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Stochastic Equations of Super-Lévy Processes with General Branching Mechanism[1]

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Abstract. In this work, the process of distribution functions of a one-dimensional super-Lévy process with general branching mechanism is characterized as the pathwise unique solution of a stochastic integral equation driven by time-space Gaussian white noises and Poisson random measures. This generalizes the recent work of Xiong (2013), where the result for a super-Brownian motion with binary branching mechanism was obtained.

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1 Introduction

It is well-known that a binary branching super-Brownian motion $\{X_t : t \geq 0\}$ over the one-dimensional Euclidean space $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure with the density process $\{X_t(x) : t > 0, x \in \mathbb{R}\}$ solving the stochastic partial differential equation

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\dot{W}_t(x), \quad t > 0, x \in \mathbb{R}, \tag{1.1}$$

where $\{\dot{W}_t(x) : t > 0, x \in \mathbb{R}\}$ is the derivative of a space-time Gaussian white noise. The above stochastic partial differential equation was first established by Konno and Shiga (1988); see also Reimers (1989). The weak uniqueness of the solution to (1.1) follows from that of a martingale problem for the super-Brownian motion. The pathwise uniqueness for the equation (1.1) still remains open. The main difficulty comes from the unbounded drift coefficient and the non-Lipschitz diffusion coefficient. See, e.g., Mytnik (2002) and Mytnik and Perkins (2011) for some important progresses in the subject. A different approach for pathwise uniqueness of the super-Brownian motion was suggested in the recent work of Xiong (2013), where a stochastic equation for the *distribution function process* of $\{X_t : t \geq 0\}$ was formulated and the strong existence and uniqueness for the equation were established. The purpose of this work is to generalize the result of Xiong (2013) to a super-Lévy process with general branching mechanism.

Let $M(\mathbb{R})$ be the space of finite Borel measures on $\mathbb{R}$ endowed with the weak convergence topology. Let $B(\mathbb{R})$ denote the Banach space of bounded Borel functions on $\mathbb{R}$ furnished with the supremum norm $\|\cdot\|$. For $\mu \in M(\mathbb{R})$ and $f \in B(\mathbb{R})$ write $\mu(f) = \int f d\mu$. Let $C(\mathbb{R})$

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be the subset of $B(\mathbb{R})$ of bounded continuous functions. Let $C_0(\mathbb{R})$ be the subset of $C(\mathbb{R})$ of functions vanishing at infinity. Let $C^2(\mathbb{R})$ be the subset of $C(\mathbb{R})$ of functions with derivatives up to the second order belonging to $C(\mathbb{R})$. Let $C_0^2(\mathbb{R})$ be the subset of $C^2(\mathbb{R})$ of functions with derivatives up to the second order belonging to $C_0(\mathbb{R})$. We use the superscript "+" to denote the subsets of positive elements of the function spaces, e.g., $B(\mathbb{R})^+$. Let $(P_t)_{t\geq 0}$ denote the transition semigroup of a one-dimensional *Lévy process* ξ with strong generator A characterized by

$$Af(x) = \beta f'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}^\circ} [f(x+z) - f(x) - f'(x)z1_{\{|z|\leq 1\}}]\nu(dz) \tag{1.2}$$

for $f \in C_0^2(\mathbb{R})$, where $\sigma \geq 0$ and β are constants, and $(1 \wedge z^2)\nu(dz)$ is a finite Borel measure on $\mathbb{R}^\circ := \mathbb{R} \setminus \{0\}$. Let ϕ be a *branching mechanism* given by

$$\phi(\lambda) = b\lambda + \frac{1}{2}c\lambda^2 + \int_0^\infty (e^{-z\lambda} - 1 + z\lambda)m(dz), \quad \lambda \geq 0, \tag{1.3}$$

where $c \geq 0$ and b are constants, and $(z \wedge z^2)m(dz)$ is a finite measure on $(0, \infty)$. By a *super-Lévy process* we mean a cádlág Markov process in $M(\mathbb{R})$ with transition semigroup $(Q_t)_{t\geq 0}$ defined by

$$\int_{M(\mathbb{R})} e^{-\nu(f)} Q_t(\mu, d\nu) = \exp\{-\mu(v_t)\}, \quad f \in B(\mathbb{R})^+, \tag{1.4}$$

where $(t, x) \mapsto v_t(x)$ is the unique locally bounded positive solution to

$$v_t(x) + \int_0^t P_{t-s}[\phi(v_s(\cdot))](x)ds = P_t f(x), \qquad x \in \mathbb{R}, t \geq 0. \tag{1.5}$$

Let $D(\mathbb{R})$ be the set of bounded right-continuous increasing functions f on $\mathbb{R}$ satisfying $f(-\infty) = 0$. Then there is a 1-1 correspondence between $D(\mathbb{R})$ and $M(\mathbb{R})$ assigning a measure to its distribution function. We endow $D(\mathbb{R})$ with the topology induced by this correspondence from the weak convergence topology of $M(\mathbb{R})$. Then for any $M(\mathbb{R})$-valued stochastic process $\{X_t : t \geq 0\}$, its *distribution function process* $\{Y_t : t \geq 0\}$ is a $D(\mathbb{R})$-valued stochastic process.

Our *first main result* in this paper, Theorem 3.1, asserts that a cádlág $D(\mathbb{R})$-valued stochastic process $\{Y_t : t \geq 0\}$ is the distribution function process of a super-Lévy process if and only if there exist, on an extension of the original probability space, a Gaussian white noise $\{W(dt, du) : t \geq 0, u > 0\}$ with intensity $dtdu$ and a compensated Poisson random measure $\{\tilde{N}_0(dt, dz, du) : t \geq 0, z > 0, u > 0\}$ with intensity $dtm(dz)du$ so that $\{Y_t : t \geq 0\}$ solves the stochastic integral equation

$$\begin{aligned} Y_t(x) \;&=\; Y_0(x) + \int_0^t A^* Y_s(x)ds + \sqrt{c}\int_0^t \int_0^{Y_{s-}(x)} W(ds, du) \\ &\quad + \int_0^t \int_0^\infty \int_0^{Y_{s-}(x)} z\tilde{N}_0(ds, dz, du) - b\int_0^t Y_s(x)ds, \end{aligned} \tag{1.6}$$

where A^* denotes the dual operator of A and $Y_{s-}(x)$ means the value of the random function $Y_{s-} \in D(\mathbb{R})$ at $x \in \mathbb{R}$. Let $\mathscr{S}(\mathbb{R})$ denote the *Schwartz space* of rapidly decreasing functions on $\mathbb{R}$. The above equation can only be understood in the following *weak sense*: for every $f \in \mathscr{S}(\mathbb{R})$,

$$\langle Y_t, f\rangle \;=\; \langle Y_0, f\rangle + \int_0^t \langle Y_s, Af\rangle ds + \sqrt{c}\int_{\mathbb{R}} f(y)dy \int_0^t \int_0^{Y_{s-}(y)} W(ds, du)$$

$$-b\int_0^t \langle Y_s, f\rangle ds + \int_{\mathbb{R}} f(y)dy \int_0^t \int_0^\infty \int_0^{Y_{s-}(y)} z\tilde{N}_0(ds,dz,du), \tag{1.7}$$

where "$\langle\cdot,\cdot\rangle$" denotes the duality between $\mathscr{S}(\mathbb{R})$ and its dual space $\mathscr{S}'(\mathbb{R})$.

We shall make the convention that a stochastic integral takes automatically a predictable version of the integrand. The technical details are given in Section 2. Then we can simply write $Y_s(x)$ instead of $Y_{s-}(x)$ in both (1.6) and (1.7). We shall use this convention in the sequel of the introduction.

Our *second main result*, Theorem 6.12, establishes the pathwise uniqueness for cádlág $D(\mathbb{R})$-valued solutions of (1.6) under a mild condition on A. Let $D^1(\mathbb{R})$ be the subset of $D(\mathbb{R})$ consisting of absolutely continuous functions. Under our condition, there is a version of the solution so that $Y_t \in D^1(\mathbb{R})$ for every $t > 0$. This makes it possible for us to connect (1.6) with a backward doubly stochastic equation. Take a constant $T > 0$ and define the Gaussian white noise $W^T(ds,dx)$ on $[0,T]\times\mathbb{R}$ by

$$W^T([0,t]\times A) = W([T-t,T]\times A), \qquad 0\le t\le T,\ A\in\mathscr{B}(0,\infty)$$

and the compensated Poisson random measure $\tilde{N}_0^T(ds,dz,du)$ on $[0,T]\times(0,\infty)^2$ by

$$\tilde{N}_0^T([0,t]\times B) = \tilde{N}_0([T-t,T]\times B), \qquad 0\le t\le T,\ B\in\mathscr{B}(0,\infty)^2.$$

From (1.6) we get the backward stochastic integral equation, for $0\le t\le T$,

$$\begin{aligned} Y_{T-t}(x) &= Y_0(x) + \int_t^T A^*Y_{T-s}(x)ds + \sqrt{c}\int_{t-}^{T-}\int_0^{Y_{T-s}(x)} W^T(\overleftarrow{ds},du) \\ &\quad - \int_t^T bY_{T-s}(x)ds + \int_{t-}^{T-}\int_0^\infty\int_0^{Y_{T-s}(x)} z\tilde{N}_0^T(\overleftarrow{ds},dz,du). \end{aligned} \tag{1.8}$$

Let $\{\xi^*(t): t\ge 0\}$ be a Lévy process with generator A^*, which is defined on a further extension of the probability space and is independent of $\{W(ds,du)\}$ and $\{\tilde{N}_0(ds,dz,du)\}$. By the Lévy-Itô representation, we have

$$\xi^*(t) = \xi^*(0) + \sigma B_t - \beta t - \int_0^t\int_{\{|z|\le 1\}} z\tilde{M}(ds,dz) - \int_0^t\int_{\{|z|>1\}} zM(ds,dz). \tag{1.9}$$

where $\{B_t: t\ge 0\}$ is a standard Brownian motion, $\{M(ds,dz): s>0, z\in\mathbb{R}^\circ\}$ is a Poisson random measure with intensity $ds\nu(dz)$ and $\{\tilde{M}(ds,dz): s>0, z\in\mathbb{R}^\circ\}$ is the compensated measure. Let $\xi_t^r = \xi^*(t)-\xi^*(r\wedge t)$ for $t,r\ge 0$. From (1.8) and (1.9) we shall derive the backward doubly stochastic equation, for $0\le r\le t\le T$ and $x\in\mathbb{R}$,

$$\begin{aligned} Y_{T-t}(\xi_t^r + x) &= Y_0(\xi_T^r + x) - b\int_t^T Y_{T-s}(\xi_s^r + x)ds + \sqrt{c}\int_{t-}^{T-}\int_0^{Y_{T-s}(\xi_s^r+x)} W^T(\overleftarrow{ds},du) \\ &\quad + \int_{t-}^{T-}\int_0^\infty\int_0^{Y_{T-s}(\xi_s^r+x)} z\tilde{N}_0^T(\overleftarrow{ds},dz,du) - \sigma\int_t^T \nabla Y_{T-s}(\xi_s^r + x)dB_s \\ &\quad - \int_t^T\int_{\mathbb{R}^\circ} [Y_{T-s}(\xi_s^r + x - z) - Y_{T-s}(\xi_s^r + x)]\tilde{M}(ds,dz). \end{aligned} \tag{1.10}$$

Compared with (1.6), the advantage of (1.10) is that it holds in the *strong or classical sense*, so its pathwise uniqueness can be treated as a one-dimensional stochastic equation. The key point of our approach to the pathwise uniqueness of (1.6) is to reduce the problem to that of (1.10).

This work is strongly influenced by Xiong (2013), where the special case with $A = \Delta/2$ and $b = m(0,\infty) = 0$ was considered. In fact, a general result on the existence and uniqueness of strong solutions to a stochastic partial differential equation was established by Xiong (2013), from which the results on the binary branching super Brownian motion and some other measure-valued diffusion processes were derived. The approach of Xiong (2013) relies heavily on the theory of weighted Sobolev norms. In our case, we cannot establish the equation (1.10) simultaneously for all $(t,x) \in [r,T] \times \mathbb{R}$ because of the lack of (right- or left-) continuity of the process $t \mapsto Y_{T-t}(\xi_t^r + x)$. It is clear that both sides of (1.10) are neither right-continuous nor left-continuous. This feature makes the treatment of the backward doubly stochastic equation technically more difficult. The establishment of (1.10) requires careful estimations for some stochastic integrals with respect to Gaussian white noises and Poisson random measures. We also prove a general pathwise uniqueness result for one-dimensional backward doubly stochastic equations, which can be applied to (1.10). We think the general result is of independent interest.

We would like to mention that the pathwise uniqueness for some similar stochastic equations of distribution functions of superprocesses was established in Dawson and Li (2012). The process of distribution functions of a generalized Fleming-Viot process over the real line with Brownian mutation was characterized as the pathwise unique solution of a jump-type stochastic partial differential equation in Li et al. (2012+), which generalizes another result of Xiong (2013). The study of backward doubly stochastic equations was initiated in the pioneer work of Pardoux and Peng (1994). We refer the reader to Bertoin (1992) and Sato (1999) for the general theory of Lévy processes.

The paper is organized as follows. In Section 2 a criterion for pathwise uniqueness for general backward doubly stochastic equations is given. In Section 3 we prove some results on the super-Lévy process and establish the stochastic equation (1.6). In Section 4 the measurability and integrability properties required by (1.10) are proved. In Section 5 we establish a useful intermediate smoothed version of the stochastic equation. The pathwise uniqueness for (1.6) is established in Section 6.

Notation: Let ∇ and Δ denote the first and the second order spatial differential operators, respectively. For notational convenience, we sometimes write $\mathbb{R}_+$ for $[0,\infty)$. Throughout this paper, we make the conventions

$$\int_x^y = -\int_y^x = \int_{(x,y]}, \quad \int_{x-}^{y-} = \int_{[x,y)} \quad \text{and} \quad \int_x^\infty = \int_{(x,\infty)}$$

for any $y \geq x \in \mathbb{R}$. We also set

$$g_\delta(z) = \frac{1}{\sqrt{2\pi\delta}} \exp\{-z^2/2\delta\}, \qquad z \in \mathbb{R},\, \delta > 0. \tag{1.11}$$

We use C to denote a positive constant whose value might change from line to line. We write C_T if the constant depends on another constant $T \geq 0$.

2 Backward doubly stochastic equations

In this section, we give a criterion of pathwise uniqueness for a general backward doubly s-tochastic equation with jumps. We formulate the result in a general abstract setting since it is of independent interest.

Suppose that E, F and U_i $(i = 0, 1)$ are Polish spaces, and that $\pi(du)$, $\nu(du)$ and $\mu_i(du)$ are σ-finite Borel measures on E, F and U_i $(i = 0, 1)$, respectively. Let $T > 0$ be a fixed constant. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space furnished with filtrations $\{\mathscr{F}_t : 0 \le t \le T\}$ and $\{\mathscr{G}_t : 0 \le t \le T\}$, which are independent and satisfy the usual hypotheses. Let $\{B_t : 0 \le t \le T\}$ be a standard $(\mathscr{F}_t)$-Brownian motion and $\{M(dt, du) : 0 \le t \le T, u \in F\}$ be an $(\mathscr{F}_t)$-Poisson random measure with intensity $dt\nu(du)$. Let $\{W(dt, du) : 0 \le t \le T, u \in E\}$ be a $(\mathscr{G}_t)$-Gaussian white noise with intensity $dt\pi(du)$. For each $i = 0, 1$ let $\{N_i(dt, du) : 0 \le t \le T, u \in U_i\}$ be a $(\mathscr{G}_t)$-Poisson random measure with intensity $dt\mu_i(du)$. Suppose that $\{N_0(dt, du)\}$ and $\{N_1(dt, du)\}$ are independent of each other. Let $\{\tilde{N}_i(dt, du)\}$ denote the compensated measure of $\{N_i(dt, du)\}$. Here we consider those *two* Poisson random measures to formulate our results in a convenient way for possible applications to immigration superprocesses in the future; see (1.7) and (2.1) in Fu and Li (2010).

Let $\mathscr{G}_t^r = \sigma(\mathscr{F}_t \cup \mathscr{G}_{T-r})$ for $0 \le r \le t \le T$. Then $\mathscr{G}_t^0 = \sigma(\mathscr{F}_t \cup \mathscr{G}_T)$ and $\mathscr{G}_T^{T-t} = \sigma(\mathscr{F}_T \cup \mathscr{G}_t)$. It is easy to see that both $\{\mathscr{G}_t^0 : 0 \le t \le T\}$ and $\{\mathscr{G}_T^{T-t} : 0 \le t \le T\}$ are filtrations satisfying the usual hypotheses. Observe also that $\{B_t : 0 \le t \le T\}$ is a standard $(\mathscr{G}_t^0)$-Brownian motion and $\{M(dt, du) : 0 \le t \le T, u \in F\}$ is a $(\mathscr{G}_t^0)$-Poisson random measure with intensity $dt\nu(du)$. We define the $(\mathscr{G}_T^{T-t})$-Gaussian white noise $\{W^T(dt, du) : 0 \le t \le T, u \in E\}$ by

$$W^T([0, t] \times A) = W([T - t, T] \times A), \qquad 0 \le t \le T,\ A \in \mathscr{B}(E).$$

For $i = 0, 1$ define the $(\mathscr{G}_T^{T-t})$-Poisson random measure $\{N_i^T(dt, du) : 0 \le t \le T, u \in U_i\}$ by

$$N_i^T([0, t] \times B) = N_i([T - t, T] \times B), \qquad 0 \le t \le T,\ B \in \mathscr{B}(U_i).$$

A real process $\{\xi_s : 0 \le s \le T\}$ is said to be *progressive* with respect to the family of σ-algebras $\{\mathscr{G}_t^r : 0 \le r \le t \le T\}$ if for any $0 \le r \le t \le T$ the restriction of $(s, \omega) \mapsto \xi_s(\omega)$ to $[r, t] \times \Omega$ is measurable with respect to the σ-algebra $\mathscr{B}[r, t] \times \mathscr{G}_t^r$. A two-parameter real process $\{\zeta_s(u) : 0 \le s \le T, u \in E\}$ is said to be *progressive* with respect to the family of σ-algebras $\{\mathscr{G}_t^r : 0 \le r \le t \le T\}$ if for any $0 \le r \le t \le T$ the restriction of $(s, u, \omega) \mapsto \zeta_s(u, \omega)$ to $[r, t] \times E \times \Omega$ is measurable with respect to the σ-algebra $\mathscr{B}[r, t] \times \mathscr{B}(E) \times \mathscr{G}_t^r$.

Let $\mathscr{P}$ denote the σ-algebra on $\Omega \times [0, T]$ generated by all real-valued left continuous processes progressive with respect to the σ-algebras $\{\mathscr{G}_t^r : 0 \le r \le t \le T\}$. A process $\{\xi_s : 0 \le s \le T\}$ is said to be *predictable* if the mapping $(\omega, s) \mapsto \xi_s(\omega)$ is $\mathscr{P}$-measurable. A two-parameter process $\{\zeta_s(u) : 0 \le s \le T, u \in E\}$ is said to be *predictable* if the mapping $(\omega, s, x) \mapsto \zeta_s(\omega, x)$ is $(\mathscr{P} \times \mathscr{B}(E))$-measurable. The reader may refer to Ikeda and Watanabe(1989, Section II.3) for the theory of time-space stochastic integrals of predictable two-parameter processes respect to point processes or random measures. The stochastic integrals respect to martingale measures were discussed in Li (2011, Section 7.3).

We now introduce the following Banach spaces of stochastic processes:

- Let $\mathscr{M}_T^\alpha$ $(\alpha > 0)$ denote the space of $(\mathscr{G}_t^r)$-progressive processes $\{\xi_s : 0 \le s \le T\}$ such that

$$\|\xi\|_0^\alpha := \mathbf{E}\Big\{\Big[\int_0^T |\xi_s|^\alpha ds\Big]^{\frac{1}{\alpha}}\Big\} < \infty.$$

- Let $\mathscr{M}_T^\alpha(E)$ $(\alpha > 0)$ denote the space of two-parameter $(\mathscr{G}_t^r)$-progressive processes $\{\xi_s(u) : 0 \le s \le T, u \in E\}$ such that

$$\|\xi\|_E^\alpha := \mathbf{E}\Big\{\Big[\int_0^T ds \int_E |\xi_s(u)|^\alpha \pi(du)\Big]^{\frac{1}{\alpha}}\Big\} < \infty.$$

- Let $\mathscr{M}_T^\alpha(F)$ ($\alpha > 0$) denote the space of two-parameter $(\mathscr{G}_t^r)$-progressive processes $\{\zeta_s(u) : 0 \leq s \leq T, u \in F\}$ such that

$$\|\zeta\|_F^\alpha := \mathbf{E}\Big\{\Big[\int_0^T\int_F |\zeta_s(u)|^\alpha M(ds,du)\Big]^{\frac{1}{\alpha}}\Big\} < \infty.$$

- For $i = 0, 1$ let $\mathscr{M}_T^\alpha(U_i)$ ($\alpha > 0$) denote the space of two-parameter $(\mathscr{G}_t^r)$-progressive processes $\{\gamma_s(u) : 0 \leq s \leq T, u \in U_i\}$ such that

$$\|\gamma\|_{U_i}^\alpha := \mathbf{E}\Big\{\Big[\int_{0-}^{T-}\int_{U_i} |\gamma_s(u)|^\alpha N_i^T(\overleftarrow{ds}, du)\Big]^{\frac{1}{\alpha}}\Big\} < \infty.$$

Here and in the sequel, the stochastic integral of a progressive process refers to that of a predictable version of the integrand. For example, a process $\{\zeta_s(u) : 0 \leq s \leq T, u \in E\}$ in the space $\mathscr{M}_T^\alpha(E)$ is clearly progressive with respect to the filtration $\{\mathscr{G}_t^0 : 0 \leq s \leq T\}$ and so a $(\mathscr{G}_t^0)$-predictable process $\{\zeta(s,u) : 0 \leq s \leq T, u \in E\}$ can be defined using

$$\zeta(s,u,\omega) = \limsup_{\delta\to 0+}\frac{1}{\delta}\int_{s-\delta}^s \zeta_{t\vee 0}(u,\omega)dt. \tag{2.1}$$

In fact, by Lebesgue's theorem, for any $(u,\omega) \in E \times \Omega$ we have $\zeta(s,u,\omega) = \zeta_s(u,\omega)$ for a.e. $s \in [0,T]$, so we can take $\{\zeta(s,u) : 0 \leq s \leq T, u \in E\}$ as a representative of $\{\zeta_s(u) : 0 \leq s \leq T, u \in E\}$. We use $W^T(\overleftarrow{ds}, du)$, $N_i^T(\overleftarrow{ds}, du)$ and $\tilde{N}_i^T(\overleftarrow{ds}, du)$ to denote the relevant backward stochastic integrals. For example, for any process $\{\gamma(s,u) : 0 \leq s \leq T, u \in U_0\}$ in the space $\mathscr{M}_T^1(U_0)$ we have

$$\int_{0-}^{T-}\int_{U_0}\gamma(s,u)N_0^T(\overleftarrow{ds}, du) = \int_0^T\int_{U_0}\gamma(T-s,u)N_0(ds,du). \tag{2.2}$$

Observe that the integrand $(s,u) \mapsto \gamma(T-s,u)$ on the right-hand side of (2.2) is progressive with respect to the filtration $\{\mathscr{G}_T^{T-s} : 0 \leq s \leq T\}$, so it has a $(\mathscr{G}_T^{T-s})$-predictable version.

We shall establish a generalized Itô-Pardoux-Peng formula for a stochastic process defined by forward and backward stochastic integrals. For this purpose, let us consider a random variable Y_T measurable with respect to the σ-algebra $\mathscr{G}_T^T$ and a multi-parameter process

$$\{(b(s), Z(s), a(s,x), \zeta(s,y), \gamma_0(s,u_0), \gamma_1(s,u_1)) : 0 \leq s \leq T, x \in E, y \in F, u_0 \in U_0, u_1 \in U_1\}$$

belonging to $\mathscr{M}_T^1 \times \mathscr{M}_T^2 \times \mathscr{M}_T^2(E) \times \mathscr{M}_T^2(F) \times \mathscr{M}_T^2(U_0) \times \mathscr{M}_T^1(U_1)$. We define the process $\{Y_t : 0 \leq t \leq T\}$ by

$$\begin{aligned} Y_t \;=\; & Y_T + \int_t^T b(s)ds + \int_t^T\int_E a(s,u)W^T(\overleftarrow{ds}, du) + \int_{t-}^{T-}\int_{U_0}\gamma_0(s,u)\tilde{N}_0^T(\overleftarrow{ds}, du) \\ & + \int_{t-}^{T-}\int_{U_1}\gamma_1(s,u)N_1^T(\overleftarrow{ds}, du) - \int_t^T Z(s)dB_s - \int_t^T\int_F \zeta(s,u)\tilde{M}(ds,du). \end{aligned} \tag{2.3}$$

In general, the sample paths of the process $\{Y_t : 0 \leq t \leq T\}$ are neither right-continuous nor left-continuous. But the left and right limits exist and the set of discontinuity points is at most countable. In fact, it is easy to see that the second, third and sixth terms on the right-hand of (2.3) are continuous, the forth and fifth terms are left-continuous with right limits, and the last term is right-continuous with left limits. Observe also that the third and the forth terms are time-reversed martingales with respect to the decreasing family of σ-algebras $\{\mathscr{G}_T^t : 0 \leq t \leq T\}$.

Proposition 2.1 (Itô-Pardoux-Peng formula) *Suppose that the process $\{Y_t : 0 \leq t \leq T\}$ defined by (2.3) is progressive with respect to the family of σ-algebras $\{\mathscr{G}_t^r : 0 \leq r \leq t \leq T\}$. Then for any $0 \leq t \leq T$ and any function f on $\mathbb{R}$ with bounded and continuous first and second derivatives we have, almost surely,*

$$\begin{aligned} f(Y_t) &= f(Y_T) + \int_t^T f'(Y_s)b(s)ds + \int_t^T \int_E f'(Y_s)a(s,u)W^T(\overleftarrow{ds}, du) \\ &\quad + \frac{1}{2}\int_t^T ds \int_E f''(Y_s)a(s,u)^2\pi(du) \\ &\quad + \int_{t-}^{T-} \int_{U_0} [f(Y_s + \gamma_0(s,u)) - f(Y_s)]\tilde{N}_0^T(\overleftarrow{ds}, du) \\ &\quad + \int_t^T ds \int_{U_0} [f(Y_s + \gamma_0(s,u)) - f(Y_s) - f'(Y_s)\gamma_0(s,u)]\mu_0(du) \\ &\quad + \int_{t-}^{T-} \int_{U_1} [f(Y_s + \gamma_1(s,u)) - f(Y_s)]N_1^T(\overleftarrow{ds}, du) \\ &\quad - \int_t^T f'(Y_s)Z(s)dB_s - \frac{1}{2}\int_t^T f''(Y_s)Z(s)^2 ds \\ &\quad - \int_t^T \int_F [f(Y_s + \zeta(s,u)) - f(Y_s)]\tilde{M}(ds, du) \\ &\quad - \int_t^T ds \int_F [f(Y_s + \zeta(s,u)) - f(Y_s) - f'(Y_s)\zeta(s,u)]\nu(du). \end{aligned} \tag{2.4}$$

Sketch of Proof. We first consider the continuous case by assuming $\mu_0(U_0) = \mu_1(U_1) = \nu(F) = 0$. Let $t = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[t, T]$ and let $\varepsilon_n = \max_{1\leq i\leq n} |t_i - t_{i-1}|$. By (2.3) we have

$$Y_{t_{i-1}} = Y_{t_i} + \int_{t_{i-1}}^{t_i} b(s)ds + \int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds}, du) - \int_{t_{i-1}}^{t_i} Z(s)dB_s.$$

By Taylor's expansion there exist $\tau_i, \sigma_i \in [t_{i-1}, t_i]$ so that

$$\begin{aligned} f(Y_{t_{i-1}}) &= f(Y_{t_i}) + f'(Y_{t_{i-1}})(Y_{t_{i-1}} - Y_{t_i}) - \frac{1}{2}f''(Y_{\tau_i})(Y_{t_{i-1}} - Y_{t_i})^2 \\ &= f(Y_{t_i}) + \int_{t_{i-1}}^{t_i} f'(Y_{t_{i-1}})b(s)ds + \int_{t_{i-1}}^{t_i}\int_E f'(Y_{t_i})a(s,u)W^T(\overleftarrow{ds}, du) \\ &\quad - \int_{t_{i-1}}^{t_i} f'(Y_{t_{i-1}})Z(s)dB_s - \frac{1}{2}f''(Y_{\tau_i})(Y_{t_{i-1}} - Y_{t_i})^2 \\ &\quad + [f'(Y_{t_{i-1}}) - f'(Y_{t_i})]\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds}, du) \\ &= f(Y_{t_i}) + \int_{t_{i-1}}^{t_i} f'(Y_{t_{i-1}})b(s)ds + \int_{t_{i-1}}^{t_i}\int_E f'(Y_{t_i})a(s,u)W^T(\overleftarrow{ds}, du) \\ &\quad - \int_{t_{i-1}}^{t_i} f'(Y_{t_{i-1}})Z(s)dB_s - \frac{1}{2}f''(Y_{\tau_i})(Y_{t_{i-1}} - Y_{t_i})^2 \\ &\quad + f''(Y_{\sigma_i})[Y_{t_{i-1}} - Y_{t_i}]\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds}, du) \\ &= f(Y_{t_i}) + \int_{t_{i-1}}^{t_i} f'(Y_{t_{i-1}})b(s)ds + \int_{t_{i-1}}^{t_i}\int_E f'(Y_{t_i})a(s,u)W^T(\overleftarrow{ds}, du) \end{aligned}$$

$$
\begin{aligned}
&-\int_{t_{i-1}}^{t_i} f'(Y_{t_{i-1}})Z(s)dB_s - \frac{1}{2}f''(Y_{\tau_i})\Big(\int_{t_{i-1}}^{t_i} b(s)ds\Big)^2\\
&-\frac{1}{2}f''(Y_{\tau_i})\Big[\Big(\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du)\Big)^2+\Big(\int_{t_{i-1}}^{t_i} Z(s)dB_s\Big)^2\Big]\\
&-f''(Y_{\tau_i})\int_{t_{i-1}}^{t_i} b(s)ds\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du)\\
&+f''(Y_{\tau_i})\int_{t_{i-1}}^{t_i} b(s)ds\int_{t_{i-1}}^{t_i} Z(s)dB_s\\
&+f''(Y_{\tau_i})\int_{t_{i-1}}^{t_i} Z(s)dB_s\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du)\\
&+f''(Y_{\sigma_i})\int_{t_{i-1}}^{t_i} b(s)ds\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du)\\
&+f''(Y_{\sigma_i})\Big(\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du)\Big)^2\\
&-f''(Y_{\sigma_i})\int_{t_{i-1}}^{t_i} Z(s)dB_s\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du).
\end{aligned}
$$

By taking the summation we obtain

$$
\begin{aligned}
f(Y_t) &= f(Y_T)+\sum_{i=1}^n\int_{t_{i-1}}^{t_i} f'(Y_{t_{i-1}})b(s)ds-\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\int_E f'(Y_{t_i})a(s,u)W^T(\overleftarrow{ds},du)\\
&-\sum_{i=1}^n\int_{t_{i-1}}^{t_i} f'(Y_{t_{i-1}})Z(s)dB_s-\frac{1}{2}\sum_{i=1}^n f''(Y_{\tau_i})\Big(\int_{t_{i-1}}^{t_i} b(s)ds\Big)^2\\
&-\frac{1}{2}\sum_{i=1}^n f''(Y_{\tau_i})\Big[\Big(\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du)\Big)^2+\Big(\int_{t_{i-1}}^{t_i} Z(s)dB_s\Big)^2\Big]\\
&-\sum_{i=1}^n f''(Y_{\tau_i})\int_{t_{i-1}}^{t_i} b(s)ds\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du)\\
&+\sum_{i=1}^n f''(Y_{\tau_i})\int_{t_{i-1}}^{t_i} b(s)ds\int_{t_{i-1}}^{t_i} Z(s)dB_s\\
&+\sum_{i=1}^n f''(Y_{\tau_i})\int_{t_{i-1}}^{t_i} Z(s)dB_s\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du)\\
&+\sum_{i=1}^n f''(Y_{\sigma_i})\int_{t_{i-1}}^{t_i} b(s)ds\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du)\\
&+\sum_{i=1}^n f''(Y_{\sigma_i})\Big(\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du)\Big)^2\\
&-\sum_{i=1}^n f''(Y_{\sigma_i})\int_{t_{i-1}}^{t_i} Z(s)dB_s\int_{t_{i-1}}^{t_i}\int_E a(s,u)W^T(\overleftarrow{ds},du).
\end{aligned}
$$

Then we can let $\varepsilon_n \to 0$ to get

$$
\begin{aligned}
f(Y_t) &= f(Y_T)+\int_t^T f'(Y_s)b(s)ds+\int_t^T\int_E f'(Y_s)a(s,u)W^T(\overleftarrow{ds},du)\\
&-\int_t^T f'(Y_s)Z(s)dB_s+\frac{1}{2}\int_t^T ds\int_E f''(Y_s)a(s,u)^2\pi(du)
\end{aligned}
$$

$$-\frac{1}{2}\int_t^T f''(Y_s)Z(s)^2 ds.$$

Based on the above formula in the continuous case, the proof of (2.4) is that in the classical case; see, e.g., Protter (2005, p.78) and Situ (2005, p.59). One can first show the equation holds for the case $\nu(F) = 0$ and then for the general case. □

Suppose that β is a Borel function on $\mathbb{R}$ and σ and g_i are Borel functions on the product spaces $\mathbb{R} \times E$ and $\mathbb{R} \times U_i$ $(i = 0, 1)$, respectively. We consider the following backward doubly stochastic integral equation:

$$\begin{aligned} Y_t &= Y_T + \int_t^T \beta(Y_s)ds + \int_t^T \int_E \sigma(Y_s, u)W^T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{T-} \int_{U_0} g_0(Y_s, u)\tilde{N}_0^T(\overleftarrow{ds}, du) + \int_{t-}^{T-} \int_{U_1} g_1(Y_s, u)N_1^T(\overleftarrow{ds}, du) \\ &\quad - \int_t^T Z_s dB_s - \int_t^T \int_F \zeta_s(u)\tilde{M}(ds, du). \end{aligned} \tag{2.5}$$

A two-parameter process $\{(Y_s, Z_s, \zeta_s(u)) : 0 \leq s \leq T, u \in F\}$ is called a *solution of* (2.5) if it is progressive with respect to the family of σ-algebras $\{\mathscr{G}_t^r : 0 \leq r \leq t \leq T\}$ and for every fixed $0 \leq t \leq T$ the equation is satisfied almost surely. The last statement includes the requirement that the multi-parameter process

$$\{(\beta(Y_s), Z_s, \sigma(Y_s, x), \zeta_s(y), g_0(Y_s, u_0), g_1(Y_s, u_1)) : 0 \leq s \leq T, x \in E, y \in F, u_0 \in U_0, u_1 \in U_1\}$$

belongs to $\mathscr{M}_T^1 \times \mathscr{M}_T^2 \times \mathscr{M}_T^2(E) \times \mathscr{M}_T^2(F) \times \mathscr{M}_T^2(U_0) \times \mathscr{M}_T^1(U_1)$. We are interested in the pathwise uniqueness of solution to (2.5). To simplify the formulation of the result, let us consider the following:

Condition 2.2 *For each $u \in U_0$, $x \mapsto g_0(x, u)$ is nondecreasing and*

$$\begin{aligned} \int_E |\sigma(x, u) - \sigma(y, u)|^2 \pi(du) &+ \int_{U_0} |g_0(x, u) - g_0(y, u)|^2 \mu_0(du) \\ &+ \int_{U_1} |g_1(x, u) - g_1(y, u)|\mu_1(du) + |\beta(x) - \beta(y)| \leq C|x - y| \end{aligned}$$

for all $x, y \in \mathbb{R}$.

It should be noticed that we do not require (2.5) holds simultaneously for all $0 \leq t \leq T$ because of the lack of left- and right-continuities of the process $\{Y_t : 0 \leq t \leq T\}$. For the same reason, the following Yamada-Watanabe type theorem only establishes the pathwise uniqueness of the solution for fixed time.

Theorem 2.3 *Suppose that Condition 2.2 is satisfied and both $\{(Y_1(t), Z_1(t), \zeta_1(t, u)) : 0 \leq t \leq T, u \in F\}$ and $\{(Y_2(t), Z_2(t), \zeta_2(t, u)) : 0 \leq t \leq T, u \in F\}$ are solutions of (2.5) with $Y_1(T) = Y_2(T)$. Then $\mathbf{P}\{Y_1(t) = Y_2(t)\} = 1$ for every $0 \leq t \leq T$ and*

$$\mathbf{E}\Big[\int_0^T |Z_1(s) - Z_2(s)|^2 ds\Big] + \mathbf{E}\Big[\int_0^T ds \int_F |\zeta_1(s, u) - \zeta_2(s, u)|^2 \nu(du)\Big] = 0. \tag{2.6}$$

Proof. This proof adapts the arguments from [2, 4, 16]. Let $Y(t) = Y_1(t) - Y_2(t)$, $Z(t) = Z_1(t) - Z_2(t)$, $\zeta(t,u) = \zeta_1(t,u) - \zeta_2(t,u)$, $b(s) = \beta(Y_1(s)) - \beta(Y_2(s))$, $a(t,u) = \sigma(Y_1(t),u) - \sigma(Y_2(t),u)$ and $\gamma_i(t,u) = g_i(Y_1(t),u) - g_i(Y_2(t),u)$ $(i = 0,1)$. In view of (2.5), we have

$$\begin{aligned} Y(t) &= \int_t^T b(s)ds + \int_t^T \int_E a(s,u)W^T(\overleftarrow{ds},du) + \int_{t-}^{T-} \int_{U_0} \gamma_0(s,u)\tilde{N}_0^T(\overleftarrow{ds},du) \\ &\quad + \int_{t-}^{T-} \int_{U_1} \gamma_1(s,u)N_1^T(\overleftarrow{ds},du) - \int_t^T Z(s)dB_s - \int_t^T \int_F \zeta(s,u)\tilde{M}(ds,du). \end{aligned} \tag{2.7}$$

For each integer $k \geq 0$ define $a_k = \exp\{-k(k+1)/2\}$. Then $a_k \to 0$ decreasingly as $k \to \infty$ and

$$\int_{a_k}^{a_{k-1}} z^{-1}dz = k, \qquad k \geq 1.$$

Let $x \mapsto g_k(x)$ be a positive continuous function supported by (a_k, a_{k-1}) so that

$$\int_{a_k}^{a_{k-1}} g_k(x)dx = 1$$

and $g_k(x) \leq 2(kx)^{-1}$ for every $x > 0$. Let

$$f_k(z) = \int_0^{|z|} dy \int_0^y g_k(x)dx, \qquad z \in \mathbb{R}.$$

It is easy to see that $|f_k'(z)| \leq 1$ and

$$0 \leq |z|f_k''(z) = |z|g_k(|z|) \leq 2k^{-1}, \qquad z \in \mathbb{R}.$$

Moreover, we have $f_k(z) \to |z|$ increasingly as $k \to \infty$. Then by (2.7) and the Itô-Pardoux-Peng formula established in Proposition 2.1 we get

$$\begin{aligned} f_k(Y(t)) &= \int_t^T f_k'(Y(s))b(s)ds - \frac{1}{2}\int_t^T f_k''(Y(s))Z(s)^2 ds \\ &\quad + \frac{1}{2}\int_t^T ds \int_E f_k''(Y(s))a(s,u)^2\pi(du) \\ &\quad + \int_{t-}^{T-} \int_{U_1} \big[f_k(Y(s) + \gamma_1(s,u)) - f_k(Y(s))\big] N_1^T(\overleftarrow{ds},du) \\ &\quad + \int_t^T ds \int_{U_0} \big[f_k(Y(s) + \gamma_0(s,u)) - f_k(Y(s)) - f_k'(Y(s))\gamma_0(s,u)\big]\mu_0(du) \\ &\quad - \int_t^T ds \int_F \big[f_k(Y(s) + \zeta(s,u)) - f_k(Y(s)) - f_k'(Y(s))\zeta(s,u)\big]\nu(du) \\ &\quad - \int_t^T f_k'(Y(s))Z(s)dB_s + \int_t^T \int_E f_k'(Y(s))a(s,u)W^T(\overleftarrow{ds},du) \\ &\quad + \int_{t-}^{T-} \int_{U_0} \big[f_k(Y(s) + \gamma_0(s,u)) - f_k(Y(s))\big]\tilde{N}_0^T(\overleftarrow{ds},du) \\ &\quad - \int_t^T \int_F \big[f_k(Y(s) + \zeta(s,u)) - f_k(Y(s))\big]\tilde{M}(ds,du). \end{aligned} \tag{2.8}$$

Here the last four terms on the right-hand side have mean zero. Using the convexity of the function f_k, one can show that the second and sixth terms are negative. Moreover, it is easy to see that

$$G_k(s) := \frac{1}{2}\int_E f_k''(Y(s))a(s,u)^2\pi(du) \leq \frac{1}{2}Cg_k(|Y(s)|)|Y(s)| \leq Ck^{-1},$$

which tends to zero as $k \to \infty$. As in the proof of Theorem 3.2 in Fu and Li (2010) one can show that, as $k \to \infty$,

$$H_k(s) := \int_{U_0} \big[f_k(Y(s)+\gamma_0(s,u)) - f_k(Y(s)) - f_k'(Y(s))\gamma_0(s,u)\big]\mu_0(du) \to 0.$$

By (2.8) and Fatou's lemma we obtain

$$\begin{aligned}\mathbf{E}[|Y(t)|] &\le \lim_{k\to\infty} \mathbf{E}\Big[f_k(Y(t)) - \int_t^T (G_k(s)+H_k(s))ds\Big] \\ &\le \int_t^T \mathbf{E}[|b(s)|]ds + \int_t^T \mathbf{E}\Big[\int_{U_1} |\gamma_1(s,u)|\mu_1(du)\Big]ds \\ &\le C\int_t^T \mathbf{E}[|Y(s)|]ds.\end{aligned}$$

By Gronwall's inequality, we have $\mathbf{E}[|Y(t)|] = 0$, and so $\mathbf{P}\{Y_1(t) = Y_2(t)\} = 1$. Returning to (2.7) we get

$$\int_0^T Z(s)dB_s + \int_0^T \int_F \zeta(s,u)\tilde{M}(ds,du) = 0.$$

Then we can take the expectation of the square of the left-hand side to obtain

$$\mathbf{E}\Big[\int_0^T Z(s)^2 ds\Big] + \mathbf{E}\Big[\int_0^T ds \int_F \zeta(s,u)^2\nu(du)\Big] = 0.$$

That gives (2.6) and completes the proof. □

3 The super-Lévy process

In this section, we establish the stochastic equation (1.6) for the distribution function process of a super-Lévy process and derive some consequences of the equation. For $s, u \ge 0$ let

$$Y_s^{-1}(u) = \inf\{x \in \mathbb{R} : Y_s(x) \ge u\}.$$

Let $(P_t)_{t\ge 0}$ be the transition semigroup of a Lévy process with strong generator A given by (1.2). Let $P_t^b = e^{-bt}P_t$ for $t \ge 0$ and $b \in \mathbb{R}$. The following main result of this section is a generalization of Theorem 1.3 in Xiong (2013).

Theorem 3.1 *A cádlág $D(\mathbb{R})$-valued stochastic process $\{Y_t : t \ge 0\}$ is the distribution function process of a super-Lévy process with transition semigroup defined by* (1.4) *and* (1.5) *if and only if there exist, on an enlarged probability space, a Gaussian white noise $\{W(ds,du) : s \ge 0, u > 0\}$ with intensity $dsdu$ and a compensated Poisson random measure $\{\tilde{N}_0(ds,dz,du) : s \ge 0, z > 0, u > 0\}$ with intensity $dsm(dz)du$ so that $\{Y_t : t \ge 0\}$ solves the stochastic equation* (1.6).

Proof. Suppose that $\{Y_t : t \ge 0\}$ is a cádlág $D(\mathbb{R})$-valued stochastic process solving the stochastic equation (1.6). Let $\{X_t : t \ge 0\}$ be the corresponding cádlág $M(\mathbb{R})$-valued process. By integration by parts, for any $f \in \mathscr{S}(\mathbb{R})$ we have $X_t(f) = -\langle Y_t, f'\rangle$, and so (1.7) yields

$$X_t(f) = -\langle Y_0, f'\rangle - \int_0^t \langle Y_s, Af'\rangle ds - \sqrt{c}\int_0^t \int_0^\infty \Big[\int_{\mathbb{R}} f'(x)1_{\{u\le Y_{s-}(x)\}}dx\Big]W(ds,du)$$

$$
\begin{aligned}
&\quad + b\int_0^t \langle Y_s, f'\rangle ds - \int_0^t\int_0^\infty\int_0^\infty \Big[\int_{\mathbb{R}} f'(x)1_{\{u\le Y_{s-}(x)\}}dx\Big] z\tilde{N}_0(ds,dz,du) \\
&= X_0(f) - \int_0^t \langle Y_s, (Af)'\rangle ds - \sqrt{c}\int_0^t\int_0^\infty \Big[\int_{Y_{s-}^{-1}(u)}^\infty f'(x)dx\Big] W(ds,du) \\
&\quad - b\int_0^t X_s(f)ds - \int_0^t\int_0^\infty\int_0^\infty \Big[\int_{Y_{s-}^{-1}(u)}^\infty f'(x)dx\Big] z\tilde{N}_0(ds,dz,du) \\
&= X_0(f) + \int_0^t X_s(Af - bf)ds + I_t^c(f) + I_t^d(f),
\end{aligned}
$$

where

$$
I_t^c(f) = \sqrt{c}\int_0^t\int_0^\infty f(Y_{s-}^{-1}(u))W(ds,du) \tag{3.1}
$$

and

$$
I_t^d(f) = \int_0^t\int_0^\infty\int_0^\infty f(Y_{s-}^{-1}(u))z\tilde{N}_0(ds,dz,du). \tag{3.2}
$$

Since $s\mapsto Y_s$ has at most countably may jumps, by Itô's formula, for any $G\in C^2(\mathbb{R})$,

$$
\begin{aligned}
G(X_t(f)) &= G(X_0(f)) + \frac{c}{2}\int_0^t ds\int_0^\infty G''(X_s(f))f(Y_{s-}^{-1}(u))^2 du \\
&\quad + \int_0^t G'(X_s(f))X_s(Af-bf)ds + \int_0^t ds\int_0^\infty du\int_0^\infty \Big[G(X_s(f) + zY_{s-}^{-1}(u)) \\
&\qquad - G(X_s(f)) - zf(Y_{s-}^{-1}(u))G'(X_s(f))\Big] m(dz) + \text{local mart.} \\
&= G(X_0(f)) + \frac{c}{2}\int_0^t G''(X_s(f))X_s(f^2)ds + \int_0^t G'(X_s(f))X_s(Af-bf)ds \\
&\quad + \int_0^t ds\int_{\mathbb{R}} X_s(dx)\int_0^\infty \Big[G(X_s(f) + zf(x)) - G(X_s(f)) \\
&\quad - zf(x)G'(X_s(f))\Big] m(dz) + \text{local mart.}
\end{aligned} \tag{3.3}
$$

By an approximation argument, one can see the above relation remains true for any $f\in\mathscr{D}(A)$, the full domain of the strong generator A. By Li (2011, Theorem 7.13) one can see $\{X_t : t\ge 0\}$ is a super-Lévy process.

Conversely, suppose that $\{Y_t : t\ge 0\}$ is the distribution function process of a cádlág super-Lévy process $\{X_t : t\ge 0\}$ with transition semigroup defined by (1.4) and (1.5). Then $Y_{t-}(x) = X_{t-}(-\infty, x]$ for $t>0$ and $x\in\mathbb{R}$. By considering a conditional probability, we may assume X_0 and Y_0 are deterministic. By Li (2011, p.153) one can see $\{X_t : t\ge 0\}$ satisfies the following martingale problem: For each $f\in C_0^2(\mathbb{R})$,

$$
X_t(f) = X_0(f) + \int_0^t X_s(Af-bf)ds + I_t^c(f) + I_t^d(f), \tag{3.4}
$$

where $t\mapsto I_t^c(f)$ is a continuous martingale with quadratic variation process

$$
\langle I^c(f)\rangle_t = \int_0^t X_s(cf^2)ds, \tag{3.5}
$$

and $t \mapsto I_t^d(f)$ is a purely discontinuous martingale. Let $K(\mu, d\nu)$ be the kernel from $M(\mathbb{R})$ to $M(\mathbb{R})^\circ := M(\mathbb{R}) \setminus \{0\}$ defined by

$$\int_{M(\mathbb{R})^\circ} F(\nu)K(\mu, d\nu) = \int_{\mathbb{R}} \mu(dx) \int_0^\infty F(u\delta_x)m(du)$$

for positive Borel functions F on $M(\mathbb{R})$. Then there is an optional random measure $N(ds, d\nu)$ on $(0, \infty) \times M(\mathbb{R})^\circ$ with predictable compensator $\hat{N}(ds, d\nu) = dsK(X_{s-}, d\nu)$ such that

$$I_t^d(f) = \int_0^t \int_{M(\mathbb{R})^\circ} \nu(f)\tilde{N}(ds, d\nu), \tag{3.6}$$

where $\tilde{N}(ds, d\nu) = N(ds, d\nu) - \hat{N}(ds, d\nu)$. As in the proof of Li (2011, Theorem 7.25), one can see that there exists an orthogonal martingale measure $\{I^c(dt, dx) : t \geq 0, x \in \mathbb{R}\}$ having covariation measure $cdtX_t(dx)$ so that

$$I_t^c(f) = \int_0^t \int_{\mathbb{R}} f(x)I^c(ds, dx).$$

For any $g \in B(\mathbb{R}_+)$ we can define a continuous martingale $t \mapsto Z_t^c(g)$ by

$$Z_t^c(g) = \int_0^t \int_{\mathbb{R}} g(Y_{s-}(x))I^c(ds, dx). \tag{3.7}$$

Since $s \mapsto X_s$ has at most countably may jumps, we have

$$\begin{aligned}\langle Z^c(g)\rangle_t &= c\int_0^t ds \int_{\mathbb{R}} g(Y_{s-}(x))^2 X_{s-}(dx) = c\int_0^t ds \int_{\mathbb{R}} g(Y_{s-}(x))^2 dY_{s-}(x) \\ &= c\int_0^t ds \int_0^{Y_{s-}(\infty)} g(u)^2 du = c\int_0^t ds \int_0^\infty g(u)^2 1_{\{u \leq Y_{s-}(\infty)\}} du.\end{aligned}$$

Observe that the family $\{Z_t^c(g) : t \geq 0, g \in B(\mathbb{R}_+)\}$ determines a martingale measure $\{Z_t^c(B) : t \geq 0, B \in \mathscr{B}(\mathbb{R}_+)\}$. By El Karoui and Méléard (1990, Theorem III-6), on some extension of the probability space one can define a Gaussian white noise $W(ds, du)$ on $(0, \infty)^2$ based on the Lebesgue measure so that

$$Z_t^c(g) = \sqrt{c}\int_0^t \int_0^\infty g(u)1_{\{u \leq Y_{s-}(\infty)\}} W(ds, du). \tag{3.8}$$

For $x \in \mathbb{R}$ it is easy to see that

$$\mathbf{E}\Big[\int_0^t ds \int_{\mathbb{R}} 1_{\{Y_{s-}(y) \leq Y_{s-}(x), y > x\}} X_s(dy)\Big] = \mathbf{E}\Big[\int_0^t ds \int_{\mathbb{R}} 1_{\{Y_s(y) = Y_s(x), y > x\}} X_s(dy)\Big] = 0.$$

Then by (3.7) and (3.8) we have

$$\begin{aligned}I_t^c(-\infty, x] &= \int_0^t \int_{\mathbb{R}} 1_{\{y \leq x\}} I^c(ds, dy) = \int_0^t \int_{\mathbb{R}} 1_{\{Y_{s-}(y) \leq Y_{s-}(x)\}} I^c(ds, dy) \\ &= \int_0^t \int_0^\infty 1_{\{u \leq Y_{s-}(x)\}} Z^c(ds, du) = \sqrt{c}\int_0^t \int_0^\infty 1_{\{u \leq Y_{s-}(x)\}} W(ds, du).\end{aligned}$$

For any $g \in B(\mathbb{R}_+^2)$ we can define a purely discontinuous martingale $t \mapsto Z_t^d(g)$ by

$$Z_t^d(g) = \int_0^t \int_{M(\mathbb{R})^\circ} \tilde{N}(ds, d\nu) \int_{\mathbb{R}} g(\nu(1), Y_{s-}(x))\nu(dx). \tag{3.9}$$

Then $\{Z_t^d(g) : t \geq 0, g \in B(\mathbb{R}_+^2)\}$ determines a martingale measure $\{Z^d(dt, dz, du) : t, y, u \geq 0\}$ with compensator $\hat{Z}^d(dt, dz, du)$ determined by

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}_+^2} g(z,u)\hat{Z}^d(ds, dz, du) &= \int_0^t \int_{M(\mathbb{R})^\circ} \hat{N}(ds, d\nu) \int_{\mathbb{R}} g(\nu(1), Y_{s-}(x))\nu(dx) \\
&= \int_0^t ds \int_{\mathbb{R}} X_{s-}(dx) \int_0^\infty g(z, Y_{s-}(x)) z m(dz) \\
&= \int_0^t ds \int_0^\infty du \int_0^\infty g(z,u) z 1_{\{u \leq Y_{s-}(\infty)\}} m(dz).
\end{aligned}$$

By Ikeda and Watanabe (1989, p.93), on an extension of the probability space, there is a Poisson random measure $N_0(ds, dz, du)$ on $(0,\infty)^3$ with intensity $dsm(dz)du$ so that

$$\int_0^t \int_{\mathbb{R}_+^2} g(z,u) Z^d(ds, dz, du) = \int_0^t \int_0^\infty \int_0^\infty g(z,u) z 1_{\{u \leq Y_{s-}(\infty)\}} \tilde{N}_0(ds, dz, du), \tag{3.10}$$

where $\tilde{N}_0(ds, dz, du) = N_0(ds, dz, du) - dsm(dz)du$. By (3.6), (3.9) and (3.10) it follows that

$$\begin{aligned}
I_t^d(-\infty, x] &= \int_0^t \int_{M(\mathbb{R})^\circ} \tilde{N}(ds, d\nu) \int_{\mathbb{R}} 1_{\{y \leq x\}} \nu(dy) \\
&= \int_0^t \int_{M(\mathbb{R})^\circ} \tilde{N}(ds, d\nu) \int_{\mathbb{R}} 1_{\{Y_{s-}(y) \leq Y_{s-}(x)\}} \nu(dy) - H_t(x) \\
&= \int_0^t \int_{\mathbb{R}_+^2} 1_{\{u \leq Y_{s-}(x)\}} Z^d(ds, dz, du) - H_t(x) \\
&= \int_0^t \int_0^\infty \int_0^{Y_{s-}(x)} z \tilde{N}_0(ds, dz, du) - H_t(x),
\end{aligned}$$

where

$$H_t(x) = \int_0^t \int_{M(\mathbb{R})^\circ} \tilde{N}(ds, d\nu) \int_{\mathbb{R}} 1_{\{Y_{s-}(y) \leq Y_{s-}(x), y > x\}} \nu(dy).$$

For any $k \geq 0$ let

$$H_k(t,x) = \int_0^t \int_{M(\mathbb{R})^\circ} 1_{\{\nu(1) \leq k\}} \tilde{N}(ds, d\nu) \int_{\mathbb{R}} 1_{\{Y_{s-}(y) \leq Y_{s-}(x), y > x\}} \nu(dy).$$

It is easy to see that

$$\mathbf{E}[H_k(t,x)^2] = \mathbf{E}\Big[\int_0^t ds \int_{\mathbb{R}} X_{s-}(dy) \int_0^k u^2 1_{\{Y_{s-}(y) \leq Y_{s-}(x), y > x\}} m(du)\Big] = 0.$$

Then by Fatou's lemma one can see $\mathbf{E}[H_t(x)^2] = 0$, and so $H_t(x) = 0$ almost surely. It follows that

$$I_t^d(-\infty, x] = \int_0^t \int_0^\infty \int_0^{Y_{s-}(x)} z \tilde{N}_0(ds, dz, du).$$

Let $\bar{f}(x) = \int_x^\infty f(y)dy$ for $f \in \mathscr{S}(\mathbb{R})$. It is simple to see that $A\bar{f} = \overline{Af}$ and $\langle Y_t, f\rangle = X_t(\bar{f})$. By stochastic Fubini's theorem,

$$\langle Y_t, f\rangle = X_0(\bar{f}) + \int_0^t X_s(A\bar{f} - b\bar{f})ds + I_t^c(\bar{f}) + I_t^d(\bar{f})$$

$$= \langle Y_0, f\rangle + \int_0^t \langle Y_s, Af\rangle ds - b\int_0^t \langle Y_s, f\rangle ds$$
$$+ \int_{\mathbb{R}} I_t^c(-\infty, x]f(x)dx + \int_{\mathbb{R}} I_t^d(-\infty, x]f(x)dx.$$

Then we have (1.7), which is the weak form of (1.6). □

Proposition 3.2 *Suppose that $\{Y_t : t \geq 0\}$ is the distribution function process of a super-Lévy process solving* (1.6). *Let $\{\mathscr{G}_t : t \geq 0\}$ be the augmented filtration generated by $\{Y_t : t \geq 0\}$ and $\{W(dt, du), \tilde{N}_0(dt, dz, du) : t \geq 0, z > 0, u > 0\}$. Then for each $t \geq 0$ the mapping $(s, x, \omega) \mapsto Y_s(x, \omega)$ restricted to $[0, t] \times \mathbb{R} \times \Omega$ is measurable relative to the σ-algebras $\mathscr{B}[0, t] \times \mathscr{B}(\mathbb{R}) \times \mathscr{G}_t$ and $\mathscr{B}(\mathbb{R}_+)$.*

Proof. By Theorem 3.1 the measure-valued process $\{X_t : t \geq 0\}$ corresponding to $\{Y_t : t \geq 0\}$ is a super-Lévy process. By the right-continuity of $\{X_s : s \geq 0\}$, for $f \in \mathscr{S}(\mathbb{R})$ the mapping $(s, \omega) \mapsto X_s(f, \omega)$ restricted to $[0, t] \times \Omega$ is measurable relative to the σ-algebras $\mathscr{B}[0, t] \times \mathscr{G}_t$ and $\mathscr{B}(\mathbb{R})$. By a monotone class argument, the measurability also holds for $f \in B(\mathbb{R})$. In particular, for $x \in \mathbb{R}$ the mapping $(s, \omega) \mapsto Y_s(x, \omega) = X_s((-\infty, x], \omega)$ restricted to $[0, t] \times \Omega$ is measurable relative to the σ-algebras $\mathscr{B}[0, t] \times \mathscr{G}_t$ and $\mathscr{B}(\mathbb{R}_+)$. By the right-continuity of $x \mapsto Y_s(x, \omega)$ one can see $(s, x, \omega) \mapsto Y_s(x, \omega)$ restricted to $[0, t] \times \mathbb{R} \times \Omega$ is measurable relative to the σ-algebras $\mathscr{B}[0, t] \times \mathscr{B}(\mathbb{R}) \times \mathscr{G}_t$ and $\mathscr{B}(\mathbb{R}_+)$. □

Proposition 3.3 (Li, 2011, p.48 and p.157) *Suppose that $\{X_t : t \geq 0\}$ is a super-Lévy process with deterministic initial state $X_0 = \mu \in M(\mathbb{R})$. Then for $t \geq 0$ and $f \in B(\mathbb{R})$ we have*

$$\mathbf{E}\{X_t(f)\} = \mu(P_t^b f) = \mu P_t^b(f) \quad \text{and} \quad \mathbf{E}\Big\{\sup_{t\in[0,T]} X_t(1)\Big\} < \infty.$$

Proposition 3.4 *Suppose that $\{X_t : t \geq 0\}$ is a super-Lévy process with distribution function process $\{Y_t : t \geq 0\}$ solving the stochastic equation* (1.6). *Then for any $t \geq 0$ and $f \in B(\mathbb{R})$ we have a.s.*

$$\begin{aligned} X_t(f) &= X_0(P_t^b f) + \sqrt{c}\int_0^t \int_0^\infty P_{t-s}^b f(Y_s^{-1}(u))W(ds, du) \\ &\quad + \int_0^t \int_0^\infty \int_0^\infty z P_{t-s}^b f(Y_s^{-1}(u))\tilde{N}_0(ds, dz, du). \end{aligned} \tag{3.11}$$

Proof. Under the conditions of the theorem, it is simple to see that (3.1) and (3.2) define two martingale measures $I^c(ds, dx)$ and $I^d(ds, dx)$ on $(0, \infty) \times \mathbb{R}$. By Li (2011, Theorem 7.26), we have

$$X_t(f) = X_0(P_t^b f) + \int_0^t \int_0^\infty P_{t-s}^b f(x)(I^c + I^d)(ds, dx),$$

which can be rewritten into the form of (3.11). □

Proposition 3.5 *Suppose that $\{X_t : t \geq 0\}$ is a super-Lévy process with deterministic initial state $X_0 = \mu \in M(\mathbb{R})$. Then for any $a \geq 0$ we have*

$$\sup_{f\in B_a} \mathbf{E}\{|X_t(f) - X_r(P_{t-r}^b f)|\} \to 0 \qquad (t \to r+ \text{ or } r \to t-) \tag{3.12}$$

where $B_a = \{f \in B(\mathbb{R}) : \|f\| \leq a\}$.

Proof. By Proposition 3.4, for any $t \geq r \geq 0$ we have

$$\begin{aligned} X_t(f) &= X_r(P^b_{t-r}f) + \sqrt{c}\int_r^t \int_0^\infty P^b_{t-s}f(Y_s^{-1}(z))W(ds,dz) \\ &\quad + \int_r^t \int_0^\infty \int_0^\infty zP^b_{t-s}f(Y_s^{-1}(u))\tilde{N}_0(ds,dz,du). \end{aligned} \tag{3.13}$$

Based on Proposition 3.3, we can give some estimates of the stochastic integrals on the right-hand side. For the first integral we have

$$\begin{aligned} &\mathbf{E}\Big\{\Big[\int_r^t \int_0^\infty P^b_{t-s}f(Y_s^{-1}(z))W(ds,dz)\Big]^2\Big\} \\ &\quad = \mathbf{E}\Big\{\int_r^t ds \int_0^\infty [P^b_{t-s}f(Y_s^{-1}(u))]^2 du\Big\} \\ &\quad = \mathbf{E}\Big\{\int_r^t ds \int_{\mathbb{R}} [P^b_{t-s}f(x)]^2 X_s(dx)\Big\} \\ &\quad = \int_r^t ds \int_{\mathbb{R}} \mu(dx) \int_{\mathbb{R}} [P^b_{t-s}f(y)]^2 P^b_s(x,dy) \\ &\quad \leq \|f\|e^{|b|t} \int_r^t ds \int_{\mathbb{R}} P^b_t|f|(x)\mu(dx) \\ &\quad \leq \|f\|^2 e^{2|b|t}\mu(1)(t-r). \end{aligned}$$

For the second integral we have

$$\begin{aligned} &\mathbf{E}\Big\{\Big[\int_r^t \int_0^1 \int_0^\infty zP^b_{t-s}f(Y_s^{-1}(u))\tilde{N}_0(ds,dz,du)\Big]^2\Big\} \\ &\quad = \mathbf{E}\Big\{\int_r^t ds \int_0^1 z^2 m(dz) \int_0^\infty [P^b_{t-s}f(Y_s^{-1}(u))]^2 du\Big\} \\ &\quad = \int_0^1 z^2 m(dz)\mathbf{E}\Big\{\int_r^t ds \int_{\mathbb{R}} [P^b_{t-s}f(x)]^2 X_s(dx)\Big\} \\ &\quad \leq \|f\|^2 e^{2|b|t}\mu(1)(t-r)\int_0^1 z^2 m(dz) \end{aligned}$$

and

$$\begin{aligned} &\mathbf{E}\Big\{\Big|\int_r^t \int_1^\infty \int_0^\infty zP^b_{t-s}f(Y_s^{-1}(u))\tilde{N}_0(ds,dz,du)\Big|\Big\} \\ &\quad \leq \mathbf{E}\Big\{\Big|\int_r^t \int_1^\infty \int_0^\infty zP^b_{t-s}f(Y_s^{-1}(u))N_0(ds,dz,du)\Big|\Big\} \\ &\qquad + \mathbf{E}\Big\{\Big|\int_r^t ds \int_1^\infty zm(dz) \int_0^\infty P^b_{t-s}f(Y_s^{-1}(u))du\Big|\Big\} \\ &\quad = 2\int_1^\infty zm(dz)\mathbf{E}\Big\{\int_r^t ds \int_0^\infty |P^b_{t-s}f(x)|X_s(dx)\Big\} \\ &\quad = 2\int_1^\infty zm(dz)\int_r^t ds \int_{\mathbb{R}} \mu(dx) \int_{\mathbb{R}} |P^b_{t-s}f(x)|P^b_s(x,dy) \\ &\quad \leq 2\int_1^\infty zm(dz)\int_r^t ds \int_{\mathbb{R}} P^b_t|f|(x)\mu(dx) \\ &\quad \leq 2\|f\|e^{|b|t}(t-r)\mu(1)\int_1^\infty zm(dz). \end{aligned}$$

Then we have (3.12). □

A super-Lévy process is typically absolutely continuous with respect to the Lebesgue measure. For the convenience of statements of the results, let us consider the following condition:

Condition 3.6 *There exists a continuous function* $(t,z)\mapsto p_t(z)$ *on* $(0,\infty)\times\mathbb{R}$ *so that*

$$P_t(x,dy)=p_t(y-x)dy, \qquad t>0,\ x,y\in\mathbb{R},$$

and

$$p_t(z)\le t^{-\alpha}C(t), \qquad t>0,\ z\in\mathbb{R}$$

for a constant $0<\alpha<1$ *and an increasing function* $t\mapsto C(t)$ *on* $[0,\infty)$.

It is well-known that the above condition is satisfied if $(P_t)_{t\ge0}$ is the transition semigroup of a stable process with index in $(1,2]$. Under the condition, we write $p_t^b(z)=e^{-bt}p_t(z)$ for $t>0$ and $z\in\mathbb{R}$.

Theorem 3.7 *Suppose that Condition 3.6 holds and* $\{X_t:t\ge0\}$ *is a super-Lévy process with distribution function process* $\{Y_t:t\ge0\}$ *defined by (1.6). Then for each* $t>0$ *the random measure* $X_t(dx)$ *is absolutely continuous with respect to the Lebesgue measure with density*

$$\begin{aligned}X_t(x) \;=\; &\int_{\mathbb{R}}p_t^b(x-z)\mu(dz)+\sqrt{c}\int_0^t\int_0^\infty p_{t-s}^b(x-Y_s^{-1}(z))W(ds,dz)\\ &+\int_0^t\int_0^\infty\int_0^\infty zp_{t-s}^b(x-Y_s^{-1}(u))\tilde{N}_0(ds,dz,du), \qquad x\in\mathbb{R}.\end{aligned} \tag{3.14}$$

Proof. By Proposition 3.4 and a stochastic Fubini's theorem one can see for each $t>0$ the random measure $X_t(dx)$ is absolutely continuous with density given by (3.14); see the proof of Li (2011, p.171). □

Proposition 3.8 *Suppose that Condition 3.6 holds and* $\{X_t:t\ge0\}$ *is a super-Lévy process with deterministic and absolutely continuous initial state* $X_0=\mu\in M(\mathbb{R})$. *Then for any* $r\ge0$ *and* $a\ge0$ *we have*

$$\lim_{t\to r}\sup_{f\in B_a}\mathbf{E}\big[|X_t(f)-X_r(f)|\big]=0 \tag{3.15}$$

with $B_a=\{f\in B(\mathbb{R}):\|f\|\le a\}$.

Proof. Let $(P_t^*)_{t\ge0}$ denote the dual semigroup of $(P_t)_{t\ge0}$. Fix a density $x\mapsto h(x)$ of the measure $\mu(dx)$. By the duality relation, we have

$$\begin{aligned}\sup_{f\in B_a}|\mu(P_r^bf-P_t^bf)| \;&=\; \sup_{f\in B_a}\Big|\int_{\mathbb{R}}f(x)[P_t^{*b}h(x)-P_r^{*b}h(x)]dx\Big|\\ &\le\; a\int_{\mathbb{R}}|P_t^{*b}h(x)-P_r^{*b}h(x)|dx.\end{aligned} \tag{3.16}$$

The right-hand side tends to zero as $t\to r+$ or $r\to t-$ by the strong continuity of $(P_t^*)_{t\ge0}$ on the Banach space $L^1(\mathbb{R})$ of integrable functions. By (3.11), for any $t\ge r\ge0$ and $f\in B(\mathbb{R})$,

$$X_r(f-P_{t-r}^bf) \;=\; \mu(P_r^bf-P_t^bf)+\sqrt{c}\int_0^r\int_0^\infty(P_{r-s}^bf-P_{t-s}^bf)(Y_s^{-1}(u))W(ds,du)$$

$$+\int_0^r\int_0^\infty\int_0^\infty z(P^b_{r-s}f-P^b_{t-s}f)(Y_s^{-1}(u))\tilde{N}_0(ds,dz,du). \tag{3.17}$$

As in the proof of Proposition 3.5, we can estimate the stochastic integrals on the right-hand side. For the first integral we have

$$\begin{aligned}\mathbf{E}\Big\{\Big|\int_0^r\int_0^\infty(P^b_{t-s}f-P^b_{r-s}f)(Y_s^{-1}(u))W(ds,du)\Big|^2\Big\}\\ &=\mathbf{E}\Big\{\int_0^r ds\int_{\mathbb{R}}[P^b_{t-s}f(x)-P^b_{r-s}f(x)]^2X_s(dx)\Big\}\\ &=\int_0^r ds\int_{\mathbb{R}}[P^b_{t-s}f(x)-P^b_{r-s}f(x)]^2\mu P^b_s(dx).\end{aligned} \tag{3.18}$$

For the second integral we have

$$\begin{aligned}\mathbf{E}\Big\{\Big|\int_0^r\int_0^1\int_0^\infty z(P^b_{t-s}f-P^b_{r-s}f)(Y_s^{-1}(u))\tilde{N}_0(ds,dz,du)\Big|^2\Big\}\\ &=\mathbf{E}\Big\{\int_0^r ds\int_0^1 z^2m(dz)\int_{\mathbb{R}}[P^b_{t-s}f(x)-P^b_{r-s}f(x)]^2X_s(dx)\Big\}\\ &=\int_0^1 z^2m(dz)\int_0^r ds\int_{\mathbb{R}}[P^b_{t-s}f(x)-P^b_{r-s}f(x)]^2\mu P^b_s(dx)\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\mathbf{E}\Big\{\Big|\int_0^r\int_1^\infty\int_0^\infty z[(P^b_{t-s}f-P^b_{r-s}f)(Y_s^{-1}(u))]\tilde{N}_0(ds,dz,du)\Big|\Big\}\\ &\le 2\,\mathbf{E}\Big\{\int_0^r ds\int_1^\infty zm(dz)\int_0^\infty|P^b_{t-s}f(x)-P^b_{r-s}f(x)|X_s(dx)\Big\}\\ &=2\int_1^\infty zm(dz)\int_0^r ds\int_{\mathbb{R}}|P^b_{t-s}f(x)-P^b_{r-s}f(x)|\mu P^b_s(dx).\end{aligned} \tag{3.20}$$

For $n\ge 1$ let $f_n(x,y)=f(x+y)1_{\{|y|\le n\}}$ and $f_n^c(x,y)=f(x+y)1_{\{|y|>n\}}$. We have

$$\begin{aligned}&\int_0^r ds\int_{\mathbb{R}}\big[P^b_{t-s}f(x)-P^b_{r-s}f(x)\big]^2\mu P^b_s(dx)\\ &\le 2e^{|b|t}\|f\|\int_0^r ds\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}f(x+y)\big[e^{-b(t-s)}p_{t-s}(y)-e^{-b(r-s)}p_{r-s}(y)\big]dy\Big|\mu P_s(dx)\\ &\le 2e^{|b|t}\|f\|\int_0^r ds\int_{\mathbb{R}}\mu P_s(dx)\int_{\mathbb{R}}|f_n(x,y)|\big|e^{-b(t-s)}p_{t-s}(y)-e^{-b(r-s)}p_{r-s}(y)\big|dy\\ &\quad+2e^{|b|t}\|f\|\int_0^r ds\int_{\mathbb{R}}\mu P_s(dx)\int_{\mathbb{R}}|f_n^c(x,y)|\big[e^{-b(t-s)}p_{t-s}(y)+e^{-b(t-s)}p_{r-s}(y)\big]dy\\ &\le 2e^{|b|t}\|f\|^2\int_0^r ds\int_{\mathbb{R}}\mu P_s(dx)\int_{\{|y|\le n\}}\big|e^{-b(t-s)}p_{t-s}(y)-e^{-b(r-s)}p_{r-s}(y)\big|dy\\ &\quad+2e^{2|b|t}\|f\|^2\int_0^r ds\int_{\mathbb{R}}\big[P_{t-s}(0,[-n,n]^c)+P_{r-s}(0,[-n,n]^c)\big]\mu P_s(dx).\end{aligned}$$

By the property of the transition semigroup $(P_t)_{t\ge 0}$ we have

$$\lim_{n\to\infty}\sup_{0\le s\le T}P_s(0,[-n,n]^c)=0,\qquad T\ge 0.$$

Then we can use Condition 3.6, the continuity of $(t,z)\mapsto e^{-bt}p_t(z)$ and dominated convergence to see that

$$\sup_{f\in B_a}\int_0^r ds\int_{\mathbb{R}}\big[P^b_{t-s}f(x)-P^b_{r-s}f(x)\big]^2\mu P^b_s(dx)\to 0\quad(t\to r+\text{ or }r\to t-).$$

This implies

$$\sup_{f\in B_a}\int_0^r ds\int_{\mathbb{R}}\big|P^b_{t-s}f(x)-P^b_{r-s}f(x)\big|\mu P^b_s(dx)\to 0\quad(t\to r+\text{ or }r\to t-).$$

From (3.16)–(3.20) it follows that

$$\sup_{f\in B_a}\mathbf{E}\big\{|X_r(f)-X_r(P^b_{t-r}f)|\big\}\to 0\qquad(t\to r+\text{ or }r\to t-).\tag{3.21}$$

Then we get (3.15) from (3.12) and (3.21). □

Theorem 3.9 *Suppose that Condition 3.6 holds and $\{Y_t: t\geq 0\}$ is the distribution function process of a super-Lévy process solving* (1.6) *with deterministic and absolutely continuous initial state $Y_0\in D^1(\mathbb{R})$. Then $\{Y_t(x): t\geq 0, x\in\mathbb{R}\}$ is continuous in $L^1(\mathbf{P})$.*

Proof. By Theorem 3.1 the measure-valued process $\{X_t: t\geq 0\}$ corresponding to $\{Y_t: t\geq 0\}$ is a super-Lévy process. For $r,t\geq 0$ and $x,y\in\mathbb{R}$ we have

$$\mathbf{E}\{|Y_r(x)-Y_t(y)|\}\leq\mathbf{E}\{|X_r(1_x)-X_t(1_x)|\}+\mathbf{E}\{X_t(|1_x-1_y|)\},\tag{3.22}$$

where $1_x(z)=1_{\{z\leq x\}}$. Suppose that $Y_0(dx)=h(x)dx$. By Proposition 3.3 we have

$$\mathbf{E}\{X_t(|1_x-1_y|)\}=\int_{\mathbb{R}}P^b_t|1_x-1_y|(z)\mu(dz)=\int_{\mathbb{R}}|1_x(z)-1_y(z)|P^{*b}_t h(z)dz,$$

which goes to zero as $(r,x)\to(t,y)$. Then the result follows by (3.22) and Proposition 3.8. □

Suppose that Condition 3.6 holds and $\{Y_t: t\geq 0\}$ is a solution of (1.6) with deterministic initial state $Y_0\in D(\mathbb{R})$. Let $\{\mathscr{G}_t: t\geq 0\}$ be the filtration generated by $\{Y_t: t\geq 0\}$. By Theorem 3.7 for every $t>0$ we have almost surely $Y_t\in D^1(\mathbb{R})$. In particular, for every $t>0$ the function $x\mapsto Y_t(x)$ is absolutely continuous on $\mathbb{R}$ almost surely. For $t>0$ a version of the derivative of $x\mapsto Y_t(x)$ is defined by (3.14). Recalling (1.11) for any $\delta>0$ we define the operator T_δ by

$$T_\delta f(x)=\int_{\mathbb{R}}f(y)g_\delta(y-x)dy=\int_{\mathbb{R}}f(x+z)g_\delta(z)dz,\qquad x\in\mathbb{R}.\tag{3.23}$$

It is easy to see that

$$\nabla Y_t(x):=\liminf_{n\to\infty}T_{1/n}X_t(x)=\liminf_{n\to\infty}\int_{\mathbb{R}}g_{1/n}(y-x)X_t(dy),\qquad t>0,\ x\in\mathbb{R}\tag{3.24}$$

defines a modification of the density field $\{X_t(x): t>0, x\in\mathbb{R}\}$ so that for each $t\geq 0$ the mapping $(s,x,\omega)\mapsto\nabla Y_s(x,\omega)1_{\{s>0\}}$ restricted to $[0,t]\times\mathbb{R}\times\Omega$ is measurable relative to the σ-algebra $\mathscr{B}[0,t]\times\mathscr{B}(\mathbb{R})\times\mathscr{G}_t$.

4 Measurability and integrability properties

In this section, we always assume Condition 3.6 is satisfied. We shall establish the measurability and integrability properties required by the stochastic integral equation (1.10). Suppose that

$\{Y_t : t \geq 0\}$ is a solution of (1.6) with deterministic initial state $Y_0 \in D(\mathbb{R})$. Let $\mu \in M(\mathbb{R})$ be the measure determined by Y_0. By Theorem 3.1 the corresponding measure-valued process $\{X_t : t \geq 0\}$ is a super-Lévy process with $X_0 = \mu$. Write $1_x(z) = 1_{\{z \leq x\}}$ for notational convenience. By Proposition 3.4 we have

$$Y_t(x) = \int_{\mathbb{R}} P_t^b 1_x(y)\mu(dy) + Z_t(x) + \tilde{H}_t(x), \tag{4.1}$$

where

$$\begin{aligned} Z_t(x) &= \sqrt{c}\int_0^t \int_0^\infty P_{t-s}^b 1_x(Y_s^{-1}(u))W(ds,du) \\ &\quad + \int_0^t \int_0^1 \int_0^\infty z P_{t-s}^b 1_x(Y_s^{-1}(u))\tilde{N}_0(ds,dz,du) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \tilde{H}_t(x) &= \int_0^t \int_1^\infty \int_0^\infty z P_{t-s}^b 1_x(Y_s^{-1}(u))N_0(ds,dz,du) \\ &\quad - \int_0^t ds \int_1^\infty z m(dz) \int_0^\infty P_{t-s}^b 1_x(Y_s^{-1}(u))du. \end{aligned} \tag{4.3}$$

Let $H_t(x)$ and $\hat{H}_t(x)$ denote the first and the second terms on the right-hand side of (4.3), respectively. According to (4.1), we can write

$$\nabla Y_t(x) = \int_{\mathbb{R}} p_t^b(x-y)\mu(dy) + \nabla Z_t(x) + \nabla \tilde{H}_t(x), \tag{4.4}$$

where

$$\begin{aligned} \nabla Z_t(x) &= \sqrt{c}\int_0^t \int_{\mathbb{R}} p_{t-s}^b(x - Y_s^{-1}(u))W(ds,du) \\ &\quad + \int_0^t \int_0^1 \int_0^\infty z p_{t-s}^b(x - Y_s^{-1}(u))\tilde{N}_0(ds,dz,du) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \nabla \tilde{H}_t(x) &= \int_0^t \int_1^\infty \int_0^\infty z p_{t-s}^b(x - Y_s^{-1}(u))N_0(ds,dz,du) \\ &\quad - \int_0^t ds \int_1^\infty z m(dz) \int_0^\infty p_{t-s}^b(x - Y_s^{-1}(u))du. \end{aligned} \tag{4.6}$$

Let $\nabla H_t(x)$ and $\nabla \hat{H}_t(x)$ denote the first and the second terms on the right-hand side of (4.6), respectively.

Let $\{\xi_t^* : t \geq 0\}$ and $\{\xi_t^r : 0 \leq r \leq t \leq T\}$ be given as in the introduction. Let $\{\mathscr{F}_t : t \geq 0\}$ be the augmented natural filtration of $\{\xi_t^* : t \geq 0\}$. Let $\{\mathscr{G}_t : t \geq 0\}$ be the augmented filtration generated by $\{Y_t : t \geq 0\}$ and $\{W(dt,du), \tilde{N}_0(dt,dz,du) : t \geq 0, z > 0, u > 0\}$. Fix the constant $T > 0$ and let $\{\mathscr{G}_t^r : 0 \leq r \leq t \leq T\}$ be the family of σ-algebras defined as in Section 2.

Proposition 4.1 *For any $0 \leq r \leq t \leq u \leq T$, the restriction to $[t,u] \times \mathbb{R} \times \Omega$ of the mapping*

$$(s,x,\omega) \mapsto (Y_{T-s}(\xi_s^r(\omega) + x, \omega), \nabla Y_{T-s}(\xi_s^r(\omega) + x, \omega)) \tag{4.7}$$

is measurable with respect to the σ-algebra $\mathscr{B}[t,u] \times \mathscr{B}(\mathbb{R}) \times \mathscr{G}_u^t$.

Proof. It is clear that the restriction to $[t,u]\times\Omega$ of $(s,\omega)\mapsto \xi_s^r(\omega)$ is measurable relative to the σ-algebras $\mathscr{B}[t,u]\times\mathscr{G}_u^t$ and $\mathscr{B}(\mathbb{R})$. Then the restriction to $[t,u]\times\mathbb{R}\times\Omega$ of $(s,x,\omega)\mapsto(s,\xi_s^r(\omega)+x,\omega)$ is measurable relative to the σ-algebras $\mathscr{B}[t,u]\times\mathscr{B}(\mathbb{R})\times\mathscr{G}_u^t$ and $\mathscr{B}[t,u]\times\mathscr{B}(\mathbb{R})\times\mathscr{G}_u^t$. By Proposition 3.2 the restriction to $[t,u]\times\mathbb{R}\times\Omega$ of $(s,y,\omega)\mapsto Y_{T-s}(y,\omega)$ is measurable relative to the σ-algebras $\mathscr{B}[t,u]\times\mathscr{B}(\mathbb{R})\times\mathscr{G}_u^t$ and $\mathscr{B}(\mathbb{R}_+)$. Then the composed mapping $(s,x,\omega)\mapsto Y_{T-s}(\xi_s^r(\omega)+x,\omega)$ is measurable relative to the σ-algebras $\mathscr{B}[t,u]\times\mathscr{B}(\mathbb{R})\times\mathscr{G}_u^t$ and $\mathscr{B}(\mathbb{R}_+)$. The required measurability of the other coordinate in (4.7) follows as a consequence. □

Lemma 4.2 *For each $T\ge 0$ there is a constant $C_T\ge 0$ so that*

$$\mathbf{E}\big[Z_{T-s}(\xi_s^r+x)^2\big]\le C_T\mu(1),\qquad 0\le r<s<T, x\in\mathbb{R}. \tag{4.8}$$

Proof. Since $\{\xi_s^r\}$ is independent of $\{Y_s(x)\}$, $\{W(ds,du)\}$ and $\{N_0(ds,dz,du)\}$, by Itô's isometry we have

$$\begin{aligned}
\mathbf{E}\big[Z_{T-s}(\xi_s^r+x)^2\big] &= c\,\mathbf{E}\Big\{\Big[\int_0^{T-s}\int_0^\infty P_{T-s-v}^b 1_{\xi_s^r+x}(Y_v^{-1}(u))W(dv,du)\Big]^2\Big\}\\
&\quad+\mathbf{E}\Big\{\Big[\int_0^{T-s}\int_0^1\int_0^\infty zP_{T-s-v}^b 1_{\xi_s^r+x}(Y_v^{-1}(u))\tilde{N}_0(dv,dz,du)\Big]^2\Big\}\\
&= C\,\mathbf{E}\Big\{\int_0^{T-s}dv\int_0^\infty[P_{T-s-v}^b 1_{\xi_s^r+x}(Y_v^{-1}(u))]^2du\Big\}\\
&= C\,\mathbf{E}\Big\{\int_0^{T-s}dv\int_{\mathbb{R}}[P_{T-s-v}^b 1_{\xi_s^r+x}(u)]^2X_v(du)\Big\}\\
&\le Ce^{2|b|T}\mathbf{E}\Big\{\int_0^{T-s}X_v(1)dv\Big\},
\end{aligned}$$

where

$$C=c+\int_0^1 z^2m(dz). \tag{4.9}$$

Then (4.8) follows by Proposition 3.3. □

Lemma 4.3 *For each $T\ge 0$ there is a constant $C_T\ge 0$ so that, for $0\le r<s<T$ and $x\le y\in\mathbb{R}$,*

$$\mathbf{E}\Big\{[Z_{T-s}(\xi_s^r+y)-Z_{T-s}(\xi_s^r+x)]^2\Big\}\le(y-x)^2\frac{C_T\mu(1)}{(T-r)^\alpha}. \tag{4.10}$$

Proof. Since $\{\xi_s^r\}$ is independent of $\{Y_s(x)\}$ and $\{W(ds,du)\}$, by Itô's isometry we have

$$\begin{aligned}
&\mathbf{E}\Big\{\Big[\int_0^{T-s}\int_0^\infty P_{T-s-v}^b(1_{\xi_s^r+y}-1_{\xi_s^r+x})(Y_v^{-1}(u))W(dv,du)\Big]^2\Big\}\\
&\quad=\mathbf{E}\Big\{\int_0^{T-s}dv\int_0^\infty[P_{T-s-v}^b(1_{\xi_s^r+y}-1_{\xi_s^r+x})(Y_v^{-1}(u))]^2du\Big\}\\
&\quad=\mathbf{E}\Big\{\int_0^{T-s}dv\int_{\mathbb{R}}[P_{T-s-v}^b(1_{\xi_s^r+y}-1_{\xi_s^r+x})(u)]^2X_v(du)\Big\}\\
&\quad=\mathbf{E}\Big\{\int_0^{T-s}dv\int_{\mathbb{R}}X_v(du)\int_{\mathbb{R}}[P_{T-s-v}^b(1_{y-\xi}-1_{x-\xi})(u)]^2p_{s-r}(\xi)d\xi\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{T-s} dv \int_{\mathbb{R}} \mu P_v^b(du) \int_{\mathbb{R}} \Big\{ \int_{\mathbb{R}} [1_{y-\xi}(\eta) - 1_{x-\xi}(\eta)] p^b_{T-s-v}(\eta - u) d\eta \Big\}^2 p_{s-r}(\xi) d\xi \\
&\leq C_T(y-x) \int_0^{T-s} \frac{dv}{(T-s-v)^\alpha} \int_{\mathbb{R}} \mu P_v(du) \int_{\mathbb{R}} p_{s-r}(\xi) d\xi \int_{x-\xi}^{y-\xi} p_{T-s-v}(\eta - u) d\eta \\
&= C_T(y-x) \int_0^{T-s} \frac{dv}{(T-s-v)^\alpha} \int_{\mathbb{R}} \mu P_v(du) \int_{\mathbb{R}} p_{s-r}(\xi) d\xi \int_{x-u}^{y-u} p_{T-s-v}(\eta - \xi) d\eta \\
&= C_T(y-x) \int_0^{T-s} \frac{dv}{(T-s-v)^\alpha} \int_{\mathbb{R}} \mu(dz) \int_{\mathbb{R}} p_v(u-z) du \int_{x-u}^{y-u} p_{T-r-v}(\eta) d\eta \\
&= C_T(y-x) \int_0^{T-s} \frac{dv}{(T-s-v)^\alpha} \int_{\mathbb{R}} \mu(dz) \int_{\mathbb{R}} p_v(u-z) du \int_x^y p_{T-r-v}(\eta - u) d\eta \\
&= C_T(y-x) \int_0^{T-s} \frac{dv}{(T-s-v)^\alpha} \int_{\mathbb{R}} \mu(dz) \int_x^y p_{T-r}(\eta - z) d\eta \\
&\leq (y-x)^2 \frac{C_T \mu(1)(T-s)^{1-\alpha}}{(1-\alpha)(T-r)^\alpha},
\end{aligned}
$$

where we have also used Condition 3.6 twice. Similarly,

$$
\begin{aligned}
&\mathbf{E}\Big\{ \Big[\int_0^{T-s} \int_0^1 \int_0^\infty z P^b_{T-s-v}(1_{\xi_s^r + y} - 1_{\xi_s^r + x})(Y_v^{-1}(u)) \tilde{N}_0(dv, dz, du) \Big]^2 \Big\} \\
&\quad = \mathbf{E}\Big\{ \int_0^{T-s} dv \int_0^1 z^2 m(dz) \int_0^\infty [P^b_{T-s-v}(1_{\xi_s^r + y} - 1_{\xi_s^r + x})(Y_v^{-1}(u))]^2 du \Big\} \\
&\quad \leq (y-x)^2 \frac{C_T \mu(1)(T-s)^{1-\alpha}}{(1-\alpha)(T-r)^\alpha}.
\end{aligned}
$$

By putting the two inequalities together we obtain (4.10). □

Lemma 4.4 *For each $T \geq 0$ there is a constant $C_T \geq 0$ so that, for $0 \leq r < s < T$ and $x \in \mathbb{R}$,*

$$
\mathbf{E}\Big\{ [\nabla Z_{T-s}(\xi_s^r + x)]^2 \Big\} \leq C_T (T-s)^{1-\alpha} \int_{\mathbb{R}} p_{T-r}(x-y) \mu(dy). \tag{4.11}
$$

Proof. Let us bound separately the two terms corresponding to the right-hand side of (4.5). Since $\{\xi_s^r\}$ is independent of $\{Y_s(x)\}$ and $\{W(ds, du)\}$, by Itô's isometry it follows that

$$
\begin{aligned}
&\mathbf{E}\Big\{ \Big[\int_0^{T-s} \int_0^\infty p^b_{T-s-v}(\xi_s^r + x - Y_s^{-1}(u)) W(dv, du) \Big]^2 \Big\} \\
&\quad = \mathbf{E}\Big\{ \int_0^{T-s} dv \int_0^\infty p^b_{T-s-v}(\xi_s^r + x - Y_v^{-1}(u))^2 du \Big\} \\
&\quad = \mathbf{E}\Big\{ \int_0^{T-s} dv \int_{\mathbb{R}} p^b_{T-s-v}(\xi_s^r + x - u)^2 X_v(du) \Big\} \\
&\quad = \mathbf{E}\Big\{ \int_0^{T-s} dv \int_{\mathbb{R}} X_v(du) \int_{\mathbb{R}} p^b_{T-s-v}(\xi - u)^2 p_{s-r}(x-\xi) d\xi \Big\} \\
&\quad = \int_0^{T-s} dv \int_{\mathbb{R}} \mu P_v^b(du) \int_{\mathbb{R}} p^b_{T-s-v}(\xi - u)^2 p_{s-r}(x-\xi) d\xi \\
&\quad \leq C_T \int_0^{T-s} \frac{dv}{(T-s-v)^\alpha} \int_{\mathbb{R}} p_v(u-y) \mu(dy) \int_{\mathbb{R}} p_{T-r-v}(x-u) du \\
&\quad \leq C_T \int_0^{T-s} \frac{dv}{(T-s-v)^\alpha} \int_{\mathbb{R}} p_{T-r}(x-y) \mu(dy)
\end{aligned}
$$

$$\leq C_T(T-s)^{1-\alpha}\int_{\mathbb{R}} p_{T-r}(x-y)\mu(dy).$$

By similar calculations,

$$\begin{aligned}
&\mathbf{E}\Big\{\Big[\int_0^{T-s}\int_0^1\int_0^\infty zp^b_{T-s-v}(x-Y_s^{-1}(u))\tilde{N}_0(dv,dz,du)\Big]^2\Big\}\\
&\quad=\mathbf{E}\Big\{\int_0^{T-s}dv\int_0^1 z^2m(dz)\int_0^\infty p^b_{T-s-v}(\xi_s^r+x-Y_v^{-1}(u))^2du\Big\}\\
&\quad\leq C_T(T-s)^{1-\alpha}\int_{\mathbb{R}} p_{T-r}(x-y)\mu(dy).
\end{aligned}$$

Then we have (4.11). □

Proposition 4.5 *For any $0\leq r<T$ and $x\in\mathbb{R}$ we have*

$$\mathbf{E}\Big\{\Big[\int_r^T|\nabla Y_{T-s}(\xi_s^r+x)|^2ds\Big]^{\frac{1}{2}}\Big\}<\infty.$$

Proof. We are going to bound separately the three terms corresponding to the right-hand side of (4.4). It is simple to see that

$$\begin{aligned}
&\mathbf{E}\Big\{\int_r^T\Big|\int_{\mathbb{R}} p^b_{T-s}(\xi_s^r+x-y)\mu(dy)\Big|^2ds\Big\}\\
&\quad=\int_r^T ds\int_{\mathbb{R}}\Big|\int_{\mathbb{R}} p^b_{T-s}(\xi-y)\mu(dy)\Big|^2p_{s-r}(x-\xi)d\xi\\
&\quad\leq C_T\mu(1)\int_r^T\frac{ds}{(T-s)^\alpha}\int_{\mathbb{R}} p_{T-s}(\xi-y)\mu(dy)\int_{\mathbb{R}} p_{s-r}(x-\xi)d\xi\\
&\quad=C_T\mu(1)\int_r^T\frac{ds}{(T-s)^\alpha}\int_{\mathbb{R}} p_{T-r}(x-y)\mu(dy)\\
&\quad\leq C_T\mu(1)^2(T-r)^{1-2\alpha}<\infty. \qquad (4.12)
\end{aligned}$$

By (4.11) we have

$$\mathbf{E}\Big\{\int_r^T|\nabla Z_{T-s}(\xi_s^r+x)|^2ds\Big\}\leq C_T\mu(1)(T-r)^{2-2\alpha}<\infty. \qquad (4.13)$$

By the Cauchy-Schwarz inequality and Proposition 3.3,

$$\begin{aligned}
&\mathbf{E}\Big\{\Big(\int_r^T|\nabla\hat{H}_{T-s}(\xi_s^r+x)|^2ds\Big)^{\frac{1}{2}}\Big\}\\
&\quad\leq C_T\mathbf{E}\Big\{\Big(\int_r^T\Big|\int_0^{T-s}ds_1\int_{\mathbb{R}} p_{T-s-s_1}(\xi_s^r+x-u)X_{s_1}(du)\Big|^2ds\Big)^{\frac{1}{2}}\Big\}\\
&\quad\leq C_T\mathbf{E}\Big\{\Big(\int_0^T X_{s_1}(1)ds_1\int_r^T ds\int_0^{T-s}ds_2\int_{\mathbb{R}} p_{T-s-s_2}(\xi_s^r+x-u)^2X_{s_2}(du)\Big)^{\frac{1}{2}}\Big\}\\
&\quad\leq C_T\Big\{\mathbf{E}\Big[\int_0^T X_{s_1}(1)ds_1\Big]\mathbf{E}\Big[\int_r^T ds\int_0^{T-s}ds_2\int_{\mathbb{R}} p_{T-s-s_2}(\xi_s^r+x-u)^2X_{s_2}(du)\Big]\Big\}^{\frac{1}{2}}\\
&\quad\leq C_T\Big\{\mathbf{E}\Big[\int_r^T ds\int_0^{T-s}ds_2\int_{\mathbb{R}} p_{T-s-s_2}(\xi_s^r+x-u)^2X_{s_2}(du)\Big]\Big\}^{\frac{1}{2}}. \qquad (4.14)
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&\mathbf{E}\Big\{\Big(\int_r^T |\nabla H_{T-s}(\xi_s^r+x)|^2 ds\Big)^{\frac{1}{2}}\Big\} \\
&\quad \le C_T\mathbf{E}\Big[\Big(\int_r^T \Big|\int_0^{T-s}\int_1^\infty\int_0^\infty z1_{\{u\le X_{s_1}(1)\}}N_0(ds_1,dz,du) \\
&\qquad \cdot \int_0^{T-s}\int_1^\infty\int_0^\infty zp_{T-s-s_1}(\xi_s^r+x-Y_{s_1}^{-1}(u))^2N_0(ds_1,dz,du)\Big|ds\Big)^{\frac{1}{2}}\Big] \\
&\quad \le C_T\mathbf{E}\Big[\Big(\int_0^T\int_1^\infty\int_0^\infty z1_{\{u\le X_{s_1}(1)\}}N_0(ds_1,dz,du)\Big)^{\frac{1}{2}} \\
&\qquad \cdot\Big(\int_r^T ds\int_0^{T-s}\int_1^\infty\int_0^\infty zp_{T-s-s_1}(\xi_s^r+x-Y_{s_1}^{-1}(u))^2N_0(ds_1,dz,du)\Big)^{\frac{1}{2}}\Big] \\
&\quad \le C_T\Big\{\mathbf{E}\Big[\int_0^T\int_1^\infty\int_0^\infty z1_{\{u\le X_{s_1}(1)\}}N_0(ds_1,dz,du)\Big]\Big\}^{\frac{1}{2}}\cdot\Big\{\mathbf{E}\Big[\int_r^T ds \\
&\qquad\qquad \int_0^{T-s}\int_1^\infty\int_0^\infty zp_{T-s-s_2}(\xi_s^r+x-Y_{s_2}^{-1}(u))^2N_0(ds_2,dz,du)\Big]\Big\}^{\frac{1}{2}} \\
&\quad \le C_T\Big\{\mathbf{E}\Big[\int_0^T ds_1\int_1^\infty zX_{s_1}(1)m(dz)\Big]\mathbf{E}\Big[\int_r^T ds\int_0^{T-s}ds_2\int_1^\infty zm(dz) \\
&\qquad\qquad \int_0^\infty p_{T-s-s_2}(\xi_s^r+x-Y_{s_2}^{-1}(u))^2du\Big]\Big\}^{\frac{1}{2}} \\
&\quad \le C_T\Big\{\mathbf{E}\Big[\int_r^T ds\int_0^{T-s}ds_2\int_{\mathbb{R}}p_{T-s-s_2}(\xi_s^r+x-u)^2X_{s_2}(du)\Big]\Big\}^{\frac{1}{2}}. \qquad (4.15)
\end{aligned}
$$

By the independence of $\{\xi_s^r\}$ and $\{X_s\}$, the square of the right-hand side of (4.14) or (4.15) is bounded above by

$$
\begin{aligned}
&C_T\int_r^T ds\int_0^{T-s}ds_2\int_{\mathbb{R}}\mu P_{s_2}(du)\int_{\mathbb{R}}p_{T-s-s_2}(\xi-u)^2p_{s-r}(x-\xi)d\xi \\
&\quad \le C_T\int_r^T ds\int_0^{T-s}(T-s-s_2)^{-\alpha}ds_2\int_{\mathbb{R}}p_{T-r}(x-y)\mu(dy) \\
&\quad \le C_T\int_r^T (T-s)^{1-\alpha}ds\int_{\mathbb{R}}p_{T-r}(x-y)\mu(dy) \\
&\quad \le C_T\mu(1)(T-r)^{2-2\alpha}<\infty. \qquad (4.16)
\end{aligned}
$$

Combining (4.4) and (4.12)–(4.16) we obtain the desired result. □

Lemma 4.6 *For any $0\le r<T$ and $x\in\mathbb{R}$ we have*

$$
\mathbf{E}\Big\{\Big[\int_r^T\int_{\{|z|\le 1\}}\big|\hat{H}_{T-s}(\xi_s^r+x-z)-\hat{H}_{T-s}(\xi_s^r+x)\big|^2M(ds,dz)\Big]^{\frac{1}{2}}\Big\}<\infty. \qquad (4.17)
$$

Proof. By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\text{l.h.s. of (4.17)} &= \mathbf{E}\Big\{\Big(\int_r^T\int_{\{|z|\le 1\}}\Big|\int_0^z\nabla\hat{H}_{T-s}(\xi_s^r+x-\eta)d\eta\Big|^2M(ds,dz)\Big)^{\frac{1}{2}}\Big\} \\
&\le \mathbf{E}\Big\{\Big(\int_r^T\int_{\{|z|\le 1\}}zM(ds,dz)\int_0^z|\nabla\hat{H}_{T-s}(\xi_s^r+x-\eta)|^2d\eta\Big)^{\frac{1}{2}}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{E}\Big\{\Big(\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z\Big|\int_0^{T-s} ds_1\int_1^\infty z_1 m(dz_1) \\
&\qquad\qquad \int_0^\infty p^b_{T-s-s_1}(\xi^r_s+x-\eta-Y^{-1}_{s_1}(u))du\Big|^2 d\eta\Big)^{\frac{1}{2}}\Big\} \\
&= C\,\mathbf{E}\Big\{\Big(\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z\Big|\int_0^{T-s} ds_1 \\
&\qquad\qquad \int_{\mathbb{R}} p^b_{T-s-s_1}(\xi^r_s+x-\eta-u)X_{s_1}(du)\Big|^2 d\eta\Big)^{\frac{1}{2}}\Big\} \\
&\le C\,\mathbf{E}\Big\{\Big(\int_0^T X_{s_1}(1)ds_1\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta\int_0^{T-s} ds_2 \\
&\qquad\qquad \int_{\mathbb{R}} p^b_{T-s-s_2}(\xi^r_s+x-\eta-u)^2 X_{s_2}(du)\Big)^{\frac{1}{2}}\Big\} \\
&\le C\Big\{\mathbf{E}\Big[\int_0^T X_{s_1}(1)ds_1\Big]\mathbf{E}\Big[\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta\int_0^{T-s} ds_2 \\
&\qquad\qquad \int_{\mathbb{R}} p^b_{T-s-s_2}(\xi^r_s+x-\eta-u)^2 X_{s_2}(du)\Big]\Big\}^{\frac{1}{2}} \\
&\le C_T\Big\{\mathbf{E}\Big[\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta\int_0^{T-s} ds_2 \\
&\qquad\qquad \int_{\mathbb{R}} p_{T-s-s_2}(\xi^r_s+x-\eta-u)^2 X_{s_2}(du)\Big]\Big\}^{\frac{1}{2}}.
\end{aligned}
$$

(The above qualities are all positive because of the product $z\int_0^z$.) Then by the independence of $\{\xi_t\}$ and $\{X_t\}$ we have

$$
\begin{aligned}
\{\text{l.h.s. of } (4.17)\}^2 &\le C_T\int_r^T ds\int_{\{|z|\le 1\}} z\nu(dz)\int_0^z d\eta\int_0^{T-s} ds_2\int_{\mathbb{R}}\mu(dy) \\
&\qquad \int_{\mathbb{R}} p_{s_2}(u-y)du\int_{\mathbb{R}} p_{s-r}(x-\eta-\xi)p_{T-s-s_2}(\xi-u)^2 d\xi \\
&\le C_T\int_r^T ds\int_0^{T-s}(T-s-s_2)^{-\alpha}ds_2\int_{\{|z|\le 1\}} z\nu(dz) \\
&\qquad \int_0^z d\eta\int_{\mathbb{R}} p_{T-r}(x-\eta-y)\mu(dy) \\
&\le \frac{C_T\mu(1)}{(T-r)^\alpha}\int_r^T (T-s)^{1-\alpha}ds\int_{\{|z|\le 1\}} z^2\nu(dz)<\infty.
\end{aligned}
$$

Then we have (4.17). □

Lemma 4.7 *For any $0\le r<T$ and $x\in\mathbb{R}$ we have*

$$
\mathbf{E}\Big\{\Big[\int_r^T\int_{\{|z|\le 1\}}\big|H_{T-s}(\xi^r_s+x-z)-H_{T-s}(\xi^r_s+x)\big|^2 M(ds,dz)\Big]^{\frac{1}{2}}\Big\}<\infty. \tag{4.18}
$$

Proof. By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\text{l.h.s. of } (4.18) &= \mathbf{E}\Big\{\Big[\int_r^T\int_{\{|z|\le 1\}}\Big|\int_0^z \nabla H_{T-s}(\xi^r_s+x-v)dv\Big|^2 M(ds,dz)\Big]^{\frac{1}{2}}\Big\} \\
&\le \mathbf{E}\Big\{\Big[\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z[\nabla H_{T-s}(\xi^r_s+x-\eta)]^2 d\eta\Big]^{\frac{1}{2}}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\,\mathbf{E}\Big\{\Big(\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z\Big|\int_0^{T-s}\int_1^\infty\int_0^\infty \\
&\qquad z_1 p^b_{T-s-s_1}(\xi^r_s+x-\eta-Y^{-1}_{s_1}(w))N_0(ds_1,dz_1,dw)\Big|^2 d\eta\Big)^{\frac12}\Big\} \\
&\le C\,\mathbf{E}\Big\{\Big(\int_0^T\int_1^\infty\int_0^\infty z_1 1_{\{w_1\le Y_{s_1}(\infty)\}}N_0(ds_1,dz_1,dw_1)\Big)^{\frac12} \\
&\qquad\cdot\Big(\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta\int_0^{T-s}\int_1^\infty\int_0^\infty \\
&\qquad z_2 p^b_{T-s-s_2}(\xi^r_s+x-\eta-Y^{-1}_{s_2}(w_2))^2 N_0(ds_2,dz_2,dw_2)\Big)^{\frac12}\Big\} \\
&\le C\Big\{\mathbf{E}\Big[\int_0^T\int_1^\infty\int_0^\infty z_1 1_{\{w_1\le Y_{s_1}(\infty)\}}N_0(ds_1,dz_1,dw_1)\Big] \\
&\qquad\cdot\mathbf{E}\Big[\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta\int_0^{T-s}\int_1^\infty\int_0^\infty \\
&\qquad z_2 p^b_{T-s-s_2}(\xi^r_s+x-\eta-Y^{-1}_{s_2}(w_2))^2 N_0(ds_2,dz_2,dw_2)\Big]\Big\}^{\frac12} \\
&\le C_T\Big\{\mathbf{E}\Big[\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta\int_0^{T-s} ds_2\int_1^\infty z_2 m(dz_2) \\
&\qquad\int_0^\infty p_{T-s-s_2}(\xi^r_s+x-\eta-Y^{-1}_{s_2}(w_2))^2 dw_2\Big]\Big\}^{\frac12} \\
&\le C_T\Big\{\mathbf{E}\Big[\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta\int_0^{T-s} ds_2 \\
&\qquad\int_{\mathbb{R}} p_{T-s-s_2}(\xi^r_s+x-\eta-w_2)^2 X_{s_2}(dw_2)\Big]\Big\}^{\frac12}.
\end{aligned}
$$

The right-hand side is finite by the proof of Lemma 4.6. □

Proposition 4.8 *For any $0\le r<T$ and $x\in\mathbb{R}$ we have*

$$
\mathbf{E}\Big\{\Big[\int_r^T\int_{\mathbb{R}^\circ}\big|Y_{T-s}(\xi^r_s+x-z)-Y_{T-s}(\xi^r_s+x)\big|^2 M(ds,dz)\Big]^{\frac12}\Big\}<\infty. \tag{4.19}
$$

Proof. Recall the decomposition (4.1). We shall bound separately the three terms corresponding to the right-hand side. We first observe

$$
\begin{aligned}
&\mathbf{E}\Big\{\int_r^T\int_{\{|z|\le 1\}}\Big[\int_{\mathbb{R}} P^b_{T-s}(1_{\xi^r_s+x-z}-1_{\xi^r_s+x})(y)\mu(dy)\Big]^2 M(ds,dz)\Big\} \\
&\quad=\int_r^T ds\int_{\{|z|\le 1\}}\nu(dz)\int_{\mathbb{R}}\Big[\int_{\mathbb{R}} P^b_{T-s}(1_\xi-1_{\xi-z})(y)\mu(dy)\Big]^2 p_{s-r}(x-\xi)d\xi \\
&\quad\le\int_r^T ds\int_{\{|z|\le 1\}}\nu(dz)\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}\mu(dy)\int_{\xi-z}^{\xi} p^b_{T-s}(\eta-y)d\eta\Big]^2 p_{s-r}(x-\xi)d\xi \\
&\quad\le C_T\mu(1)\int_r^T\frac{ds}{(T-s)^\alpha}\int_{\{|z|\le 1\}} z\nu(dz)\int_{\mathbb{R}}\mu(dy)\int_{\mathbb{R}} p_{s-r}(x-\xi)d\xi\int_{\xi-z}^{\xi} p_{T-s}(\eta-y)d\eta \\
&\quad= C_T\mu(1)\int_r^T\frac{ds}{(T-s)^\alpha}\int_{\{|z|\le 1\}} z\nu(dz)\int_{\mathbb{R}}\mu(dy)\int_{\mathbb{R}} p_{T-s}(\eta-y)d\eta\int_{\eta}^{\eta+z} p_{s-r}(x-\xi)d\xi \\
&\quad= C_T\mu(1)\int_r^T\frac{ds}{(T-s)^\alpha}\int_{\{|z|\le 1\}} z\nu(dz)\int_{\mathbb{R}}\mu(dy)\int_{\mathbb{R}} p_{T-s}(\eta-y)d\eta\int_{x-z}^{x} p_{s-r}(\zeta-\eta)d\zeta
\end{aligned}
$$

$$
\begin{aligned}
&= C_T\mu(1)\int_r^T \frac{ds}{(T-s)^\alpha}\int_{\{|z|\le 1\}} z\nu(dz)\int_{\mathbb{R}}\mu(dy)\int_{x-z}^x p_{T-r}(\zeta-y)d\zeta \\
&\le \frac{C_T\mu(1)^2}{(T-r)^\alpha}\int_r^T \frac{ds}{(T-s)^\alpha}\int_{\{|z|\le 1\}} z^2\nu(dz). \qquad (4.20)
\end{aligned}
$$

By Lemma 4.3 we have

$$
\begin{aligned}
&\mathbf{E}\Big\{\int_r^T\int_{\{|z|\le 1\}}\big|Z_{T-s}(\xi_s^r+x-z)-Z_{T-s}(\xi_s^r+x)\big|^2 M(ds,dz)\Big\} \\
&\quad = \int_r^T ds\int_{\{|z|\le 1\}} \mathbf{E}\{[Z_{T-s}(\xi_s^r+x-z)-Z_{T-s}(\xi_s^r+x)]^2\}\nu(dz) \\
&\quad \le C_T\mu(1)(T-r)^{1-\alpha}\int_{\{|z|\le 1\}} z^2\nu(dz). \qquad (4.21)
\end{aligned}
$$

By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\mathbf{E}\Big\{\Big[\int_r^T\int_{\{|z|>1\}}\big|Y_{T-s}(\xi_s^r+x-z)-Y_{T-s}(\xi_s^r+x)\big|^2 M(ds,dz)\Big]^{\frac{1}{2}}\Big\} \\
&\le \mathbf{E}\Big\{\Big[\sup_{s\in[0,T]} X_{T-s}(1)\Big]^{\frac{1}{2}}\cdot\Big[\int_r^T\int_{\{|z|>1\}} X_{T-s}(1)M(ds,dz)\Big]^{\frac{1}{2}}\Big\} \\
&\le \Big\{\mathbf{E}\Big[\sup_{s\in[0,T]} X_{T-s}(1)\Big]\cdot\int_r^T ds\int_{\{|z|>1\}}\mathbf{E}\big[X_{T-s}(1)\big]\nu(dz)\Big\}^{\frac{1}{2}} < \infty.
\end{aligned}
$$

Then the result follows from (4.20), (4.21) and Lemmas 4.6 and 4.7. □

By Propositions 4.5 and 4.8 the last two terms on the right-hand side of (1.10) are well-defined. To see the second and the third terms make sense, it suffices to observe, for $0 \le r \le t \le T$ and $x \in \mathbb{R}$,

$$
\mathbf{E}\Big\{\int_t^T Y_{T-s}(\xi_s^r+x)ds\Big\} \le \mathbf{E}\Big\{\int_r^T X_{T-s}(1)ds\Big\} < \infty.
$$

On the other hand, we have

$$
\begin{aligned}
&\mathbf{E}\Big\{\Big|\int_{t-}^{T-}\int_0^1\int_0^{Y_{T-s}(\xi_s^r+x)} z\tilde{N}_0^T(\overleftarrow{ds},dz,du)\Big|^2\Big\} \\
&\quad = \mathbf{E}\Big\{\int_t^T ds\int_0^1 Y_{T-s}(\xi_s^r+x)z^2 m(dz)\Big\} \\
&\quad \le \mathbf{E}\Big\{\int_r^T X_{T-s}(1)ds\int_0^1 z^2 m(dz)\Big\} < \infty.
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{E}\Big\{\Big|\int_{t-}^{T-}\int_1^\infty\int_0^{Y_{T-s}(\xi_s^r+x)} zN_0^T(\overleftarrow{ds},dz,du)\Big|\Big\} \\
&\quad = \mathbf{E}\Big\{\int_t^T ds\int_1^\infty Y_{T-s}(\xi_s^r+x)zm(dz)\Big\} \\
&\quad \le \mathbf{E}\Big\{\int_r^T X_{T-s}(1)ds\int_1^\infty zm(dz)\Big\} < \infty.
\end{aligned}
$$

Then the fourth term on the right-hand side of (1.10) is also well-defined.

5 A smoothed stochastic equation

The purpose of this section is to establish a smoothed form of the stochastic equation (1.10). We shall use the settings of the last section. More specifically, we consider a deterministic and absolutely continuous initial state $Y_0 \in D^1(\mathbb{R})$. Let $U_t(x) = Y_{T-t}(x)$ for $0 \le t \le T$ and $x \in \mathbb{R}$. For any $\delta > 0$ and any function h on $\mathbb{R}$ let $h^\delta = T_\delta h$ if the right-hand side is well-defined. Then letting $f(y) = g_\delta(y - x)$ in (1.8) we have

$$\begin{aligned} U_t^\delta(x) &= U_T^\delta(x) + \int_t^T A^* U_s^\delta(x)ds + \sqrt{c}\int_{\mathbb{R}} g_\delta(y-x)dy \int_{t-}^{T-}\int_0^{U_s(y)} W^T(\overleftarrow{ds}, du) \\ &\quad - \int_t^T bU_s^\delta(x)ds + \int_{\mathbb{R}} g_\delta(y-x)dy \int_{t-}^{T-}\int_0^\infty \int_0^{U_s(y)} z\tilde{N}_0^T(\overleftarrow{ds}, dz, du). \end{aligned} \tag{5.1}$$

Clearly, each term in (5.1) has a version smooth in $x \in \mathbb{R}$, so we can fix $0 \le t \le T$ and assume the equation holds simultaneously for all $x \in \mathbb{R}$ in the strong sense. The main result of this section is the following:

Proposition 5.1 *For any $x \in \mathbb{R}$, $\delta > 0$ and $0 \le r \le t \le T$ we almost surely have*

$$\begin{aligned} U_t^\delta(\xi_t^r + x) &= U_T^\delta(\xi_T^r + x) - b\int_t^T U_s^\delta(\xi_s^r + x)ds - \sigma\int_t^T \nabla U_s^\delta(\xi_s^r + x)dB_s \\ &\quad - \int_t^T \int_{\mathbb{R}^\circ} [U_s^\delta(\xi_s^r + x - z) - U_s^\delta(\xi_s^r + x)]\tilde{M}(ds, dz) \\ &\quad + \sqrt{c}\int_{t-}^{T-}\int_0^\infty \Big[\int_{\mathbb{R}} g_\delta(z) 1_{\{u \le U_s(\xi_s^r + x + z)\}} dz\Big] W^T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{T-}\int_0^\infty\int_0^\infty \Big[\int_{\mathbb{R}} g_\delta(v) 1_{\{u \le U_s(\xi_s^r + x + v)\}} dv\Big] z\tilde{N}_0^T(\overleftarrow{ds}, dz, du). \end{aligned} \tag{5.2}$$

We give the proof of this proposition by a number of lemmas. Let $r = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[r, T]$ and let $v_i = t_i \vee t$. Let $\varepsilon_n = \max_{1 \le i \le n} |t_i - t_{i-1}|$. By (1.9) and Itô's formula, for any $f \in C^2(\mathbb{R})$,

$$f(\xi^*(t)) = f(\xi^*(0)) + \int_0^t A^* f(\xi^*(s))ds + M_t, \tag{5.3}$$

where A^* denotes the weak form of the dual generator and

$$M_t(f) = \sigma\int_0^t f'(\xi^*(s-))dB_s + \int_0^t\int_{\mathbb{R}^\circ} [f(\xi^*(s-) - z) - f(\xi^*(s-))]\tilde{M}(ds, dz) \tag{5.4}$$

is a martingale. We can write $\xi^*(s)$ instead of $\xi^*(s-)$ on the right-hand side of (5.4) with the convention that a stochastic integral takes automatically a predictable version of the integrand. In view of (5.1), (5.3) and (5.4), using a stochastic Fubini's theorem we have

$$\begin{aligned} &U_t^\delta(\xi_t^r + x) - U_T^\delta(\xi_T^r + x) \\ &\quad = \sum_{i=1}^n [U_{v_{i-1}}^\delta(\xi_{v_{i-1}}^r + x) - U_{v_{i-1}}^\delta(\xi_{v_i}^r + x)] + \sum_{i=1}^n [U_{v_{i-1}}^\delta(\xi_{v_i}^r + x) - U_{v_i}^\delta(\xi_{v_i}^r + x)] \\ &\quad = -\sum_{i=1}^n \int_{v_{i-1}}^{v_i} A^* U_{v_{i-1}}^\delta(\xi_s^r + x)ds - \sigma\sum_{i=1}^n \int_{v_{i-1}}^{v_i} \nabla U_{v_{i-1}}^\delta(\xi_s^r + x)dB_s \end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{n}\int_{v_{i-1}}^{v_i}\int_{\mathbb{R}^\circ}[U^\delta_{v_{i-1}}(\xi^r_s+x-z)-U^\delta_{v_{i-1}}(\xi^r_s+x)]\tilde{M}(ds,dz)\\
&+\sum_{i=1}^{n}\int_{v_{i-1}}^{v_i}[A^*U^\delta_s(\xi^r_{v_i}+x)-bU^\delta_s(\xi^r_{v_i}+x)]ds\\
&+\sqrt{c}\sum_{i=1}^{n}\int_{v_{i-1}-}^{v_i-}\int_0^\infty\Big[\int_{\mathbb{R}}g_\delta(z)1_{\{u\le U_s(\xi^r_{v_i}+x+z)\}}dz\Big]W^T(\overleftarrow{ds},du)\\
&+\sum_{i=1}^{n}\int_{v_{i-1}-}^{v_i-}\int_0^\infty\int_0^\infty\Big[\int_{\mathbb{R}}g_\delta(v)1_{\{u\le U_s(\xi^r_{v_i}+x+v)\}}dv\Big]z\tilde{N}^T_0(\overleftarrow{ds},dz,du)\\
=&-b\int_t^T\sum_{i=1}^{n}I_i(s-)U^\delta_s(\xi^r_{v_i}+x)ds-\sigma\int_t^T\sum_{i=1}^{n}I_i(s-)\nabla U^\delta_{v_{i-1}}(\xi^r_s+x)dB_s\\
&-\int_t^T\int_{\mathbb{R}^\circ}\sum_{i=1}^{n}I_i(s-)[U^\delta_{v_{i-1}}(\xi^r_s+x-z)-U^\delta_{v_{i-1}}(\xi^r_s+x)]\tilde{M}(ds,dz)\\
&-\int_t^T\sum_{i=1}^{n}I_i(s-)[A^*U^\delta_{v_{i-1}}(\xi^r_s+x)-A^*U^\delta_s(\xi^r_{v_i}+x)]ds\\
&+\sqrt{c}\int_{t-}^{T-}\int_0^\infty\sum_{i=1}^{n}I_i(s)\Big[\int_{\mathbb{R}}g_\delta(z)1_{\{u\le U_s(\xi^r_{v_i}+x+z)\}}dz\Big]W^T(\overleftarrow{ds},du)\\
&+\int_{t-}^{T-}\int_0^\infty\int_0^\infty\sum_{i=1}^{n}I_i(s)\Big[\int_{\mathbb{R}}g_\delta(v)1_{\{u\le U_s(\xi^r_{v_i}+x+v)\}}dv\Big]z\tilde{N}^T_0(\overleftarrow{ds},dz,du), \qquad (5.5)
\end{aligned}
$$

where $I_i(s)=1_{[v_{i-1},v_i)}(s)$ and $I_i(s-)=1_{(v_{i-1},v_i]}(s)$. Then (5.2) follows from (5.5) and the lemmas to follow.

Lemma 5.2 *For any $\delta>0$ and $T\ge 0$ there is positive constant $C_T(\delta)$ so that*

$$
\mathbf{E}\Big\{\sup_{t\in[0,T]}\Big[\|U_t\|+\|U^\delta_t\|+\|\nabla U^\delta_t\|+\|\triangle U^\delta_t\|+\|A^*U^\delta_t\|\Big]\Big\}\le C_T(\delta).
$$

Proof. Since the measure-valued process $\{X_t:t\ge 0\}$ corresponding to $\{Y_t:t\ge 0\}$ is a super-Lévy process, using the contraction property of the operator T_δ we see

$$
\mathbf{E}\Big\{\sup_{t\in[0,T]}\|U^\delta_t\|\Big\}\le\mathbf{E}\Big\{\sup_{t\in[0,T]}\|U_t\|\Big\}=\mathbf{E}\Big\{\sup_{t\in[0,T]}X_{T-t}(1)\Big\}.
$$

By Proposition 3.3 the right-hand side is bounded. Note that $\nabla g_\delta(z)=-zg_\delta(z)/\delta$. Then we have

$$
\begin{aligned}
\mathbf{E}\Big\{\sup_{t\in[0,T]}\|\nabla U^\delta_t\|\Big\} &= \mathbf{E}\Big\{\sup_{(t,x)\in[0,T]\times\mathbb{R}}\Big|\frac{1}{\delta}\int_{\mathbb{R}}U_t(x+z)zg_\delta(z)dz\Big|\Big\}\\
&\le \frac{1}{\delta}\mathbf{E}\Big\{\sup_{t\in[0,T]}\|U_t\|\Big\}\int_{\mathbb{R}}|z|g_\delta(z)dz,
\end{aligned}
$$

where the right-hand side is bounded. Similarly one can prove the assertions for ΔU^δ_t and $A^*U^\delta_t$. □

Lemma 5.3 *The two-parameter process*

$$
\{(U_t(x),U^\delta_t(x),\nabla U^\delta_t(x),\Delta U^\delta_t(x),A^*U^\delta_t(x)):0\le t\le T,x\in\mathbb{R}\} \qquad (5.6)
$$

is continuous in $L^1(\mathbf{P})$.

Proof. The assertion for $(t,x)\mapsto U_t(x)$ follows from Theorem 3.9 and that for $(t,x)\mapsto U_t^\delta(x)$ follows by Proposition 3.3 and the contraction property of T_δ. For $r,t\ge 0$ and $x,y\in\mathbb{R}$ we get

$$\begin{aligned}\mathbf{E}\{|\nabla U_r^\delta(x)-\nabla U_t^\delta(y)|\} &= \frac{1}{\delta}\mathbf{E}\Big\{\Big|\int_{\mathbb{R}}[U_r(x+z)-U_t(y+z)]zg_\delta(z)dz\Big|\Big\}\\ &\le \frac{1}{\delta}\int_{\mathbb{R}}\mathbf{E}\{|U_r(x+z)-U_t(y+z)|\}|z|g_\delta(z)dz.\end{aligned}$$

By Theorem 3.9 and dominated convergence the right-hand side goes to zero as $(r,x)\to(t,y)$. The arguments for the last two coordinators in (5.6) are similar. □

Lemma 5.4 *For any $x\in\mathbb{R}$, $\delta>0$ and $0\le r\le t\le T$, as $\varepsilon_n\to 0$ we have*

$$\mathbf{E}\Big\{\int_t^T\sum_{i=1}^n I_i(s-)|U_s^\delta(\xi_{v_i}^r+x)-U_s^\delta(\xi_s^r+x)|ds\Big\}\to 0. \tag{5.7}$$

Proof. By the independence of $\{\xi_s^r\}$ and $\{U_s(x)\}$,

$$\text{l.h.s. of (5.7)}\le\int_r^T\sum_{i=1}^n I_i(s-)\mathbf{E}\{f(s,\xi_{v_i}^r+x,\xi_s^r+x)\}ds, \tag{5.8}$$

where

$$(s,x,y)\mapsto f(s,x,y):=\mathbf{E}\{|U_s^\delta(x)-U_s^\delta(y)|\}$$

is a bounded continuous function on $[0,T]\times\mathbb{R}^2$ by Lemmas 5.2 and 5.3. Then by the right-continuity of $s\mapsto\xi_s^r$ and dominated convergence, the right-hand side of (5.8) goes to zero as $\varepsilon_n\to 0$. □

Lemma 5.5 *For any $x\in\mathbb{R}$, $\delta>0$ and $0\le r\le t\le T$, as $\varepsilon_n\to 0$ we have*

$$\mathbf{E}\Big\{\int_t^T\sum_{i=1}^n I_i(s-)|A^*U_{v_{i-1}}^\delta(\xi_s^r+x)-A^*U_s^\delta(\xi_{v_i}^r+x)|ds\Big\}\to 0. \tag{5.9}$$

Proof. By the independence of $\{\xi_s^r\}$ and $\{U_s(x)\}$,

$$\begin{aligned}\text{l.h.s. of (5.9)} &\le \int_r^T\sum_{i=1}^n I_i(s-)\mathbf{E}\Big\{|A^*U_{v_{i-1}}^\delta(\xi_s^r+x)-A^*U_s^\delta(\xi_s^r+x)|\Big\}ds\\ &\quad+\int_r^T\sum_{i=1}^n I_i(s-)\mathbf{E}\Big\{|A^*U_s^\delta(\xi_s^r+x)-A^*U_s^\delta(\xi_{v_i}^r+x)|\Big\}ds\\ &= \int_r^T\sum_{i=1}^n I_i(s-)ds\int_{\mathbb{R}}\mathbf{E}\Big\{|A^*U_{v_{i-1}}^\delta(\xi)-A^*U_s^\delta(\xi)|\Big\}P_{s-r}^*(x,d\xi)\\ &\quad+\int_r^T\sum_{i=1}^n I_i(s-)\mathbf{E}\Big\{|A^*U_s^\delta(\xi_s^r+x)-A^*U_s^\delta(\xi_{v_i}^r+x)|\Big\}ds.\end{aligned} \tag{5.10}$$

Observe that

$$(s,\xi,\eta)\mapsto f_s(\xi,\eta):=\mathbf{E}\{|A^*U_s^\delta(\xi)-A^*U_s^\delta(\eta)|\}$$

is a bounded continuous function on $[0,T]\times\mathbb{R}^2$ by Lemmas 5.2 and 5.3. By dominated convergence and the right-continuity of $s\mapsto\xi_s^r$, the right-hand side of (5.10) goes to zero as $\varepsilon_n\to 0$. □

Lemma 5.6 *For any $x \in \mathbb{R}$, $\delta > 0$ and $0 \le r \le t \le T$, as $\varepsilon_n \to 0$ we have*

$$\mathbf{E}\Big\{\Big|\int_t^T \sum_{i=1}^n I_i(s-)[\nabla U^\delta_{v_{i-1}}(\xi^r_s + x) - \nabla U^\delta_s(\xi^r_s + x)]dB_s\Big|\Big\} \to 0. \tag{5.11}$$

Proof. By the Hölder inequality and the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
\text{l.h.s. of (5.11)} &\le C\,\mathbf{E}\Big\{\Big[\int_r^T \sum_{i=1}^n I_i(s-)|\nabla U^\delta_{v_{i-1}}(\xi^r_s + x) - \nabla U^\delta_s(\xi^r_s + x)|^2 ds\Big]^{\frac{1}{2}}\Big\} \\
&\le C\Big\{\mathbf{E}\Big[\sup_{t\in[r,T]} \|\nabla U^\delta_t\|\Big] \int_r^T \sum_{i=1}^n I_i(s-)\mathbf{E}[m(v_{i-1}, s, \xi^r_s)]ds\Big\}^{\frac{1}{2}} \\
&\le C\Big\{\mathbf{E}\Big[\sup_{t\in[r,T]} \|\nabla U^\delta_t\|\Big] \int_r^T ds \int_{\mathbb{R}} \sum_{i=1}^n I_i(s-)m(v_{i-1}, s, \xi)P^*_{s-r}(x, d\xi)\Big\}^{\frac{1}{2}},
\end{aligned}$$

where

$$(t, s, x) \mapsto m(t, s, x) := \mathbf{E}\{|\nabla U^\delta_t(x) - \nabla U^\delta_s(x)|\}$$

is a bounded continuous function on $[0, T]^2 \times \mathbb{R}$ by Lemmas 5.2 and 5.3. Then we get (5.11) by dominated convergence. □

Lemma 5.7 *For any $x \in \mathbb{R}$, $\delta > 0$ and $0 \le r \le t \le T$, as $\varepsilon_n \to 0$ we have*

$$\begin{aligned}
\mathbf{E}\Big\{\Big|\int_t^T \int_{\mathbb{R}^\circ} \sum_{i=1}^n I_i(s-)\{[U^\delta_{v_{i-1}}(\xi^r_s + x - z) - U^\delta_{v_{i-1}}(\xi^r_s + x)] \\
-[U^\delta_s(\xi^r_s + x - z) - U^\delta_s(\xi^r_s + x)]\}\tilde{M}(ds, dz)\Big|\Big\} \to 0.
\end{aligned} \tag{5.12}$$

Proof. By the Burkholder-Davis-Gundy inequality,

$$\text{l.h.s. of (5.12)} \le C\,\mathbf{E}\Big\{\Big[\int_r^T \int_{\mathbb{R}^\circ} \sum_{i=1}^n I_i(s-)f_i(s, \xi^r_s + x, z)^2 M(ds, dz)\Big]^{\frac{1}{2}}\Big\},$$

where

$$f_i(s, \xi, z) = \big[U^\delta_{v_{i-1}}(\xi - z) - U^\delta_{v_{i-1}}(\xi)\big] - \big[U^\delta_s(\xi - z) - U^\delta_s(\xi)\big].$$

For any $s \ge 0$ and $z, \xi \in \mathbb{R}$ we have $\mathbf{E}\{|f_i(s, \xi, z)|\} \to 0$ as $\varepsilon_n \to 0$ by Lemma 5.3. Moreover, we have

$$|f_i(s, \xi, z)| \le 2(|z| \wedge 1) \sup_{t\in[0,T]} \big[\|U^\delta_t\| + \|\nabla U^\delta_t\|\big]$$

Then by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\{\text{l.h.s. of (5.12)}\}^2 \le C\,\Big\{\mathbf{E}\Big(\sup_{t\in[0,T]} \big[\|U^\delta_t\| + \|\nabla U^\delta_t\|\big]^{\frac{1}{2}} \Big[\int_r^T \int_{\mathbb{R}^\circ} \sum_{i=1}^n I_i(s-) \\
|f_i(s, \xi^r_s + x, z)|(|z| \wedge 1)M(ds, dz)\Big]^{\frac{1}{2}}\Big)\Big\}^2
\end{aligned}$$

$$\leq C\,\mathbf{E}\Big\{\sup_{t\in[0,T]}\big[\|U_t^\delta\|+\|\nabla U_t^\delta\|\big]\Big\}\mathbf{E}\Big\{\int_r^T\int_{\mathbb{R}^\circ}\sum_{i=1}^n I_i(s-)$$

$$|f_i(s,\xi_s^r+x,z)|(|z|\wedge 1)M(ds,dz)\Big\}$$

$$= C\int_r^T ds\int_{\mathbb{R}^\circ}\sum_{i=1}^n I_i(s-)\mathbf{E}\big\{|f_i(s,\xi_s^r+x,z)|\big\}(|z|\wedge 1)\nu(dz)$$

$$= C\int_r^T ds\int_{\mathbb{R}} P_{s-r}^*(x,d\xi)\int_{\mathbb{R}^\circ}\sum_{i=1}^n I_i(s-)\mathbf{E}\{|f_i(s,\xi,z)|\}(|z|\wedge 1)\nu(dz),$$

which tends to zero as $\varepsilon_n\to 0$ by dominated convergence. □

Lemma 5.8 *For any $x\in\mathbb{R}$, $\delta>0$ and $0\le r\le t\le T$, as $\varepsilon_n\to 0$ we have*

$$\mathbf{E}\Big\{\Big|\int_{t-}^{T-}\int_0^\infty\sum_{i=1}^n I_i(s)\Big[\int_{\mathbb{R}} g_\delta(z)(1_{\{u\le U_s(\xi_{v_i}^r+x+z)\}}$$

$$-1_{\{u\le U_s(\xi_s^r+x+z)\}})dz\Big]W^T(\overleftarrow{ds},du)\Big|^2\Big\}\to 0. \tag{5.13}$$

Proof. By Itô's isometry, we have

$$\text{l.h.s. of (5.13)} \le \int_r^T ds\int_0^\infty \mathbf{E}\Big\{\Big|\sum_{i=1}^n I_i(s)\int_{\mathbb{R}}(1_{\{u\le U_s(\xi_{v_i}^r+x+z)\}}$$

$$-1_{\{u\le U_s(\xi_s^r+x+z)\}})g_\delta(z)dz\Big|^2\Big\}du$$

$$\le \int_r^T ds\int_{\mathbb{R}}\mathbf{E}\Big\{\int_0^\infty\sum_{i=1}^n I_i(s)|1_{\{u\le U_s(\xi_{v_i}^r+x+z)\}}$$

$$-1_{\{u\le U_s(\xi_s^r+x+z)\}}|^2du\Big\}g_\delta(z)dz$$

$$= \int_r^T ds\int_{\mathbb{R}}\mathbf{E}\Big\{\sum_{i=1}^n I_i(s)|U_s(\xi_{v_i}^r+x+z)$$

$$-U_s(\xi_s^r+x+z)|\Big\}g_\delta(z)dz$$

$$= \int_r^T ds\int_{\mathbb{R}}\sum_{i=1}^n I_i(s)\mathbf{E}\{h_s(\xi_{v_i}^r+x+z,\xi_s^r+x+z)\}g_\delta(z)dz, \tag{5.14}$$

where

$$(s,\xi,\eta)\mapsto h_s(\xi,\eta):=\mathbf{E}\{|U_s(\xi)-U_s(\eta)|\}$$

is a bounded continuous function on $[0,T]\times\mathbb{R}^2$ by Lemmas 5.2 and 5.3. By dominated convergence and the right-continuity of $s\mapsto\xi_s^r$, the right-hand side of (5.14) tends to zero as $\varepsilon_n\to 0$. □

Lemma 5.9 *For any $x\in\mathbb{R}$, $\delta>0$ and $0\le r\le t\le T$, as $\varepsilon_n\to 0$ we have*

$$\mathbf{E}\Big\{\Big|\int_{t-}^{T-}\int_0^\infty\int_0^\infty\sum_{i=1}^n I_i(s)z\Big[\int_{\mathbb{R}} g_\delta(v)1_{\{u\le U_s(\xi_{v_i}^r+x+v)\}}dv$$

$$-\int_{\mathbb{R}} g_\delta(v)1_{\{u\le U_s(\xi_s^r+x+v)\}}dv\Big]\tilde N_0^T(\overleftarrow{ds},dz,du)\Big|\Big\}\to 0. \tag{5.15}$$

Proof. Let $h_s(\xi,\eta)$ be defined as in the last proof. One can see

$$\begin{aligned}\mathbf{E}\Big\{\Big|\int_{t-}^{T-}\int_0^1\int_0^\infty\sum_{i=1}^n I_i(s)z\Big[\int_{\mathbb{R}} g_\delta(v)1_{\{u\le U_s(\xi^r_{v_i}+x+v)\}}dv\\ -\int_{\mathbb{R}} g_\delta(v)1_{\{u\le U_s(\xi^r_s+x+v)\}}dv\Big]\tilde{N}^T_0(\overleftarrow{ds},dz,du)\Big|^2\Big\}\\ \le\int_r^T ds\int_0^1 z^2m(dz)\int_{\mathbb{R}}\sum_{i=1}^n I_i(s)\mathbf{E}\{h_s(\xi^r_{v_i}+x+v,\xi^r_s+x+v)\}g_\delta(v)dv\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\Big\{\Big|\int_{t-}^{T-}\int_1^\infty\int_0^\infty\sum_{i=1}^n I_i(s)z\Big[\int_{\mathbb{R}} g_\delta(v)1_{\{u\le U_s(\xi^r_{v_i}+x+v)\}}dv\\ -\int_{\mathbb{R}} g_\delta(v)1_{\{u\le U_s(\xi^r_s+x+v)\}}dv\Big]\tilde{N}^T_0(\overleftarrow{ds},dz,du)\Big|\Big\}\\ \le\mathbf{E}\Big\{\int_{r-}^{T-}\int_1^\infty\int_0^\infty\sum_{i=1}^n I_i(s)z\Big[\int_{\mathbb{R}} g_\delta(v)|1_{\{u\le U_s(\xi^r_{v_i}+x+v)\}}\\ -1_{\{u\le U_s(\xi^r_s+x+v)\}}|dv\Big]N^T_0(\overleftarrow{ds},dz,du)\Big\}\\ +\mathbf{E}\Big\{\int_r^T ds\int_1^\infty m(dz)\int_0^\infty\sum_{i=1}^n I_i(s)z\Big[\int_{\mathbb{R}} g_\delta(v)|1_{\{u\le U_s(\xi^r_{v_i}+x+v)\}}\\ -1_{\{u\le U_s(\xi^r_s+x+v)\}}|dv\Big]du\Big\}\\ \le 2\int_r^T ds\int_1^\infty zm(dz)\int_{\mathbb{R}}\sum_{i=1}^n I_i(s)\mathbf{E}\{h_s(\xi^r_{v_i}+x+v,\xi^r_s+x+v)\}g_\delta(v)dv.\end{aligned}$$

The right-hand sides of both inequalities tend to zero as $\varepsilon_n\to 0$. □

6 Backward equation and pathwise uniqueness

In this section, we shall prove the pathwise uniqueness for the stochastic integral equation (1.6). The key step is to establish the backward doubly stochastic integral equation (1.10). We shall use the settings of the last two sections. In particular, we assume Condition 3.6 is satisfied. In the following Lemmas 6.2–6.10, we also assume $Y_0\in D^1(\mathbb{R})$.

Lemma 6.1 *For any $t\ge 0$ the mappings $x\mapsto Z_t(x)$ and $x\mapsto\nabla Z_t(x)$ are continuous in $L^2(\mathbf{P})$.*

Proof. We only show the result for $x\mapsto\nabla Z_t(x)$ since that for $x\mapsto Z_t(x)$ is simpler. By (4.5) and Itô's isometry, for any $t\ge 0$ and $x,y\in\mathbb{R}$ we have

$$\begin{aligned}\mathbf{E}\Big\{[\nabla Z_t(x)-\nabla Z_t(y)]^2\Big\} &= C\,\mathbf{E}\Big\{\int_0^t ds\int_0^\infty[p^b_{t-s}(x-Y^{-1}_s(u))-p^b_{t-s}(y-Y^{-1}_s(u))]^2du\Big\}\\ &= C\,\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}}[p^b_{t-s}(x-z)-p^b_{t-s}(y-z)]^2X_s(dz)\Big\}\\ &\le Ce^{2|b|t}\int_0^t\frac{ds}{(T-s)^\alpha}\int_{\mathbb{R}}|p_{t-s}(x-z)-p_{t-s}(y-z)|\mu P_s(dz), \quad (6.1)\end{aligned}$$

where C is defined by (4.9). By dominated convergence one can see the right-hand side of (6.1) tends to zero as $x\to y$. □

Lemma 6.2 *For $x \in \mathbb{R}$ and $0 \le r \le t \le T$, as $\delta \to 0$ we have*

$$\mathbf{E}\Big\{\Big|\int_t^T [\nabla Z^\delta_{T-s}(\xi^r_s + x) - \nabla Z_{T-s}(\xi^r_s + x)]dB_s\Big|^2\Big\} \to 0. \tag{6.2}$$

Proof. Since $\{Z_s(x)\}$ is independent of $\{\xi^r_s\}$ and $\{B_s\}$, by Itô's isometry we have

$$\begin{aligned} \text{l.h.s. of (6.2)} \;&\le\; \int_r^T \mathbf{E}\Big\{[\nabla Z^\delta_{T-s}(\xi^r_s + x) - \nabla Z_{T-s}(\xi^r_s + x)]^2\Big\}ds \\ &\le\; \int_r^T ds \int_{\mathbb{R}} h^r_s(x,z) g_\delta(z)dz, \end{aligned} \tag{6.3}$$

where

$$h^r_s(x,z) = \mathbf{E}\Big\{[\nabla Z_{T-s}(\xi^r_s + x + z) - \nabla Z_{T-s}(\xi^r_s + x)]^2\Big\}.$$

By (4.11) it is easy to see that

$$\begin{aligned} h^r_s(x,z) \;&\le\; 2\mathbf{E}\Big\{[\nabla Z_{T-s}(\xi^r_s + x + z)]^2 + [\nabla Z_{T-s}(\xi^r_s + x)]^2\Big\} \\ &\le\; \frac{C_T}{(T-s)^\alpha} \sup_{z\in\mathbb{R}} \int_{\mathbb{R}} p_{T-r}(z-y)\mu(dy) \le \frac{C_T\mu(1)}{(T-r)^\alpha (T-s)^\alpha}. \end{aligned}$$

By Lemma 6.1 and the independence of $\{\xi^r_s\}$ and $\{U_s(x)\}$ it follows that $(x,z) \mapsto h^r_s(x,z)$ is bounded and continuous on $\mathbb{R}^2$. Then we can apply dominated convergence in (6.3) to obtain (6.2). □

Lemma 6.3 *For $x \in \mathbb{R}$ and $0 \le r \le t \le T$, as $\delta \to 0$ we have*

$$\mathbf{E}\Big\{\Big|\int_t^T [\nabla \hat{H}^\delta_{T-s}(\xi^r_s + x) - \nabla \hat{H}_{T-s}(\xi^r_s + x)]dB_s\Big|\Big\} \to 0. \tag{6.4}$$

Proof. Since $\{Y_s(x)\}$ is independent of $\{\xi^r_s\}$ and $\{B_s\}$, by the Burkholder-Davis-Gundy inequality and the Cauchy-Schwarz inequality we have

$$\begin{aligned} \text{l.h.s. of (6.4)} \;&\le\; C\,\mathbf{E}\Big\{\Big(\int_r^T \Big|\int_0^{T-s} ds_1 \int_1^\infty z m(dz) \int_0^\infty [T_\delta p^b_{T-s-s_1}(\xi^r_s + x - Y^{-1}_{s_1}(u)) \\ &\qquad\qquad - p^b_{T-s-s_1}(\xi^r_s + x - Y^{-1}_{s_1}(u))]du\Big|^2 ds\Big)^{\frac{1}{2}}\Big\} \\ &=\; C_T\mathbf{E}\Big\{\Big(\int_r^T \Big|\int_0^{T-s} ds_1 \int_{\mathbb{R}} [T_\delta p_{T-s-s_1}(\xi^r_s + x - u) \\ &\qquad\qquad - p_{T-s-s_1}(\xi^r_s + x - u)] X_{s_1}(du)\Big|^2 ds\Big)^{\frac{1}{2}}\Big\} \\ &\le\; C_T\mathbf{E}\Big\{\Big(\int_0^T X_{s_1}(1)ds_1\Big)^{\frac{1}{2}}\Big(\int_r^T ds \int_0^{T-s} ds_2 \int_{\mathbb{R}} [T_\delta p_{T-s-s_2}(\xi^r_s + x - u) \\ &\qquad\qquad - p_{T-s-s_2}(\xi^r_s + x - u)]^2 X_{s_2}(du)\Big)^{\frac{1}{2}}\Big\} \\ &\le\; C_T\Big\{\mathbf{E}\Big[\int_0^T X_{s_1}(1)ds_1\Big]\mathbf{E}\Big[\int_r^T ds \int_0^{T-s} ds_2 \int_{\mathbb{R}} [T_\delta p_{T-s-s_2}(\xi^r_s + x - u) \\ &\qquad\qquad - p_{T-s-s_2}(\xi^r_s + x - u)]^2 X_{s_2}(du)\Big]\Big\}^{\frac{1}{2}} \end{aligned}$$

$$\leq C_T\Big\{\mathbf{E}\Big[\int_r^T ds\int_0^{T-s} ds_2\int_{\mathbb{R}}[T_\delta p_{T-s-s_2}(\xi_s^r+x-u) - p_{T-s-s_2}(\xi_s^r+x-u)]^2 X_{s_2}(du)\Big]\Big\}^{\frac{1}{2}}.$$

Then by the independence of $\{\xi_s^r\}$ and $\{X_s\}$ we have

$$\begin{aligned}\{\text{l.h.s. of (6.4)}\}^2 &\leq C_T\int_r^T ds\int_0^{T-s} ds_2\int_{\mathbb{R}}\mu(dy)\int_{\mathbb{R}} p_{s_2}(u-y)du \\ &\qquad \int_{\mathbb{R}} p_{s-r}(x-\xi)[T_\delta p_{T-s-s_2}(\xi-u)-p_{T-s-s_2}(\xi-u)]^2 d\xi \\ &\leq C_T\int_r^T ds\int_0^{T-s}(T-s-s_2)^{-\alpha}ds_2\int_{\mathbb{R}} f_\delta(r,s,s_2,x,y)\mu(dy),\end{aligned}$$

where

$$f_\delta(r,s,s_2,x,y)=\int_{\mathbb{R}} p_{s_2}(u-y)du\int_{\mathbb{R}} p_{s-r}(x-\xi)|T_\delta p_{T-s-s_2}(\xi-u)-p_{T-s-s_2}(\xi-u)|d\xi. \quad (6.5)$$

It is easy to see that $f_\delta(r,s,s_2,x,y)\to 0$ as $\delta\to 0$ and

$$\begin{aligned}f_\delta(r,s,s_2,x,y) &\leq \int_{\mathbb{R}} p_{s-r}(x-\xi)d\xi\int_{\mathbb{R}} p_{s_2}(u-y)[T_\delta p_{T-s-s_2}(\xi-u)+p_{T-s-s_2}(\xi-u)]du \\ &= \int_{\mathbb{R}} p_{s-r}(x-\xi)d\xi\int_{\mathbb{R}}[p_{T-s}(\xi-y+v)+p_{T-s}(\xi-y)]g_\delta(v)dv \\ &= \int_{\mathbb{R}}[p_{T-r}(x-y+v)+p_{T-r}(x-y)]g_\delta(v)dv\leq C_T(T-r)^{-\alpha}.\end{aligned}$$

Then we get (6.4) by dominated convergence. □

Lemma 6.4 *For $x\in\mathbb{R}$ and $0\leq r\leq t\leq T$, as $\delta\to 0$ we have*

$$\mathbf{E}\Big\{\Big|\int_t^T[\nabla H_{T-s}^\delta(\xi_s^r+x)-\nabla H_{T-s}(\xi_s^r+x)]dB_s\Big|\Big\}\to 0. \quad (6.6)$$

Proof. Since $\{H_s(x)\}$ is independent of $\{\xi_s^r\}$ and $\{B_s\}$, by the Burkholder-Davis-Gundy inequality and the Cauchy-Schwarz inequality we have

$$\begin{aligned}\text{l.h.s. of (6.6)} &\leq C\,\mathbf{E}\Big\{\Big(\int_r^T\Big|\int_0^{T-s}\int_1^\infty\int_0^\infty z[T_\delta p_{T-s-s_1}^b(\xi_s^r+x-Y_{s_1}^{-1}(u)) \\ &\qquad - p_{T-s-s_1}^b(\xi_s^r+x-Y_{s_1}^{-1}(u))]N_0(ds_1,dz,du)\Big|^2 ds\Big)^{\frac{1}{2}}\Big\} \\ &\leq C_T\mathbf{E}\Big\{\Big(\int_0^T\int_1^\infty\int_0^\infty z1_{\{u\leq Y_{s_1}(\infty)\}}N_0(ds_1,dz,du)\Big)^{\frac{1}{2}} \\ &\qquad \cdot\Big(\int_r^T ds\int_0^{T-s}\int_1^\infty\int_0^\infty z|T_\delta p_{T-s-s_2}(\xi_s^r+x-Y_{s_2}^{-1}(u)) \\ &\qquad - p_{T-s-s_2}(\xi_s^r+x-Y_{s_2}^{-1}(u))|^2 N_0(ds_2,dz,du)\Big)^{\frac{1}{2}}\Big\} \\ &\leq C_T\Big\{\mathbf{E}\Big[\int_0^T\int_1^\infty\int_0^\infty z1_{\{u\leq Y_{s_1}(\infty)\}}N_0(ds_1,dz,du)\Big] \\ &\qquad \cdot\mathbf{E}\Big[\int_r^T ds\int_0^{T-s}\int_1^\infty\int_0^\infty z|T_\delta p_{T-s-s_2}(\xi_s^r+x-Y_{s_2}^{-1}(u))\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - p_{T-s-s_2}(\xi_s^r + x - Y_{s_2}^{-1}(u))|^2 N_0(ds_2, dz, du)\Big]\Big\}^{\frac{1}{2}} \\
&\le\; C_T\Big\{\mathbf{E}\Big[\int_0^T X_{s_1}(1)ds_1 \int_1^\infty zm(dz)\Big] \cdot \mathbf{E}\Big[\int_t^T ds \int_0^{T-s} ds_2 \\
&\qquad \int_1^\infty zm(dz) \int_0^\infty |T_\delta p_{T-s-s_2}(\xi_s^r + x - Y_{s_2}^{-1}(u)) \\
&\qquad\qquad - p_{T-s-s_2}(\xi_s^r + x - Y_{s_2}^{-1}(u))|^2 du\Big]\Big\}^{\frac{1}{2}} \\
&\le\; C_T\Big\{\mathbf{E}\Big[\int_r^T ds \int_0^{T-s} ds_2 \int_{\mathbb{R}} |T_\delta p_{T-s-s_2}(\xi_s^r + x - u) \\
&\qquad\qquad - p_{T-s-s_2}(\xi_s^r + x - u)|^2 X_{s_2}(du)\Big]\Big\}^{\frac{1}{2}},
\end{aligned}
$$

which goes to zero as $\delta \to 0$ by the proof of Lemma 6.3. □

Lemma 6.5 *For $x \in \mathbb{R}$ and $0 \le r \le t \le T$, as $\delta \to 0$ we have*

$$
\begin{aligned}
&\mathbf{E}\Big\{\Big|\int_t^T \int_{\{|z|\le 1\}} \{[Z_{T-s}^\delta(\xi_s^r + x - z) - Z_{T-s}^\delta(\xi_s^r + x)] \\
&\qquad - [Z_{T-s}(\xi_s^r + x - z) - Z_{T-s}(\xi_s^r + x)]\}\tilde{M}(ds, dz)\Big|^2\Big\} \to 0.
\end{aligned} \tag{6.7}
$$

Proof. Since $\{Z_s(x)\}$ is independent of $\{\xi_s^r\}$ and $\{M(ds, dz)\}$, by Itô's isometry we have

$$
\begin{aligned}
\text{l.h.s of (6.7)} \;&\le\; \int_r^T ds \int_{\{|z|\le 1\}} \mathbf{E}\Big\{\Big|[Z_{T-s}^\delta(\xi_s^r + x - z) - Z_{T-s}^\delta(\xi_s^r + x)] \\
&\qquad -[Z_{T-s}(\xi_s^r + x - z) - Z_{T-s}(\xi_s^r + x)]\Big|^2\Big\}\nu(dz) \\
&\le\; \int_r^T ds \int_{\{|z|\le 1\}} \nu(dz) \int_{\mathbb{R}} h_s(x, y, z) g_\delta(y) dy,
\end{aligned} \tag{6.8}
$$

where

$$
\begin{aligned}
h_s^r(x, y, z) \;=\; &\mathbf{E}\Big\{\Big|[Z_{T-s}(\xi_s^r + x + y - z) - Z_{T-s}(\xi_s^r + x + y)] \\
&\qquad - [Z_{T-s}(\xi_s^r + x - z) - Z_{T-s}(\xi_s^r + x)]\Big|^2\Big\}.
\end{aligned}
$$

By (4.10) we have

$$
h_s^r(x, y, z) \le \frac{C_T \mu(1) z^2}{(T-r)^\alpha}.
$$

On the other hand, by Lemma 6.1 it is easy to see that the function $y \mapsto h_s^r(x, y, z)$ is continuous and vanishes at $y = 0$. By dominated convergence, the right-hand side of (6.8) tends to zero as $\delta \to 0$. □

Lemma 6.6 *For $x \in \mathbb{R}$ and $0 \le r \le t \le T$, as $\delta \to 0$ we have*

$$
\begin{aligned}
&\mathbf{E}\Big\{\Big|\int_t^T \int_{\{|z|\le 1\}} \{[\hat{H}_{T-s}^\delta(\xi_s^r + x - z) - \hat{H}_{T-s}^\delta(\xi_s^r + x)] \\
&\qquad - [\hat{H}_{T-s}(\xi_s^r + x - z) - \hat{H}_{T-s}(\xi_s^r + x)]\}\tilde{M}(ds, dz)\Big|\Big\} \to 0.
\end{aligned} \tag{6.9}
$$

Proof. Since $\{Y_s(x)\}$ is independent of $\{\xi_s^r\}$ and $\{M(ds,dz)\}$, by the Burkholder-Davis-Gundy inequality and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\text{l.h.s. of (6.9)} \;\le\;& C\,\mathbf{E}\Big\{\Big(\int_r^T\int_{\{|z|\le 1\}}\Big|[\hat H^\delta_{T-s}(\xi_s^r+x-z)-\hat H^\delta_{T-s}(\xi_s^r+x)] \\
&\qquad -[\hat H_{T-s}(\xi_s^r+x-z)-\hat H_{T-s}(\xi_s^r+x)]\Big|^2 M(ds,dz)\Big)^{\frac12}\Big\} \\
=\;& C\,\mathbf{E}\Big\{\Big(\int_r^T\int_{\{|z|\le 1\}}\Big|\int_0^z[\nabla\hat H^\delta_{T-s}(\xi_s^r+x-\eta) \\
&\qquad -\nabla\hat H_{T-s}(\xi_s^r+x-\eta)]d\eta\Big|^2 M(ds,dz)\Big)^{\frac12}\Big\} \\
\le\;& C\,\mathbf{E}\Big\{\Big(\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z|\nabla\hat H^\delta_{T-s}(\xi_s^r+x-\eta) \\
&\qquad -\nabla\hat H_{T-s}(\xi_s^r+x-\eta)|^2 d\eta\Big)^{\frac12}\Big\} \\
\le\;& C\,\mathbf{E}\Big\{\Big(\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z\Big|\int_0^{T-s}ds_1\int_1^\infty zm(dz) \\
&\qquad \int_0^\infty [T_\delta p^b_{T-s-s_1}(\xi_s^r+x-\eta-Y_{s_1}^{-1}(u)) \\
&\qquad -p^b_{T-s-s_1}(\xi_s^r+x-\eta-Y_{s_1}^{-1}(u))]du\Big|^2 d\eta\Big)^{\frac12}\Big\} \\
\le\;& C_T\mathbf{E}\Big\{\Big(\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z\Big|\int_0^{T-s}ds_1 \\
&\qquad \int_{\mathbb{R}}[T_\delta p_{T-s-s_1}(\xi_s^r+x-\eta-u) \\
&\qquad -p_{T-s-s_1}(\xi_s^r+x-\eta-u)]X_{s_1}(du)\Big|^2 d\eta\Big)^{\frac12}\Big\} \\
\le\;& C_T\mathbf{E}\Big\{\Big(\int_0^T X_{s_1}(1)ds_1\Big)^{\frac12}\Big(\int_t^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta \\
&\qquad \int_0^{T-s}ds_2\int_{\mathbb{R}}|T_\delta p_{T-s-s_2}(\xi_s^r+x-\eta-u) \\
&\qquad -p_{T-s-s_2}(\xi_s^r+x-\eta-u)|^2 X_{s_2}(du)\Big)^{\frac12}\Big\} \\
\le\;& C_T\Big\{\mathbf{E}\Big[\int_0^T X_{s_1}(1)ds_1\Big]\mathbf{E}\Big[\int_t^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta \\
&\qquad \int_0^{T-s}ds_2\int_{\mathbb{R}}|T_\delta p_{T-s-s_2}(\xi_s^r+x-\eta-u) \\
&\qquad -p_{T-s-s_2}(\xi_s^r+x-\eta-u)|^2 X_{s_2}(du)\Big]\Big\}^{\frac12} \\
\le\;& C_T\Big\{\mathbf{E}\Big[\int_r^T\int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta \\
&\qquad \int_0^{T-s}ds_2\int_{\mathbb{R}}|T_\delta p_{T-s-s_2}(\xi_s^r+x-\eta-u) \\
&\qquad -p_{T-s-s_2}(\xi_s^r+x-\eta-u)|^2 X_{s_2}(du)\Big]\Big\}^{\frac12}.
\end{aligned}$$

Then by the independence of $\{\xi_t\}$ and $\{X_t\}$ we have

$$\{\text{l.h.s. of (6.9)}\}^2 \;\le\; C_T\int_r^T ds\int_{\{|z|\le 1\}} z\nu(dz)\int_0^z d\eta\int_0^{T-s}ds_2\int_{\mathbb{R}}\mu(dy)$$

$$\int_{\mathbb{R}} p_{s_2}(u-y)du \int_{\mathbb{R}} p_{s-r}(x-\eta-\xi)|T_\delta p_{T-s-s_2}(\xi-u) - p_{T-s-s_2}(\xi-u)|^2 d\xi$$
$$\le C_T \int_r^T ds \int_0^{T-s} (T-s-s_2)^{-\alpha} ds_2 \int_{\{|z|\le 1\}} z\nu(dz) \int_0^z d\eta \int_{\mathbb{R}} f_\delta(r,s,s_2,x-\eta,y)\mu(dy),$$

where the function f_δ is defined by (6.5). By the proof of Lemma 6.3 we have (6.9). □

Lemma 6.7 *For $x \in \mathbb{R}$ and $0 \le r \le t \le T$, as $\delta \to 0$ we have*

$$\mathbf{E}\Big\{\Big|\int_t^T \int_{\{|z|\le 1\}} \{[H^\delta_{T-s}(\xi^r_s + x - z) - H^\delta_{T-s}(\xi^r_s + x)] - [H_{T-s}(\xi^r_s + x - z) - H_{T-s}(\xi^r_s + x)]\}\tilde{M}(ds,dz)\Big|\Big\} \to 0. \tag{6.10}$$

Proof. Since $\{Y_s(x)\}$ is independent of $\{\xi^r_s\}$ and $\{M(ds,dz)\}$, by the Burkholder-Davis-Gundy inequality and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\text{l.h.s. of (6.10)} &\le C\,\mathbf{E}\Big\{\Big(\int_r^T \int_{\{|z|\le 1\}} \Big|[H^\delta_{T-s}(\xi^r_s + x - z) - H^\delta_{T-s}(\xi^r_s + x)] \\
&\qquad -[H_{T-s}(\xi^r_s + x - z) - H_{T-s}(\xi^r_s + x)]\Big|^2 M(ds,dz)\Big)^{\frac{1}{2}}\Big\} \\
&= C\,\mathbf{E}\Big\{\Big(\int_r^T \int_{\{|z|\le 1\}} \Big|\int_0^z [\nabla H^\delta_{T-s}(\xi^r_s + x - \eta) \\
&\qquad -\nabla H_{T-s}(\xi^r_s + x - \eta)]d\eta\Big|^2 M(ds,dz)\Big)^{\frac{1}{2}}\Big\} \\
&\le C\,\mathbf{E}\Big\{\Big(\int_r^T \int_{\{|z|\le 1\}} zM(ds,dz) \int_0^z |\nabla H^\delta_{T-s}(\xi^r_s + x - \eta) \\
&\qquad -\nabla H_{T-s}(\xi^r_s + x - \eta)|^2 d\eta\Big)^{\frac{1}{2}}\Big\} \\
&\le C\,\mathbf{E}\Big\{\Big(\int_r^T \int_{\{|z|\le 1\}} zM(ds,dz) \\
&\qquad \int_0^z \Big|\int_0^{T-s}\int_1^\infty\int_0^\infty z_1[T_\delta p^b_{T-s-s_1}(\xi^r_s + x - \eta - Y^{-1}_{s_1}(w)) \\
&\qquad - p^b_{T-s-s_1}(\xi^r_s + x - \eta - Y^{-1}_{s_1}(w))]N(ds_1,dz_1,dw)\Big|^2 d\eta\Big)^{\frac{1}{2}}\Big\} \\
&\le C_T\mathbf{E}\Big\{\Big(\int_0^T\int_1^\infty\int_0^\infty z_1 1_{\{w_1\le Y_{s_1}(\infty)\}} N(ds_1,dz_1,dw_1)\Big)^{\frac{1}{2}} \\
&\qquad \cdot\Big(\int_r^T \int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta \\
&\qquad \int_0^{T-s}\int_1^\infty\int_0^\infty z_2|T_\delta p_{T-s-s_2}(\xi^r_s + x - \eta - Y^{-1}_{s_2}(w_2)) \\
&\qquad - p_{T-s-s_2}(\xi^r_s + x - \eta - Y^{-1}_{s_2}(w_2))|^2 N(ds_2,dz_2,dw_2)\Big)^{\frac{1}{2}}\Big\} \\
&\le C_T\Big\{\mathbf{E}\Big[\int_0^T\int_1^\infty\int_0^\infty z_1 1_{\{w_1\le Y_{s_1}(\infty)\}} N(ds_1,dz_1,dw_1)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\cdot \mathbf{E}\Big[\int_r^T \int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta \\
&\qquad \int_0^{T-s}\int_1^\infty \int_0^\infty z_2|T_\delta p_{T-s-s_2}(\xi_s^r+x-\eta-Y_{s_2}^{-1}(w_2)) \\
&\qquad -p_{T-s-s_2}(\xi_s^r+x-\eta-Y_{s_2}^{-1}(w_1))|^2 N(ds_2,dz_2,dw_2)\Big]\Big\}^{\frac{1}{2}} \\
\le\ & C_T\Big\{\mathbf{E}\Big[\int_r^T \int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta\int_0^{T-s} ds_2\int_1^\infty z_2 m(dz_2) \\
&\qquad \int_0^\infty |T_\delta p_{T-s-s_2}(\xi_s^r+x-\eta-Y_{s_2}^{-1}(w_2)) \\
&\qquad -p_{T-s-s_2}(\xi_s^r+x-\eta-Y_{s_2}^{-1}(w_2))|^2 dw_2\Big]\Big\}^{\frac{1}{2}} \\
\le\ & C_T\Big\{\mathbf{E}\Big[\int_r^T \int_{\{|z|\le 1\}} zM(ds,dz)\int_0^z d\eta\int_0^{T-s} ds_2 \\
&\qquad \int_{\mathbb{R}} |T_\delta p_{T-s-s_2}(\xi_s^r+x-\eta-w_2) \\
&\qquad -p_{T-s-s_2}(\xi_s^r+x-\eta-w_2)|^2 X_{s_2}(dw_2)\Big]\Big\}^{\frac{1}{2}}.
\end{aligned}
$$

The right-hand side tends to zero as $\delta\to 0$ by the proof of Lemma 6.6. □

Lemma 6.8 *For $x\in\mathbb{R}$ and $0\le r\le t\le T$, as $\delta\to 0$ we have*

$$
\begin{aligned}
\mathbf{E}\Big\{\Big|\int_t^T\int_{\{|z|>1\}}&\{[Y_{T-s}^\delta(\xi_s^r+x-z)-Y_{T-s}^\delta(\xi_s^r+x)] \\
&-[Y_{T-s}(\xi_s^r+x-z)-Y_{T-s}(\xi_s^r+x)]\}\tilde{M}(ds,dz)\Big|\Big\}\to 0.
\end{aligned}\tag{6.11}
$$

Proof. Since $\{Y_t(x)\}$ is independent of $\{\xi_t\}$ and $\{M(ds,dz)\}$, we have

$$
\begin{aligned}
\text{l.h.s. of (6.11)}\ \le\ & 2\int_r^T ds\int_{\{|z|>1\}}\mathbf{E}\Big\{\Big|[Y_{T-s}^\delta(\xi_s^r+x-z)-Y_{T-s}^\delta(\xi_s^r+x)] \\
&\qquad -[Y_{T-s}(\xi_s^r+x-z)-Y_{T-s}(\xi_s^r+x)]\Big|\Big\}\nu(dz) \\
\le\ & 2\int_r^T ds\int_{\mathbb{R}} P_{s-r}^*(x,dy)\int_{\{|z|>1\}}\nu(dz)\int_{\mathbb{R}} h_s(y,z,v)g_\delta(v)dv,
\end{aligned}
$$

where

$$
h_s(y,z,v)=\mathbf{E}\Big\{\Big|[Y_{T-s}(y-z+v)-Y_{T-s}(y-z)]-[Y_{T-s}(y+v)-Y_{T-s}(y)]\Big|\Big\}.
$$

By Theorem 3.9 or Lemma 5.3 one sees $(y,z,v)\mapsto h_s(y,z,v)$ is a bounded continuous function on $\mathbb{R}^3$ vanishing at $v=0$. Then (6.11) holds by dominated convergence. □

Lemma 6.9 *For $x\in\mathbb{R}$ and $0\le r\le t\le T$, as $\delta\to 0$ we have*

$$
\begin{aligned}
\mathbf{E}\Big\{\Big|\int_{t-}^{T-}\int_0^\infty\Big[&\int_{\mathbb{R}} g_\delta(z)1_{\{u\le Y_{T-s}(\xi_s^r+x+z)\}}dz \\
&-1_{\{u\le Y_{T-s}(\xi_s^r+x)\}}\Big]W^T(\overleftarrow{ds},du)\Big|^2\Big\}\to 0.
\end{aligned}\tag{6.12}
$$

Proof. Since $\{\xi_s^r\}$ is independent of $\{Y_s(x)\}$ and $\{W^T(ds,du)\}$, by Itô's isometry and the Hölder inequality we have

$$\begin{aligned}\text{l.h.s. of (6.12)} &= \int_r^T ds \int_0^\infty \mathbf{E}\Big\{\Big|\int_{\mathbb{R}}[1_{\{u\le Y_{T-s}(\xi_s^r+x+z)\}} - 1_{\{u\le Y_{T-s}(\xi_s^r+x)\}}]g_\delta(z)dz\Big|^2\Big\}du \\ &\le \int_r^T ds \int_{\mathbb{R}} \mathbf{E}\Big\{\int_0^\infty [1_{\{u\le Y_{T-s}(\xi_s^r+x+z)\}} - 1_{\{u\le Y_{T-s}(\xi_s^r+x)\}}]^2 du\Big\} g_\delta(z)dz \\ &\le \int_r^T ds \int_{\mathbb{R}} \mathbf{E}\{|Y_{T-s}(\xi_s^r+x+z) - Y_{T-s}(\xi_s^r+x)|\} g_\delta(z)dz \\ &= \int_r^T ds \int_{\mathbb{R}} P_{r-s}^*(x,d\xi)\int_{\mathbb{R}} \mathbf{E}\{|Y_{T-s}(\xi+z) - Y_{T-s}(\xi)|\} g_\delta(z)dz.\end{aligned}$$

By Theorem 3.9 or Lemma 5.3 one can see

$$(s,\xi,z)\mapsto \mathbf{E}\{|Y_{T-s}(\xi-z)-Y_{T-s}(\xi)|\}$$

is a bounded continuous function on $[0,T]\times\mathbb{R}^2$. By dominated convergence we get (6.12). □

Lemma 6.10 *For $x\in\mathbb{R}$ and $0\le r\le t\le T$, as $\delta\to 0$ we have*

$$\begin{aligned}\mathbf{E}\Big\{\Big|\int_{t-}^{T-}\int_0^\infty\int_0^\infty z\Big[\int_{\mathbb{R}} g_\delta(y)1_{\{u\le Y_{T-s}(\xi_s^r+x+y)\}}dy \\ -1_{\{u\le Y_{T-s}(\xi_s^r+x)\}}\Big]\tilde{N}_0^T(\overleftarrow{ds},dz,du)\Big|\Big\}\to 0.\end{aligned} \tag{6.13}$$

Proof. Since $\{\xi_s^r\}$ is independent of $\{Y_s(x)\}$ and $\{N_0^T(ds,dz,du)\}$, by similar calculations as in the last proof, we have

$$\begin{aligned}&\mathbf{E}\Big\{\Big|\int_{t-}^{T-}\int_0^1\int_0^\infty z\Big[\int_{\mathbb{R}} g_\delta(z)1_{\{u\le Y_{T-s}(\xi_s^r+x+z)\}}dz - 1_{\{u\le Y_{T-s}(\xi_s^r+x)\}}\Big]\tilde{N}_0^T(\overleftarrow{ds},dz,du)\Big|^2\Big\} \\ &\quad\le \int_0^1 z^2 m(dz)\int_r^T ds\int_{\mathbb{R}} P_{s-r}^*(x,d\xi)\int_{\mathbb{R}}\mathbf{E}\{|Y_{T-s}(\xi+z)-Y_{T-s}(\xi)|\}g_\delta(z)dz,\end{aligned}$$

which tends to zero as $\delta\to 0$. On the other hand, one can see

$$\begin{aligned}&\mathbf{E}\Big\{\Big|\int_{t-}^{T-}\int_1^\infty\int_0^\infty z\Big[\int_{\mathbb{R}} g_\delta(z)1_{\{u\le Y_{T-s}(\xi_s^r+x+z)\}}dz - 1_{\{u\le Y_{T-s}(\xi_s^r+x)\}}\Big]\tilde{N}_0^T(\overleftarrow{ds},dz,du)\Big|\Big\} \\ &\quad\le 2\int_1^\infty z m(dz)\int_r^T ds\int_{\mathbb{R}} P_{s-r}^*(x,d\xi)\int_{\mathbb{R}}\mathbf{E}\{|Y_{T-s}(\xi+z)-Y_{T-s}(\xi)|\}g_\delta(z)dz.\end{aligned}$$

The right-hand side clearly goes to zero as $\delta\to 0$. □

Theorem 6.11 *For any $x\in\mathbb{R}$ and $0\le r\le t\le T$, the backward doubly stochastic integral equation* (1.10) *almost surely holds.*

Proof. By considering a conditional probability, we may assume that Y_0 is deterministic. We first assume $Y_0\in D^1(\mathbb{R})$. By Theorem 3.9 or Lemma 5.3 it is easy to see that, as $\delta\to 0$,

$$\mathbf{E}\Big\{\int_t^T |Y_{T-s}^\delta(\xi_s^r+x)-Y_{T-s}(\xi_s^r+x)|ds\Big\}\to 0.$$

By Lemmas 6.2–6.4 we have, as $\delta \to 0$,

$$\mathbf{E}\Big\{\Big|\int_t^T [\nabla Y^\delta_{T-s}(\xi^r_s + x) - \nabla Y_{T-s}(\xi^r_s + x)]dB_s\Big|\Big\} \to 0.$$

By Lemmas 6.5–6.8 we have, as $\delta \to 0$,

$$\begin{aligned}\mathbf{E}\Big\{\Big|\int_t^T \int_{\mathbb{R}^\circ} \{[Y^\delta_{T-s}(\xi^r_s + x - z) - Y^\delta_{T-s}(\xi^r_s + x)] \\ - [Y_{T-s}(\xi^r_s + x - z) - Y_{T-s}(\xi^r_s + x)]\}\tilde{M}(ds, dz)\Big|^2\Big\} \to 0.\end{aligned}$$

Combining those with Lemmas 6.9 and 6.10, we get the desired equation from (5.2) by letting $\delta \to 0$. For a solution of with general initial state $Y_0 \in D(\mathbb{R})$, we let $Y^\epsilon_t = Y_{\epsilon+t}$ for $t \geq 0$ and $0 < \epsilon \leq T$. By Theorem 3.7 we have $Y^\epsilon_0 \in D^1(\mathbb{R})$ almost surely. Then we can apply (1.10) to $T - \epsilon > 0$ and $(t, x) \mapsto Y^\epsilon_t(x)$ to get

$$\begin{aligned}Y_{T-t}(\xi^r_t + x) &= Y_\epsilon(\xi^r_T + x) + \sqrt{c}\int_{t-}^{(T-\epsilon)-}\int_0^\infty 1_{\{u \leq Y_{T-s}(\xi^r_s + x)\}} W^T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{(T-\epsilon)-}\int_0^\infty\int_0^\infty 1_{\{u \leq Y_{T-s}(\xi^r_s + x)\}} z\tilde{N}^T_0(\overleftarrow{ds}, dz, du) \\ &\quad - b\int_t^{T-\epsilon} Y_{T-s}(\xi^r_s + x)ds - \sigma\int_t^{T-\epsilon} \nabla Y_{T-s}(\xi^r_s + x)dB_s \\ &\quad - \int_t^{T-\epsilon}\int_{\mathbb{R}^\circ} [Y_{T-s}(\xi^r_s + x - z) - Y_{T-s}(\xi^r_s + x)]\tilde{M}(ds, dz). \qquad (6.14)\end{aligned}$$

By Proposition 3.8 we have

$$\mathbf{E}\{|Y_\epsilon(\xi^r_T + x) - Y_0(\xi^r_T + x)|\} = \int_{\mathbb{R}} \mathbf{E}\{|Y_\epsilon(y) - Y_0(y)|\} P^*_{T-r}(x, dy),$$

which tends to zero as $\epsilon \to 0$. Then by letting $\epsilon \to 0$ in (6.14) we obtain the desired equation. □

Theorem 6.12 *The pathwise uniqueness holds for cádlág $D(\mathbb{R})$-valued solutions of the stochastic equation* (1.6).

Proof. Suppose that $\{Y^1_t : t \geq 0\}$ and $\{Y^2_t : t \geq 0\}$ are two cádlág $D(\mathbb{R})$-valued solutions of (1.6) with $Y^1_0 = Y^2_0 \in D(\mathbb{R})$. By considering a conditional probability, we may assume that the initial state is deterministic. Since both $(t, x) \mapsto Y^1_{T-t}(\xi^r_t + x)$ and $(t, x) \mapsto Y^2_{T-t}(\xi^r_t + x)$ satisfy (1.10), by Theorem 2.3 for any $0 \leq r \leq t \leq T$ and $x \in \mathbb{R}$ we have $\mathbf{P}\{Y^1_{T-t}(\xi^r_t + x) = Y^2_{T-t}(\xi^r_t + x)\} = 1$. In particular, for any $0 \leq t \leq T$ and $x \in \mathbb{R}$ we have almost surely

$$Y^1_{T-t}(x) = Y^1_{T-t}(\xi^t_t + x) = Y^2_{T-t}(\xi^t_t + x) = Y^2_{T-t}(x).$$

Then the continuity of $x \mapsto Y^1_t(x)$ and $x \mapsto Y^2_t(x)$ imply $\mathbf{P}\{Y^1_t(x) = Y^2_t(x) \text{ for all } x \in \mathbb{R}\} = 1$ for every $t \geq 0$. It follows that $\langle Y^1_t, f\rangle = \langle Y^2_t, f\rangle$ almost surely for every $t \geq 0$ and $f \in \mathscr{S}(\mathbb{R})$. By the right-continuity of the processes we obtain $\mathbf{P}\{\langle Y^1_t, f\rangle = \langle Y^2_t, f\rangle \text{ for } t \geq 0\} = 1$ for every $f \in \mathscr{S}(\mathbb{R})$. Considering a suitable sequence $\{f_1, f_2, \cdots\} \subset \mathscr{S}(\mathbb{R})$ we can conclude $\mathbf{P}\{Y^1_t(x) = Y^2_t(x) \text{ for } t \geq 0 \text{ and } x \in \mathbb{R}\} = 1$. That gives the pathwise uniqueness for (1.6). □

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