^\circ}\nu\cdot 1_{\{\langle 1,\nu\rangle\ge l\}}N(ds,d\nu).$$

By Theorem 2.3,

$$
\begin{aligned}
\langle\phi,X_t^L\rangle = \langle\phi,\mu\rangle+\int_0^t\langle a\phi'',\omega_s\rangle ds+M_t^c(\phi) &+\int_0^{t+}\int_{M(\mathbb{R})^\circ}\langle\phi,\nu\rangle 1_{\{\langle 1,\nu\rangle<l\}}\tilde{N}(ds,d\nu)\\
&-\int_0^{t+}\int_{M(\mathbb{R})^\circ}\langle\phi,\nu\rangle 1_{\{\langle 1,\nu\rangle\ge l\}}\hat{N}(ds,d\nu). \quad (2.37)
\end{aligned}
$$

Thus if $F(\mu)=f(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_m,\mu\rangle)\in\mathcal{D}(\mathcal{L})$, then by Itô's formula

$$
\begin{aligned}
I_t := F(X_t^L) &+ \frac{1}{2}\sum_{i=1}^m\int_0^t ds\int_{\mathbb{R}}\omega_s(dx)\int_l^\infty\gamma(x,d\xi)f^i(\langle\phi_1,X_s^L\rangle,\cdots,\langle\phi_n,X_s^L\rangle)\xi\phi_i(x)\\
&-\frac{1}{2}\sum_{i=1}^m\int_0^t f^i(\langle\phi_1,X_s^L\rangle,\cdots,\langle\phi_n,X_s^L\rangle)\langle a\phi_i'',\omega_s\rangle ds\\
&-\frac{1}{2}\sum_{i,j=1}^m\int_0^t f^{ij}(\langle\phi_1,X_s^L\rangle,\cdots,\langle\phi_n,X_s^L\rangle)d\langle M^c(\phi_i),M^c(\phi_j)\rangle s\\
&-\int_0^t ds\int_{\mathbb{R}}\omega_s(dx)\int_0^l\gamma(x,d\xi)\big\{f(\langle\phi_1,X_s^L\rangle+\xi\phi_1(x),\cdots,\langle\phi_n,X_s^L\rangle+\xi\phi_n(x))\\
&\qquad -f(\langle\phi_1,X_s^L\rangle,\cdots,\langle\phi_n,X_s^L\rangle)-\xi\sum_{i=1}^m\phi_i(x)f^i(\langle\phi_1,X_s^L\rangle,\cdots,\langle\phi_n,X_s^L\rangle)\big\}
\end{aligned}
$$

is a local martingale under $\mathbf{Q}_\mu$.

Let $\tau^1=\inf\{t\ge 0:\langle 1,\omega_t\rangle\ge l+\langle 1,\mu\rangle\}\wedge T$ and $\tau^2=\inf\{t\ge 0:|\langle 1,\omega_t\rangle-\langle 1,\omega_{t-}\rangle|\ge l\}$. Set $\tau=\tau^1\wedge\tau^2$. The following lemma gives another martingale characterization for X^L.

Lemma 2.4 *Let $\mathbf{P}_\mu$ be a probability measure on $(\Omega, \mathcal{F})$ such that $\mathbf{P}_\mu(\omega_0 = \mu) = 1$. Then*

$$\begin{aligned} I_t(\phi) := \exp\Big\{ - \langle \phi, X^L_{t\wedge\tau}\rangle \; &+ \int_0^{t\wedge\tau} [\langle a\phi'', \omega_s\rangle - \int_{\mathbb{R}} \langle h(z-\cdot)\phi', \omega_s\rangle^2 dz] ds \\ &- \int_0^{t\wedge\tau} ds \int_{\mathbb{R}} \omega_s(dx) \int_l^\infty \xi\phi(x)\gamma(x, d\xi) \\ &- \int_0^{t\wedge\tau} ds \int_{\mathbb{R}} \omega_s(dx) \int_0^l (e^{-\xi\phi(x)} - 1 + \xi\phi(x))\gamma(x, d\xi)\Big\} \end{aligned} \tag{2.38}$$

is a $\mathbf{P}_\mu$-martingale for every $\phi \in C^2(\mathbb{R})^{++}$ if and only if $\{I_{t\wedge\tau}\}$ is a $\mathbf{P}_\mu$-martingale for each $F \in \mathcal{D}(\mathcal{L})$.

Proof. The desired result follows from the formula of integration by parts and the same argument as in the proof of Théorèm 7 of [7]. □

The next two theorems are analogous to Theorem (3.1) and Theorem (3.3) of [16].

Theorem 2.4 *Given a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ such that $\mathbf{P}(\omega(0) = \mu) = 1$ and $\{I(t \wedge \tau) : t \geq 0\}$ is a $\mathbf{P}$-martingale. Define*

$$\mathbf{S}_\omega = \delta_\omega \otimes \mathbf{Q}'_{\{\tau(\omega), X^L_{\tau(\omega)}\}}$$

and

$$\mathbf{P}'(A) = \mathbf{P}[\mathbf{S}_\omega(A)], \quad A \in \mathcal{F},$$

where $\mathbf{S}_\omega$ is a measure on $(\Omega, \mathcal{F})$ satisfying

$$\mathbf{S}_\omega(A_1 \cap A_2) = 1_{A_1}(\omega)\mathbf{Q}'_{\{\tau(\omega), X^L_{\tau(\omega)}\}}(A_2)$$

for $A_1 \in \sigma(\bigcup_{0\leq s<\tau(\omega)} \mathcal{F}_s)$ and $A_2 \in \mathcal{F}^{\tau(\omega)}$. Define $\mathcal{F}_{\tau-} = \sigma\{X^L_{t\wedge\tau} : t \geq 0\}$. Then $\mathbf{P}'$ is also a solution of $(\mathcal{L}', \mu)$-martingale problem and $\mathbf{P} = \mathbf{Q}'_\mu$ on $\mathcal{F}_{\tau-}$. In particular, we can take $\mathbf{P} = \mathbf{Q}_\mu$.

Proof. Let $0 \leq t_1 < t_2$ and $A \in \mathcal{F}_{t_1}$. Given $\omega \in \Omega$, for this proof only, let $y(t, \omega)$ denote the position of ω at time t for convenient. Let $F \in \mathcal{D}(\mathcal{L})$. Then

$$\begin{aligned} \mathbf{P}'[1_A F(y_{t_2})] \;&=\; \mathbf{P}[1_{A\cap\{\tau>t_2\}} F(X^L_{t_2})] + \mathbf{P}[1_{A\cap\{t_1<\tau\leq t_2\}} \mathbf{Q}'_{\tau(\omega), X^L_{\tau(\omega)}}[F(y_{t_2})]] \\ &+ \mathbf{P}[1_{\{\tau\leq t_1\}} \mathbf{S}_\omega[1_A F(y_{t_2})]] = I_1 + I_2 + I_3. \end{aligned}$$

By the martingale formula of $\mathbf{Q}'$

$$I_2 = \mathbf{P}[1_{A\cap\{t_1<\tau\leq t_2\}} F(X^L_\tau)] + \mathbf{P}'[1_{A\cap\{t_1<\tau\leq t_2\}} \int_\tau^{t_2} \mathcal{L}' F(y_u) du],$$

and

$$\begin{aligned} I_1 + I_2 \;&=\; \mathbf{P}[1_{A\cap\{\tau>t_1\}} F(X^L_{\tau\wedge t_2})] + \mathbf{P}'[1_{A\cap\{t_1<\tau\leq t_2\}} \int_\tau^{t_2} \mathcal{L}' F(y_u) du] \\ &= \mathbf{P}[1_{A\cap\{\tau>t_1\}} F(X^L_{t_1})] + \mathbf{P}[1_{A\cap\{\tau>t_1\}} \int_{t_1}^{\tau\wedge t_2} \mathcal{L}' F(y_u) du] \\ &+ \mathbf{P}'[1_{A\cap\{\tau>t_1\}} \int_{\tau\wedge t_2}^{t_2} \mathcal{L}' F(y_u) du] \end{aligned}$$

$$= \mathbf{P}'[1_{A\cap\{\tau>t_1\}}F(y_{t_1})] + \mathbf{P}'[1_{A\cap\{\tau>t_1\}}\int_{t_1}^{t_2}\mathcal{L}'F(y_u)du],$$

where the second equality follows from that $\{I_{t\wedge\tau}\}$ is a martingale and the fact that $F(X_t^L)-I_t = \int_0^t \mathcal{L}'F(\omega_s)ds$ for $\tau > t$. On the other hand,

$$I_3 = \mathbf{P}'[1_{A\cap\{\tau\le t_1\}}F(y_{t_1})] + \mathbf{P}'[1_{A\cap\{\tau\le t_1\}}\int_{t_1}^{t_2}\mathcal{L}'F(y_u)du].$$

Thus $\mathbf{P}'$ solves the $(\mathcal{L}',\mu)$-martingale problem. Then the desired conclusion follows from the uniqueness of the $(\mathcal{L}',\mu)$-martingale problem. □

Theorem 2.5 *Let $M_l(\mathbb{R}) = \{\nu : \langle 1,\nu\rangle \ge l\}$. There is a $\mathcal{F}_{\tau-}$-measurable function $\tau' : \Omega \to [0,T]$ such that for $\Gamma \in \mathfrak{B}(M_l(\mathbb{R}))$,*

$$\mathbf{Q}_\mu[\int_0^{\tau+} N(ds,\Gamma)|\mathcal{F}_{\tau-}] = \int_0^{\tau'} \exp\{-\int_0^t ds\int_{\mathbb{R}} X_{s\wedge\tau}^L(dx)\int_l^\infty \gamma(x,d\xi)\}K(X_{t\wedge\tau}^L,\Gamma)dt \quad (2.39)$$

holds for any solution $\mathbf{Q}_\mu$ to the $(\mathcal{L},\mu)$-martingale problem. In particular, $\mathbf{Q}_\mu$ is uniquely determined on $\mathcal{F}_\tau$.

Proof. In accordance with the notation used in Theorem 2.3, we have

$$\int_0^{t+} N(ds,\Gamma) = \int_0^{t+}\tilde{N}(ds,\Gamma) + \int_0^t \hat{N}(ds,\Gamma), \quad (2.40)$$

where $\hat{N}(ds,\Gamma)$ is determined by (2.35). An application of Itô's formula and integration by parts shows that

$$J_t^\alpha := \exp[\alpha\int_0^{t+} N(ds,\Gamma) - \int_0^t (e^\alpha - 1)\hat{N}(ds,\Gamma)]$$

is a $\mathbf{Q}_\mu$-martingale for all $\alpha \in \mathbb{R}$. Combing (2.37) and (2.40) together and using Itô's formula and integration by parts again we see $I_t(\phi)J_t^\alpha$ is a $\mathbf{Q}_\mu$-martingale for all $\phi \in C^2(\mathbb{R})^{++}$. By Theorem 2.4 and Lemma 2.4, $I_t(\phi)$, J_t^α, $\mathbf{Q}'$, $\mathbf{Q}_\mu$ and $\mathcal{F}_{\tau-}$ satisfy the requirement of Theorem (3.2) in [16]. Hence, for any bounded stopping time t_0,

$$\mathbf{Q}_\mu[J_{t_0}^\alpha|\mathcal{F}_{\tau-}] = 1 \quad (a.s., \mathbf{Q}_\mu). \quad (2.41)$$

Since τ^1 is a stopping time and $\tau^1 \le T$, we can find a measurable function $f : (M(\mathbb{R}))^{\mathbf{N}} \to [0,T]$ and $0 \le t_1 < \cdots < t_n < \cdots \le T$ such that

$$\tau^1 = f(\omega_{t_1},\cdots,\omega_{t_n},\cdots).$$

Define

$$\tau' = f(X_{t_1\wedge\tau}^L,\cdots,X_{t_n\wedge\tau}^L,\cdots).$$

Note that $\tau^1 = \tau'$ if $\tau^1 < \tau^2$. On the other hand,

$$\begin{aligned}\mathbf{Q}_\mu[\tau \le t|\mathcal{F}_{\tau-}] &= 1_{[0,t]}(\tau')\mathbf{Q}_\mu[\tau^2 > \tau^1|\mathcal{F}_{\tau-}] + \mathbf{Q}_\mu[\tau^2 \le \tau^1 \wedge t|\mathcal{F}_{\tau-}] \\ &= 1_{[0,t]}(\tau')\mathbf{Q}_\mu[1 - \int_0^{\tau+} N(ds, M_l(\mathbb{R}))|\mathcal{F}_{\tau-}]\end{aligned}$$

$$+\mathbf{Q}_\mu[\int_0^{(t\wedge\tau)+} N(ds, M_l(\mathbb{R}))|\mathcal{F}_{\tau-}].$$

According to (2.41),

$$\begin{aligned}
&\mathbf{Q}_\mu[\int_0^{(t\wedge\tau)+} N(ds,\Gamma)|\mathcal{F}_{\tau-}] \\
&= \mathbf{Q}_\mu[\int_0^{t\wedge\tau} \hat{N}(ds,\Gamma)|\mathcal{F}_{\tau-}] \\
&= \int_0^t \mathbf{Q}_\mu[\tau > s|\mathcal{F}_{\tau-}] \int_{\mathbb{R}} X^L_{s\wedge\tau}(dx) \int_0^\infty \gamma(x,d\xi) 1_{\{\xi\delta_x\in\Gamma\}} ds
\end{aligned} \tag{2.42}$$

for any $\Gamma \in \mathfrak{B}(M_l(\mathbb{R}))$. Thus

$$\begin{aligned}
\mathbf{Q}_\mu[\tau \le t|\mathcal{F}_{\tau-}] \;=\; & 1_{[0,t]}(\tau')\left(1 - \int_0^t \mathbf{Q}_\mu[\tau > s|\mathcal{F}_{\tau-}] \int_{\mathbb{R}} X^L_{s\wedge\tau}(dx) \int_0^\infty \gamma(x,d\xi) 1_{\{\xi\delta_x\in M_l(\mathbb{R})\}} ds\right) \\
& + \int_0^t \mathbf{Q}_\mu[\tau > s|\mathcal{F}_{\tau-}] \int_{\mathbb{R}} X^L_{s\wedge\tau}(dx) \int_0^\infty \gamma(x,d\xi) 1_{\{\xi\delta_x\in M_l(\mathbb{R})\}} ds
\end{aligned}$$

and so

$$\mathbf{Q}_\mu[\tau > t|\mathcal{F}_{\tau-}] = 1_{(t,\infty)}(\tau') \exp\{-\int_0^t ds \int_{\mathbb{R}} X^L_{s\wedge\tau}(dx) \int_l^\infty \gamma(x,d\xi)\}.$$

Plugging this back into (2.42) and setting $t = T$, we obtain (2.39).

Finally, since $\omega_\tau = X^L_\tau + \int_0^{\tau+}\int \nu 1_{\{\langle 1,\nu\rangle\ge l\}} N(ds,d\nu)$, we see that the distribution of ω_τ under $\mathbf{Q}_\mu$ given $\mathcal{F}_{\tau-}$ is uniquely determined, and, therefore $\mathbf{Q}_\mu$ is uniquely determined on $\mathcal{F}_\tau$. □

Lemma 2.5 *Let $\mathbf{Q}_\mu$ be a solution of $(\mathcal{L},\mu)$-martingale problem. Given a finite stopping time β, let $\mathcal{Q}_\omega$ be a regular conditional probability distribution of $\mathbf{Q}_\mu|\mathcal{F}_\beta$. Then there is an $N \in \mathcal{F}_\beta$ such that $\mathbf{Q}_\mu(N) = 0$ and when $\omega \notin N$*

$$F(\omega'_{t\vee\beta(\omega)}) - F(\omega'_{\beta(\omega)}) - \int_{\beta(\omega)}^{t\vee\beta(\omega)} \mathcal{L}F(\omega'_s)ds$$

under $\mathcal{Q}_\omega$ is a martingale for $F \in \mathcal{D}_0(\mathcal{L})$. In particular, it is a local martingale for all $F \in \mathcal{D}(\mathcal{L})$.

Proof. The argument in this proof is exactly the same as that in Theorem 6.1.3 of [17]. We omit it here. □

Now, we come to our main theorem in this subsection.

Theorem 2.6 *Suppose that for $l > 1$, the $(\mathcal{L}',\mu)$-martingale problem is well-posed. Then uniqueness hold for $(\mathcal{L},\mu)$-martingale problem.*

Proof. Suppose $\mathbf{Q}_\mu$ is a solution of $(\mathcal{L},\mu)$-martingale problem. Define $\beta_0 = 0$ and

$$\beta_{n+1} = (\inf\{t \ge \beta_n : |\langle 1,\omega_t\rangle - \langle 1,\omega_{t-}\rangle| \ge l \text{ or } \langle 1,\omega_t\rangle - \langle 1,\omega_{\beta_n}\rangle \ge l\}) \wedge (\beta_n + l).$$

Then for each $n \ge 1$, β_n is a stopping time bounded by nl. By Lemma 2.5 and Theorem 2.5, we can prove by induction that $\mathbf{Q}_\mu$ is uniquely determined on $\mathcal{F}_{\beta_n}$ for all $n \ge 1$. In order to get the desired conclusion we only need to show that $\mathbf{Q}_\mu(\beta_n \le t) \to 0$ as $n \to \infty$ for each $t > 0$.

Let $\beta_0^1 = 0$ and $\beta_0^2 = 0$. Define

$$\beta_{n+1}^1 = \inf\{t \geq \beta_n^1 : \langle 1, \omega_t\rangle - \langle 1, \omega_{\beta_n^1}\rangle \geq l\}$$

and

$$\beta_{n+1}^2 = \inf\{t \geq \beta_n^2 : \langle 1, \omega_t\rangle - \langle 1, \omega_{t-}\rangle \geq l\}.$$

It is easy to see that in order to get the desired conclusion it suffices to show that $\mathbf{Q}_\mu(\beta_n^1 \leq t) \to 0$ and $\mathbf{Q}_\mu(\beta_n^2 \leq t) \to 0$ as $n \to \infty$. First, by Lemma 2.3, we can deduce that

$$\lim_{n\to\infty} \mathbf{Q}_\mu(\beta_n^1 \leq t) = 0.$$

Then

$$\begin{aligned}\sum_{0<s\leq t} 1_{\{\langle 1,\Delta\omega_s\rangle \geq l\}} &\leq \sum_{0<s\leq t} \langle 1, \Delta\omega_s\rangle 1_{\{\langle 1,\Delta\omega_s\rangle\geq l\}} \\ &= \int_0^{t+}\int_{M(\mathbb{R})^\circ} \langle 1, \nu\rangle 1_{\{\langle 1,\nu\rangle\geq l\}} N(ds, d\nu).\end{aligned}$$

But according to the *Step 4* in the proof of Theorem 2.3,

$$\mathbf{Q}_\mu\left[\int_0^{t+}\int_{M(\mathbb{R})^\circ} \langle 1, \nu\rangle 1_{\{\langle 1,\nu\rangle\geq l\}} N(ds, d\nu)\right] < \infty,$$

which yields that

$$\lim_{n\to\infty} \mathbf{Q}_\mu(\beta_n^2 \leq t) = 0.$$

□

3 Existence

3.1 Interacting-branching particle system

We first give a formulation of the interacting-branching particle system. Then we construct a solution of the $(\mathcal{L}', \mu)$-martingale problem by using particle system approximation. We recall that

$$G^m := \frac{1}{2}\sum_{i=i}^m a(x_i)\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i,j=1,i\neq j}^m \rho(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Suppose that $X_t = (x_1(t), \cdots, x_m(t))$ is a Markov process in $\mathbb{R}^m$ generated by G^m. By Lemma 2.3.2 of [2] we know that $X_t = (x_1(t), \cdots, x_m(t))$ is an exchangeable Feller process. Let $N(\mathbb{R})$ denote the space of integer-valued measures on $\mathbb{R}$. For $\theta > 0$, let $M_\theta(\mathbb{R}) = \{\theta^{-1}\sigma : \sigma \in N(\mathbb{R})\}$. Let ζ be the mapping from $\cup_{m=1}^\infty \mathbb{R}^m$ to $M_\theta(\mathbb{R})$ defined by

$$\zeta(x_1, \cdots, x_m) = \frac{1}{\theta}\sum_{i=1}^m \delta_{x_i}, \quad m \geq 1.$$

By Proposition 2.3.3 of [2] we know that $\zeta(X_t)$ is a Feller Markov process in $M_\theta(\mathbb{R})$ with generator $\mathcal{A}_\theta$ given by

$$\mathcal{A}_\theta F(\mu) = \frac{1}{2}\int_{\mathbb{R}} a(x)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) + \frac{1}{2\theta}\int_{\mathbb{R}^2} c(x)c(y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu(dx)$$

$$+\frac{1}{2}\int_{\mathbb{R}^2}\rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy). \tag{3.1}$$

In particular, if

$$F(\mu)=f\left(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle\right),\quad \mu\in M_\theta(\mathbb{R}), \tag{3.2}$$

for $f\in C^2(\mathbb{R}^n)$ and $\{\phi_i\}\subset C^2(\mathbb{R})$, then

$$\begin{aligned}\mathcal{A}_\theta F(\mu) &= \frac{1}{2}\sum_{i=1}^n f^i\left(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle\right)\langle a\phi_i'',\mu\rangle\\ &\quad+\frac{1}{2\theta}\sum_{i,j=1}^n f^{ij}\left(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle\right)\langle c^2\phi_i'\phi_j',\mu\rangle\\ &\quad+\frac{1}{2}\sum_{i,j=1}^n f^{ij}\left(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle\right)\int_{\mathbb{R}^2}\rho(x-y)\phi_i'(x)\phi_j'(y)\mu(dx)\mu(dy). \end{aligned}\tag{3.3}$$

Now we introduce a branching mechanism to the interacting particle system. Suppose that for each $x\in\mathbb{R}$ we have a discrete probability distribution $p(x)=\{p_i(x): i=0,1,\cdots\}$ such that each $p_i(\cdot)$ is a Borel measurable function on $\mathbb{R}$. This serves as the distribution of the offspring number produced by a particle that dies at site $x\in\mathbb{R}$. We assume that

$$\sum_{i=1}^\infty ip_i(x)\leq 1, \tag{3.4}$$

and

$$\sigma_p(x):=\sum_{i=1}^\infty i^2p_i(x) \tag{3.5}$$

is bounded in $x\in\mathbb{R}$. For $0\leq z\leq 1$, let

$$g(x,z):=\sum_{i=0}^\infty p_i(x)z^i. \tag{3.6}$$

Let $\Gamma_\theta(\mu,d\nu)$ be the probability kernel on $M_\theta(\mathbb{R})$ defined by

$$\int_{M_\theta(\mathbb{R})}F(\nu)\Gamma_\theta(\mu,d\nu)=\frac{1}{\langle 1,\mu\rangle}\left\langle\sum_{j=0}^\infty p_j(x)F\left(\mu+(j-1)\theta^{-1}\delta_x\right),\mu\right\rangle, \tag{3.7}$$

where $\mu\in M_\theta(\mathbb{R})$ is given by

$$\mu=\frac{1}{\theta}\sum_{i=1}^{\theta\langle 1,\mu\rangle}\delta_{x_i}.$$

For a constant $\lambda>0$, we define the bounded operator $\mathcal{B}_\theta$ on $B(M_\theta(\mathbb{R})$ by

$$\mathcal{B}_\theta F(\mu)=\lambda\theta(\theta\wedge\langle 1,\mu\rangle)\int_{M_\theta(\mathbb{R})}[F(\nu)-F(\mu)]\Gamma_\theta(\mu,d\nu). \tag{3.8}$$

For $\mathcal{A}_\theta$ generates a Markov process on $M_\theta(\mathbb{R})$, then $\mathcal{L}_\theta:=\mathcal{A}_\theta+\mathcal{B}_\theta$ also generates a Markov process; see Problem 4.11.3 of [6]. By martingale inequality and Theorem 4.3.6 of [6], we obtain that the corresponding Markov process has a modification with sample paths in $D([0,\infty),M_\theta(\mathbb{R}))$. We shall call the process generated by $\mathcal{L}_\theta$ an interacting-branching particle system with parameter $(a,\rho,\gamma,\lambda,p)$ and unit mass $1/\theta$.

3.2 Particle system approximation

Recall that

$$\Psi_0(x,z) := \frac{1}{2}\sigma(x)z^2 + \int_l^\infty \xi\gamma(x,d\xi)z + \int_0^l (e^{-z\xi} - 1 + z\xi)\gamma(x,d\xi). \tag{3.9}$$

According to the conditions (i) and (iii) on the σ and $\gamma(x,d\xi)$, $\Psi_0(x,\phi(x)) \in C(\mathbb{R})$ can be extended continuously to $\hat{\mathbb{R}}$ for $\phi \in C^2_\partial(\mathbb{R})^{++}$. And, if

$$F(\mu) = f(\langle \phi_1, \mu\rangle, \cdots, \langle \phi_n, \mu\rangle), \qquad \mu \in M(\mathbb{R}), \tag{3.10}$$

for $f \in C^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C^2(\mathbb{R})$, then

$$\begin{aligned} \mathcal{A}F(\mu) &= \frac{1}{2}\sum_{j=1}^n f^i(\langle \phi_1, \mu\rangle, \cdots, \langle \phi_n, \mu\rangle)\langle a\phi_i'', \mu\rangle \\ &\quad + \frac{1}{2}\sum_{i,j=1}^n f^{ij}(\langle \phi_1, \mu\rangle, \cdots, \langle \phi_n, \mu\rangle)\int_{\mathbb{R}^2} \rho(x-y)\phi_i'(x)\phi_j'(y)\mu^2(dxdy) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \mathcal{B}'F(\mu) &= \frac{1}{2}\sum_{i,j=1}^n f^{ij}(\langle \phi_1, \mu\rangle, \cdots, \langle \phi_n, \mu\rangle)\langle \sigma\phi_i\phi_j, \mu\rangle \\ &\quad - \int_{\mathbb{R}} \mu(dx)\int_l^\infty \xi\gamma(x,d\xi)\sum_{i=1}^n f^i(\langle \phi_1, \mu\rangle, \cdots, \langle \phi_n, \mu\rangle)\phi_i(x) \\ &\quad + \int_{\mathbb{R}} \mu(dx)\int_0^l \{f(\langle \phi_1, \mu\rangle + \xi\phi_1(x), \cdots, \langle \phi_n, \mu\rangle + \xi\phi_n(x)) \\ &\quad - f(\langle \phi_1, \mu\rangle, \cdots, \langle \phi_n, \mu\rangle) - \xi\sum_{i=1}^n f^i(\langle \phi_1, \mu\rangle, \cdots, \langle \phi_n, \mu\rangle)\phi_i(x)\}\gamma(x,d\xi) \end{aligned} \tag{3.12}$$

Suppose $\{X_t^{(k)} : t \geq 0\}$ is a sequence of *cádlág* interacting-branching particle systems with parameters $(a, \rho, \gamma, \lambda_k, p^{(k)})$ and unit mass $1/k$ and initial states $X_0^k = \mu_k \in M_k(\mathbb{R})$. We can regard $\{X_t^{(k)} : t \geq 0\}$ as a process with state space $M(\hat{\mathbb{R}})$. Let σ_p^k and g_k be defined by (3.5) and (3.6) respectively with p_i replaced by $p_i^{(k)}$. Let

$$\psi_k(x,z) := k\lambda_k[g_k(x, 1 - z/k) - (1 - z/k)], \quad 0 \leq z \leq k. \tag{3.13}$$

We have that $\frac{d}{dz}\psi_k(x,0+) = \lambda_k[1 - \frac{d}{dz}g_k(x,1)]$ and $\frac{d^2}{dz^2}\psi_k(x,0+) = \lambda_k\sigma_p^k/k$.

Lemma 3.1 *Suppose that the sequence $\{\lambda_k\sigma_p^k/k\}$ and $\{\langle 1, \mu_k\rangle\}$ are bounded. Then $\{X_t^{(k)} : t \geq 0\}$ form a tight sequence in $D([0,+\infty), M(\hat{\mathbb{R}}))$.*

Proof. By (3.4), it is easy to see that $\{\langle 1, X_t^{(k)}\rangle : t \geq 0\}$ is a supermartingale. By using martingale inequality, one can check that $\{X_t^{(k)} : t \geq 0\}$ satisfies the compact containment condition. Let $\mathcal{L}_k$ denote the generator of $\{X_t^{(k)} : t \geq 0\}$ and let F be given by (3.10) with $f \in C_0^2(\mathbb{R}^n)$ and with each $\phi_i \in C^2_\partial(\mathbb{R})^{++}$. Then

$$F(X_t^{(k)}) - F(X_0^{(k)}) - \int_0^t \mathcal{L}_kF(X_s^{(k)})ds, \quad t \geq 0,$$

is a martingale and the desired tightness result follows from Theorem 3.9.4 of Ethier and Kurtz [6]. □

In the sequel of this subsection, we assume $\{\phi_i\} \subset C^2_\partial(\mathbb{R})$. In this case, (3.10), (3.11) and (3.12) can be extended to continuous functions on $M(\hat{\mathbb{R}})$. Let $\hat{\mathcal{A}}F(\mu)$ and $\hat{\mathcal{B}}'F(\mu)$ be defined respectively by the right hand side of the (3.11) and (3.12) and let $\hat{\mathcal{L}}'F(\mu) = \hat{\mathcal{A}}F(\mu) + \hat{\mathcal{B}}'F(\mu)$, all defined as continuous functions on $M(\hat{\mathbb{R}})$.

Lemma 3.2 *Let $\mathcal{D}_0(\hat{\mathcal{L}}')$ be the totality of all functions of the form (3.10) with $f \in C^2_0(\mathbb{R}^n)$ and with each $\phi_i \in C^2_\partial(\mathbb{R})^{++}$. Suppose that $\mu_k \to \mu \in M(\hat{\mathbb{R}})$ as $k \to +\infty$ and the sequence $\{\lambda_k \sigma_p^k / k\}$ is bounded. If for each $h \geq 0$, $\psi_k(x,z) \to \Psi_0(x,z)$ uniformly on $\mathbb{R} \times [0,h]$ and $\frac{d}{dz}\psi_k(x,0+) \to \frac{d}{dz}\Psi_0(x,0)$ uniformly on $\mathbb{R}$ as $k \to +\infty$, then for each $F \in \mathcal{D}_0(\hat{\mathcal{L}}')$,*

$$F(\omega_t) - F(\omega_0) - \int_0^t \hat{\mathcal{L}}'F(\omega_s)ds, \quad t \geq 0, \tag{3.14}$$

is a martingale under any limit point $\mathbf{Q}_\mu$ of the distributions of $\{X_t^{(k)} : t \geq 0\}$, where $\{\omega_t : t \geq 0\}$ denotes the coordinate process of $D([0,\infty), M(\hat{\mathbb{R}}))$.

Proof. By passing to a subsequence if it is necessary, we may assume that the distribution of $\{X_t^{(k)} : t \geq 0\}$ on $D([0,+\infty), M(\hat{\mathbb{R}}))$ converges to $\mathbf{Q}_\mu$. Using Skorokhod's representation, we may assume that the processes $\{X_t^{(k)} : t \geq 0\}$ are defined on the same probability space and the sequence converges almost surely to a càdlàg process $\{X_t : t \geq 0\}$ with distribution $\mathbf{Q}_\mu$ on $D([0,\infty), M(\hat{\mathbb{R}}))$ ([6], p.102). Let $K(X) = \{t \geq 0 : \mathbf{P}\{X_t = X_{t-}\} = 1\}$. By Lemma 3.7.7 of [6], the complement of the set $K(X)$ is at most countable and by Proposition 3.5.2 of [6], for each $t \in K(X)$ we have a.s. $\lim_{k\to\infty} X_t^{(k)} = X_t$. Our proof will be divided into 3 steps.

Step 1. We shall show that

$$M_t(\phi) := \langle \phi, X_t \rangle - \langle \phi, X_0 \rangle - \frac{1}{2}\int_0^t \langle a\phi'', X_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} X_s(dx)\phi(x) \int_l^\infty \xi\gamma(x,d\xi), \quad t \geq 0, \tag{3.15}$$

is a square-integrable martingale with $\phi \in C^2_\partial(\mathbb{R})$. First, Fatou's Lemma tells us $\mathbf{E}\langle 1, X_t \rangle \leq \liminf_{k\to\infty} \mathbf{E}\langle 1, X_t^{(k)} \rangle$. On the other hand, for $\mu_k \in M_k(\mathbb{R})$ we can get that

$$\mathcal{L}_k \langle \phi, \mu_k \rangle = \frac{1}{2}\langle a\phi'', \mu_k \rangle - \frac{k \wedge \mu_k(1)}{\mu_k(1)} \langle \frac{d}{dz}\psi_k(x,0+)\phi(x), \mu_k \rangle.$$

Then for $t \in K(X)$

$$\mathbf{E}\langle 1, X_t \rangle \leq \liminf_{k\to\infty} \mathbf{E}\langle 1, X_t^{(k)} \rangle \leq \liminf_{k\to\infty} \mathbf{E}\langle 1, X_0^{(k)} \rangle \leq \langle 1, X_0 \rangle \tag{3.16}$$

and a.s.

$$\lim_{k\to\infty} \mathcal{L}_k \langle \phi, X_t^{(k)} \rangle = \hat{\mathcal{L}}' \langle \phi, X_t \rangle = \frac{1}{2}\langle a\phi'', X_t \rangle - \int_{\mathbb{R}} X_t(dx)\phi(x) \int_l^\infty \xi\gamma(x,d\xi).$$

Suppose that $\{H_i\}_{i=1}^n \subset C(M(\hat{\mathbb{R}}))$ and $\{t_i\}_{i=1}^{n+1} \subset K(X)$ with $0 \leq t_1 < \cdots < t_n < t_{n+1}$. Then

$$\mathbf{E}\big\{\big[\langle \phi, X_{t_{n+1}} \rangle - \langle \phi, X_{t_n} \rangle - \int_{t_n}^{t_{n+1}} \hat{\mathcal{L}}' \langle \phi, X_s \rangle ds\big] \prod_{i=1}^n H_i(X_{t_i})\big\}$$

$$\begin{aligned}
&= \mathbf{E}\{\langle\phi, X_{t_{n+1}}\rangle \prod_{i=1}^n H_i(X_{t_i})\} - \mathbf{E}\{\langle\phi, X_{t_n}\rangle \prod_{i=1}^n H_i(X_{t_i})\} \\
&\quad - \int_{t_n}^{t_{n+1}} \mathbf{E}\{\hat{\mathcal{L}}'\langle\phi, X_s\rangle \prod_{i=1}^n H_i(X_{t_i})\} ds \\
&= \lim_{k\to\infty} \mathbf{E}\{\langle\phi, X^{(k)}_{t_{n+1}}\rangle \prod_{i=1}^n H_i(X^{(k)}_{t_i})\} - \lim_{k\to\infty} \mathbf{E}\{\langle\phi, X^{(k)}_{t_n}\rangle \prod_{i=1}^n H_i(X^{(k)}_{t_i})\} \\
&\quad - \lim_{k\to\infty} \int_{t_n}^{t_{n+1}} \mathbf{E}\{\mathcal{L}_k\langle\phi, X^{(k)}_s\rangle \prod_{i=1}^n H_i(X^{(k)}_{t_i})\} ds \\
&= \lim_{k\to\infty} \mathbf{E}\{[\langle\phi, X^{(k)}_{t_{n+1}}\rangle - \langle\phi, X^{(k)}_{t_n}\rangle - \int_{t_n}^{t_{n+1}} \mathcal{L}_k\langle\phi, X^{(k)}_s\rangle ds] \prod_{i=1}^n H_i(X^{(k)}_{t_i})\} \\
&= 0.
\end{aligned}$$

Since $\{X_t : t \geq 0\}$ is right continuous, the equality

$$\mathbf{E}\{[\langle\phi, X_{t_{n+1}}\rangle - \langle\phi, X_{t_n}\rangle - \int_{t_n}^{t_{n+1}} \hat{\mathcal{L}}'\langle\phi, X_s\rangle ds] \prod_{i=1}^n H_i(X_{t_i})\} = 0$$

holds without the restriction $\{t_i\}_{i=1}^{n+1} \subset K(X)$. That is (3.15) is a martingale. Observe that if $F(\mu) = f(\langle 1, \mu\rangle)$ with $f \in C_0^2(\mathbb{R})$, then $\mathcal{A}_\theta F(\mu) = 0$ and $\mathcal{B}_\theta\mathcal{F}(\mu)$ is equal to

$$\frac{\lambda[\theta \wedge \langle 1, \mu\rangle]}{2\theta\langle 1, \mu\rangle} \sum_{j=1}^{+\infty} (j-1)^2 \langle p_j f''(\langle 1, \mu\rangle + \xi_j), \mu\rangle \tag{3.17}$$

for some constant $0 < \xi_j < (j-1)/\theta$. This follows from Taylor's expansion. Recall that the sequence $\{\lambda_k \sigma_p^k / k\}$ and $\{\langle 1, \mu_k\rangle\}$ are bounded. By the same argument as in the proof of Lemma 2.1, we have

$$\sup_k \mathbf{E}\langle 1, X^{(k)}_s\rangle^2 < \infty.$$

It follows from the Fatou's Lemma that $\mathbf{E}\langle 1, X_t\rangle^2$ ia a locally bounded function of $t \geq 0$. Thus (3.15) is a square-integrable martingale.

Step 2. We shall show that under $\mathbf{Q}_\mu$

$$\exp\{-\langle\phi, \omega_t\rangle\} - \exp\{-\langle\phi, \omega_0\rangle\} - \int_0^t \hat{\mathcal{L}}' \exp\{\langle\phi, \omega_s\rangle\} ds, \quad t \geq 0, \tag{3.18}$$

is a martingale for $\phi \in C_\partial^2(\mathbb{R})^{++}$. Let $\mu_k \in M_k(\mathbb{R})$ is given by

$$\mu_k = \frac{1}{k} \sum_{i=1}^{k\langle 1, \mu_k\rangle} \delta_{x_i}.$$

Note that

$$\begin{aligned}
\mathcal{A}_k \exp\{-\langle\phi, \mu_k\rangle\} &= -\frac{1}{2} \exp\{-\langle\phi, \mu_k\rangle\}\langle a\phi'', \mu_k\rangle + \frac{1}{2k} \exp\{-\langle\phi, \mu_k\rangle\}\langle (c\phi')^2, \mu_k\rangle \\
&\quad + \frac{1}{2} \exp\{-\langle\phi, \mu_k\rangle\} \int_{\mathbb{R}^2} \rho(x-y)\phi'(x)\phi'(y)\mu_k(dx)\mu_k(dy)
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\mathcal{B}_k \exp\{-\langle \phi, \mu_k\rangle\} \\
&= \frac{k\lambda_k(k\wedge \mu_k(1))}{\mu_k(1)}\Big\langle [\sum_{j=0}^{\infty} p_j(x) e^{-\langle \phi,\mu_k\rangle - \frac{j-1}{k}\phi(x)} - \sum_{j=0}^{\infty} p_j(x) e^{-\langle \phi,\mu_k\rangle}], \mu_k\Big\rangle \\
&= \exp\{-\langle \phi, \mu_k\rangle\}\Big\langle \frac{k\lambda_k(k\wedge \mu_k(1))}{\mu_k(1)}[\sum_{j=0}^{\infty} p_j(x)(e^{-\frac{j-1}{k}\phi(x)} - 1)], \mu_k\Big\rangle \\
&= \exp\{-\langle \phi, \mu_k\rangle\}\Big\langle \frac{(k\wedge \mu_k(1))}{\mu_k(1)}\psi_k(x, k - ke^{-\phi(x)/k})e^{\phi(x)/k}, \mu_k\Big\rangle. \qquad (3.20)
\end{aligned}$$

Since for each $h \geq 0$, $\psi_k(x,z) \to \Psi_0(x,z)$ uniformly on $\mathbb{R}\times[0,h]$, we conclude for $t \in K(X)$ a.s. $\lim_{k\to\infty} \mathcal{L}_k \exp\{-\langle \phi, X_t^{(k)}\rangle\} = \hat{\mathcal{L}}' \exp\{-\langle \phi, X_t\rangle\}$ boundedly by (3.16), (3.19), (3.20) and the definition of $\hat{\mathcal{L}}'$. By the same argument as in *Step 1* we can get that (3.18) is a martingale. That is

$$W_t(\phi) := e^{-\langle \phi, X_t\rangle} - \int_0^t e^{-\langle \phi, X_s\rangle}[-\frac{1}{2}\langle a\phi'', X_s\rangle + \frac{1}{2}\int_{\hat{\mathbb{R}}} \langle h(z-\cdot)\phi', X_s\rangle^2 dz + \langle \Psi_0(\phi), X_s\rangle]ds, \ t\geq 0, \qquad (3.21)$$

is a martingale with $\phi \in C_\partial^2(\mathbb{R})^{++}$, where $\Psi_0(\phi) := \Psi_0(x, \phi(x))$. Then $\{\exp\{-\langle \phi, X_t\rangle\} : t \geq 0\}$ is a special semi-martingale with $\phi \in C_\partial^2(\mathbb{R})^{++}$.

Step 3. Let $S(\hat{\mathbb{R}})$ denote the space of finite signed Borel measures on $\hat{\mathbb{R}}$ endowed with the σ-algebra generated by the mappings $\mu \mapsto \langle 1, \mu\rangle$ for all $f \in C(\hat{\mathbb{R}})$. Let $S(\hat{\mathbb{R}})^\circ = S(\hat{\mathbb{R}}) \setminus \{0\}$. We define the optional random measure $N(ds, d\nu)$ on $[0,\infty)\times S(\hat{\mathbb{R}})^\circ$ by

$$N(ds, d\nu) = \sum_{s>0} 1_{\{\Delta X_s \neq 0\}} \delta_{(s,\Delta X_s)}(ds, d\nu),$$

where $\Delta X_s = X_s - X_{s-} \in S(\hat{\mathbb{R}})$. Let $\hat{N}(ds, d\nu)$ denote the predictable compensator of $N(ds, d\nu)$ and let $\tilde{N}(ds, d\nu)$ denote the corresponding measure. By the same argument as in the proof of Theorem 2.3, we can obtain that for $\phi \in C_\partial^2(\mathbb{R})$

$$\begin{aligned}
\langle \phi, X_t\rangle &= \langle \phi, \mu\rangle + \int_0^t \langle a\phi'', X_s\rangle ds + M_t^c(\phi) + \int_0^{t+}\int_{S(\hat{\mathbb{R}})} \nu(\phi)\tilde{N}(ds, d\nu) \\
&\quad - \int_0^t ds \int_{\hat{\mathbb{R}}} X_s(dx)\phi(x) \int_l^\infty \xi\gamma(x, d\xi), \qquad (3.22)
\end{aligned}$$

where $M_t^c(\phi)$ is a continuous local martingale. We also conclude that the jump measure of the process X has compensator

$$\hat{N}(ds, d\nu) = dsX_s(dx)1_{\{0<\xi<l\}}\gamma(x, d\xi)\cdot\ \delta_{\xi\delta_x}(d\nu), \quad \nu \in M(\hat{\mathbb{R}})\setminus\{0\}, \qquad (3.23)$$

and for $\{\phi_i\}_{i=1}^2 \subset C_\partial^2(\mathbb{R})^{++}$,

$$\begin{aligned}
\langle M^c(\phi_1), M^c(\phi_2)\rangle_t &= \frac{1}{2}\int_0^t \int_{\hat{\mathbb{R}}^2} \rho(x-y)\phi_1'(x)\phi_2'(y)X_s(dx)X_s(dy)ds \\
&\quad + \frac{1}{2}\int_0^t \int_{\hat{\mathbb{R}}^2} \rho(x-y)\phi_2'(x)\phi_1'(y)X_s(dx)X_s(dy)ds \\
&\quad + \int_0^t \langle \sigma\phi_1\phi_2, X_s\rangle ds. \qquad (3.24)
\end{aligned}$$

Let $f\in C_0^2(\mathbb{R}^n)$ and $\{\phi_i\}_{i=1}^n\subset C_\partial^2(\mathbb{R})^{++}$. By (3.22), (3.23), (3.24) and Itô's formula, we obtain

$$\begin{aligned}
& f(\langle\phi_1,X_t\rangle,\cdots,\langle\phi_n,X_t\rangle)\\
= & f(\langle\phi_1,X_0\rangle,\cdots,\langle\phi_n,X_0\rangle)+\frac{1}{2}\sum_{i=1}^n\int_0^t f^i(\langle\phi_1,X_s\rangle,\cdots,\langle\phi_n,X_s\rangle)\langle a\phi_i'',X_s\rangle ds\\
& +\frac{1}{2}\sum_{i,j=1}^n\int_0^t f^{ij}(\langle\phi_1,X_s\rangle,\cdots,\langle\phi_n,X_s\rangle)d\langle M^c(\phi_i),M^c(\phi_j)\rangle_t\\
& -\frac{1}{2}\sum_{i=1}^n\int_0^t ds\int_{\hat{\mathbb{R}}}X_s(dx)\int_l^\infty\gamma(x,d\xi)f^i(\langle\phi_1,X_s\rangle,\cdots,\langle\phi_n,X_s\rangle)\xi\phi_i(x)\\
& +\int_0^t ds\int_{\hat{\mathbb{R}}}X_s(dx)\int_0^l\gamma(x,d\xi)\{f(\langle\phi_1,X_s\rangle+\xi\phi_1(x),\cdots,\langle\phi_n,X_s\rangle+\xi\phi_n(x))\\
& \quad -f(\langle\phi_1,X_s\rangle,\cdots,\langle\phi_n,X_s\rangle)-\xi\sum_{i=1}^n\phi_i(x)f^i(\langle\phi_1,X_s\rangle,\cdots,\langle\phi_n,X_s\rangle)\}\\
& +(loc.mart.).
\end{aligned}$$

Hence

$$F(X_t)-F(X_0)-\int_0^t\hat{\mathcal{L}}'F(X_s)ds,\quad t\geq 0,$$

is a local martingale for each $F\in\mathcal{D}_0(\hat{\mathcal{L}}')$. Since $f\in C_0^2(\mathbb{R}^n)$ and $\phi_i\in C_\partial^2(\mathbb{R})^{++}$, both F and $\hat{\mathcal{L}}'F$ are bounded functions on $M(\hat{\mathbb{R}})$. Thus (3.14) is martingale. We complete the proof. □

Lemma 3.3 *Let $\mathcal{D}_0(\hat{\mathcal{L}}')$ be as in Lemma 3.2. Then for each $\mu\in M(\hat{\mathbb{R}})$, there is a probability measure $\mathbf{Q}_\mu$ on $D([0,\infty),M(\hat{\mathbb{R}}))$ under which (3.14) is a martingale for each $F\in\mathcal{D}_0(\hat{\mathcal{L}}')$.*

Proof. We only need to construct a sequence $\psi_k(x,z)$ such that for each $h\geq 0$, $\psi_k(x,z)\to\Psi_0(x,z)$ uniformly on $\mathbb{R}\times[0,h]$, and $\frac{d}{dz}\psi_k(x,0+)\to\frac{d}{dz}\Psi_0(x,0)$ uniformly on $\mathbb{R}$ as $k\to+\infty$. Moreover, $\{\frac{d^2}{dz^2}\psi_k(x,0+)\}$ should be a bounded sequence.

Let $\Psi_1(x,z)=\frac{1}{2}\sigma(x)z^2+\int_0^l(e^{-z\xi}-1+z\xi)\gamma(x,d\xi)$. We first define the sequences

$$\lambda_{1,k}=1+k||\sigma||+\sup_x\int_0^l\xi(1-e^{-k\xi})\gamma(x,d\xi)$$

and

$$g_{1,k}(x,z)=z+\frac{\Psi_1(x,k(1-z))}{k\lambda_{1,k}}.$$

It is easy to check that $g_{1,k}(x,1)=1$ and

$$\frac{d^n}{dz^n}g_{1,k}(x,z)\geq 0,\quad x\in\mathbb{R},\ 0\leq z\leq 1,$$

for all integer $n\geq 0$. Consequently, $g_{1,k}(x,\cdot)$ is a probability generating function. Let $\psi_{1,k}(x,z)$ be defined by (3.13) with (λ_k,g_k) replaced by $(\lambda_{1,k},g_{1,k})$. Then

$$\psi_{1,k}(x,z)=\Psi_1(x,z)\ \text{ for }\ 0\leq z\leq k.$$

Let $b(x):=\int_l^\infty\xi\gamma(x,d\xi)$. Suppose $||b||>0$. Set

$$g_{2,k}(x,z)=z+||b||^{-1}b(x)(1-z).$$

Then $g_{2,k}(x,\cdot)$ is a probability generating function. Let $\lambda_{2,k} = ||b||$ and let $\psi_{2,k}(x,z)$ be defined by (3.13) with (λ_k, g_k) replaced by $(\lambda_{2,k}, g_{2,k})$. Then we have

$$\psi_{2,k}(x,z) = b(x)z.$$

Finally we let $\lambda_k = \lambda_{1,k} + \lambda_{2,k}$ and $g_k = \lambda_k^{-1}(\lambda_{1,k}g_{1,k} + \lambda_{2,k}g_{2,k})$. Then the sequence ψ_k defined by (3.13) is equal to $\psi_{1,k} + \psi_{2,k}$ which satisfies the required conditions obviously. □

Theorem 3.1 *Let $\{\omega_t : t \geq 0\}$ denote the coordinate process of $D([0,\infty), M(\mathbb{R}))$. Then for each $\mu \in M(\mathbb{R})$ there is a probability measure $\mathbf{Q}_\mu$ on $D([0,\infty), M(\mathbb{R}))$ such that $\{\omega_t : t \geq 0\}$ under $\mathbf{Q}_\mu$ is a solution of the $(\mathcal{L}', \mu)$-martingale problem.*

Proof. For each $\mu \in M(\mathbb{R})$, let $\mathbf{Q}_\mu$ be the probability measure on $D([0,\infty), M(\hat{\mathbb{R}}))$ provided by Lemma 3.2. We claim that for any $T > 0$

$$\mathbf{Q}_\mu\{\omega_t(\{\partial\}) = 0 \text{ for all } t \in [0,T]\} = 1.$$

Consequently, $\mathbf{Q}_\mu$ is supported by $D([0,\infty), M(\mathbb{R}))$. In fact, for any $\phi \in C^2_\partial(\mathbb{R})^+$, by *Step* 1 in the proof of Lemma 3.2,

$$M_t(\phi) := \langle \phi, \omega_t\rangle - \langle \phi, \mu\rangle - \frac{1}{2}\int_0^t \langle a\phi'', \omega_s\rangle ds + \int_0^t ds \int_{\hat{\mathbb{R}}} \omega_s(dx)\phi(x)\int_l^\infty \xi\gamma(x, d\xi), \quad t \geq 0, \tag{3.25}$$

is a càdlàg square-integrable martingale with quadratic variation process given by

$$\langle M(\phi)\rangle_t = \int_0^t \langle(\sigma + \int_0^l \xi^2\gamma(\cdot, d\xi))\phi^2, \omega_s\rangle ds + \int_0^t ds \int_{\hat{\mathbb{R}}} \langle h(z-\cdot)\phi', \omega_s\rangle^2 dz.$$

For $k \geq 1$, let

$$\phi_k(x) = \begin{cases} \exp\{-\frac{1}{|x|^2 - k^2}\}, & \text{if } |x| > k, \\ 0, & \text{if } |x| \leq k. \end{cases}$$

One can check that $\{\phi_k\} \subset C^2_\partial(\mathbb{R})$ such that $\lim_{|x|\to\infty} \phi_k(x) = 1$, $\lim_{|x|\to\infty} \phi_k(x)' = 0$ and $\phi_k(\cdot) \to 1_{\{\partial\}}(\cdot)$ boundedly and pointwise. $||\phi_k'|| \to 0$ and $||\phi_k''|| \to 0$ as $k \to \infty$. Let $\sigma_0 = \sigma + \int_0^l \xi^2\gamma(\cdot, d\xi)$. By Theorem 1.6.10 of [8], we have

$$\begin{aligned} &\mathbf{Q}_\mu\{\sup_{0\leq t\leq T} |M_t(\phi_k) - M_t(\phi_j)|^2\} \\ \leq\quad & 4\int_0^T \mathbf{Q}_\mu\{\langle \sigma_0(\phi_k - \phi_j)^2, \omega_s\rangle\} ds + 4\int_0^T ds \int_{\hat{\mathbb{R}}} \mathbf{Q}_\mu\{\langle h(z-\cdot)(\phi_k' - \phi_j'), \omega_s\rangle^2\} dz. \end{aligned}$$

By dominated convergence theorem, $\mathbf{Q}_\mu\{\sup_{0\leq t\leq T} |M_t(\phi_k) - M_t(\phi_j))|^2\} \to 0$ as $k, j \to 0$. Therefore, there exists $M^\partial = (M_t^\partial)_{t\geq 0}$ such that for every $t > 0$,

$$\mathbf{Q}_\mu\{|M_t(\phi_k) - M_t^\partial|^2\} \to 0$$

and

$$\sup_{0\leq s\leq t} |M_s(\phi_k) - M_s^\partial| \to 0 \quad \text{in probabilty}$$

as $k \to \infty$. We obtain M^∂ has càdlàg path. By Lemma 2.1.2 of [8], M^∂ is a square-integrable martingale with zero mean. It follows from (3.25) that

$$M_t^\partial := \omega_t(\{\partial\}) + \int_0^t ds\omega_s(\{\partial\}) \int_l^\infty \xi\gamma(\partial, d\xi)$$

is a càdlàg square-integrable martingale with zero mean. Thus $\mathbf{Q}_\mu(\omega_t(\{\partial\})) = 0$. Then the claim follows from the right continuity of $\{\omega_t(\{\partial\}) : t \geq 0\}$. We have

$$F(\omega_t) - F(\omega_0) - \int_0^t \mathcal{L}' F(\omega_s) ds, \quad t \geq 0,$$

is martingale for $F \in \mathcal{D}_0(\hat{\mathcal{L}}')$. Thus by Remark 2.1, it is a local martingale for $F \in \mathcal{D}(\mathcal{L})$. □

Combining Theorem 2.2 and Theorem 3.1 we get that the $(\mathcal{L}', \mu)$-martingale problem is well-posed. The following theorem will show that the existence of solutions to $(\mathcal{L}, \mu)$-martingale problem.

Theorem 3.2 *For each $\mu \in M(\mathbb{R})$ there is a probability measure $\mathbf{Q}_\mu$ on $(\Omega, \mathcal{F})$ such that $\mathbf{Q}_\mu$ is a solution of the $(\mathcal{L}, \mu)$-martingale problem.*

Proof. Let $\lambda_n(\mu) = 1_{\{\langle 1,\mu\rangle < n\}} \int \mu(dx) \int_l^\infty \gamma(x, d\xi)$ and define a transition function on $M(\mathbb{R}) \times \mathfrak{B}(M(\mathbb{R}))$ by

$$\Gamma(\mu, d\nu) := \begin{cases} \delta_\mu(d\nu), & \int \mu(dx) \int_l^\infty \gamma(x, d\xi) = 0, \\ (\int \mu(dx) \int_l^\infty \gamma(x, d\xi))^{-1} \mu(dx)\gamma(x, d\xi)\delta_{\mu+\xi\delta_x}(d\nu), & \textit{otherwise}. \end{cases}$$

Define $\mathcal{B}_n$ on $B(M(R))$ by

$$B_n F(\mu) := \lambda_n(\mu) \int (F(\nu) - F(\mu))\Gamma(\mu, d\nu) = 1_{\{\langle 1,\mu\rangle < n\}} \int \mu(dx) \int_l^\infty (F(\mu + \xi\delta_x) - F(\mu))\gamma(x, d\xi).$$

Since the $(\mathcal{L}', \mu)$-martingale problem is well-posed, there exists a semigroup $(Q_t')_{t\geq 0}$ on $B(M(R))$ with transition function given by (2.19) and full generator denoted by $\mathcal{L}_0'$. We can follow from Problem 4.11.3 of [6] to conclude that there exists a Markov process denoted by $X^n = \{X_t^n : t \geq 0\}$ whose transition semigroup has full generator given by $\mathcal{L}_0' + \mathcal{B}_n$. In the following we assume that $X_0^n = \mu$ *a.s.*. Thus $(\mathcal{L}_0' + \mathcal{B}_n, \mu)$-martingale problem is well-posed. Since $\mathcal{L}' + \mathcal{B}_n \subset \mathcal{L}_0' + \mathcal{B}_n$, X^n is also a solution of $(\mathcal{L}' + \mathcal{B}_n, \mu)$-martingale problem. Let $U_n := \{\mu \in M(\mathbb{R}) : \langle 1, \mu\rangle < n\}$. According to Theorem 4.3.6 of [6], there is a modification of X^n with sample path in $D([0, \infty), M(\mathbb{R}))$. Set

$$\tau^n := \inf\{t \geq 0 : \langle 1, X_t^n\rangle \geq n \text{ or } \langle 1, X_{t-}^n\rangle \geq n\}$$

and $\tilde{X}^n = X^n_{\cdot\wedge\tau^n}$. Then $\tilde{X}^n$ is a solution of the stopped martingale problem for $(\mathcal{L}, U_n)$ and by Theorem 4.6.1 of [6], $\tilde{X}^n$ is the unique solution of the stopped martingale problem for $(\mathcal{L}_0' + \mathcal{B}_n, \delta_\mu, U_n)$. Put

$$\tau_k^n := \inf\{t \geq 0 : \langle 1, \tilde{X}_t^n\rangle \geq k \text{ or } \langle 1, \tilde{X}_{t-}^n\rangle \geq k\}.$$

For $k < n$, $\tilde{X}^n_{\cdot\wedge\tau_k^n}$ is a solution of the stopped martingale problem for $(\mathcal{L}_0' + \mathcal{B}_k, \delta_\mu, U_k)$ and hence has the same distribution as $\tilde{X}^k$. On the other hand, since $\tilde{X}^n$ is a solution of the stopped martingale problem for $(\mathcal{L}, U_n)$, it follows from Lemma 2.3 that

$$\sup_n \mathbf{E} \sup_{0\leq s\leq t} \langle 1, \tilde{X}_s^n\rangle < \infty.$$

Thus for each $t > 0$,

$$\lim_{n\to\infty} \mathbf{P}\{\tau^n \leq t\} = 0.$$

For any $k,\ m \geq 1$, let $Y^k,\ Y^m$ be two $D([0,\infty), M(\mathbb{R}))$-valued random variables such that they have same distributions with $\tilde{X}^k$ and $\tilde{X}^m$ respectively and $Y^k(t) = Y^m(t)$ for $t \leq \tau^{k\wedge m}$. Thus the Skorohod distance between Y^k and Y^m is less than $e^{-\tau^{k\wedge m}}$. By Corollary 3.1.6 of [6], we conclude that there exist a process X^∞ such that $\tilde{X}^n \Rightarrow X^\infty$. Let

$$\tau_n^\infty = \inf\{t \geq 0 : \langle 1, X_t^\infty\rangle \geq n \text{ or } \langle 1, X_{t-}^\infty\rangle \geq n\}.$$

Since the distribution of $\tilde{X}^m_{\cdot\wedge\tau_n^m}$ does not depend on $m \geq n$, $X^\infty_{\cdot\wedge\tau_n^\infty}$ has the same distribution with $\tilde{X}^n$. Therefore,

$$\mathbf{P}\{\tau_n^\infty \leq t\} = \mathbf{P}\{\tau^n \leq t\}$$

and for each $F \in \mathcal{D}(\mathcal{L})$

$$F(X^\infty_{t\wedge\tau_n^\infty}) - \int_0^{t\wedge\tau_n^\infty} \mathcal{L}F(X_s^\infty)ds$$

is a martingale for each n. We see X^∞ is a solution of the $(\mathcal{L}, \mu)$-martingale problem. □

Combining Theorem 2.6 and Theorem 3.2, we have that the $(\mathcal{L}, \mu)$-martingale problem is well-posed. Thus we complete the construction of SDSM with general branching mechanism.The next theorem gives another martingale characterization of SDSM which is a direct consequence of Theorem 2.3 and Itô's formula.

Theorem 3.3 *Let $\{\omega_t : t \geq 0\}$ denote the coordinate process of $D([0,\infty), M(\mathbb{R}))$. Then a probability measure $\boldsymbol{Q}_\mu$ on $D([0,\infty), M(\mathbb{R}))$ is a solution of $(\mathcal{L}, \mu)$-martingale problem if and only if for $\mu \in M(\mathbb{R})$ and $\phi \in C_c^2(\mathbb{R})^+$, $\{\langle\phi, \omega_t\rangle\}$ is a semimartingale which has canonical decomposition given by*

$$\begin{aligned}\langle\phi, \omega_t\rangle \;=\; &\langle\phi, \mu\rangle + \int_0^t \langle a\phi'', \omega_s\rangle ds + M_t^c(\phi) + \int_0^{t+}\int_{M(\mathbb{R})^\circ} \langle\phi, \nu\rangle 1_{\{\langle 1,\nu\rangle<1\}}\tilde{N}(ds, d\nu)\\ &+ \int_0^{t+}\int_{M(\mathbb{R})^\circ} \langle\phi, \nu\rangle 1_{\{\langle 1,\nu\rangle\geq 1\}}N(ds, d\nu) - \int_0^t ds\int_{\mathbb{R}} \omega_s(dx)\int_1^\infty \xi\gamma(x, d\xi)\phi(x),\end{aligned} \tag{3.26}$$

where $\{M_t^c(\phi) : t \geq 0\}$ is a continuous local martingale with quadratic variation process given by

$$\langle M^c(\phi)\rangle_t = \int_0^t \langle\sigma\phi^2, \omega_s\rangle ds + \int_0^t ds\int_{\mathbb{R}} \langle h(z-\cdot)\phi', \omega_s\rangle^2 dz, \tag{3.27}$$

and

$$N(ds, d\nu) = \sum_{s>0} 1_{\{\Delta\omega_s\neq 0\}}\delta_{(s,\Delta\omega_s)}(ds, d\nu)$$

is an optional random measure on $[0,\infty)\times M(\mathbb{R})^\circ$, where $\Delta\omega_s = \omega_s - \omega_{s-} \in M(\mathbb{R})$ and $\tilde{N}(ds, d\nu)$ denotes the corresponding martingale measure. The predictable compensator of $N(ds, d\nu)$ is given by $\hat{N}(ds, d\nu) = dsK(\omega_s, d\nu)$, where $K(\mu, d\nu)$ is determined by

$$\int_{M(\mathbb{R})^\circ} F(\nu)K(\mu, d\nu) = \int_{\mathbb{R}} \mu(dx)\int_0^\infty F(\xi\delta_x)\gamma(x, d\xi)$$

for $F \in B(M(\mathbb{R}))$.

4 Moment formulas, mean and spatial covariance measures

In this section, we construct a dual process for SDSM and investigate some properties of SDSM. In accordance with the notation used in Subsection 2.2, we can define a function-valued Markov process by

$$Y_t' = P_{t-\tau_k}^{M_{\tau_k}} \Gamma_k P_{\tau_k-\tau_{k-1}}^{M_{\tau_{k-1}}} \Gamma_{k-1} \cdots P_{\tau_2-\tau_1}^{M_{\tau_1}} \Gamma_1 P_{\tau_1}^{M_0} Y_0, \quad \tau_k \le t < \tau_{k+1}, \quad 0 \le k \le M_0 - 1. \tag{4.1}$$

Let $X = \{X_t : t \ge 0\}$ be an SDSM which is the unique solution of the martingale problem for $\mathcal{L}$. If for $m \ge 2$, $\sup_x [\int_0^\infty \xi^m \gamma(x, d\xi)] < \infty$, then by the same argument as in the proof of Lemma 2.1 and martingale inequality, we have that

$$\mathbf{E} \sup_{0 \le s \le t} \langle 1, \omega_s \rangle^m < \infty.$$

Then it follows from the same argument of Theorem 2.1 that

$$\mathbf{E} \langle f, X_t^m \rangle = \mathbf{E}_{m,f}^{\sigma,\gamma} \big[\langle Y_t', \mu^{M_t} \rangle \exp \big\{ \int_0^t (2^{M_s} + \frac{M_s(M_s-1)}{2} - M_s - 1) ds \big\} \big] \tag{4.2}$$

for any $t \ge 0$ and $f \in B(\mathbb{R}^m)$.

Skoulakis and Adler [15] computed moments as a limit of moments for the particle picture; see Section 3 of [15]. Stimulated by [15], in this section, we compute moments via the dual relationship (4.2). In fact, by the construction (4.1) of $\{Y_t' : t \ge 0\}$ we have

$$\begin{aligned}
& \mathbf{E}_{m,f}^{\sigma,\gamma} \left[\langle Y_t', \mu^{M_t} \rangle \exp\{ \int_0^t (2^{M_s} + \frac{M_s(M_s-1)}{2} - M_s - 1) ds \} \right] \\
= \; & \langle P_t^m f, \mu^m \rangle \\
& + \frac{1}{2} \sum_{i,j=1, i \ne j}^{m} \int_0^t \mathbf{E}_{m-1, \Psi_{ij} P_u^m f}^{\sigma,\gamma} \left[\langle Y_{t-u}', \mu^{M_{t-u}} \rangle \exp\{ \int_0^{t-u} (2^{M_s} + \frac{M_s(M_s-1)}{2} - M_s - 1) ds \} \right] du \\
& + \sum_{a=2}^{m} \Big(\sum_{\{a\}}^{m} \int_0^t \mathbf{E}_{m-k+1, \Phi_{i_1,\cdots,i_a} P_u^m f}^{\sigma,\gamma} \big[\langle Y_{t-u}', \mu^{M_{t-u}} \rangle \\
& \qquad \times \exp\{ \int_0^{t-u} (2^{M_s} + \frac{M_s(M_s-1)}{2} - M_s - 1) ds \} \big] du \Big),
\end{aligned} \tag{4.3}$$

where $\{a\} = \{1 \le i_1 < i_2 < \cdots < i_a \le m\}$. We remark that if $\inf_{x \in \mathbb{R}} |c(x)| \ge \epsilon > 0$, the semigroup $(P_t^m)_{t>0}$ is uniformly elliptic and has density $p_t^m(x, y)$ satisfying

$$p_t^m(x, y) \le \text{const} \cdot g_{\varepsilon t}^m(x, y), \quad t > 0, x, y \in \mathbb{R}^m,$$

where $g_t^m(x, y)$ denotes the transition density of the m-dimensional standard Brownian motion (see Theorem 0.5 of [5]). In the following we always assume that $\sup_x [\int \xi^2 \gamma(x, d\xi)] < \infty$.

Theorem 4.1 *Suppose that $(\Omega, X_t, \mathbf{Q}_\mu)$ is a realization of the SDSM with parameters (a, ρ, Ψ) with $\inf_x |c(x)| \ge \epsilon > 0$. Let $f \in B(\mathbb{R})$ and $t > 0$. Then we have the first moment formula for X as follows:*

$$\mathbf{E}(\langle f, X_t \rangle) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) p_t(x, y) dy \mu(dx), \tag{4.4}$$

and $\forall\, 0 < s \le t$, $f \in B(\mathbb{R})$ and $g \in B(\mathbb{R})$, we have the second order moment formula

$$\mathbf{E}(\langle f, X_s \rangle \langle g, X_t \rangle)$$

$$
\begin{aligned}
&= \mathbf{E}(\langle f, X_s\rangle\langle P_{t-s}g, X_s\rangle)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(y_1)\left(\int_{\mathbb{R}} g(z)p_{t-s}(y_2,z)dz\right)p_s^2(x,y;y_1,y_2)dy_1dy_2\mu(dy)\mu(dx)\\
&\quad+\int_0^s du\int_{\mathbb{R}}\mu(dx)\int_{\mathbb{R}}dy\int_{\mathbb{R}^2}dy_1dy_2p_{s-u}(x,y)\sigma(y)p_u^2(y,y;y_1,y_2)\\
&\quad\times f(y_1)\left(\int_{\mathbb{R}}p_{t-s}(y_2,z)g(z)dz\right)\\
&\quad+\int_0^s du\int_{\mathbb{R}}\mu(dx)\int_{\mathbb{R}}dy\int_{\mathbb{R}^2}dy_1dy_2p_{s-u}(x,y)\left(\int_0^\infty \xi^2\gamma(y,d\xi)\right)p_u^2(y,y;y_1,y_2)\\
&\quad\times f(y_1)\left(\int_{\mathbb{R}}p_{t-s}(y_2,z)g(z)dz\right). \qquad (4.5)
\end{aligned}
$$

Proof. (4.4) is a direct conclusion of (4.3). Using (4.4) and Markov property of X we have $\mathbf{E}(\langle f, X_s\rangle\langle g, X_t\rangle) = \mathbf{E}(\langle f, X_s\rangle\langle P_{t-s}g, X_s\rangle)$. Then (4.5) is also a direct conclusion of (4.3). □

Following [15], we define two deterministic measures as follows:

1. The *mean measure* m_t defined on $\mathcal{B}(\mathbb{R})$ by

$$m_t(A) = \mathbf{E}(X_t(A)).$$

2. The *spatial measure* s_t defined on $\mathcal{B}(\mathbb{R}\times\mathbb{R})$ by

$$s_t(A_1\times A_2) = \mathbf{E}(X_t(A_1)X_t(A_2)).$$

By Theorem 4.1, we have following proposition.

Proposition 4.1 *For all $t>0$ the measures m_t and s_t have densities with respect to Lebesgue measure,denoted by $m(t;y)$ and $s(t;y_1,y_2)$, respectively. We have that*

$$m(t;y) = \int_{\mathbb{R}} p_t(x,y)\mu(dx)$$

for all $y\in\mathbb{R}$ and

$$
\begin{aligned}
s(t;y_1,y_2) &= \int_{\mathbb{R}^2} p_t^2(y,z;y_1,y_2)\mu(dy)\mu(dz)\\
&\quad+\int_0^t ds\int_{\mathbb{R}}\mu(dy)\int_{\mathbb{R}}dz\sigma(z)p_s^2(z,z;y_1,y_2)p_{t-s}(y,z)\\
&\quad+\int_0^t ds\int_{\mathbb{R}}\mu(dy)\int_{\mathbb{R}}dz\int_0^\infty \xi^2\gamma(z,d\xi)p_s^2(z,z;y_1,y_2)p_{t-s}(y,z) \qquad (4.6)
\end{aligned}
$$

for all $y_1,y_2\in\mathbb{R}$.

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